Dynamical zeta function for tree maps

Mathieu Baillif [*]

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Abstract: We study piecewise monotone and piecewise continuous maps f from a rooted oriented tree to itself, with weight functions either piecewise constant or of bounded variation. We define kneading coordinates for such tree maps. We show that the Milnor-Thurston relation holds between the weighted reduced zeta function and the weighted kneading determinant of f. This generalizes a result known for piecewise monotone interval maps.

1 Introduction

Let $f : [0,1] \to [0,1]$ be strictly monotone on the subintervals defined by the *turning points* $0 = a_0 < a_1 < \ldots < a_N = 1$ (with N finite). Let $g : [0,1] \to \mathbb{C}$ be a weight. We denote by $f^{\circ n}$ the map $f \circ f \circ \ldots \circ f$ composed n times. If each iterate of f has finitely many fixed points, it is natural to define the *weighted Ruelle zeta function* :

$$\zeta_{f,g}(t) = \exp \sum_{n \geq 1} \frac{t^n}{n} \sum_{x : f^{\circ n} x = x} \prod_{k=0}^{n-1} g(f^{\circ k} x) \, .$$

If $g \equiv 1$ we simply have the *unweighted zeta function*

$$\zeta_f(t) = \exp \sum_{n \geq 1} \frac{t^n}{n} \# \{ x \, : \, f^{\circ n} x = x \}$$

which is the "classical" zeta function of Artin-Mazur (see [1]).
For more general situations we need to introduce a *reduced* zeta function — cf Sections 2 and 3 below,

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[*] Université de Genève, Section de Mathématiques, 2-4 rue du Lièvre, CP 240, CH-1211 Genève 24, Switzerland. *e-mail:* Mathieu.Baillif@math.unige.ch

and analogously [5], [6]. In the early seventies, Milnor and Thurston were able to relate a reduced unweighted zeta function $\zeta_f(t)$ (in the case where f is continuous in $[0,1]$ and strictly monotone on finitely many subintervals of $[0,1]$) to the determinant $\Delta_f(t)$ of a finite *kneading matrix* whose coefficients are power series in t. In their reduced zeta function, Milnor and Thurston considered only the fixed points of negative type, i.e. the fixed points x where $f^{\circ n}$ is decreasing throughout a neighborhood of x, but they "counted" them two times, "since between any two fixed points of negative type there must be at least one other fixed point." The power series are obtained by looking at the orbits of the righthand and lefthand limits of the turning points (the endpoints of the subintervals cited above) under the iterates of f. The relation between ζ_f and Δ_f, which we will call *the Milnor-Thurston relation* is

$$\zeta_f(t)\Delta_f(t) = 1 + p(t) \tag{1}$$

with $p(t)$ a polynomial correcting factor (See [11]).

Baladi and Ruelle [5] later considered f piecewise continuous functions strictly monotone in the subintervals defined by the turning points, extended the Milnor-Thurston kneading matrix and the reduced zeta function to a weighted case, and they showed that the Milnor-Thurston relation was still valid, *under the assumption that the weight g be constant on each monotonicity interval* — i.e., g locally constant.

Baladi [2] then showed that (1) still holds when g is of bounded variation (plus some technical assumptions), by considering infinite (countable) matrices. More precisely, she built finite matrices with determinant $\Delta_{f,g^{(n)}}$ and zeta functions $\zeta_{f,g^{(n)}}$ corresponding to increasingly better approximations $g^{(n)}$ of g and showed that these two sequences converge as analytic — respectively meromorphic — functions in a disk, to $\Delta_{f,g}$ and $\zeta_{f,g}$ for which the formula is thus verified.

For further results, see also [4], [6], [12]. The aim of this present paper is to extend the results of [5] and [2] to functions $f : T \to T$, where T is a rooted oriented tree with finitely many edges. Sections 2 and 3 are devoted to the case of g being piecewise constant. To define the zeta function and the kneading matrix of f, we define the *address* of a point x in the tree T, following the strategy of [11]. We give here two possible definitions (there are certainly others), which each have advantadges and disadvantages : the first definition is quite elegant and natural but leads to a correcting factor $p(t)$ in the formula (1). This factor is not present when we take our second definition of the address, which is however more artificial. In the case where T is homeomorphic to the interval, both definitions reduce to the address, and thus the determinant of [2] [5].

Our main result is the Milnor-Thurston relation (1) for a map f of a rooted, oriented tree and a locally constant weight g.

In Section 4 we briefly describe how to adapt [2] to extend to weights g of bounded variation.

Generalising from interval to trees could have applications to the study of two-dimensional dynamics, as we now explain. In [9], Cvitanović, Gunaratne, and Procaccia introduced the concept of *pruning fronts* and conjectured that, suitably pruning Smale's horseshoe map yields to a complete topological description of the dynamics of the Hénon family — this they called the *Pruning Front Conjecture.* De Carvalho [7] later formalized this, giving a mathematical definition of pruning front

and constructing a family $\mathcal{P}$ of plane homeomorphisms which, conjecturally, contains the Hénon family up to semi-conjugacy. De Carvalho also shows [8] that the family P is in 1-1 correspondence with a family of tree and dendrite endomorphisms. Being able to compute zeta functions for tree endomorphisms is a first step towards computing the zeta functions of the homeomorphisms in de Carvalho's family $\mathcal{P}$ and, if the Pruning Front Conjecture is verified, of all maps in the Hénon family.
A further natural step will consist on extending our results to infinite trees, which are involved in the de Carvalho's family $\mathcal{P}$.

The author wishes to emphasize that once the address is defined one proceeds very much along the lines of and [2] [5]. Many thanks to Pierre de la Harpe, Robyn Curtis and David Cimasoni for their remarks, and especially to Viviane Baladi for her help and patience and to André de Carvalho for his comments.

2 Definitions and statement of results

2.1 The tree T

Let T be a finite tree, $V(T)$ its set of vertices and $E(T)$ its set of edges. Recalling that the valence of $a \in V(T)$ is the number of edges incident to a, we define $V_1(T) := \{a \in V(T) : val(a) = 1\}$. We fix an arbitrary $a_1 \in V_1(T)$ and call it the root of T. We consider T "embedded in the plane", each edge being a line segment and the topology on T being the induced topology. We view the edges of T as closed (i.e. the endpoints of an edge also belong to this edge). Let $x, y \in T$, $x \neq y$. We say that x is *younger* than y, which is thus *older* than x, if x is in the convex hull in T of a_1 and y. We orientate each edge from the youngest point to the oldest. From now on, we fix this orientation.
We say that an order '$\prec$' on T (as a set) is *compatible with the orientation* if $\forall x, y \in T$, $x \prec y \Rightarrow x$ is younger than y. Let us define such a compatible order : For this, we will describe a walk (with some jumps) through T and say that $x \prec y$ $(x, y \in T)$ if we visit x before visiting y. We start from a_1, follow the orientation, choosing to visit the leftmost edge each time that we have the choice. After a finite time, we will arrive at a vertex a_s of valence 1. Let us call a_{s-1} the vertex we visited immediately before a_s, and I_s the leftmost edge incident to a_{s-1} which we have not visited. We *jump* to I_s, follow again the orientation, and proceed by the same principle until we have visited every vertex. Writing $N := \#V(T)$, we denote the vertices of T by $a_1, \ldots, a_N$ and its edges by $I_1, \ldots, I_{N-1}$, so that $i < j \Leftrightarrow a_i \prec a_j$, and similarly for the I_j. See Figure 1.
There are other natural ways to define an order on T compatible with orientation which lead to the same results, but we will only consider this one to avoid unnecessary complication.

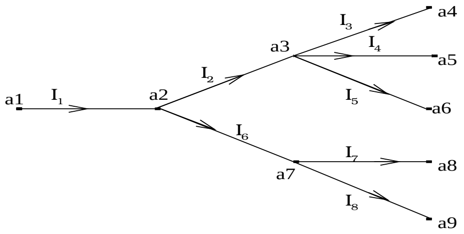

Figure 1: Order on a tree T

2.2 The map f and the weight g

Let T be a tree as in 2.1 and let $I_i \in E(T)$. We say that $h : I_i \to T$ is strictly increasing — respectively decreasing — if $\forall x, y \in I_i$ with $x \prec y$, we have $h(x) \prec h(y)$ — respectively $h(y) \prec h(x)$. If h is strictly monotone (i.e. strictly increasing or strictly decreasing) for our order, then it is strictly monotone for any order compatible with the orientation, hence monotonicity depends only on the orientation of the tree.
Let $f_i : I_i \to T$, $i = 1, \ldots, N-1$ be continuous, strictly monotone maps. By abuse of notation, we denote by f the multivalued transformation whose graph in $T \times T$ is the union of the graphs of the f_i. We let $\epsilon_i = \pm 1$ depending on whether f_i is increasing or decreasing. Let $g_i \in \mathbb{C}$, $i = 1, \ldots, N-1$ and define the functions ϵ, g such that they have the constant value ϵ_i, g_i on the interior of I_i and value 0 on the a_j's.
Let $a_k \in V(T)$, $k \neq 1$. We denote by a_k- the lefthand limit of a_k in the following sense :

$$h(a_k-) = \lim_{x \to a_k, x \prec a_k} h(x) \text{ for } h : I_{k-1} \to T.$$

If $a_k \notin V_1(T)$ or $a_k = a_1$, then it has righthand limit(s). We denote them by $a_k+_1, a_k+_2, \ldots, a_k+_{val(a_k)-1}$, turning clockwise around a_k (see Figure 2).
We will need to define the notion of a differentiable tree map. We say that a multivalued fonction $u : T \to T$, $u = (u_1, \cdots, u_{N-1})$ with $u_i : I_i \to T$ is differentiable if :
•Each u_i is piecewise differentiable, and the derivative of u at $x \in I_i$ is the derivative of u_i at x (when it exists),
•$\{x \in T |\ u \text{ is not differentiable at } x\} = V(T) \cup u^{-1}(V(T))$, and is therefore finite,
•all the righthand and lefthand limits of the derivative of u at x exist $\forall x \in T \backslash V_1(T)$. (Note that the vertices can have more than one righthand limits).
Note : The above conditions are the best possible, since the tree T is not $\mathcal{C}^\infty$-embedded in the plane. Hence, it makes sense to talk about differentiability only in the interior of the edges (which are line segments).

We will sometimes need to use another notation for the edges, the weights g_i and the signs ϵ_i : We will denote by $I_{(k,l)}$ the edge "containing" a_k+_l, and by $g_{(k,l)}$ and $\epsilon_{(k,l)}$ the corresponding g_i and ϵ_i. (This notation is needed especially for Lemma 3.1.) Just keep in mind that the letters i, j are used

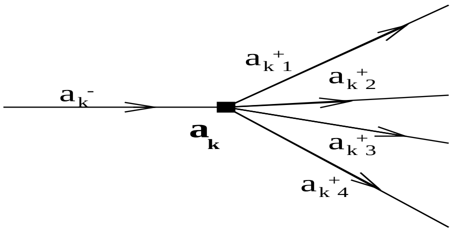

Figure 2: The limits of a_k

in the first notation and the letters k, l, m in the second one. Note that for $k_1 < k_2$, or $k_1 = k_2$ and $l < m$, if $x \in I_{(k_1,l)}, y \in I_{(k_2,m)}$ we have $x \prec y$.

2.3 The addresses $\alpha^{(r)}(x)$

We will now define the *address* of $x \in T$, which will be a vector. As mentioned before, there are several ways to proceed. Here we give two possibilities. It is important to notice that they are completely independent, and the reader might believe that we would need to prove each result involving the concept of address twice. But in fact, as we will see later, the differences between the first and second address play a role only in the proof of Lemma 3.1. The reader can therefore skip one of the definitions and proceed straight the proofs.

The first address $\alpha^{(1)}(x)$

To define the *first address* of $x \in T$, we will first add 'ghost' edges and vertices to T in the following manner : One glues a new edge I_0 to a_1 (the root) joining a new vertice a_0 and another new edge I_N to a_N (the maximal point of our tree for the order "$\prec$") joining the new vertice a_{N+1}. We orientate the two new edges from a_0 to a_1 and from a_N to a_{N+1} respectively. The order "$\prec$" on the new edges is given by the orientation. We then have a new rooted and oriented tree $T^{(1)}$ with root a_0. To extend our functions f, g, ϵ to $T^{(1)}$ we put ϵ_0, g_0 and ϵ_N, g_N equal to zero, and $f_0 : I_0 \to T$ and $f_N : I_N \to T$ arbitrary (we shall see that these functions will be cancelled in the zeta function and the kneading matrix).
The first address $\alpha^{(1)}(x) \in \{0, 1, -1\}^N$ of $x \in T$ is defined by :

$$\alpha_i^{(1)}(x) = \begin{cases} 1 & \text{if } x \succ a_i, \\ 0 & \text{if } x = a_i, \quad ; i = 1, \ldots, N \\ -1 & \text{if } x \prec a_i. \end{cases}$$

We put $N^{(1)} := N$.

The second address $\alpha^{(2)}(x)$

To define the *second address* of $x \in T$ we will also need to 'haunt' our tree with some ghosts edges and vertices, but in a slightly different way : To each vertex $a \in V_1(T) \backslash \{a_1\}$ we attach one new ghost edge joining a new ghost point (we do not need to name them). We orientate this new edge from a to the new point. We do the same with a_1, except that we orient the edge from the new point (the new root of our tree, which will be called a_0) to a_1. The new tree obtained will be christened $T^{(2)}$.

Now, each original vertex in $V(T)$ has a valence in $T^{(2)}$ at least two. The second address of $x \in T$ will be a vector in $\{0, 1, -1\}^{N^{(2)}}$. To each $a \in V(T)$ we associate exactly $val_{T^{(2)}}(a) - 1$ coefficients in $\alpha^{(2)}(x)$, each coefficient corresponding to one of the righthand limits of a. Hence, $\alpha^{(2)}(x)$ is made of N 'blocks of coefficients'. We will note the coefficient corresponding to a_k+_l by $\alpha^{(2)}_{k,l}$.

The second address $\alpha^{(2)}(x) \in Z^{N^{(2)}}$ of $x \in T \backslash V(T)$ is defined by :

$$\alpha^{(2)}_{k,l}(x) = \begin{cases} 0 & \text{if } x = a_k, \\ 1 & \text{if the shortest path from } a_0 \text{ to } x \text{ passes through } a_k \text{ and exits in } I_{k,l}, \\ -1 & \text{if the shortest path from } a_0 \text{ to } x \text{ does not pass through } a_k, \\ & \text{or if it does not pass through } I_{k,l}. \end{cases}$$

Lemma 2.1:

$$N^{(2)} = \sum_{a \in V(T)} (val_{T^{(2)}}(a) - 1) \;=\; N + \#V_1(T) - 2$$

We leave the proof to the reader.

Note:

•If T is homeomorphic to an interval, both the first and second address coincide with the weighted address defined in [5], which was adapted from the unweighted address implicit in [11].

•The idea of adding ghost edges comes from [2]. One could manage without it, but in this case the correcting factor in (1) would be more complex (see [5]). On the other hand, using ghost edges means that we have to work with kneading matrices of higher dimension.

2.4 The invariant coordinate and the kneading determinant

For $r = 1$ or 2 the *invariant coordinate* of $x \in T \backslash V(T)$ is the $N^{(r)}$-tuple of formal power series in t

$$\theta^{(r)}(x) = \theta^{(r)}_f(g)(x)(t) = \sum_{n=0}^{\infty} t^n \left(\prod_{k=0}^{n-1} (\epsilon g)(f^{\circ k} x)\right) \alpha^{(r)}(f^{\circ n} x)$$

Note that θ is single-valued because if $f^{\circ k} x \in \{a_1, \ldots, a_N\}$ for some
$k < n$ then $(\epsilon g)(f^{\circ k} x) = 0$. For each $a_k \in V(T)$ so that $val_{T^{(r)}}(a_k) \geq 2$ and $l = 1, \ldots, val_{T^{(r)}}(a_k) - 1$ we let :

$$K^{(r)}_{k,l}(t) = \frac{1}{2}[\theta^{(r)}(a_k+_l) - \theta^{(r)}(a_k-)] = (K^{(r)}_{(k,l),1}(t), \ldots, K^{(r)}_{(k,l),N^{(r)}}(t))$$

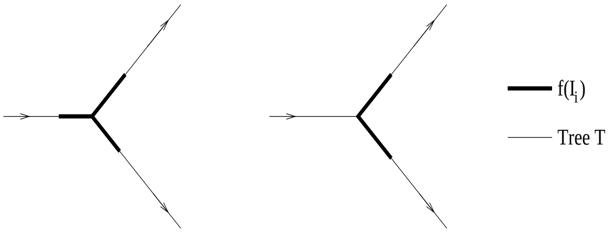

Figure 3: Forbidden cases for piecewise monotone and continuous f

We let $v^{(r)}(a_k) = val_{T^{(r)}}(a_k) - 1$ and then build the *kneading matrix* :

$$K^{(r)}(t) := \begin{pmatrix} K^{(r)}_{1,1}(t) \\ K^{(r)}_{2,1}(t) \\ \vdots \\ K^{(r)}_{2,v^{(r)}(a_2)}(t) \\ K_{3,1} \\ \vdots \\ K_{N,1} \end{pmatrix}$$

Lemma 2.2: *The matrix $K^{(r)}(t)$ is square.*
We again leave the proof to the reader.

So, let us define the *kneading determinant* to be the determinant of $K^{(r)}(t)$:

$$\Delta^{(r)}_{f,g}(t) := \det K^{(r)}(t)$$

Note: If T is homeomorphic to the interval then the kneading determinant coincides with that of [5]. Moreover, if $g \equiv 1$ the kneading determinant is a slight adaptation of [11] .

2.5 The reduced zeta function $\zeta_{f,g}(t)$

For $m \geq 1$ we denote by Fix$f^{\circ m}$ the set of fixed points of $f^{\circ m}$ (the multivalued map $f \circ f \circ \ldots \circ f$ composed m times) which have an (multi)orbit disjoint from $\{a_1, \ldots, a_N\}$ (note that for such points, $f^{\circ n}$ is single-valued for each n). We assume for the moment that for each $m \geq 1$ the set Fix$f^{\circ m}$ is finite. This assumption will be removed after Lemma 3.2.
Let $x \in$ Fix$f^{\circ m}$, $x \in I_i$. The conditions of monotonicity and continuity of each f_i imply that $f_i(I_i) \subset T_i$ with T_i homeomorphic to an interval and containing a_1. (This follows from the impossibility of cases such as those shown in Figure 3). We call T_i *the javelin of* f_i. Hence, $f^{\circ m}$ *near* x can be regarded as a map of the interval and we can set (following [5])

$$L(x, f^{\circ m}) = 0 \text{ if the graph of } f^{\circ m} \text{ doesn't cross the diagonal at } x.$$

If the graph does cross the diagonal at x, we can define

$$L(x, f^{\circ m}) = \lim_{y \to x} \frac{\mathrm{sgn}(f^m(y) - y)}{\mathrm{sgn}(x - y)} ,$$

where $\mathrm{sgn}(x)$ is the sign of x. We also set

$$\nu(x, f^{\circ m}) = -L(x, f^{\circ m}) \cdot \prod_{k=0}^{m-1} \epsilon(f^{\circ k} x) \, \in \{0, 1, -1\}$$

Let $x \in \mathrm{Fix}\,(f^{\circ m})$. We say that x is attractive — respectively repulsive – if there exists a neigbourhood U of x such that $\forall V \subset U, x \in V$ we have $f^{\circ m}(V) \subset V$ — respectively $f^{\circ m}(V) \supset V$.

Lemma 2.3:
a) If $f^{\circ m}$ *is decreasing at* $x \in$*Fix*$f^{\circ m}$*, then the assumption that* $\#$*Fix*$f^{\circ m} < \infty$ *for all* m *implies that* x *is either an attracting or repelling fixed point for* $f^{\circ m}$*.*
b) $\nu(x, f^{\circ m}) = -1 \iff f^{\circ m}$ *is increasing and attracting at the fixed point* $f^{\circ m}(x) = x$*.*

The proof of this lemma is left to the reader.

We extend now the set $\mathrm{Fix} f^{\circ m}$ to a set $\mathrm{Fix}{\star} f^{\circ m}$ obtained by adding some symbols $x*$ where $x \in T$ and $*$ is $+$, $-$, or $+_j$ for some j if $x \in V(T)$:

$$\mathrm{Fix} \star f^{\circ m} = \mathrm{Fix} f^{\circ m} \bigcup \left\{ x* \;\middle|\; \begin{matrix} f^{\circ m}(x*) = x \ , \ f^{\circ n}(x*) = a_k \text{ for some } n, k \ , \\ \text{and } \prod_{s=0}^{m-1} \epsilon(f^{\circ s}(x*)) = +1 \end{matrix} \right\}$$

If $x* \in \mathrm{Fix} \star f^{\circ m} \backslash \mathrm{Fix} f^{\circ m}$, we let

$$L(x*, f^{\circ m}) = \begin{cases} 0 & \text{if } x* \text{ is (one-sided) repelling,} \\ 1 & \text{if } x* \text{ is (one-sided) attracting,} \end{cases}$$

and

$$\nu(x*, f^{\circ m}) = -L(x*, f^{\circ m}) \in \{0, 1, -1\}.$$

Finally, we define the *"reduced" zeta function*

$$\zeta_{f,g}(t) = \zeta_f(g)(x)(t) = \exp \sum_{m=1}^{\infty} \frac{t^m}{m} \sum_{\xi \in \mathrm{Fix}{\star} f^{\circ m}} \nu(\xi, f^{\circ m}) \cdot \prod_{k=0}^{m-1} g(f^{\circ k} \xi) \, \in \mathbb{C}[[t]].$$

If $f^{\circ m}(a_i \mp_k) \neq a_i$ for all i and all $m \geq 1$, we may replace $\mathrm{Fix} \star f^{\circ m}$ by $\mathrm{Fix} f^{\circ m}$ in the above formula.

Denote by Per the set of periodic orbits of f which do not contain any of the a_i, and by Per$\star$ the extended set of periodic orbits formed of elements of $\cup_m \mathrm{Fix}{\star} f^{\circ m}$.

If $\gamma \in \mathrm{Per}\star$ is of period $p(\gamma)$ (i.e., $p(\gamma) = \min\{m \geq 1 \,:\, \gamma \subset \mathrm{Fix}\star f^{\circ m}\}$) and contains the element ξ_γ we let

$$Z(\gamma) = \prod_{k=0}^{p(\gamma)-1} g(f^{\circ k}\xi_\gamma) \ \in \mathbb{C}$$

With this notation, it is not difficult to verify that we have

$$\zeta_{f,g}(t) = \prod_{\gamma \in \mathrm{Per}\star} F(\gamma)$$

where

$$F(\gamma) = \exp \sum_{m=1}^{\infty} \frac{t^m}{m} \nu(\xi_\gamma, f^{\circ m \cdot p(\gamma)}) Z(\gamma)^m \ \in \mathbb{C}[[t]]$$

Let $\gamma \in \mathrm{Per}$, with $p(\gamma) = p$, and $x_\gamma \in \gamma$, then
(a) $F(\gamma) = 1$ if the graph of $f^{\circ p}$ does not cross the diagonal at x_γ;
(b) $F(\gamma) = \frac{1}{1-Z(\gamma)}$ if $f^{\circ p}$ is increasing at x_γ and γ repelling;
(c) $F(\gamma) = 1 - Z(\gamma)$ if $f^{\circ p}$ is increasing at x_γ and γ attracting;
(d) $F(\gamma) = 1 + Z(\gamma)$ if $f^{\circ p}$ is decreasing at x_γ and γ attracting;
(e) $F(\gamma) = \frac{1}{1-Z(\gamma)}$ if $f^{\circ p}$ is decreasing at x_γ and γ repelling.
Let $\gamma \in \mathrm{Per}\star \setminus \mathrm{Per}$, then
(f) $F(\gamma) = 1$ if γ is one-sided repelling;
(g) $F(\gamma) = 1 - Z(\gamma)$ if γ is one-sided attracting.

Notice that the cases **(a)** and **(d)** where γ is repelling give the same expression for $F(\gamma)$. If all periodic points are repelling, and if $f^{\circ m} a_i{*} \neq a_i$, for all $m \geq 1$ and $0 \leq i \leq N$, then we recover the usual (weighted) dynamical zeta function of the tree map f.

2.6 Milnor-Thurston relation for tree maps

We refer to Lemma 3.2 below for a definition of $\zeta_{f,g}(t)$ where $f^{\circ m}$ can have infinitely many fixed points.

Theorem 1: *With the above notations and assumptions we have :*

$$a) \qquad \zeta_{f,g}(t)\Delta^{(1)}_{f,g}(t) = 1, \tag{2}$$

$$b) \quad \zeta_{f,g}(t)\Delta^{(2)}_{f,g}(t) = 1 + r_g(T), \tag{3}$$

where

$$r_g(T) = \frac{1}{2} \sum_{a_n \in V_{\geq 2}(T)} (val(a_n) - 2)\epsilon_{n-1} g_{n-1},$$

with $V_{\geq 2}(T) = \{a_n | val(a_n) \geq 2\}$.

3 Proofs

3.1 Checking the Milnor-Thurston formula on a particular case

We use the homotopy argument from [11] modified in [5]. First, let us check the formula on a 'simple' example :

Lemma 3.1: *Let* $f^0 = (f_1^0, \dots, f_{N-1}^0)$ *be a piecewise continuous and piecewise monotone multivalued tree map as before. Suppose further that* $f_i^0(I_i)$ *is contained within interior of* I_1*, for* $i = 1, \dots, N$ *and that* $f_1^0 : I_1 \to T$ *is differentiable with* $|(f_1^0)'| < 1$*. Then*

$$\begin{aligned} a) \qquad \zeta_{f^0,g}(t)\Delta_{f,g}^{(1)}(t) &= 1, \\ b) \quad \zeta_{f^0,g}(t)\Delta_{f,g}^{(2)}(t) &= 1 + r_g(T), \end{aligned}$$

where $r_g(T)$ *is defined in Theorem 1.*

Proof: Our assumptions imply that the set of periodic points of f^0 consists of a single fixed point x which is attractive (the orbit of each point in I_1 converges to x). The fixed point x is either of type **(b)** (if $\epsilon_1 = +1$) or of type **(c)** (if $\epsilon_1 = -1$). Therefore, in either case we have

$$\zeta_{f^0,g}(t) = 1 - \epsilon_1 g_1 t.$$

Let us check a) first, i.e. the formula for the first address.

With the first address and the second notation of Section 2.2, we have :

$$\theta(a_k+_l) = (\dots, +1, \dots, +1, -1, \dots) + \epsilon_{(k,l)} g_{(k,l)} t \cdot \sum_{n=1}^{\infty} (\epsilon_1 g_1)^{n-1} t^{n-1} \cdot (1, -1, \dots, -1),$$

and (with the first notation)

$$\theta(a_i-) = (\dots, +1, -1, \dots, -1, \dots) + \epsilon_{i-1} g_{i-1} t \cdot \sum_{n=1}^{\infty} (\epsilon_1 g_1)^{n-1} t^{n-1} \cdot (1, -1, \dots, -1),$$

which is our first collapse of notation. Note that we do not have $(-1, -1, \dots, -1)$ in the sum because of the ghost edge that we added to the left of a_1. Hence, if $i = k$ we have :

$$K_{k,l}(t) = (0, \dots, 0, 1, \dots, 1, 0, \dots, 0) + \frac{1}{2}(\epsilon_{(k,l)} g_{(k,l)} - \epsilon_i g_i) \cdot \frac{t}{1 - \epsilon_1 g_1 t} \cdot (1, -1, \dots, -1).$$

The number of 1's in the constant term of $K_{k,l}$ is exactly the number of vertices between a_k- and a_k+_l according to the order "$\prec$". (See Figures 1 and 2). In other words, if a_k+_l enters edge I_j, then this number is just $(j - k + 1)$.

Our method will be as follows :

Since we are working with the determinant of the matrix $K_{(k,l),m}$, we are permitted to permute lines and to add or substract lines (rows) to other lines (rows). We will use this in order to obtain a new matrix $K'(t)$ with $K'(t) = 1 - \Psi(t)$, where $\Psi(t)$ will be of rank one. Hence

$$\Delta_{f^0,g}(t) = \det(1 - \Psi(t)) = \sum_{n=0}^{N-1} (-1)^n \mathrm{Tr} \wedge^n \Psi(t) = 1 - \mathrm{Tr}\Psi(t)$$

We will call *jump* of $a_k +_l$ the number of 1's in the constant term of $K_{k,l}$. Note that the jump of $a_k +_1$ is always 1. We now proceed inductively to construct $\Psi(t)$.

Let us call 'first branch' the subtree of T obtained by walking from a_0 to a_s passing only through the $a_k +_1$, and stopping at the first vertex a_s of valence one. For the rows associated to the vertices of the first branch, except a_s, the constant term is exactly what we require, and the diagonal term of Ψ is $-\frac{1}{2}(\epsilon_i g_i - (\pm\epsilon_{i-1} g_{i-1})) \cdot t \cdot \frac{1}{1-\epsilon_1 g_1 t}$, for $i = 1, \ldots, s-1$. The sign $\pm$ is always $+1$ except in the first column, i.e. except for a_1. If we sum those diagonal terms, they will all vanish, except $-\epsilon_1 g_1$ because of the sign, and $-\frac{1}{2}\epsilon_{s-1} g_{s-1}$ because it appears only once in the sum. Hence, the 'intermediate trace' of Ψ up to $K_{s-1,1}$ is $(-\epsilon_1 g_1 - \frac{1}{2}\epsilon_{s-1} g_{s-1}) \cdot t \cdot \frac{1}{1-\epsilon_1 g_1 t}$
Let us now consider the next edge I after a_s (according to our order). $I = I_{r,2}$ for some $1 < r < s$. The jump of $a_r +_2$ is $s - r$, because the number of vertices between a_r- and $a_r +_2$ is exactly $s - r$, so there are exactly $s - r$ 1's in the constant term of $\Psi_{r,2}$. The diagonal term of $\Psi_{r,2}$ is $\frac{1}{2}(\epsilon_{r,2} g_{r,2} - \epsilon_{r-1} g_{r-1}) \cdot t \cdot \frac{1}{1-\epsilon_1 g_1 t}$. Notice that $\epsilon_{r,2} g_{r,2} = \epsilon_s g_s$ in our second notation.
In order to obtain $(0, \ldots, 0, \ldots, 0, 1, 0, \ldots)$ from

$$(0, \ldots, 0, \overbrace{1, \ldots, 1}^{s-r}, 0, \ldots, 0),$$

we will subtract from $K_{r,2}$ all of the K_v for $v = r+1, \ldots, s-1$. The new diagonal term of $K' - 1$ is

$$\begin{aligned} & -\tfrac{1}{2}t \cdot ((\epsilon_s g_s - \epsilon_{r-1} g_{r-1}) - \epsilon_{s-1} g_{s-1} + \epsilon_{s-2} g_{s-2} - + \ldots - \epsilon_r g_r + \epsilon_{r-1} g_{r-1}) \\ = \; & \tfrac{1}{2}(\epsilon_{s-1} g_{s-1} - \epsilon_s g_s) \cdot t. \end{aligned}$$

So, the 'intermediate trace' up to $K_{r,2}$ is $(-\epsilon_1 g_1 - \frac{1}{2}\epsilon_s g_s) \cdot t \cdot \frac{1}{1-\epsilon_1 g_1 t}$.
Using the same trick (i.e. subtracting the $K_{u,v}$ from $K_{k,l}$ for u, v such that $a_u +_v$ is 'between' $a_k +_l$ and a_k-), we arrive to the last vertice a_N, and the intermediate trace up to a_{N-1} is $(-\epsilon_1 g_1 - \frac{1}{2}\epsilon_{N-1} g_{N-1}) \cdot t \cdot \frac{1}{1-\epsilon_1 g_1 t}$
Because we added a ghost edge to the right of a_N, the term $-\frac{1}{2}\epsilon_{N-1} g_{N-1}$ vanishes when we sum up to N and finally we obtain :

$$\Delta_{f^0,g}(t) = 1 - \mathrm{Tr}\Psi(t) = 1 + \frac{\epsilon_1 g_1 t}{1 - \epsilon_1 g_1 t} = \frac{1}{1 - \epsilon_1 g_1 t}.$$

Now, let us consider b), i.e., the second address.

We use exactly the same method : We consider the trace of a matrix $\Psi(t)$ with $K(t) = 1 - \Psi(t)$. With the second address, and $k = i$ we have

$$K_{k,l}(t) = (0, \dots, 0, 1, 0, \dots, 0) + \frac{1}{2}(\epsilon_{k,l} g_{k,l} - \epsilon_i g_i) \cdot \frac{1}{1 - \epsilon_1 g_1} \cdot (1, -1, \dots, -1).$$

We do not need to add or substract lines as in a), because the constant term is already what we require. Hence,

$$\Psi_{k,l}(t) = \frac{1}{2}(\epsilon_{k,l} g_{k,l} - \epsilon_i g_i) \cdot \frac{1}{1 - \epsilon_1 g_1} \cdot (1, -1, \dots, -1).$$

The correcting factor $1 + r_g(T)$ appears because $\epsilon_i g_i$ will be counted negatively $\mathrm{val}(a_i) - 2$ times in the trace of $\Psi(t)$. (Look at Figures 1 and 2, more details can only be confusing). □

Note : The reader has probably seen that in the proof of a) we reproduce exactly the walk that we used to define the order on T. Hence, the argument would be the same if we take any other order compatible with the orientation and which can be produced by a walk of the same type, to define the first address. Such orders are those for which, for a_k a vertex of valence greater than 1, the points "just greater than a_k" must be on an edge incident to a_k.

3.2 Technical lemmas

The method to prove the theorem is very simple, let us see first the simpler case where T is an interval. Let f be a piecewise monotone transformation of this interval. It is possible to choose a function f^0 as in Lemma 3.1 such that

$$f^\lambda := (1 - \lambda) f^0 + \lambda f$$

is piecewise strictly monotone for all $\lambda \in [0, 1]$. We already proved the Theorem 1 for f^0. The key of the theorem is to show that when we push f^0 to f through f^λ, (2) and (3) of Theorem 1 still hold. (See [5] and [11]). Now, let us look at an arbitrary tree T. Recalling that the javelin T_i of f_i is a subtree of T homeomorphic to the interval and containing $f_i(I_i) \cup \{a_1\}$, we shall proceed exactly as in [5] but considering the f_i's separately, and 'pushing' each f_i through its javelin. From now on, the differences between first and second address will not be of any importance, so we will simply ignore them and speak of *the* kneading matrix, *the* kneading determinant, etc. To simplify notation, and when it is not misleading, we will write Δ_f, ζ_f rather than $\Delta_{f,g}, \zeta_{f,g}$, throughout the rest of this section.

It is convenient to use the C^1 topology (for the definition of differentiability given in 2.2). For $r = 0$ or 1, let $P^r = P^r(N, (a_i), (\epsilon_i))$ denote the set of N-tuples $f = (f_1, \dots, f_N)$ where each $f_i : I_i \to T$ is C^r and strictly monotone increasing or monotone decreasing according to whether ϵ_i equals $+1$ or -1. If $r = 1$ we further impose that $f_i'(a_k) = 0$ for a_k the endpoints of f_i (vanishing of the lefthand

and righthand limit of the derivative at interval endpoints). The d_r distance between f and g in P^r is given by the sum of the C^r distances between the f_i's and g_i's. As already mentioned, we view f and its iterates $f^{\circ m}$ as multivalued maps $T \to T$. For the definition of ζ_f, we require that the sets $\mathrm{Fix}\, f^{\circ m}$ be finite, but this condition after Lemma 3.2. Let P^r_M consist of those $f \in P^r$ such that $f^{\circ m}(a_i *) \neq a_i$ whenever $1 \leq m \leq M$, $0 \leq i \leq N$ and $* = \pm$ (or $+_j$).

Lemma 3.2: *P^r_M is an open subset of P^r. If J_{M+1} is the ideal of elements of order $\geq M+1$ in$\mathbb{C}[[t]]$, i.e., the ideal generated by polynomials of degree $M+1$, the map*

$$f \mapsto \zeta_f \quad (mod J_{M+1})$$

defined on the set $\{f \in P^0_M \, : \, \# Fix\, f^{\circ m} < \infty \, , \, \forall m \geq 1\}$ extends to a locally constant map

$$P^0_M \to \mathbb{C}[[t]]/(J_{M+1} \cap \mathbb{C}[[t]])\,.$$

($J_{M+1} \cap \mathbb{C}[[t]]$ is the ideal generated by polynomials of degree $M+1$ in$\mathbb{C}[[t]]$.)

Proof: P^r_M is defined by a finite number of open conditions, hence is open in P^r. For general $f \in P^0$, we may define $L(f_{\ell_m} \circ \cdots \circ f_{\ell_1})$ to be:

- 0 if $f_{\ell_m} \circ \cdots \circ f_{\ell_1}$ has no fixed points,
- 0 if the domain of $f_{\ell_m} \circ \cdots \circ f_{\ell_1}$ is empty or reduced to a point,
- -1 if the left end of the graph of $f_{\ell_m} \circ \cdots \circ f_{\ell_1}$ is $<$ the diagonal and the right end $>$ the diagonal,
- $+1$ if $f_{\ell_m} \circ \cdots \circ f_{\ell_1}$ is increasing and the left end of the graph is $\geq$ the diagonal and the right end is $\leq$ the diagonal,
- $+1$ if $f_{\ell_m} \circ \cdots \circ f_{\ell_1}$ is decreasing and the left end of the graph is $>$ the diagonal and the right end is $<$ the diagonal,
- 0 in all other cases.

Note that if $f \in P^0_M$ then neither the left nor the right end of the graph of $f_{\ell_m} \circ \cdots \circ f_{\ell_1}$ can intersect the diagonal for $1 \leq m \leq M$.
When $\mathrm{Fix}\, f^{\circ m}$ is finite we have thus

$$\begin{aligned} \sum_{\xi \in \mathrm{Fix}_\star f^{\circ m}} \nu(\xi, f^{\circ m}) &= \sum_{\ell_1, \ldots, \ell_m} \sum_{\xi \in \mathrm{Fix}_\star f_{\ell_m} \circ \cdots \circ f_{\ell_1}} \nu(\xi, f_{\ell_m} \circ \cdots \circ f_{\ell_1}) \\ &= -\sum_{\ell_1, \ldots, \ell_m} (\epsilon_{\ell_1} \cdots \epsilon_{\ell_m}) L(f_{\ell_m} \circ \cdots \circ f_{\ell_1}) \qquad (4) \end{aligned}$$

We leave to the reader the details of how terms do vanish in (4). (Notice that one of the reasons for which we had to reduce ζ lies in this computation.) Note that the $L(f_{\ell_m} \circ \cdots \circ f_{\ell_1})$ for $1 \leq m \leq M$ are locally constant on P^0_M, and that

$$\{f \in P^0_M \, : \, \mathrm{Fix} f^{\circ m} \text{ is finite for all } m \geq 1\}$$

is dense in P^0_M [approximation by N-tuples of non-affine polynomials]. Finally, note that the map $\xi \mapsto \prod_{k=0}^{m-1} g(f^{\circ k}\xi)$ is constant on $\mathrm{Fix}\star f_{\ell_m}\circ\cdots\circ f_{\ell_1}$. Therefore the lemma results from the definition of ζ_f. □

Formula (4) yields a natural definition of $\zeta_{f,g} \pmod{J_{M+1}}$ for general $f \in P^0$ — i.e., without assuming that $\mathrm{Fix} f^{\circ n}$ be finite for all n. We shall use this definition to prove Theorem 1. (Note that our definition of Δ did not use the condition that the sets $\mathrm{Fix}\, f^{\circ m}$ be finite.)

Lemma 3.3 : *Let $M \geq 1$ and $\tilde{f}$ be a sufficiently small perturbation of f in P^0 such that the "shrinking condition"*

$$\text{range } \tilde{f}_{\ell_m}\circ\cdots\circ\tilde{f}_{\ell_1} \subset \text{range } f_{\ell_m}\circ\cdots\circ f_{\ell_1}$$

holds when $m \leq M$. Then

$$\begin{aligned} \zeta_{\tilde{f}}(t) &= \zeta_f(t) \pmod{J_{M+1}}, \\ \Delta_{\tilde{f}}(t) &= \Delta_f(t) \pmod{J_{M+1}}, \end{aligned}$$

Furthermore, if we have "strict shrinking"

$$\text{range } \tilde{f}_{\ell_m}\circ\cdots\circ\tilde{f}_{\ell_1} \subset \text{ interior range } f_{\ell_m}\circ\cdots\circ f_{\ell_1},$$

for $m \leq M$, then $\tilde{f}^{\circ m}(a_i\pm) \notin \{a_0,\ldots,a_N\}$ for all $0 \leq i \leq N$ and $m \leq M$, in particular $\tilde{f} \in P^0_M$.

Proof: Cf [5], Lemma 2.3. The topology of the tree plays no role, because the range of $f_{\ell_m}\circ\cdots\circ f_{\ell_1}$ is contained on a subtree homeomorphic to the interval. □

3.3 Proof of Theorem 1 - *homotopy argument*

Given f, we may take $\tilde{f}$ to be arbitrarily close to f in P^0, such that

1) $\tilde{f}_i[a_{i-1},a_i] \subset f_i(a_{i-1},a_i)$;

2) the $\tilde{f}_i$ and f_i's differ only in a δ-neighborhood of a_i (and only in the edges containing the image of f_i).

This will imply the strict shrinking condition of Lemma 3.3 for some fixed $M \geq 1$ if $\delta = \delta(M)$ is sufficiently small. We have $\tilde{f} = (\tilde{f}_1,\ldots,\tilde{f}_{N-1})$ with $\tilde{f}_i : I_i \to T$. As emphasized before, $\tilde{f}_i(I_i) \subset T_i$ with T_i subtree of T homeomorphic to the interval, $a_1 \in T_i$. Because the edges are line segments, for each i it is possible to choose a piecewise affine homeomorphism Φ_i from T_i to an interval which preserves the distances. In particular, Φ_i is piecewise $\mathcal{C}^\infty$.

Now let $f^1 \in P^1_M$ be an N-tuple of polynomials (i.e. $\Phi_i \circ f^1_i \circ \Phi_i^{-1}$ are polynomials) sufficiently close to $\tilde{f} \in P^0_M$; then the invariant coordinates of the a_i* are the same for f^1 and $\tilde{f}$ up to terms of order $> M$. Therefore

$$\begin{aligned} \zeta_f &= \zeta_{f^1} \pmod{J_{M+1}}, \\ \Delta_f &= \Delta_{f^1} \pmod{J_{M+1}}, \end{aligned}$$

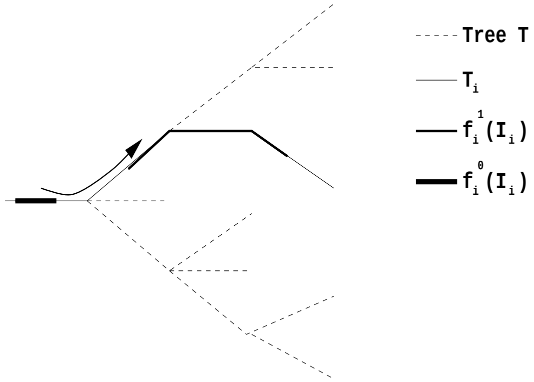

Figure 4: Pushing the graph of f_i

and the proof of the theorem reduces to showing that

$$\zeta_{f^1}\,\Delta_{f^1} = 1 + r_g(T) \pmod{J_{M+1}}\,.$$

We have now $f^1 = (f_1^1, \ldots, f_{N-1}^1)$ with $f_i^1 : I_i \to T_i \subset T$. T_i being (piecewise) $\mathcal{C}^\infty$-diffeomorphic to an interval through Φ_i, it is possible to choose $f^0 : T \to T$, $f^0 = (f_1^0, \ldots, f_{N-1}^0)$ such as in Lemma 3.2 and to "push" the graph of f^0 to the graph of f^1. More precisely : for each f_i^1 we choose $f_i^0 : I_i \to T_i$, with $f_i^0(I_i)$ is contained within the interior of I_1.

$$f_i^\lambda := \Phi_i^{-1}\Big((1-\lambda)\Phi_i \circ f_i^0 + \lambda \Phi_i \circ f_i^1\Big) \in P^1$$

(Note that $f_i^0(I_i) \subset I_0 \subset T_i$). If f_i^0 is chosen correctly, f_i^λ is strictly monotone for each λ. See Figures 4 and 5.

Now, looking at f_i^λ as functions $I_i \to T$, we can define $f^\lambda := (f_1^\lambda, \ldots, f_{N-1}^\lambda)$. We then proceed along the lines of [5] to slightly change the family (f^λ) to a new family (h^λ) (see [5], proof of Theorem 1.1), which has the following properties :

•(a) $h^\lambda \in P^1$ and $h^\lambda(T) \subset (T \backslash V_1(T))$, the map $(x, \lambda) \mapsto h^\lambda(x)$ is C^∞ (for the definition of differentiability given in 2.2), $g^0 = f^0$, $g^1 = f^1$;

•(b) for λ outside of a finite set Λ, $(h^\lambda)^{\circ m}(a_i*) \neq a_j$, if $1 \leq m \leq M$, $1 \leq i \leq N-1$, $1 \leq j \leq N-1$;

•(c) if $\lambda \in G$, there is a single (a_k*) and a single l such that $(h^\lambda)^{\circ p}(a_k*) = a_l$, for some $p \in \{1, \ldots, M\}$;

•(d) if $\lambda \in G$, $\mathrm{Fix}\,(h^\lambda)^{\circ m}$ is finite for $m \leq M$ and the points $x \in \mathrm{Fix}\,(h^\lambda)^{\circ m}$ are not neutral. To prove the theorem it suffices now (under the assumptions (a),(b),(c),(d)) to show that $1/\zeta_{h^\lambda}$ and Δ_{h^λ} are multiplied by the same factor $(\mathrm{mod}\, J_{M+1})$ when λ crosses a point in Λ. This is done in the following subsection.

• **Note:** In [5], Baladi and Ruelle made strong use of the fact that the family of interval maps (f^λ) are polynomials. Here, we must be a little bit more careful, because we composed f with

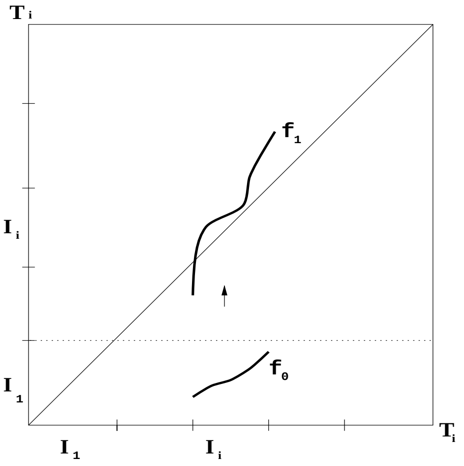

Figure 5: Pushing the graph of f_i

some homeomorphism, which could destroy the polygonal structure of f. However, since the homeomorphisms Φ_i are piecewise affine, their argument applies also here.

3.4 Key bifurcation lemma

Key lemma : *Let $M \geq 1$ and let $f \in P^1$, with $f(T) \subset (T \backslash V_1(T))$ be such that the sets $Fix\, f^{\circ m}$ are finite for $1 \leq m \leq M$, and such that there is a single a_i* ($* = -, +,$ or $+_\ell$) and a single j for which*

$$f^{\circ p}(a_i*) = a_j$$

with $1 \leq p \leq M$. If $j = i$, we take the smallest permissible p and write

$$\hat{\epsilon} = \prod_{k=0}^{p-1} \epsilon(f^{\circ k}(a_i*)) \ \ , \ \ Z(\gamma) = \prod_{k=0}^{p-1} g(f^{\circ k}(a_i*)) \cdot t^p \, .$$

We assume that the f-periodic points with period at most M are not neutral. Then, given $u, v \in P^1_M$ sufficiently close to f and such that $a_j \prec u^{\circ p}(a_i*)$, $v^{\circ p}(a_i*) \prec a_j$, such that f_i, u_i and v_i differ only in the javelin T_i of f_i, we have (mod J_{M+1}), and reading the signs $+_\ell$ as $+$:

$$\begin{array}{lll} \zeta_u = \zeta_v, & \Delta_u = \Delta_v & \text{if } j \neq i \\ \zeta_u = \zeta_v \cdot (1 - \hat{\epsilon} Z(\gamma))^{*1}, & \Delta_u = \Delta_v / (1 - \hat{\epsilon} Z(\gamma))^{*1} & \text{if } j = i \end{array}$$

Proof : The proofs for the case $i \neq j$ are easy (see [5], Lemma 2.4). Let $i = j$. If $* = +_k$ we have to consider T_ℓ with I_ℓ corresponding to $I_{(i,k)}$ in the other notation. T_ℓ being homeomorphic to the

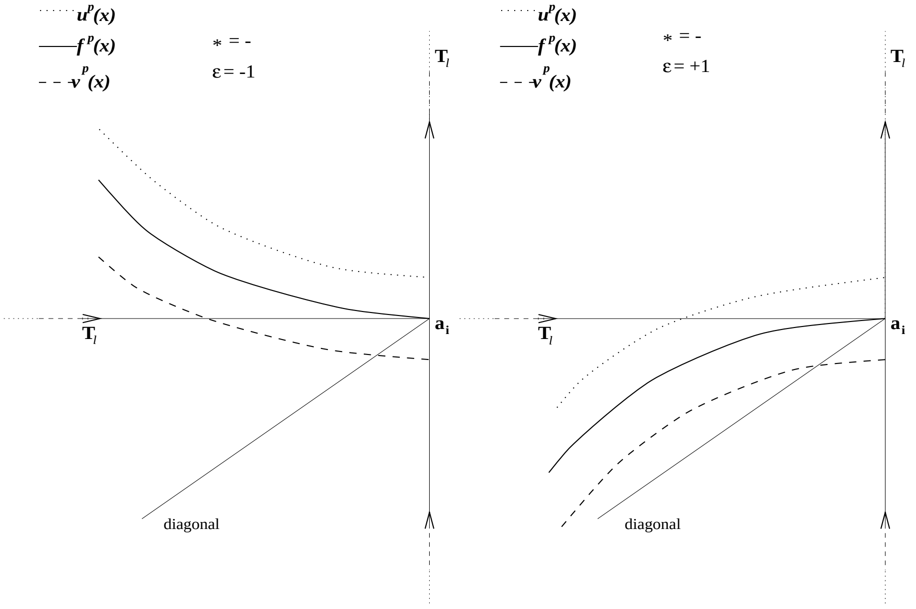

Figure 6: Different situations when f crosses the diagonal

interval, we may ignore the other edges incident to a_i and replace $+_k$ by $+$. If $*$ is $-$, we then work with f_i, and hence we can restrict to T_i. We then are reduced to the case where f, u and v are interval maps, and the proof given in [5] (Lemma 2.4, pages 629-631) applies here. Figures 6 and 7 may help the reader. $\square$.

•**Note :** The assumptions that $\mathrm{Fix}\, f^{\circ m}$ is finite for $1 \leq m \leq M$ and that no periodic point of period at most M is neutral have been inserted for convenience. By using the definition of the zeta function with the $L(f_{\ell_m} \circ \cdots \circ f_{\ell_1})$ given after Lemma 3.2, one could manage without them.

4 The case of g of bounded variation

In this paragraph we briefly explain how to adapt [2] to extend the result to the case of weights of bounded variation (instead of locally constant weights).

4.1 Definitions

Let T,$f : T \to T$ be as defined in subsection 2.2. Let $h : I \to \mathbb{C}$ with I an interval. Let ϱ be the set of all finite partitions of I. We recall that the variation of h is

$$\mathrm{var}(h) = \sup_{\varrho} \left\{ \sum_{\chi \in \varrho} |h(x_i) - h(x_{i+1})| \right\}$$

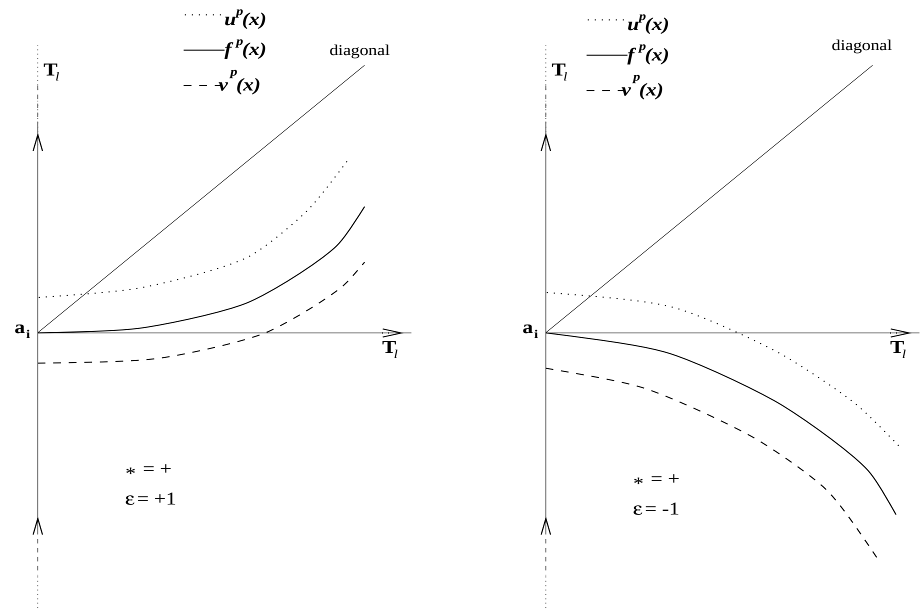

Figure 7: Different situations when f crosses the diagonal

with x_i and x_{i+1} two consecutive endpoints of the partition χ. We say that h is of bounded variation if $\text{var}(h)$ is finite.
So, let $g_i : I_i \to \mathbb{C}$, $i = 1, \cdots, N-1$ be functions of bounded variation. Let $g = (g_1, \cdots, g_{N-1})$ be the multivalued map $T \to \mathbb{C}$. Let $Z = Z_1 = E(T)$. By abuse of notation, we define $\text{var}(g)$ as $\sum \text{var}(g_i)$. Let $Z = Z_1 = E(T)$. For $n \geq 2$ let Z_n be the n^{th} refinement of Z under f, i.e. : $I_0 \cap f^{-1} I_1 \cap \ldots \cap f^{-n+1} I_{n-1}$ with $I_j \in Z$.
We can then define $T^{(n)}$ by setting : $E(T^{(n)}) = Z_n$ and $V(T^{(n)})$ to be the set of endpoints of the partition Z_n. It is important to note that each $a \in V(T^{(n)}) \backslash V(T)$ is of valence 2 and therefore, all $T^{(n)}$ are homeomorphic to T. Moreover, T and $T^{(n)}$, as sets, are the same. Hence, f can also be considered as a map $T^{(n)} \to T^{(n)}$.
We say that J is a *Z-homterval* if J is a maximal nontrivial subtree of T which is closed and has the property that $\forall n \geq 1$ there is $I \in Z_n$ with $J \subset I$. Notice that a Z-homterval is always homeomorphic to an interval (because it is a closed connex subset of an edge).
We make the additional assumption that g is constant on each Z-homterval of f.
For $n \geq 1$ and $I \in Z_n$, let $\{x_{I,m}, m \geq 1\} \subset I$ be such that $\lim_{m \to \infty} |g(x_{I,m})| = \text{ess inf}_I |g|$ and such that $\lim_{m \to \infty} g(x_{I,m})$ exists. We set $g^{(n)}$ to be the multivalued function whose restriction on each $I \in E(T^{(n)}) = Z_n$ is equal to the constant $\lim_{m \to \infty} g(x_{I,m})$.
Each $T^{(n)}$ is a finite tree, and $g^{(n)}$ is constant on the edges of $T^{(n)}$. Hence, we can built the kneading matrix as in 2.4 (we now denote it by $M_{(n)}^{(r)}(t)$, with $r = 1, 2$ depending on the address chosen), and we note its determinant by $\Delta_{f,g^{(n)}}^{(r)}$.

We note by $\zeta_{f,g^{(n)}}(t)$ the reduced zeta function of f with a weight $g^{(n)}$, and by $\zeta_{f,g}(t)$ the one with weight g. Let

$$\rho^{(n)} = \rho(g^{(n)}) = \lim_{n\to\infty}\left(\sup_x \left|\prod_{k=0}^{n-1} g^{(n)}(f^{\circ k}x)\right|\right)^{1/n}$$

and $\bar\rho := \lim_{n\to\infty}\rho^{(n)}$. Notice that for each n, (2) and (3) of Theorem 1 hold.

4.2 Results

Theorem 2 : *Let f and g be as defined before. With the above notation, we have :*

(1) for each $n \geq 1, \Delta^{(r)}_{f,g^{(n)}}(t)$ — respectively $\zeta_{f,g^{(n)}}(t)$ — is analytic — respectively meromorphic — in the open disk D_n of radius $1/\rho^{(n)}$, and the following relations hold :

$$\begin{aligned} a)\quad & \zeta_{f,g^{(n)}}(t)\Delta^{(1)}_{f,g^{(n)}}(t) = 1 \\ b)\quad & \zeta_{f,g^{(n)}}(t)\Delta^{(2)}_{f,g^{(n)}}(t) = 1 + r_g(T) \end{aligned} \tag{5}$$

with $r_g(T)$ as in theorem 1.
Assume additionally that g is continuous at each point of $Fix(f^{\circ m})$ and one-sided continuous at each point of $Fix\star(f^{\circ m})\backslash Fix(f^{\circ m})$ for all $m \geq 1$. Then :

(2) The sequence $\Delta^{(r)}_{f,g^{(n)}}(t)$ of analytic functions in D_n converges to a function $\Delta^{(r)}_{f,g}(t)$, analytic in the disk $D \subset D_n$ of radius $1/\bar\rho$. Between analytic functions in D we have the identities :

$$\begin{aligned} a)\quad & \zeta_{f,g}(t)\Delta^{(1)}_{f,g}(t) = 1 \\ b)\quad & \zeta_{f,g}(t)\Delta^{(2)}_{f,g}(t) = 1 + r_g(T) \end{aligned} \tag{6}$$

The complete statement of the theorem in the case of an interval map f should be read in [2] .
The proof consists of two points :
Firstly we have to prove (5). This has been already done in [5] for interval maps and Theorem 1 for tree maps. For the details about the radius of convergence, see [3] and [10]. The correcting factor appears in this computation (as in Theorem 1), and then the address does not play any further role. We will then write $M_{(n)}(t)$ besides $M^{(r)}_{(n)}(t)$.
Secondly, to prove (6), we must examinate the convergence. For this, Baladi uses a crucial lemma (The Key lemma in [2]) which gives a condition on a sequence of matrices of increasing size under which the sequence of their determinants is convergent in a suitable disk. The condition is that the

sum in each column of $M_{(n)}(t)$ is uniformly bounded. Let us note by $m_{i,j}^{(n)}(t)$ the coefficient in the i-th row and the j-th column of $M_{(n)}(t)$. We would like to have the following inequality :

$$m_{i,j}^{(n)}(t) \leq {v_i}^{(n)} = C \cdot |g^{(n)}(u_i +_k) - g^{(n)}(u_i -)| \text{ for all } t \text{ in the disk of radius } 1/\bar{\rho} \tag{7}$$

with C a constant and u_i an endpoint of the partition Z_n (i.e., $u_i \in V(T^{(n)})$. Clearly, the sum over i of such terms would then be $\leq C \cdot \mathrm{var}(g)$, and then the sum on each column of $M_{(n)}(t)$ would be uniformly bounded. Remember that if $u_i \notin V(T)$, it then has valence two.

In [2], Baladi shows this equality in two steps : she first considers the matrix $M^{(1)}(t)$, and shows that (7) holds for some constant C_1. The computation was made for interval maps, but since the matrix $M^{(1)}(t)$ is finite, a slight notational adaptation of the bound given in [2] (page 238) suffices. A trivial observation is that $M^{(1)}$ is a submatrix of $M^{(n)}$ for all n (after performing some elementary operations on the rows and colums of $M^{(n)}$.) Hence, summing the terms in a column in $M^{(n)}$ is the same as summing the terms in the corresponding column in $M^{(1)}$, plus some extra terms not involving the vertices of $V(T)$. (Remember that the coefficients of the kneading matrix $M^{(n)}$ are power series involving the orbits of the vertices of $T^{(n)}$). The second step is to show that (7) still holds (for another constant C_2) for the extra terms. But as we emphasized before, the vertices of $V(T^{(n)})$ which are not in $V(T)$ have valence two, and then they behave for g as interval points, and then we can conclude as in [2]. □

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