Global Level Spacings Distribution for Large Random Matrices from Classical Compact Groups: Gaussian Fluctuations [*]

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Abstract

We study the level-spacings distribution for eigenvalues of large $N \times N$ matrices from the Classical Compact Groups in the scaling limit when the mean distance between nearest eigenvalues equals 1.
Defining by $\eta_N(s)$ the number of nearest neighbors spacings, greater than $s > 0$ (smaller than $s > o$) we prove functional limit theorem for the process $(\eta_N(s) - \mathbb{E}\eta_N(s))/N^{1/2}$, giving weak convergence of this distribution to some Gaussian random process on $[0, \infty)$.
The limiting Gaussian random process is universal for all Classical Compact Groups. It is Hölder continuous with any exponent less than $1/2$. Numerical results suggest it not to be a standard Brownian bridge.
Our methods can be also applied to study n-level spacings distribution.

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1 Introduction and Formulation of Main Results

The idea that statistical behavior of eigenvalues of large random matrices would give an information about spectra of heavy nuclei was proposed by E.Wigner in fifties ([1]). Since then, random matrices have been intensively studied by F.J.Dyson, M.L.Mehta, C.E.Porter, N.Rosenzweig, M.Gaudin, L.Pastur, L.Girko and many others. [2] contains an extensive collection of early papers on this subject.

One of the most popular ensembles of random matrices, the so-called Circular Unitary Ensemble (C.U.E.) was brought into investigation by Freeman J.Dyson in [3] for studying quantum systems without time reversal symmetry. C.U.E. is the unitary group $U(N)$ with the normalized translation invariant (Haar) measure. It is a classical result ([5]) that the joint probability distribution of the eigenvalues $\{\exp(i\theta_j)\}_{j=1}^N$ in the unitary ensemble is given by the density

$$P_{N,\beta}(\theta_1,\dots\theta_N) = const_{N,\beta} \prod_{1\le k<j\le N} |\exp(i\theta_k) - \exp(i\theta_j)|^\beta \tag{1}$$

where the eigenvalues are ordered by increasing their angular coordinates

$$-\pi \le \theta_1 \le \dots \le \theta_N \le \pi \tag{2}$$

(here and further we are using the segment $[-\pi, \pi]$ with the coinciding ends as the representation for the unitary circle).

Curcular Unitary Ensemble corresponds to the case

$$\beta = 2, \qquad const_{N,2} = (2\pi)^{-N}$$

which is the simpliest one from the mathematical point of view among all possible choices of β. Two other cases with clear physical meaning, $\beta = 1$ and $\beta = 4$, correspond to the so-called Circular Orthogonal Ensemble (C.O.E) and Circular Symplectic Ensemble (C.S.E.) (no relation to the distribution of eigenvalues in Orthogonal Group $O(N)$ and Unitary Symplectic Group $USp(2N)$, which will be studied later). It is also worth mentioning that from the statistical mechanics point of view one can think about (1) as an equilibrium distribution at the temperature $T = 1/\beta$ of the Coulomb gas of N unit charges, confined to the infinitely thin circular conducting wire of radius 1, repelling each other according to the Coulomb law of two-dimensional electrostatics, i.e. with a potential energy

$$W = - \sum_{1 \le k < j \le N} \log |\exp(i\theta_k) - \exp(i\theta_j)|$$

Due to the logarithmic repulsion, typical configurations of the particles are very regulary distributed on the unit circle. For example, if we consider the number of particles hitting the interval

$$(-x, x) \subset [-\pi, \pi] \ , \qquad\qquad \mu_N(x) = \#\{j : |\theta_j| < x\}$$

then the mathematical expectation of $\mu_N(X)$ is proportional to the number N of all particles, $\mathbb{E}\mu_N(x) = Nx/\pi$, but the variance Var $\mu_N(x)$ grows only logarithmically,

$$\text{Var } \mu_N(x) = \frac{2 \log N}{\pi^2 \beta} + O(1), \qquad \beta = 1, 2, 4$$

and after the normalization, the random variable

$$(\mu_N(x) - \mathbb{E}\mu_N(x)) / \left(\text{Var } \mu_N(x)\right)^{1/2}$$

converges to the standard gaussian random variable. This and similar results can be found in the papers by O.Costin,J.Lebowitz [6] ,K.Johansson [7],[8] , H.Spohn [9] , P.Diaconis and M.Shahshahani [10] . With the exception of [7] ,the results have been obtained so far only for $\beta = 2$ or $\beta = 1, 2, 4$.

The main goal of our paper is to study the statistical behavior of level spacings for the Circular Unitary Ensemble ($\beta = 2$) . [1] After ordering in (2) the eigenvalues by increase in their angular coordinates, the nearest neighbor spacings can be defined as

$$\tau_j = \theta_{j+1} - \theta_j, \quad j = 1, \dots, N-1; \qquad \tau_N = \theta_1 + 2\pi - \theta_N \tag{3}$$

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[1] We learned from N.Katz and P.Sarnak [11] that our methods can be also applied to study other classical compact groups:

$$SO(2N), SO(2N+1), O(2N), O(2N+1), USp(2N), SU(N), O_-(2N)$$

(see section 4 for the corresponding results).

The n-point correlation functions $(n = 1, \ldots N)$ of our ensemble

$$\rho_n^{(N)}(x_1, \ldots x_n) = \frac{1}{(N-n)!} \int_{-\pi}^{\pi} \cdots \int_{-\pi}^{\pi} P_{N,2}(x_1, \ldots x_N) dx_{n+1} \ldots dx_N$$

(we extend the domain of definition of $P_{N,2}$ by symmetry to the whole N-dimensional torus) ,
have the following probabilistic meaning: Let $[x_1, x_1 + dx_1], \ldots [x_n, x_n + dx_n]$ be n infinitesimally small disjoint intervals,then $\rho_n^{(N)}(x_1, \ldots x_n) dx_1 \ldots dx_n$ is the probability to find eigenvalues in each of them.
For $\beta = 1, 2, 4$ n-point correlation functions have been calculated explicitly by F.Dyson (see [3], [4]) and in the case of C.U.E.

$$\rho_n^{(N)}(x_1, \ldots x_n) = (1/2\pi)^n \det \left(\frac{\sin N(x_i - x_j)/2}{\sin (x_i - x_j)/2} \right)_{i,j=1,\ldots n} \tag{4}$$

The conditional probability of having no eigenvalues in the interval $(0, u]$ provided there is an eigenvalue at the origin (that is the probability of nearest neighbor spacing τ to be greater than u) can be calculated using inclusion-exclusion principle :

$$\begin{aligned} \mathbb{P}_N(\tau > u) &= \Big(\rho_1^{(N)}(0) - \int_0^u \rho_2^{(N)}(0, x_2) dx_2 + \frac{1}{2!} \int_0^u \int_0^u \rho_3^{(N)}(0, x_2, x_3) dx_2 dx_3 - \\ &- \frac{1}{3!} \int_0^u \int_0^u \int_0^u \rho_4^{(N)}(0, x_2, x_3, x_4) dx_2 dx_3 dx_4 + \cdots \Big) / \rho_1^{(N)}(0) \end{aligned} \tag{5}$$

The mean distance between the nearest eigenvalues in C.U.E. is equal to $2\pi/N$. After a suitable rescaling (extension by $N/(2\pi)$ times the segment $[-\pi, \pi]$) , this distance becomes equal to 1 . In the new coordinates

$$y_k = N/2 + N\ \theta_k/(2\pi), \qquad k = 1, \ldots N$$

the rescaled n-point correlation functions

$$(2\pi/N)^n \rho_n^{(N)}(2\pi y_1/N, \ldots 2\pi y_n/N) = \det \left(\frac{\sin \pi(y_i - y_j)}{N \sin(\pi(y_i - y_j)/N)} \right)_{i,j=1,\ldots n} \tag{6}$$

have a finite limit as N tends to infinity :

$$\lim_{N\to\infty}(2\pi/N)^n\rho_n^{(N)}(2\pi y_1/N,\ldots 2\pi y_n/N)=:\rho_n^{(\infty)}(y_1,\ldots y_n)=\det\left(\frac{\sin\pi(y_i-y_j)}{\pi(y_i-y_j)}\right)_{i,j=1,\ldots n} \tag{7}$$

and respectively for

$$F_N(s):=\mathbb{P}_N(\tau>2\pi s/N)$$

$$\lim_{N\to\infty}F_N(s)=:F(s)=\sum_{n=0}^{\infty}\frac{(-1)^n}{n!}\int_{[0,s]^n}\rho_{n+1}^{(\infty)}(0,y_1,\ldots y_n)dy_1\ldots dy_n \tag{8}$$

Remark The limiting correlation functions (7) define a random point field on the real line, i.e. the probability measure on the Borel σ -algebra of the space of locally finite point configurations

$$\Omega=\Big\{(x_i)_{i=-\infty,\ldots+\infty}\ :\ \forall L>0\ \ \#\{x_i:|x_i|<L\}<\infty\Big\}$$

in the following way : if we fix m disjoint intervals $[a_{2j-1},a_{2j}]_{j=1,\ldots m}$ and define random variables $\mu_1,\ldots\mu_m$ to be the numbers of particles hitting each interval, then the generating function

$$\varphi(z_1,\ldots z_m):=\mathbb{E}\prod_{j=1}^{m}z_j^{\mu_j}$$

is given by the Fredholm determinant of the integral operator acting on $L^2(R^1)$, with the kernel

$$\sum_{i=1}^{m}(z_j-1)\frac{\sin\pi(x-y)}{\pi(x-y)}\mathbf{J}_j(y)$$

where $\mathbf{J}_j$ are indicators of the segments $[a_{2j-1},a_{2j}]$ (see [12]).

Remark Function $F(s)$ decays at infinity very fast :

$$\log F(s)=-\pi^2s^2/8+O(s)$$

(see [13],chapter 12; also [12] ,[14]) .

Recently, Z.Rudnick and P.Sarnak ([15]) showed that after suitable rescaling the n-point

correlation functions for zeroes of the Riemann Zeta Function on the critical line $\Re e\, z = 1/2$ are given exactly by the same formula (7).
These results are valid in a restricted range , see also the early paper on pair-correlations by H.L.Montgomery ([16]), and numerical results by A.M.Odlyzko on the spacings distribution of zeroes ([17] , [18]). We finish this section with the formulations of our main results :

Theorem 1.1 *Consider an arbitary subinterval* I_N *of the unit circle such that the average number of eigenvalues hitting subinterval tends to infinity as* $N \to \infty$: $|I_N| N/(2\pi) \to \infty$ *Let us define* $\eta(I_N, s)$ *to be the number of eigenvalues belonging to* I_N*, for which the distance to the nearest right neighbor is greater (smaller) than* $2\pi\ s/N$:

$$\eta(I_N, s) := \#\{j : \theta_j \in I_N, \quad \tau_j = \theta_{j+1} - \theta_j > \ (<) \ 2\pi\ s/N\}$$

Then

$$I\!\!E\ \eta(I_N, s) = \frac{N|I_N|}{2\pi} I\!\!P_N(\tau > \ (<) \ 2\pi\ s/N)$$

and finite dimensional distributions of the normalized random process

$$\xi_N(s) = (\eta(I_N, s) - I\!\!E\ \eta(I_N, s))/\ (N|I_N|/2\pi)^{1/2}$$

converge to the disributions of Gaussian random process with $I\!\!E\xi(s) = 0$ *and* $b(s,t) := I\!\!E\xi(s)\xi(t)$ *given by the formulas (28), (29), (17) in the section 2.*

To formulate the results about functional convergence we have to define the continuous approximation of $\xi_N(s)$. The realizations of $\eta(I_N, s)$ have discontinuities at points

$$\tau_j = \theta_{j+1} - \theta_j, \quad \theta_j \in I_N : \quad \eta(I_N, \tau_j + 0) - \eta(I_N, \tau_j) = 1$$

We define the graph of $\tilde{\eta}(I_N, s)$ by lineary connecting the neighboring vertices $(\tau_j, \eta(I_N, \tau_j))$, $\theta_j \in I_N$, and

$$\tilde{\xi}_N(s) := (\tilde{\eta}(I_N, s) - \mathbb{E}\eta(I_N, s))/(N|I_N|/(2\pi))^{1/2}$$

The distribution of $\tilde{\xi}_N(\cdot)$ defines a probability measure $\mathcal{P}_N$ on the space of continuous functions $C[0,\infty)$ (infinity point is not included !).

Theorem 1.2 *$\mathcal{P}_N$ weakly converges to the distribution of the Gaussian process $\xi(\cdot)$.*

Of course in both theorems we can take I_N to be $[-\pi,\pi]$. In this case $\eta_N([-\pi,\pi],s)$ will count all nearest-neighbor spacings greater(smaller) than s.

Corollary 1.3 *If we consider disjoint intervals*

$$I_N^{(1)},\dots I_N^{(m)}$$

such that

$$0 < const_1 \le |I_N^{(i)}|/\ |I_N^{(j)}| \le const_2 < \infty; \quad i,j = 1,\dots m$$

and

$$N|I_N^{(i)}|/(2\pi) \to \infty \qquad as \quad N \to \infty$$

then random vector

$$\Big((\eta(I_N^{(i)},s_i) - \mathbb{E}\eta(I_N^{(i)},s_i))/(\,Var\ \eta(I_N^{(i)},s_i))^{1/2}\Big)_{i=1}^m$$

converges in distribution to the standard gaussian random vector with independent components.

Corollary 1.4 *For any finite $T > 0$,*

$$sup_{s\in[0,T]} \quad |\eta(I_N,s) - \mathbb{E}\eta(I_N,s)|/(N|I_N|/(2\pi))^{1/2} \tag{9}$$

converges in distribution to

$$\sup_{s\in[0,T]} |\xi(s)|$$

Remark Since

$$sup_{[0,\infty)}|\mathbb{E}\eta(I_N,s) - N|I_N|\ F(s)/(2\pi)| = o(|I_N|N^{\varepsilon}/(2\pi))$$

for any $\varepsilon > 0$ (see (61)),
one can replace in (9) $\mathbb{E}\eta(I_N,s)$ by $F(s)\,N\ |I_N|/\ (2\pi)$.
We have not been able to prove the result of Corollary 1.4 for $T = \infty$ (the functional convergence of probability distributions is proven for $C[0,\infty)$, not for $C[0,\infty]$!). Therefore we settle for a weaker version :

Corollary 1.5 *With probability 1*

$$sup_{[0,\infty)}|\eta(I_N,s)-I\!E\eta(I_N,s)| = o\Big((N|I_N|/(2\pi))^{1/2\ +\varepsilon}\Big) \tag{10}$$

for any $\varepsilon > 0$.

Remark The first to study the descrepancy at the l.h.s. of (10) were N.Katz and P.Sarnak who did it in connection with the theory of geometric Zeta-functions over finite fields (see [11]). They proved the estimate

$$\mathbb{E}\ sup_{[0,\infty)}|\eta(I_N,s)-\mathbb{E}\eta(I_N,s)| = o\Big((N|I_N|/(2\pi))^{5/6\ +\varepsilon}\Big)$$

to show that for typical geometric Zeta functions the empirical distribution functions of the normalized spacings converge to the Gaudin law $F(s)$.

Remark Again we can replace $\mathbb{E}\eta(I_N,s)$ by $F(s)\ N|I_N|/(2\pi)$.

As usual for the C.U.E. similar results also hold for the limiting random point field (7) :

Theorem 1.6 *Consider the number of particles,hitting the interval* $[0,L]$,*for which the distance to the nearest right neighbor is greater than s,*

$$\eta(L,s) = \#\{x_i : 0 \le x_i \le L\ ,\quad dist(x_i, rightngb\ (x_i)) > s\}$$

Then $\qquad I\!E\eta(L,s) = LF(s)$ *and*

$$\xi_L(s) = (\eta(L,s) - LF(s))/L^{1/2}$$

converges in finite dimensional distributions to the Gaussian random process of Theorem 1.1.

Again we can define piecewise linear continuous approximation $\tilde{\xi}_L(s)$ of $\xi_L(s)$ such that

$$|\tilde{\xi}_L(s) - \xi_L(s)| \le L^{-1/2}$$

and as the analogue of Theorem 1.2 we have :

Theorem 1.7 *The distribution of* $\tilde{\xi}_L(\cdot)$ *on* $C[0,\infty)$ *weakly converges to the distribution of* $\xi(\cdot)$.

Remark We do not know simple "explicit" formulas for the covariance function $\mathbb{E}\xi(s)\xi(t)$ of the limiting Gaussian process. Since

$$\mathbb{E}(\xi(t+\delta t)-\xi(t))^2 = O(|\delta t|)$$

uniformly on any finite interval $t \in [0,T]$, $\xi(s)$ is Hölder continuous with any exponent $\alpha < 1/2$. The numerical results by S.Miller ([19]) suggest that $\xi(s)$ is not a standard Brownian bridge.

As it is usually the case, the proofs of Theorems 1.1 and 1.6, Theorems 1.2 and 1.7 are almost identical. In the next section we will prove Theorem 1.1 . The proofs of Theorem 1.2 and Corollary 1.5 are given in section 3. Results similar to Theorems 1.1 and 1.2 are valid in the cases $\beta = 1,4$ and Orthogonal and Symplectic Groups.
Minor changes ,required in the formulations and proofs of the theorems are discussed in section 4.

I would like to express my sincere gratitude to my advisor Ya.Sinai and to P.Sarnak and H.Spohn for many useful discussions. I would also like to thank N.Katz and P.Sarnak for providing me with their notes [11] on the subject prior to the publication.

2 Proof of Theorem 1.1

We prove Theorem 1.1 by computing all (to be more precise ,first N) moments of random variable $\eta(I_N, s) - \mathbb{E}\eta(I_N, s)$. Without a loss of generality we will assume the interval I_N to be the unit circle. In the rescaled coordinates

$$\{ \ y_i = N\theta_i/(2\pi) \ + N/2 \ \}_{i=1}^N$$

the N-dimensional probability density (1) and n-point correlation functions are given by the formulas

$$\begin{aligned} P_{N,2}(y_1, \dots y_N) &= N^{-N} \prod_{1\le k<j\le N} |\exp(i\pi y_j/N) - \exp(i\pi y_k/N)|^2 \\ &= \det\left(\frac{\sin\pi(y_i - y_j)}{N\sin(\pi(y_i-y_j)/N)}\right)_{i,j=1,\dots N} \end{aligned} \tag{11}$$

$$0 \le y_1 \le \dots \le y_N \le N$$

and

$$\rho_n^{(N)}(y_1, \dots y_n) = \det\left(\frac{\sin\pi(y_i - y_j)}{N\sin(\pi(y_i-y_j)/N)}\right)_{i,j=1,\dots n} \tag{12}$$

We will omit the index N in the notation for n-point correlation functions if it does not lead to ambiguity ; we also consider all variables y_i by modulo N .
To calcilate the moments of

$$\eta_N(s) := \eta([-\pi, \pi], s)$$

we introduce a representation of $\eta_N(s)$ as " a sum of infinitesimally small random variables ". This representation will be used throughout the whole proof. Consider the interval $[0, N]$ as the disjoint union of infinitesimally small subintervals $[x_i, x_i + dx_i]$:

$$[0, N] = \bigcup_i [x_i, x_i + dx_i] \quad , x_{i+1} = x_i + dx_i$$

and for each subinterval denote by $\chi(x_i, dx_i, s)$ the indicator of the event to have an eigenvalue in $[x_i, x_i + dx_i]$ and no eigenvalues in $[x_i + dx_i, x_i + s]$. Then

$$\eta_N(s) = \int_0^N \chi(x, dx, s) \tag{13}$$

More rigorously, (13) means that $\eta_N(s)$ is the integral of the discrete measure $\chi(dx)$ which has unit atoms at the points y_i, s.t. $y_{i+1} - y_i > s$ (we assume $y_{N+1} = y_1 + N$) .
The representation of $\eta_N(s)$ as "the sum of weakly dependent random variables" [2] gives us a natural setting for the Central Limit Theorem.
Using inclusion-exclusion principle, one can calculate the mathematical expectation of the products of $\chi(x_i, dx_i, s), \quad i = 1, \ldots m$:

$$\begin{aligned}\mathbb{E}\chi(x_1, dx_1, s) &= \Big(\rho_1^{(N)}(x_1) - \int_{x_1}^{x_1+s} \rho_2^{(N)}(x_1, x_2)dx_2 \\ &+ \frac{1}{2!}\int_{x_1}^{x_1+s}\int_{x_1}^{x_1+s} \rho_3^{(N)}(x_1, x_2, x_3)dx_2dx_3 - \ldots\Big)dx_1 \\ &= F_N(s)dx_1. \end{aligned} \tag{14}$$

To calculate $\mathbb{E}\ \chi(x_1, dx_1, s)\chi(x_2, dx_2, s)$ we have to consider two cases : $|x_1 - x_2|_1 \leq s$ and $|x_1 - x_2|_1 > s$. In the former , the mathematical expectation of the product is zero, by definition of $\chi(x, dx, s)$, and in the latter

$$\mathbb{E}\chi(x_1, dx_1, s)\chi(x_2, dx_2, s) = \Big(\sum_{m=0}^{N-2} \frac{(-1)^m}{m!} \int \rho_{2+m}(x_1, x_2; \ldots x_{m+2})dx_3 \ldots dx_{m+2}\Big)dx_1dx_2 \tag{15}$$

where each variable $x_3, \ldots x_{m+2}$ is integrated over the union of two intervals $[x_1, x_1 + s]$ and $[x_2, x_2 + s]$.
The key combinatorial observation used in the proof, can be first seen when we calculate the covariance of $\chi(x_1, dx_1, s)$, $\chi(x_2, dx_2, s)$. We are going to use the cluster structure of n-point correlation functions (12). Consider the m^{th} term in (15) and fix the variables of integration $x_3, \ldots x_{m+2}$.
Some of $x_i \quad i = 1, \ldots m + 2$, say k of them, $1 \leq k < m + 2$, belong to the interval $[x_1, x_1 + s]$, we will denote the indeces of those variables by

$$i_1, \ldots i_k; \qquad 1 = i_1 < \ldots < i_k \leq m + 2.$$

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[2] we will be able to show that

$$\begin{aligned} &\mathrm{Cov}(\chi(x_1, dx_1, s), \chi(x_2, dx_2, s)) = g_N(s, |x_1 - x_2|_1)dx_1dx_2 \\ \text{where} \quad &|g_N(s, x)| \leq const(s, \varepsilon)/(1 + |x|^{2-\varepsilon}) \quad \text{for any} \quad \varepsilon > 0 \\ \text{and} \quad &|x_1 - x_2|_1 := \min(|x_1 - x_2|, N - |x_1 - x_2|) \end{aligned}$$

We will denote the indeces of the remaining variables by

$$j_1, \dots j_{m+2-k}; \qquad 2 = j_1 < \dots < j_{m+2-k} \leq m+2.$$

It is clear that

$$x_{j_l} \in [x_2, x_2 + s], \qquad l = 1, \dots m+2-k$$

It follows from (12) that

$$\rho_{2+m}^{(N)}(\bar{x}) = \sum_{\sigma \in S_{m+2}} (-1)^{\sigma} \prod_{i=1}^{m+2} \frac{\sin \pi (x_i - x_{\sigma(i)})}{N \sin(\pi (x_i - x_{\sigma(i)})/N)} \tag{16}$$

(here we use the notations $\bar{x}$ for the vector $(x_1, \dots x_{m+2})$ and σ for permutations from the symmetric group S_{m+2}).
Now we decompose the sum (16) into two, where the first corresponds to the "interaction" between the particles x_1 and x_2, and is the sum over such $\sigma \in S_{m+2}$,that

$$\sigma(\{i_1, \dots i_k\}) \cap \{j_1, \dots j_{m+2-k}\} \neq \emptyset$$

and the second is over all other σ. Denoting the first sum by $\rho_{2+m,2}$ we have

$$\rho_{2+m}(x_1, x_2, \dots x_{2+m}) = \rho_{2+m,2}(x_1, x_2, \dots x_{2+m}) + \rho_k(x_{i_1}, \dots x_{i_k}) \cdot \rho_{2+m-k}(x_{j_1}, \dots x_{j_{2+m-k}}) \tag{17}$$

Formulas (14,15,17) imply for $|x_1 - x_2|_1 > s$:

$$\mathrm{Cov}(\chi(x_1, dx_1, s)\chi(x_2, dx_2, s)) = \Big(\sum_{m=0}^{N-2} \frac{(-1)^m}{m!} \int \rho_{2+m,2}(x_1, x_2; \dots x_{m+2}) dx_3 \dots dx_{m+2} + \mathcal{R} \Big) dx_1 dx_2 \tag{18}$$

where the remainder term $\mathcal{R}$ would vanish if the summation in (14),(15),(17) were from zero to infinity, and in our case

$$|\mathcal{R}| \leq \sum_{0 \leq k_1, k_2 \leq N,\, k_1 + k_2 > N} \frac{1}{k_1!} \frac{1}{k_2!} \int \rho_{k_1}(x_{i_1}, \dots x_{i_{k_1}}) dx_{i_2} \dots dx_{i_{k_1}} \int \rho_{k_2}(x_{j_1}, \dots x_{j_{k_2}}) dx_{j_2} \dots dx_{j_{k_2}} \tag{19}$$

In (18) the variables $x_3, \dots x_{2+m}$ are integrated over $[x_1, x_1+s] \cup [x_2, x_2+s]$; in (19) the variables $x_{i_2}, \dots x_{i_k}$ are integrated over $[x_1, x_1+s]$, and the variables $x_{j_2}, \dots x_{j_{m+2-k}}$ are integrated over $[x_2, x_2+s]$.

Since

$$0 \leq \det \left(\frac{\sin \pi(x_i - x_j)}{N \sin(\pi(x_i - x_j)/N)} \right)_{i,j=1,\dots n} \leq 1 \quad , \tag{20}$$

s is bounded throughout the proof of Theorems 1.1 ,1.2 , and we use these arguments only for $s < (\log N)^{1/2}$ in the proof of Corollary 1.5 ,
we obtain the estimate:

$$|\mathcal{R}| \leq N^2 (2s)^N 2^N / N! \leq \left(\frac{const \log N}{N} \right)^N N^2$$

This inequality shows that we can neglect $\mathcal{R}$ throughout the proof. Formula (20) is the general property of n-dimensional positive defined matrices with the trace less or equal to n .
It follows from (17) ,(20) that

$$|\rho_{2+m,2}(x_1, x_2, \dots x_{2+m})| \leq 2 \tag{21}$$

and the sums (14), (15), (18) are uniformly convergent as $N \to \infty$.

We want to know how fast

$$\mathrm{Cov}(\chi(x_1, dx_1, s), \chi(x_2, dx_2, s)) =: g_N(s, |x_1 - x_2|_1) dx_1 dx_2$$

decays as $|x_1 - x_2|_1$ goes to infinity.
With this question in mind we remark that $\rho_{2+m,2}(x_1, x_2, \dots x_{2+m})$ is the sum of at most $(2+m)!$ products

$$(-1)^\sigma \prod_{i=1}^{m+2} \frac{\sin \pi(x_i - x_{\sigma(i)})}{N \sin(\pi(x_i - x_{\sigma(i)})/N)}$$

each containing at least two factors

$$\frac{\sin \pi(x_i - x_{\sigma(i)})}{N \sin(\pi(x_i - x_{\sigma(i)})/N)}$$

with $x_i,\ x_{\sigma(i)}$ belonging to different intervals $[x_1, x_1+s],\ [x_2, x_2+s]$
Thus

$$|\rho_{2+m,2}(\bar{x})| \le (2+m)!\frac{const(s)}{1+|x_1-x_2|_1^2} \tag{22}$$

Here and further in our calculations we use different constants, depending on s (but not on N). Usually we will denote all of them $const(s)$. The only property which we need from these constants is their uniform boundedness on every finite interval $s \in [0,T]$.
(21) and (22) give us the desired estimate of $g_N(s,x)$:

$$\begin{aligned} |g_N(s,x)| &\le \sum_{m=0}^{\infty} \frac{1}{m!}(2s)^m \min\{2, const(s)(2+m)!/(1+x^2)\} \\ &\le \sum_{0\le m\le const_1(s)\log x/\log(\log x)} const(s)^m/(1+x^2) + \sum_{m>const_1(s)\log x/\log(\log x)} 2(2s)^m/m! \\ &\le const(s,\varepsilon)/(1+|x|^{2-\varepsilon}) \end{aligned} \tag{23}$$

for any $\varepsilon > 0$
As N tends to infinity, $g_N(s,x)$ converges to the limit uniformly in x:

$$g(s,x) := \lim_{N\to\infty} g_N(s,x) = \sum_{m=0}^{\infty} \frac{(-1)^m}{m!} \int \rho^{(\infty)}_{2+m,2}(0,x;\dots x_{m+2})dx_3 \dots dx_{m+2}$$

where the variables $x_3, \dots x_{2+m}$ are integrated over $[0,s] \cup [x, x+s]$ and $\rho^{(\infty)}_{2+m,2}$, $m \ge 0$ is defined in (17) with limiting point correlation functions,given by (7):

$$\rho^{(\infty)}_n(y_1, \dots y_n) = \det\left(\frac{\sin \pi(y_i-y_j)}{\pi(y_i-y_j)}\right)_{i,j=1,\dots n}.$$

The estimate (23) holds for $g(s,x)$ as well:

$$|g(s,x)| \le const(s,\varepsilon)/(1+|x|^{2-\varepsilon})$$

Now we are in position to write down the formula for the variance of $\eta_N(s)$:

$$\mathrm{Var}\ \eta_N(s) = \mathbb{E}\ \Big(\int_0^N (\chi(x_1, dx_1, s) - \mathbb{E}\chi(x_1, dx_1, s)) \quad \int_0^N (\chi(x_2, dx_2, s) - \mathbb{E}\chi(x_2, dx_2, s))\Big)$$

$$
\begin{aligned}
&= \mathbb{E}\int_{0}^{N}\int_{\substack{0\\|x_1-x_2|_1>s}}^{N} (\chi(x_1,dx_1,s)-\mathbb{E}\chi(x_1,dx_1,s))(\chi(x_2,dx_2,s)-\mathbb{E}\chi(x_2,dx_2,s))\\
&+ \mathbb{E}\int_{0}^{N}\int_{\substack{0\\0<|x_1-x_2|_1\le s}}^{N} (\chi(x_1,dx_1,s)-\mathbb{E}\chi(x_1,dx_1,s))(\chi(x_2,dx_2,s)-\mathbb{E}\chi(x_2,dx_2,s))\\
&+ \mathbb{E}\int_{0}^{N}\int_{\substack{0\\x_1=x_2}}^{N} (\chi(x_1,dx_1,s)-\mathbb{E}\chi(x_1,dx_1,s))(\chi(x_2,dx_2,s)-\mathbb{E}\chi(x_2,dx_2,s))\\
&= \int_{0}^{N}\int_{\substack{0\\|x_1-x_2|_1>s}}^{N} g_N(s,|x_1-x_2|_1)dx_1dx_2 - \int_{0}^{N}\int_{\substack{0\\0<|x_1-x_2|_1\le s}}^{N} F_N(s)^2 dx_1dx_2\\
&+ \int_0^N\int_0^N F_N(s)\delta(x_1-x_2)dx_1dx_2\\
&= b_N(s,s)N + O(N^{\varepsilon}). \qquad (24)
\end{aligned}
$$

where

$$
b_N(s,s) = \int_{|x|>s} g_N(s,x)dx - 2sF_N^2(s) + F_N(s) \qquad (25)
$$

Similar calculations give us the formula for the covariance of $\eta_N(s), \eta_N(t)$:

$$
\mathrm{Cov}(\eta_N(s),\eta_N(t)) = b_N(s,t)N + O(N^{\varepsilon}).
$$

with

$$
\begin{aligned}
b_N(s,t) &= \int_{-\infty}^{-t} g_N(s,t,x)dx + \int_{s}^{+\infty} g_N(s,t,x)dx\\
&- (s+t)F_N(s)F_N(t) + F_N(s\vee t). \qquad (26)
\end{aligned}
$$

where the function $g_N(s,t,x)$ is defined as

$$
g_N(s,t,x) = \sum_{m=0}^{N-2}\frac{(-1)^m}{m!}\int \rho_{2+m,2}^{(N)}(0,x;\ldots x_{m+2})dx_3\ldots dx_{m+2}
$$

the variables $x_3, \dots x_{2+m}$ are integrated over $[0,s] \cup [x, x+t]$; and $s \vee t := \max(s,t)$. Similar to (23)

$$|g_N(s,t,x)| \le const(s,t,\varepsilon)/(1+|x|^{2-\varepsilon}) \quad \text{for any} \quad \varepsilon > 0. \tag{27}$$

As $N \to \infty$ $g_N(s,t,x)$ uniformly in x converges to the limit

$$\begin{aligned} g(s,t,x) &= \lim_{N\to\infty} g_N(s,t,x) \\ &= \sum_{m=0}^{\infty} \frac{(-1)^m}{m!} \int \rho^{(\infty)}_{2+m,2}(0,x;\dots x_{m+2}) dx_3 \dots dx_{m+2}. \end{aligned} \tag{28}$$

the variables $x_3, \dots x_{2+m}$ are integrated over $[0,s] \cup [x, x+t]$.
The covariance function $b(s,t)$ of the limiting Gaussian process $\xi(s)$ is defined as

$$\begin{aligned} b(s,t) &= \lim_{N\to\infty} b_N(s,t) = \int_s^{+\infty} g(s,t,x)dx \\ &+ \int_{-\infty}^{-t} g(s,t,x)dx - (s+t)F(s)F(t) + F(s \vee t). \end{aligned} \tag{29}$$

It is a matter of lengthy, but simple calculations to show that at the origin

$$\begin{aligned} b(s,s) &= \text{Var } \xi(s) = \pi^2 s^3/9 + O(s^4) \\ F(s) &= 1 - \pi^2 s^3/9 + O(s^4). \end{aligned}$$

The functions $F(s)$, $b(s,t) - F(s \vee t)$ are analytic ,
which implies the Hölder continuity of $\xi(s)$ with any exponent less than $1/2$.

To proceed with the proof of Theorem 1, we are looking for the formulas for

$$\mathbb{E} \prod_1^{2k} (\chi(x_i, dx_i, s) - \mathbb{E}\chi(x_i, dx_i, s))$$

similar to (17),(18).
We will use again the special cluster structure of point correlation functions (12). In this

way we will be able to prove that

$$\begin{aligned}\mathbb{E}\ \prod_1^{2k}(\chi(x_i,dx_i,s)-\mathbb{E}\chi(x_i,dx_i,s)) &= \sum\prod_{(i,j)}\mathrm{Cov}(\chi(x_i,dx_i,s),\chi(x_j,dx_j,s)) \\ &+ \ R_{2k}(x_1,\ldots x_{2k})dx_1\ldots dx_{2k}. \qquad (30)\end{aligned}$$

where the summation $\sum$ is over all partitions of $\{1,\ldots 2k\}$ into pairs (i,j) and for any $\varepsilon>0$

$$\int_0^N\cdots\int_0^N|R_{2k}(x_1\ldots x_{2k})|dx_1\ldots dx_{2k}=O(N^{k-1+\varepsilon}) \qquad (31)$$

Formulas (30), (31) are the key ingredients of the proof of Theorem 1.1.
Again, we will consider the contributions to $\mathbb{E}\ (\eta_N(s)-\mathbb{E}\ \eta_N(s))^{2k}$ from the "off-diagonal" terms

$$\min_{i\neq j}|x_i-x_j|_1>s, \qquad (32)$$

"near-diagonal" $0<\min_{i\neq j}|x_i-x_j|_1\leq s$ and diagonal terms $x_i=x_j$ separately.
In calculations to follow we restrict ourselves to the case of even moments. However, all arguments work in the case of odd moments as well.
Let us consider first the "off-diagonal" case (32). By inclusion- exclusion principle

$$\mathbb{E}\ \prod_1^{2k}(\chi(x_i,dx_i,s) =$$

$$\Big(\sum_{m=0}^{N-2k}\frac{(-1)^m}{m!}\int\int\rho_{2k+m}^{(N)}(x_1,\ldots x_{2k};\ldots x_{m+2k})dx_{1+2k}\ldots dx_{m+2k}\Big)\ dx_1\ldots dx_{2k} \qquad (33)$$

the integral is over $(\bigcup_1^{2k}\ [x_i,x_i+s]\ )^m$
and

$$\rho_{2k+m}^{(N)}(\bar{x})=\sum_{\sigma\in S_{2k+m}}(-1)^\sigma\prod_{i=1}^{2k+m}\frac{\sin\pi(x_i-x_{\sigma(i)})}{N\sin(\pi(x_i-x_{\sigma(i)})/N)} \qquad (34)$$

As in (17) we decompose the sum (34) into two, the first subsum corresponds to the interaction between the particles $x_1,\ldots,x_{2k}$, where each particle interacts with at least one

other particle.
The formal definition of the first subsum is the following:

Let us denote by $x_{i_1}^{(j)},\dots x_{i_{p_j}}^{(j)}$ the variables among $x_1,\dots,x_{2k+m}$ belonging to the interval $[x_j, x_j+s]$ $\quad j=1,\dots 2k$.

We will also reserve the notations $\bar{x}^{(j)}$ for the vector $(x_{i_1}^{(j)},\dots x_{i_{p_j}}^{(j)})$ and $n(x_j)=p_j$ for the number of variables belonging to $[x_j, x_j+s]$.
We define the first subsum as the sum over such $\sigma \in S_{2k+m}$, that for any $j=1,\dots 2k$ (i.e. for any particle x_j) , there exists another index $1 \le l \le 2k$ (there exists another particle x_l), that

$$\sigma(\{i_1^{(j)},\dots i_{p_j}^{(j)}\}) \cap \{i_1^{(l)},\dots i_{p_l}^{(l)}\} \neq \emptyset \tag{35}$$

(particles x_j and x_l interact with each other). We denote this sum by $\rho_{2k+m,2k}$.
To deal with the second sum, we single out the particles not interacting with any others. Iterating, we arrive at the formula :

$$\begin{aligned} & \rho_{2k+m}(x_1,\dots x_{2k},\dots x_{2k+m}) = \rho_{2k+m,2k}(x_1,\dots x_{2k},\dots x_{2k+m}) \\ + \quad & \sum_{\emptyset\neq\mathcal{A}\subset\{1,\dots 2k\}} \Big(\prod_{j\in\mathcal{A}} \rho_{n(x_j)}(\bar{x}^{(j)})\Big) \cdot \\ \cdot \quad & \rho_{\,2k+m-\sum_{j\in\mathcal{A}} n(x_j),\ 2k-|\mathcal{A}|}(\bar{x}\setminus \cup_{j\in\mathcal{A}}\, \bar{x}^{(j)}). \end{aligned} \tag{36}$$

Since the following formula

$$\mathbb{E}\prod_1^{2k}\nu_i = \sum_{\mathcal{A}\subset\{1,\dots 2k\}} \prod_{j\in\mathcal{A}} \mathbb{E}\ \nu_i \ \cdot\ \mathbb{E}\prod_{l\notin\mathcal{A}}(\nu_l - \mathbb{E}\ \nu_l)$$

is valid for arbitary random variables ν_i , (14), (33)and (36) imply

Proposition 2.1 *Let* $\min_{i\neq j}|x_i - x_j| > s$, *and* $s \le (\log N)^{1/2}$. *Then*

$$\mathbb{E}\prod_1^{2k}(\chi(x_i,dx_i,s) - \mathbb{E}\chi(x_i,dx_i,s)) =$$

$$
\begin{aligned}
= \quad & dx_1 \cdots dx_{2k} \Big(\sum_{m=0}^{N-2k} \frac{(-1)^m}{m!} \int \rho_{2k+m,2k}^{(N)}(x_1, \ldots x_{2k}; \ldots x_{2k+m}) dx_{2k+1} \ldots dx_{2k+m} \\
+ \quad & O\big(\big(\frac{const(s) \log^k N}{N}\big)^N \cdot N^{2k}\big)\Big) \qquad (37)
\end{aligned}
$$

the variables $x_{2k+1}, \ldots x_{2k+m}$ *are integrated over* $\left(\bigsqcup_1^{2k} [x_j, x_j + s] \right)^m$

Remark The remainder term is of the same nature as in (18) and is treated similaly

The Proposition will play the central role in our proof, leading to (30),(31).
To make our arguments more visible, we associate with any permutation $\sigma \in S_{2k+m}$ an oriented graph $\mathcal{J}(\sigma)$. By definition the vertices of $\mathcal{J}(\sigma)$ are integers $1, \ldots 2k$ (particles $x_1, \ldots x_{2k}$) and there is a directed bond from the j^{th} particle to the l^{th} particle , $l \neq j$, iff (35) is satisfied.
Then in our notations

$$
\rho_{2k+m,2k}^{(N)}(\bar{x}) = \sum_{\sigma \in S_{2k+m}}^{*} (-1)^\sigma \prod_{i=1}^{2k+m} \frac{\sin \pi(x_i - x_{\sigma(i)})}{N \sin(\pi(x_i - x_{\sigma(i)})/N)}
$$

where $\sum^{*}$ is the sum over such σ that any maximal connected component of $\mathcal{J}(\sigma)$ has at least two elements. For future considerations it is also useful to define $\tilde{\rho}_{2k+m,2k}(x_1, \ldots x_{2k}, \ldots x_{2k+m})$ as the sum over σ for which $\mathcal{J}(\sigma)$ is connected. We claim that the main contribution to (37) comes from the interaction between the pairs of particles. Representing $\mathcal{J}(\sigma)$ as a disjoint union of maximal connected components

$$
\mathcal{A}_1, \ldots \mathcal{A}_p \quad : \quad \bigsqcup_1^p \mathcal{A}_q \;=\; \{1, \ldots 2k\}
$$

and denoting $(x_i)_{i \in \mathcal{A}_q}$ by $\bar{x}(\mathcal{A}_q)$ we obtain the representation of $\rho_{2k+m,2k}(\bar{x})$ as the sum of products

$$\sum_{\substack{(\mathcal{A}_1,\dots\mathcal{A}_p)\\ |\mathcal{A}_q|\geq 2\ q=1,\dots p}} \prod_1^p \tilde{\rho}_{\sum_{j\in\mathcal{A}_q} n(x_j),\ |\mathcal{A}_q|} \Big(\bigcup_{j\in\mathcal{A}_q} \bar{x}^{(j)} \Big) \tag{38}$$

(37) and (38) give us

$$\begin{aligned} &\mathbb{E} \prod_1^{2k} (\chi(x_i, dx_i, s) - \mathbb{E}\chi(x_i, dx_i, s)) = \\ = & \prod_1^{2k} dx_i \Big(\sum^{**}_{\substack{(\mathcal{A}_1,\dots\mathcal{A}_p)\\ |\mathcal{A}_q|\geq 2, q=1,\dots p}} \prod_1^p \\ \cdot & \Big(\sum_{m=0}^{N-|\mathcal{A}_q|} \frac{(-1)^m}{m!} \int\limits_{(\sqcup_{j\in\mathcal{A}_q}[x_j, x_j+s])^m} \tilde{\rho}_{|\mathcal{A}_q|+m, |\mathcal{A}_q|}(\bar{x}(\mathcal{A}_q), y_1, \dots y_m) dy_1 \dots y_m \Big) \\ + & O\big(\big(\frac{const(s) \log^k N}{N} \big)^N \cdot N^{2k} \big) \Big) \end{aligned} \tag{39}$$

where the sum $\overset{**}{\sum}$ is over all partitions of $\{1, \dots 2k\}$ into two- and more element subsets.

Since $\tilde{\rho}_{2+m,2}(\bar{x}) = \rho_{2+m,2}(\bar{x})$
the sum over partitions into the two-element subsets is exactly

$$\sum_{\substack{partitions\\ into\ pairs}} \prod_{(i,j)} \text{Cov}\ (\chi(x_i, dx_i, s), \chi(x_j, dx_j, s)) \tag{40}$$

To estimate the remaining part, let us introduce the notation

$$\begin{aligned} & r_{\ |\mathcal{A}_q|}\ (\bar{x}(\mathcal{A}_q), s) := \\ = & \sum_{m=0}^{N-|\mathcal{A}_q|} \frac{(-1)^m}{m!} \int\limits_{(\sqcup_{j\in\mathcal{A}_q}[x_j, x_j+s])^m} \tilde{\rho}_{\ |\mathcal{A}_q|+m, |\mathcal{A}_q|}\ (\bar{x}(\mathcal{A}_q), y_1, \dots y_m) dy_1 \dots y_m \end{aligned} \tag{41}$$

The next lemma,together with (39) clearly implies (30), (31) :

Lemma 2.2

$$\int\limits_{\substack{[0,N]^l \\ \min_{i\neq j}|x_i-x_j|_1>s}} r_l(x_1,\dots x_l,s)dx_1\dots dx_l = \begin{cases} \int\limits_{|x|>s} g_N(s,x)dx\cdot N + o(N^\varepsilon) & if \quad l=2 \ , \\ o(N^{1+\varepsilon}) & if \quad l>2 \end{cases} \tag{42}$$

for any $\varepsilon > 0$.

Proof:

The case $l=2$ was considered above when we calculated the variance of $\eta_N(s)$. Assume now $l>2$ and denote by $r_{l,m}$ the m-th term in (41).

We are looking for the estimates on $\tilde{\rho}_{l+m,l}$ similar to (21) , (22). Since $\tilde{\rho}_{l+m,l}$ can be obtained by addition, substraction and multiplication the finite number of times (depending only on l) the determinants of the form

$$\det\Big(\frac{\sin\pi(x_i-x_j)}{N\sin(\pi(x_i-x_j)/N)}\Big)$$

(20) provides the estimates

$$|\tilde{\rho}_{l+m,l}(x_1,\dots x_l,\dots x_{l+m})| < const_l \tag{43}$$

and

$$\int\limits_{[0,N]^l} |r_{l,m}\ (x_1,\dots x_l)|\ dx_1\dots dx_l \ \le\ const_l\frac{(l\ s)^m}{m!}\ N^l \tag{44}$$

To get an estimate, similar to (22) we write by definition :

$$\tilde{\rho}^{(N)}_{l+m,l}\ (x_1,\dots x_l,\dots x_{l+m}) = \sum_{\substack{\sigma\in S_{l+m} \\ \mathcal{J}(\sigma)\ is\ connected}} (-1)^\sigma \prod_{i=1}^{l+m} \frac{\sin\pi(x_i-x_{\sigma(i)})}{N\sin(\pi(x_i-x_{\sigma(i)})/N)}$$

Consider an arbitary term from this sum. Our goal is to estimate the absolute value of

$$\prod_{i=1}^{l+m} \frac{\sin \pi (x_i - x_{\sigma(i)})}{N \sin(\pi (x_i - x_{\sigma(i)})/N)}$$

by

$$const^{l+m}(s) \prod_{1}^{l} \frac{2}{1 + |x_j - x_{\tau(j)}|} \tag{45}$$

where τ is some cyclic permutation of integers $1, \dots l$ (particles $x_1, \dots x_l$), depending on σ and partition (35).

To do this we will replace

$$\frac{\sin \pi (x_i - x_{\sigma(i)})}{N \sin(\pi (x_i - x_{\sigma(i)})/N)}$$

by 1

whenever x_i, $x_{\sigma(i)}$ belong to the same segment $[x_j, x_j + s]$, $j = 1, \dots l$, and we will replace it by $2/(1 + |x_i - x_{\sigma(i)}|)$ in the opposite case.

If we write σ as a product of disjoint cyclic permutations

$$\sigma = \sigma_1 \cdot \dots \sigma_m$$

each σ_p , $p = 1, \dots m$ determines some cyclic excursion on the graph $\mathcal{J}(\sigma)$, with every step of the excursion corresponding to some term $2/(1 + |x_i - x_{\sigma(i)}|)$ in our estimate. Since the graph $\mathcal{J}(\sigma)$ is connected, the path of every excursion intersects the path of some other excursion, and after several switches we can go from one path to another (otherwise we would have a nontrivial maximal connected component of $\mathcal{J}$).

Therefore we can combine these paths into one big cyclic path (with possible selfintersections), going along which we will visit all vertices of $\mathcal{J}$. Again, to each step of the path, $j \longrightarrow l$ there corresponds a term

$$\frac{2}{1 + |x_i - x_{\sigma(i)}|} \quad : \quad x_i \in [x_j, x_j + s], \quad x_{\sigma(i)} \in [x_l, x_l + s]$$

in our estimate.

The whole number of steps of the constructed path is at most [3] $l + m$. Now we will

---

[3] See Remark after the end of the Lemma's proof

eliminate all possible selfintersections. If j is the current position of our walk, n is the previous one and l is the next one, and vector j has been already visited before, we replace two steps $n \longrightarrow j$, $j \longrightarrow l$ by one $n \longrightarrow l$ in the new ,modified walk.
Using the inequalities

$$2/(1+|x_i - x_{\sigma(i)}|) \le const(s)\ 2/(1+|x_j - x_l|)$$
$$2/(1+|x-y|)\ 2/(1+|y-z|) \le 2\cdot 2/(1+|x-z|)$$

and subsequently eliminating from the path sites visited before, we finally obtain the path without selfintersections,given by some cyclic permutation $\tau \in S_l$.
This leads to (45) and the inequality

$$\int\limits_{(\sqcup_{j=1}^{l}\ [x_j,x_j+s]\ )^m} \left| \prod_{i=1}^{l+m} \frac{\sin \pi(x_i - x_{\sigma(i)})}{N \sin(\pi(x_i - x_{\sigma(i)})/N)} \right| dx_{l+1}\dots dx_{l+m}$$
$$\le \quad (l\cdot s)^m (const(s))^{l+m} \sum_{\tau\in S_l} \prod_{1}^{l} \frac{2}{1+|x_j - x_{\tau(j)}|}$$

The inequality

$$\int\limits_0^N 2/(1+|x-y|)\cdot 2/(1+|y-z|)\ dy \le\ const\ \log(N+1)\ 2/(1+|x-z|)$$

implies

$$\int\limits_{[0,N]^l} \prod_{j=1}^{l} 2/(1+|x_j - x_{\tau(j)}|)\ dx_1\dots dx_l \le\ const\ N\log^{l-2}(N+1)$$

and

$$|\int\limits_{[0,N]^l} r_{l,m}\ (x_1,\dots x_l)\ dx_1\dots dx_l| \ \le\ \frac{1}{m!} l!(l+m)!(l\ s)^m (const(s))^{l+m} N\log^{l-2}(N+1) \quad (46)$$

Finally,to get an estimate (42) of

$$|\int\limits_{[0,N]^l} r_l\ (x_1,\dots x_l)\ dx_1\dots dx_l| \ \le \sum_0^{\infty} \int\limits_{[0,N]^l} |r_l\ (x_1,\dots x_l)|\ dx_1\dots dx_l$$

one can use (46) for $m < const(s,l,\varepsilon)\log(N+1)$
where $const(s,l,\varepsilon)$ is small enough , and (44) for $m \geq const(s,l,\varepsilon)\log(N+1)$
∎

(39) ,(40) ,(42) give us the following result :

$$\mathbb{E} \int\limits_{\substack{[0,N]^{2k} \\ \min_{i\neq j}|x_i-x_j|_1>s}} \prod_1^{2k}(\chi(x_i,dx_i,s)-\mathbb{E}\chi(x_i,dx_i,s))$$

$$= \quad (2k-1)!!\ (\int\limits_{|x|>s} g_N(s,x)dx\ )^k\ N^k\ +\ O\ \Big(\ const(s,\varepsilon)N^{k-1+\varepsilon}\Big) \tag{47}$$

Remark The trivial bound for the number of steps of the path (at most $l+m$) is enough for our purposes now. However in the proof of Corollary 1.5 we need an estimate that the number of steps is bounded by some number, depending only on l .
To acomplish this we have to eliminate some loops of the path (i.e. replace the corresponding multipliers by 1). Namely we eliminate a loop, if after throwing it out, we still have a closed path,visiting all vertices of $\mathcal{J}$. After such procedure completed we arrive at the path with at most $\sum_1^{l-1} j\ + l - 1 = l(l+1)/2\ -1$ steps.

In the second part of the proof of Theorem 1.1 we take into account the contributions to $\mathbb{E}\ (\eta_N(s) - \mathbb{E}\ \eta_N(s))^{2k}$ from the diagonal ($x_i = x_j$) and "near-diagonal"
($0 < |x_i - x_j|_1 \leq s$) terms.
We introduce an equivalence relation on the set of particles $\{x_1, \ldots x_{2k}\}$, calling $x_i,\ x_j$ the "neighbors", if there is a sequence of particles

$$x_{i_0} = x_i, x_{i_1}, \ldots x_{i_p} = x_j$$

such that

$$\max_{r=o,\ldots p-1}\ |x_{i_{r+1}} - x_{i_r}|_1 \leq s. \tag{48}$$

We claim that the contributions to $\mathbb{E}\ (\eta_N(s) - \mathbb{E}\ \eta_N(s))^{2k}$ of order of N^k appear only when each equivalence class of (48) contains one or two particles.
Assume that we have l two-element equivalence classes, say

$$\{x_1, x_2\}, \ldots \{x_{2l-1}, x_{2l}\}$$

and $2k-2l$ one-element equivalence classes $\{x_{2l+1}\}, \ldots \{x_{2k}\}$.
Since

$$\chi(x_i, dx_i, s)\,\chi(x_j, dx_j, s) = 0$$

if $0 < |x_i - x_j|_1 \leq s$ and always

$$\chi^2(x_i, dx_i, s) = \chi(x_i, dx_i, s)$$

$$\begin{aligned}
&\mathbb{E} \int\limits_{\substack{[0,N]^{2k} \\ \{x_1,x_2\},\ldots\{x_{2l-1},x_{2l}\} \\ \{x_{2l+1}\},\ldots\{x_{2k}\}}} \prod_{1}^{2k} \left(\chi(x_i, dx_i, s) - \mathbb{E}\chi(x_i, dx_i, s)\right) \\
= \ &\mathbb{E} \int\limits_{\substack{[0,N]^{2k} \\ \{x_1,x_2\},\ldots\{x_{2l-1},x_{2l}\} \\ \{x_{2l+1}\},\ldots\{x_{2k}\}}} \prod_{i=1}^{l} \Big[-\mathbb{E}\chi(x_{2i-1}, dx_{2i-1}, s)\ \mathbb{E}\ \chi(x_{2i}, dx_{2i}, s) \\
&\qquad - \left(\chi(x_{2i-1}, dx_{2i-1}, s) - \mathbb{E}\chi(x_{2i-1}, dx_{2i-1}, s)\right)\ \mathbb{E}\ \chi(x_{2i}, dx_{2i}, s) \\
&\qquad - \left(\chi(x_{2i}, dx_{2i}, s) - \mathbb{E}\chi(x_{2i}, dx_{2i}, s)\right)\ \mathbb{E}\ \chi(x_{2i-1}, dx_{2i-1}, s) \\
&\qquad + \mathbb{E}\ \chi(x_{2i}, dx_{2i}, s)\ \delta(x_{2i-1} - x_{2i})\ dx_{2i-1} \Big] \cdot \\
&\qquad \cdot \prod_{j=2l+1}^{2k} \left(\chi(x_j, dx_j, s) - \mathbb{E}\chi(x_j, dx_j, s)\right) \\
= \ &\left(F_N(s) - 2sF_N^2(s)\right)^l (2k-2l-1)!!\ \Big(\int\limits_{|x|>s} g_N(s,x)dx\Big)^{k-l} N^k + O\left(const(s)N^{k-1+\varepsilon}\right)
\end{aligned}$$

All such choises of equivalence classes produce

$$\sum_{l=0}^{k} \frac{(2k)!}{(2l)!(2k-2l)!} \frac{(2l)!}{l!\ 2^l} \left(F_N(s) - 2sF_N^2(s)\right)^l \frac{(2k-2l)!}{(k-l)!\ 2^{k-l}} \Big(\int\limits_{|x|>s} g_N(s,x)dx\Big)^{k-l} \cdot N^k$$

$$
\begin{aligned}
&+ \quad O(const(s,\varepsilon)N^{k-1+\varepsilon}) \\
&= \quad (2k-1)!! \left(F_N(s) - 2sF_N^2(s) + \int\limits_{|x|>s} g_N(s,x)dx \right)^k N^k \\
&+ \quad O(const(s,\varepsilon)N^{k-1+\varepsilon})
\end{aligned}
\tag{49}
$$

If some equivalence classes have three elements or more, the contributing terms to $\mathbb{E}\ (\eta_N(s) - \mathbb{E}\ \eta_N(s))^{2k}$ will be bounded by some power (say l) of $F_N(s) \int\limits_0^N 1dx$,multiplied by the n-dimensional integral ($n < 2k - 2l$)

$$
\int\limits_{\substack{[0,N]^{2k} \\ \min_{i\neq j} |x_i - x_j|_1 > s}} |\mathbb{E} \prod_1^n (\chi(x_i, dx_i, s) - \mathbb{E}\chi(x_i, dx_i, s))|
$$

and multiplied by the areas of some polyhedrons of size s . It follows from (30) , (31) that they are of order of $O(N^{k-1})$. Thus

$$
\begin{aligned}
\mathbb{E} \left((\eta_N(s) - \mathbb{E}\ \eta_N(s)) / N^{1/2} \right)^{2k} &= (2k-1)!!\ (b_N(s,s))^k + O(N^{-1+\varepsilon}) \\
&= (2k-1)!!\ (b(s,s))^k + o(1)
\end{aligned}
\tag{50}
$$

Similar calculations yield

$$
\mathbb{E} \left((\eta_N(s) - \mathbb{E}\ \eta_N(s)) / N^{1/2} \right)^{2k+1} = o(N^{-1/2+\varepsilon})
$$

The convergence of the mixed moments

$$
\mathbb{E} \prod_1^n \left((\eta_N(s_i) - \mathbb{E}\ \eta_N(s_i)) / N^{1/2} \right)
$$

to the moments of Gaussian random process $\xi(s)$ can be proven in the same way. Since the convergence of all moments to the gaussian ones implies the convergence of finite dimensional distributions, Theorem 1 is proven. ■

3 Proof of Theorem 1.2 and Corollaries

We start with the proof of Theorem 1.2 . Since

$$\left|\tilde{\xi}_N(s) - \xi_N(s)\right| \leq N^{-1/2} \tag{51}$$

the finite-dimensional distributions of $\tilde{\xi}_N(\cdot)$ also converge to those of limiting Gaussian process $\xi_N(\cdot)$ as $N \to \infty$, and for the functional convergence of probability distributions of $\tilde{\xi}_N(\cdot)$ on $C\,[0,\infty)$ it is enough to prove the tightness (relative compactness) of any sequence of distributions of $\tilde{\xi}_{N_n}(\cdot)$ on $C\,[0,T]$ as $N_n \to \infty$, T is arbitary large ([20]).

Let us define for any $f \in C\,[0,T]$ and $\delta > 0$, the modulus of continuity of the function f as

$$\omega_f(\delta) = \sup\,\{\ |f(s) - f(t)|\ :\ 0 \leq s,t \leq T,\ |s-t| < \delta\ \}$$

The classical criterion of relative compactness ([20]) tells that the family $\{\mathcal{P}\}$ of probability measures on $C[0,T]$ is relatively compact iff

(i) for each arbitary small positive α there exists an $A(\alpha)$, such that

$$\mathcal{P}\,\{f:\ |f(0)| > A\ \} < \alpha\ \ ,\ \ for\ \ any\ \ \mathcal{P}. \tag{52}$$

(ii) for each $\alpha,\ \beta > 0$ there exists some $\delta(\alpha,\ \beta)$ such that

$$\mathcal{P}\,\{f:\ \omega_f(\delta) > \beta\ \} <\ \alpha\ \ ,\ \ for\ \ any\ \ \mathcal{P}\ . \tag{53}$$

The results of O.Costin and J.Lebowitz ([6]) tell us that

$$\tilde{\xi}_N(0) = O\left((\log N/N)^{1/2}\right)$$

which gives us (i). To prove (ii) we need

Lemma 3.1 *There exist some constants* c_1, c_2 *depending on* $T > 0$, *such that*

$$\begin{aligned} \mathcal{P}_N \quad & \left\{\ |\tilde{\xi}_N(t) - \tilde{\xi}_N(s)| <\ c_1(|t-s|^{1/20} + N^{-1/4})\ ,\ \forall\ t,s \in [0,T] : |t-s| < \delta\ \right\} \\ > \quad & 1 -\ c_2\,(\delta^{4/5} + N^{-1/20}\log N) \end{aligned} \tag{54}$$

Assuming Lemma 3.1 is proven , we can quickly finish the proof of Theorem 1.2 :
Let us choose N_*, δ_* such that

$$\begin{array}{rcl} c_1 N_*^{-1/4} < \beta/2 & , & c_1 \delta_*^{1/20} < \beta/2 \\ c_2 \log N_* \ N_*^{-1/20} < \alpha/2 & , & c_2 \delta_*^{4/5} < \alpha/2 \end{array}$$

For any fixed probability distribution $\mathcal{P}$ on $C[0,T]$ and arbitary $\alpha\ ,\beta > 0$ one can find some $\delta(\alpha, \beta, \mathcal{P}) > 0$ such that

$$\mathcal{P}\ \{f : \ \omega_f(\delta) > \beta\ \} < \ \alpha$$

Let us choose such a δ for any $\tilde{\xi}_{N_i}$, $N_i < N_*$, and define the final δ as the minimum of such $\delta' s$ and δ_* .
Condition (ii) is satisfied .The family of probability disributions given by $\tilde{\xi}_{N_n}(\cdot)$ is tight.
Now we will prove Lemma 3.1 .
First, let us note that if t, s belong to some interval of the length $c_3 N^{-3/4}$, $c_3 \leq 1$: $s, t \in [s', s' + c_3\, N^{-3/4}]$ then

$$|\tilde{\xi}_N(t) - \tilde{\xi}_N(s)| \leq |\tilde{\xi}_N(s' + c_3\, N^{-3/4}) - \tilde{\xi}_N(s')| + const\, N^{-1/4} \tag{55}$$

Indeed, by definition, for $s' \leq s \leq t \leq s' + c_3\, N^{-1/4}$

$$\begin{array}{rcl} 0 & \leq & \tilde{\xi}_N(t) - \tilde{\xi}_N(s) + F_N(t)\, N^{1/2} - F_N(s)\, N^{1/2} \\ & \leq & \tilde{\xi}_N(s' + c_3 N^{-3/4}) - \tilde{\xi}_N(s') + F_N(s' + c_3 N^{-3/4})\, N^{1/2} - F_N(s')\, N^{1/2} \end{array}$$

which implies

$$|\tilde{\xi}_N(t) - \tilde{\xi}_N(s)| \leq |\tilde{\xi}_N(s' + c_3\, N^{-3/4}) - \tilde{\xi}_N(s')| + N^{1/2} \cdot 2\ Variation_{[s', s' + c_3 N^{-3/4}]}\Big(F_N(s)\Big)$$

Since $F_N(s)$ are uniformly continuously differentiable functions of s , the last inequality proves (55) . Now we divide the segment $[0, T]$ into 2^k disjoint subsegments

$$\Delta_l^{(k)} = [l\ T/2^k, (l+1)T/2^k], \qquad l = 0, 1, \ldots 2^k - 1$$

with k ranging from 1 to $\left[(\log T + \frac{3}{4}\log N)/\log 2\right] + 1$ (i.e. $|\Delta_l^{(k)}| \geq \frac{1}{2}N^{-3/4}$). Using Chebyshev inequality and (50) :

$$\mathcal{P}_N\left(|\xi_N(t) - \xi_N(s)| > |t-s|^{1/20}\right) \leq \frac{\mathbb{E}|\xi_N(t) - \xi_N(s)|^4}{|t-s|^{1/5}}$$
$$\leq \frac{3\left(b_N(t,t) - b_N(s,t) - b_N(t,s) - b_N(s,s)\right)^2 + const(T,\varepsilon)N^{-1+\varepsilon}}{|t-s|^{1/5}}$$

Since partial derivatives of $b_N(s,t) - F_N(s \vee t)$, $F_N(s)$ are uniformly bounded on any finite interval, we arrive at the inequality (consider $\varepsilon = 1/20$) :

$$\mathcal{P}_N\left(|\xi_N(t) - \xi_N(s)| > |t-s|^{1/20}\right) \leq const(T)\left((t-s)^2 + N^{-19/20}\right)/|t-s|^{1/5}$$

where $t, s \in [0,T]$
which gives us (we denote all constants appearing in our calculations by $const(T)$) :

$$\mathcal{P}_N \left(\bigsqcup_{l=0}^{2^k-1}\left\{|\xi_N((l+1)T/2^k) - \xi_N(l\,T/2^N)| > (T/2^k)^{1/20}\right\}\right) \leq$$
$$\leq 2^k\, const(T)\,\frac{(T/2^k)^2 + N^{-19/20}}{(T/2^k)^{1/5}} \leq const(T)\ \left((2^{-k})^{4/5} + N^{-19/20}(2^k)^{6/5}\right)$$
$$\leq const(T)\ \left((2^{-k})^{4/5} + N^{-19/20}N^{18/20}\right) \leq const(T)\ \left((2^{-k})^{4/5} + N^{-1/20}\right)$$

and

$$\mathcal{P}_N \left(\bigsqcup_{k=k_0}^{[\frac{\log T + \frac{3}{4}\log N}{\log 2}]+1}\ \bigsqcup_{l=0}^{2^k-1}\left\{|\xi_N((l+1)T/2^k) - \xi_N(l\,T/2^N)| > (T/2^k)^{1/20}\right\}\right)$$
$$\leq const(T)\left((2^{-k_0})^{4/5} + N^{-1/20}\log N\right) \tag{56}$$

Choosing k_0 such that $2^{-k_0+1} \leq \delta \leq 2^{-k_0+2}$ and combining (56) with (55) and (51) , we finish the proof. ∎

Corollary 1.3 can be proven by using the same machinery as in Theorem 1.1. To prove Corollary 1.4 consider the continuous functional on $C[0,\infty)$:

$$G_T(f) = \sup_{s\in[0,T]} |f(s)|$$

apply Theorem 1.2 and take into account

$$\lim_{N\to\infty} \sup_{s\in[0,T]} |\tilde{\xi}_N(s) - \xi_N(s)| = 0$$

Before we proceed with the proof of Corollary 1.5 , we want to obtain an estimate on

$$\sup_{[0,\infty)} |\mathbb{E}\ \eta(I_N, s) - F(s)\ \frac{N\,|I_N|}{2\pi}|$$

Consider first the case

$$s \le (\log N)^{1/2} \qquad : \tag{57}$$

$$\begin{aligned} |F_N(s) - F(s)| &= \sum_{n=0}^{N-1} \frac{(-1)^n}{n!} \int_{[0,s]^n} \rho_{n+1}^{(N)}(0,\bar{x}) - \rho_{n+1}^{(\infty)}(0,\bar{x})\ d\bar{x} \\ &+ \sum_{n=N}^{\infty} \frac{(-1)^{n-1}}{n!} \int_{[0,s]^n} \rho_{n+1}^{(\infty)}(0,\bar{x})\ d\bar{x} \\ &\le \sum_{n=0}^{N-1} \frac{1}{n!}\ Area\ ([0,s]^n) \sup_{\bar{x}\in[0,s]^n} |\rho_{n+1}^{(N)}(0,\bar{x}) - \rho_{n+1}^{(\infty)}(0,\bar{x})| \\ &+ \sum_{n=N}^{\infty} \frac{1}{n!}\ Area\ ([0,s]^n)\cdot 1 \end{aligned}$$

Using the inequality

$$\sup_{|x|\le s} \left|\frac{\sin \pi x}{N\sin(\pi x/N)} - \frac{\sin \pi x}{\pi x}\right| < const\ (s/N)^2$$

and the representation of $\rho_{n+1}^{(N)}$, $\rho_{n+1}^{(\infty)}$ as determinants (6) , (7) we have

$$\sup_{\bar{x}\in[0,s]^n} |\rho_{n+1}^{(N)}(0,\bar{x}) - \rho_{n+1}^{(\infty)}(0,\bar{x})| \le const\ (n+1)!\,(n+1)\,(s/N)^2$$

Since $\rho_{n+1}^{(N)}, \ \rho_{n+1}^{(\infty)}$ are not greater than 1 by absolute value, we finally arrive at

$$\begin{aligned} |F_N(s) - F(s)| &\leq \sum_{n=0}^{N-1} \frac{s^n}{n!} \min\Big(2, \ const \ (n+1)!\,(n+1)\,(s/N)^2\Big) + \sum_{N}^{\infty} \frac{s^n}{n!} \\ &= o((s/N)^{1-\varepsilon}) + O(s^N / \, N!) \end{aligned}$$

for any $\varepsilon > 0$. Thus

$$\sup_{s \leq (\log N)^{1/2}} |F_N(s) - F(s)| = o(N^{-1+\varepsilon}) \tag{58}$$

The function $F(s)$ decays at infinity superexponentially :

$$\log F(s) = -\pi^2 s^2/8 + O(s) \tag{59}$$

(see [13],[12],[14]),that gives us

$$F((\log N)^{1/2}) = o(N^{-9/8}) \tag{60}$$

Since $0 \leq F_N(s) \leq F_N((\log N)^{1/2})$, $0 \leq F(s) \leq F((\log N)^{1/2})$ *for any* $s \geq (\log N)^{1/2}$ (58) , (59) establish

$$\begin{aligned} &\sup_{[0,\infty)} |F_N(s) - F(s)| = o(N^{-1+\varepsilon}) \\ &\sup_{[0,\infty)} |\mathbb{E}\ \eta(I_N, s) - F(s)\ \frac{N|I_N|}{2\pi}| = o(N^{\varepsilon} \frac{|I_N|}{2\pi}) \end{aligned} \tag{61}$$

We finish this section with the proof of Corollary 1.5 .
Since the tails of distribution functions $F(s)$, $F_N(s)$ are small enough (see (58) , (60)), we have to prove only that

$$\sup_{0 \leq s \leq (\log N)^{1/2}} |\eta(I_N, s) - \mathbb{E}\ \eta(I_N, s)| = o\left((N\, \frac{|I_N|}{2\pi})^{1/2\ +\varepsilon} \right)$$

To do this we will estimate the moments of $\eta(I_N, s) - \mathbb{E}\ \eta(I_N, s)$, when s is slightly growing up , $s \leq (\log N)^{1/2}$.

Lemma 3.2 *For any even integer* $2k$ *, arbitary small* $\varepsilon > 0$ *and* $0 \leq s \leq (\log N)^{1/2}$ *there exists some constant* $c(\varepsilon, 2k)$ *, depending only on* ε *and* $2k$ *, such that*

$$\mathbb{E}\ (\eta(I_N, s) - \mathbb{E}\ \eta(I_N, s))^{2k} \ \leq c(\varepsilon, 2k)\ \left(N\ \frac{|I_N|}{2\pi}\right)^{k+\varepsilon}$$

Proof:
Again, without a loss in generality, we can assume I_N to be a unit circle, $I_N = [-\pi, \pi]$.

Examinating the proof of Theorem 1.1 we realize that all what we need is the following generalization of estimates (42) from Lemma 2.2 :

$$\sup_{0 \leq s \leq (\log N)^{1/2}} \Bigg| \int\limits_{\substack{[0,N]^l \\ \min_{i \neq j} |x_i - x_j|_1 > s}} r_l(x_1, \dots x_l, s) dx_1 \dots dx_l \Bigg| \leq const(\varepsilon, l)\ N^{l/2\ +\varepsilon} \quad , l \geq 2 \tag{62}$$

Going along the lines of calculations from Theorem 1.1 ,we will clarify the dependence on s of the constants appearing there. We remind that we defined r_l in (41) as

$$r_l(x_1, \dots x_l, s) = \sum_{m=0}^{N-l} \frac{(-1)^m}{m!} \int \tilde{\rho}^{(N)}_{l+m,l}(x_1, \dots x_l; \dots x_{l+m}) dx_{l+1} \dots dx_{l+m} \tag{63}$$

(the integration is over $\left(\sqcup_1^l\ [x_i, x_i + s]\ \right)^m$).
For simplicity we will consider cases $l = 2$ and $l > 2$ separately.
Let us first l equal 2. Rewrite (63) :

$$r_l(x_1, x_2, s) = \sum_{m=0}^{N-2} \frac{(-1)^m}{m!} \int\limits_{(\sqcup_1^2 [x_i, x_i+s])^m} \rho^{(N)}_{2+m,2}(x_1, x_2, \dots x_{2+m}) dx_3 \dots dx_{2+m}$$

The estimates (21) , (22) give us

$$|\rho_{2+m,2}(\bar{x})| \leq \min\left(2,\ (m+2)!\ \frac{2}{1 + \max\left(|x_1 - x_2|_1 - s,\ 0\right)^2}\right)$$

Using the inequality

$$1/\left(1+\max(x-s,0)^2\right) \leq (2+s^2)/(1+x^2) \tag{64}$$

we obtain

$$\begin{aligned}
|\rho_{2+m,2}\ (\bar{x})| &\leq \ \min\left(2;\ (m+2)!\ \frac{2\ (2+s^2)}{1+|x_1-x_2|_1^2}\right) \\
&\leq \ 2\ (2+\log N)\ \min\left(1;\ \frac{(m+2)!}{1+|x_1-x_2|_1^2}\right) \\
|r_2(x_1,x_2)| &\leq \ \sum_{m=0}^{N-2} \frac{1}{m!}(2s)^m\ 2\ (2+\log N)\ \min\left(1;\ \frac{(m+2)!}{1+|x_1-x_2|_1^2}\right)
\end{aligned}$$

Fix some $\varepsilon > 0$. If $|x_1-x_2|_1 < N^{\varepsilon/2}$, we arrive at

$$|r_2(x_1,x_2)| \leq 2\ (2+\log N)\ \exp(2s) = 2\ (2+\log N)\ \exp(2(\log N)^{1/2}) \tag{65}$$

If $|x_1-x_2|_1 \geq N^{\varepsilon/2}$, then for any $\varepsilon_1 > 0$

$$(2s)^m = o(|x_1-x_2|_1^{1+\varepsilon_1}) \quad if \quad (m+2)! \sim\ 1+|x_1-x_2|_1^2$$

and

$$|r_2(x_1,x_2)| \leq 2\ (2+\log N)\ const(\varepsilon)\left(\frac{1}{1+|x_1-x_2|_1^2}\right)^{1/2-\varepsilon/4} \tag{66}$$

Integrating the r.h.s. of (65) , (66) over

$$\{x_1,x_2 \in [0,N],\ \ |x_1-x_2| < (\geq) N^{\varepsilon/2}\}$$

we obtain the desired result. The case $l \geq 2$ will be treated in a similar fashion. Again the inequalities (43) , (45) are crucial in our calculations. We keep the former and slightly refine the latter. Namely, using the procedure described on the pages 17-18 , section 2 and Remark from the page 19 , we can estimate each term in $\tilde{\rho}_{l+m,l}$ as

$$\left|\prod_{1}^{l+m} \frac{\sin\pi(x_i-x_{\sigma(i)})}{N\sin(\pi(x_i-x_{\sigma(i)})/N)}\right| \leq \prod_{k=1}^{K} \frac{2}{1+\max\left(|x_{j_{k+1}}-x_{j_k}|_1-s,0\right)}$$

where $\sigma \in S_{l+m}$ is such, that $\mathcal{J}(\sigma)$ is connected, and

$$j_1 \rightarrow j_2 \ldots \rightarrow j_k \rightarrow j_1$$

is some closed path on $\mathcal{J}(\sigma)$ with possible intersections, visiting all vertices $\{1, \ldots l\}$, with the number of steps K not greater than $l(l+1)/2 - 1$. Using the inequality (64) and eliminating possible selfintersections as explained in section 2,we arrive at

$$\left|\prod_{i=1}^{l+m} \frac{\sin \pi(x_i - x_{\sigma(i)})}{N \sin(\pi(x_i - x_{\sigma(i)})/N)}\right| \leq (2\ (2+s^2))^{-1+(l+1)l/2} \prod_{j=1}^{l} \frac{2}{1+|x_j - x_{\tau(j)}|_1} \tag{67}$$

where τ is some cyclic permutation of integers $1, \ldots l$.
The estimates (43) , (67) imply

$$|\tilde{\rho}_{l+m,l}| \leq \sum_{\tau \in S_l} \min \left(const(l)\ ,\ (m+2)!\ (2\ (2+\log N))^{-1+(l+1)l/2} \prod_{j=1}^{l} \frac{2}{1+|x_j - x_{\tau(j)}|_1} \right)$$

and

$$\begin{aligned} |r_l(x_1, \ldots x_l, s)| \leq\ & const(l)\ (2\ (2+\log N))^{-1+(l+1)l/2} \cdot \\ \cdot \sum_{\tau \in S_l} \sum_{m=0}^{\infty} & \frac{1}{m!}\ (l\ s)^m\ \min(\ 1;\ (m+2)! \prod_{j=1}^{l} \frac{2}{1+|x_j - x_{\tau(j)}|_1}\ ) \end{aligned} \tag{68}$$

Fix $\tau \in S_l$. If

$$\prod_{j=1}^{l} (\ 1+|x_j - x_{\tau(j)}|_1) < N^{\varepsilon/(2l)} \tag{69}$$

the corresponding summand in (68) will be estimated from above by

$$const(l)\ (4+2\log N)^{(l+1)l/2-1}\ \exp(l\ (\log N)^{1/2}) \tag{70}$$

If

$$\prod_{j=1}^{l} (\ 1+|x_j - x_{\tau(j)}|_1) \geq N^{\varepsilon/(2l)} \tag{71}$$

then for any $\varepsilon_1 > 0$

$$(l\ s)^m = o(\prod_{j=1}^{l} (1 + |x_j - x_{\tau(j)}|_1)^{1/2+\varepsilon_1}) \ \ ,if \ \ (m+l)! \sim \prod_{j=1}^{l} (\frac{2}{1+|x_j - x_{\tau(j)}|_1})$$

and the corresponding summand is not greater than

$$const(l)\ (4 + 2\log N)^{(l+1)l/2-1}\ const(\varepsilon_1)\ (\ \prod_{j=1}^{l} \frac{1}{1+|x_j - x_{\tau(j)}|_1})^{1/2\ -\varepsilon_1} \tag{72}$$

for any $\varepsilon_1 > 0$. Integrating (70) , (72) over the domains (69) , (71) we obtain the desired result. Lemma 3.2 is proven . ∎

Via the Chebyshev inequality, Lemma 3.2 asserts that for any $n > 0,\ \varepsilon > 0$ there exists some constant, depending only on ε , n such that for any fixed s , $0 \leq s \leq (\log N)^{1/2}$

$$\mathcal{P}_N\ \{\ |\eta_N(s) - \mathbb{E}\ \eta_N(s)| > 1/3\ N^{1/2\ +\varepsilon}\} < \ const(\varepsilon, n)\ N^{-n} \tag{73}$$

Dividing the segment $[0, (\log N)^{1/2}]$ into $M = [(\log N)^{1/2} N^{3/4}]$ segments

$$[s_i, s_{i+1}] \quad , \ s_i = (\log N)^{1/2}\ i/M\ ;\ i = 0, \ldots M-1$$

we estimate the probability

$$\mathcal{P}_N\ \{\ \sup_{s_i} |\eta_N(s_i) - \mathbb{E}\ \eta_N(s_i)| > 1/2\ N^{1/2\ +\varepsilon}\}$$

by the sum of probabilities :

$$\mathcal{P}_N\ \{\ \sup_{s_i} |\eta_N(s_i) - \mathbb{E}\ \eta_N(s_i)| > 1/2\ N^{1/2\ +\varepsilon}\} < const(\varepsilon, n)\ (\log N)^{1/2}\ N^{-n\ +3/4} \tag{74}$$

To finish the proof we claim that variations of $\eta_N(s)$ on the segments $[s_i, s_{i+1}]$ are negligibly small,as well as the tails of $\eta_N(s),\ \mathbb{E}\ \eta_N(s)$, $when\ \ s \geq (\log N)^{1/2}$. Namely,similar to (55)

$$\begin{aligned} & Variation_{[s_i,s_{i+1}]}\ \eta_N(s) \leq |\eta_N(s_{i+1}) - \eta_N(s_i)| \\ \leq \ \ & |\eta_N(s_{i+1}) - \mathbb{E}\ \eta_N(s_{i+1})| + |\eta_N(s_i) - \mathbb{E}\ \eta_N(s_i)| + \ Variation_{[s_i,s_{i+1}]}\ \mathbb{E}\ \eta_N(s) \end{aligned} \tag{75}$$

and by (58) , (59) and smoothness of $F(s)$

$$
\begin{aligned}
Variation_{[s_i,s_{i+1}]}\ \mathbb{E}\ \eta_N(s) \ &\le \sup_{[0,(\log N)^{1/2}]} |\ \mathbb{E}\ \eta_N(s) - \ N\ F(s)| + \ N\ Variation_{[s_i,s_{i+1}]}\ F(s) \\
&= \ o(N^{\varepsilon}) + O(N^{1/4}) \qquad (76)
\end{aligned}
$$

Estimates (74) , (75) , (76) imply

$$
\mathcal{P}_N\ \{ \sup_{0\le s\le(\log N)^{1/2}} |\eta_N(s) - \mathbb{E}\ \eta_N(s)| > N^{1/2\ +\varepsilon}\} < const(\varepsilon, n)\ N^{-n+3/4}(\log N)^{1/2}
$$

Choosing $n > 7/4$ we have

$$
\sum_{N=1}^{\infty} \mathcal{P}\ \{ \sup_{[0,(\log N)^{1/2}]} |\eta_N(s) - \mathbb{E}\ \eta_N(s)| > N^{1/2\ +\varepsilon}\} < \ \infty
$$

and applying Borel-Cantelli lemma we find that with probability 1 there exists some integer N_0 , such that for any $N > N_0$

$$
\sup_{[0,(\log N)^{1/2}]} |\eta_N(s) - \mathbb{E}\ \eta_N(s)| \ \le N^{1/2\ +\varepsilon} \qquad (77)
$$

Since for $s > (\log N)^{1/2}$ we have

$$
\begin{aligned}
&\eta_N(s) \le \eta((\log N)^{1/2}) \ , \\
&\mathbb{E}\ \eta_N(s) \ = N\ F_N(s) \ \le N\ F_N((\log N)^{1/2}) \ , \\
&N\ F(s) \ \le N\ F((\log N)^{1/2})
\end{aligned}
$$

and

$$
\begin{aligned}
&N\ F((\log N)^{1/2}) = o(N^{-1/8}) \ , \\
&N\Big(F((\log N)^{1/2}) - F_N((\log N)^{1/2})\Big) \ = \ o(N^{\varepsilon})
\end{aligned}
$$

(see (60) , (61)) ,we can extend the sup in (77) to the whole real axis :

$$
\sup_{[0,\infty)} |\eta_N(s) - \mathbb{E}\ \eta_N(s)| \ \le const(\varepsilon)\ N^{1/2\ +\varepsilon}
$$

Corollary 1.5 is proven. ∎

4 Orthogonal and Symplectic Groups. Conclusive Remarks

The results of our paper are valid for the other Classical Compact Groups as well. The key factor here is the Vandermonde determinant nature of the density of the distribution function of eigenvalues. Formulas for the disribution of the eigenvalues with respect to the normalized Haar measure are classical(see ([5])).
However it has been noted only recently by N.Katz and P.Sarnak ([11]) that the corresponding n-point correlation functions have the form of determinants, similar to (6) . For the Unitary Group it was known for more than thirty years, back to pioneering papers by F.Dyson ,M.Gaudin and M.L.Mehta ([2] ,[4]). Below we write down the formulas for the distibution of the eigenvalues and n-point correlation functions for SO(2N), SO(2N+1), USp(2N),O$_-$(2N+2) .

The SO(2N) case

The eigenvalues of an element M in SO(2N) can be arranged in pairs :

$$\begin{aligned}&\exp(i\theta_1)\ ,\ \exp(-i\theta_1)\ ,\ldots\exp(i\theta_N)\ ,\ \exp(-i\theta_N)\\ &0\le\theta_1\le\theta_2\le\ldots\theta_N\le\pi\end{aligned}\tag{78}$$

The probability distribution of eigenvalues is defined by its density :

$$P_N(\theta_1,\ldots\theta_N)=2\ \left(\frac{1}{2\pi}\right)^N\prod_{1\le i<j\le N}(2\cos\theta_i-2\cos\theta_j)^2\tag{79}$$

In the rescaled coordinates

$$x_i=(2N-1)\frac{\theta_i}{2\pi}\ \ ;\ \ 0\le x_1\le\ldots x_N\le N-1/2$$

n-point correlation functions are equal to

$$\begin{aligned}&\mathcal{R}_n^{(N)}(x_1,\ldots x_n)\\ =\ &\det\left(\frac{\sin\pi(x_i-x_j)}{(2N-1)\sin(\pi(x_i-x_j)/(2N-1))}+\frac{\sin\pi(x_i+x_j)}{(2N-1)\sin(\pi(x_i+x_j)/(2N-1))}\right)_{i,j=1,\ldots n}\end{aligned}\tag{80}$$

Note the similarity of (80) to (6) . Since

$$\frac{\sin\pi(x_i+x_j)}{(2N-1)\sin(\pi(x_i+x_j)/(2N-1))}$$

is small when $x_i, x_j \gg 1$, n-point correlation function (80) can be considered as a small perturbation of

$$\det\left(\frac{\sin\pi(x_i-x_j)}{(2N-1)\sin(\pi(x_i-x_j)/(2N-1))}\right)_{i,j=1,\ldots n} \tag{81}$$

The SO(2N+1) case

The first $2N$ eigenvalues of the matrix M from SO(2N+1) can be arranged in pairs as in (78) . The $2N+1^{th}$ eigenvalue equals 1. The probability distribution of eigenvalues is defined by its density :

$$P_N(\theta_1,\ldots\theta_N) = (2/\pi)^N \prod_{1\le i<j\le N}(2\cos\theta_i - 2\cos\theta_j)^2 \prod_{i=1}^{N}\sin^2(\theta_i/2) \tag{82}$$

In the rescaled coordinates

$$x_i = N\theta_i/\pi \ \ ; \ \ 0\le x_1\le\ldots x_N\le N$$

n-point correlation functions are given by the formula

$$\mathcal{R}_n^{(N)}(x_1,\ldots x_n) = \det\left(\frac{\sin\pi(x_i-x_j)}{2N\sin(\pi(x_i-x_j)/2N)} - \frac{\sin\pi(x_i+x_j)}{2N\sin(\pi(x_i+x_j)/2N)}\right)_{i,j=1,\ldots n} \tag{83}$$

The USp(2N) case

The eigenvalues of an element M in USp(2N) can be arranged in pairs :

$$\exp(i\theta_1)\ ,\ \exp(-i\theta_1)\ ,\ldots\exp(i\theta_N)\ ,\ \exp(-i\theta_N)$$
$$0\le\theta_1\le\theta_2\le\ldots\theta_N\le\pi \tag{84}$$

The probability distribution of eigenvalues is defined by its density :

$$P_N(\theta_1, \dots \theta_N) = (2/\pi)^N \prod_{1 \le i < j \le N} (2\cos\theta_i - 2\cos\theta_j)^2 \prod_{i=1}^{N} \sin^2(\theta_i) \tag{85}$$

In the rescaled coordinates

$$x_i = (2N+1)\theta_i/(2\pi) \ \ ; \ \ 0 \le x_1 \le \dots x_N \le (2N+1)/2$$

n-point correlation functions are equal to

$$\begin{aligned} & \mathcal{R}_n^{(N)}(x_1, \dots x_n) \\ = & \ \det\left(\frac{\sin \pi(x_i - x_j)}{(2N+1)\sin(\pi(x_i - x_j)/(2N+1))} - \frac{\sin \pi(x_i + x_j)}{(2N+1)\sin(\pi(x_i + x_j)/(2N+1))}\right)_{i,j=1,\dots n} \end{aligned} \tag{86}$$

The $O_-(2N+2)$ case

The first 2N eigenvalues can be arranged in pairs, similar to (78) , the $(2N+1)^{th}$ and $(2N+2)^{th}$ eigenvalues are +1 and -1. The formulas for $P_N(\theta_1, \dots \theta_N)$, $\mathcal{R}_n^{(N)}(x_1, \dots x_n)$ coincide with those from the USp(2N) case.
The following universal result is valid for the all cases, considered above.

Proposition 4.1 *Let I_N be an arbitary subinterval of $[0, \pi]$ ($[-\pi, 0]$) ,such that the average number of eigenvalues hitting I_N tends to infinity (i.e. $N \ |I_N|/\pi \to \infty$). Then $(\eta(I_N, s) - \mathbb{E}\ \eta(I_N, s))/(N\ |I_N|/\pi)^{1/2}$ converges in finite-dimensional distributions to the Gaussian random process of Theorem 1.1. The analogues of Theorem 1.2 and Corollaries also hold.*

We have to examine two aspects of the proof of Theorem 1.1 : combinatorial and analytical. Since n-point correlation functions (80) , (83) , (86) still have the form

$$\det(K_N(x_i, x_j))_{i,j=1,\dots n}$$

all combinatorial considerations remain the same.From the analytical point of view, we will treat

$$\mathcal{R}_n^{(N)}(x_1, \dots x_n) = \det\left(\frac{\sin\pi(x_i - x_j)}{(2N+p)\sin(\pi(x_i - x_j)/(2N+p))} - \frac{\sin\pi(x_i + x_j)}{(2N+p)\sin(\pi(x_i + x_j)/(2N+p))} \right)$$

$p = -1, 0, 1$ as a small perturbation of

$$\rho_n^{(2N+p)}(x_1, \dots x_n) = \det\left(\frac{\sin\pi(x_i - x_j)}{(2N+p)\sin(\pi(x_i - x_j)/(2N+p))} \right)_{i,j=1,\dots n}$$

Namely, if $x_2, \dots x_{1+m} \in [x_1, x_1 + s]$, then

$$|\mathcal{R}_{1+m}^{(N)} - \rho_{1+m}^{(2N+p)}| \le \min\ (\ 2; (m+1)!\ (m+1)\ 2/(1+|x_1|)\ )$$

and

$$\begin{aligned}
\mathbb{E}\ \eta_N([0,\pi], s) \quad &= \int\limits_0^{\frac{2N+p}{2}-s} dx_1 \sum_{m=0}^{N-1} \frac{(-1)^m}{m!} \int\limits_{[x_1, x_1+s]^m} \mathcal{R}_{1+m}^{(N)}(x_1, x_2, \dots x_{1+m})\ dx_2 \dots dx_{1+m} \\
&+ \int\limits_{\frac{2N+p}{2}-s}^{\frac{2N+p}{2}} dx_1 \sum_{m=0}^{N-1} \frac{(-1)^m}{m!} \int\limits_{[x_1, (2N+p)/2]^m} \mathcal{R}_{1+m}^{(N)}(x_1, x_2, \dots x_{1+m})\ dx_2 \dots dx_{1+m} \\
&= \int\limits_0^{\frac{2N+p}{2}} dx_1 \sum_{m=0}^{N-1} \frac{(-1)^m}{m!} \int\limits_{[x_1, x_1+s]^m} \rho_{1+m}^{(2N+p)}(x_1, x_2, \dots x_{1+m})\ dx_2 \dots dx_{1+m} \\
&+ \quad \text{Remainder term.}
\end{aligned}$$

where the remainder term is estimated as

$$\begin{aligned}
|\text{Remainder term}| \quad &\le \int\limits_0^{\frac{2N+p}{2}} dx_1 \sum_{m=0}^{N-1} \frac{1}{m!}\ s^m\ \min(\ 2; (m+1)!\ (m+1)\ 2/(1+|x_1|_1)) \\
&+ \quad s\ \exp(s)
\end{aligned}$$

where $|x|_1 = \min(\ |x|, |N\ + p/2\ - x|\ )$, which implies

$$|\mathbb{E}\ \eta_N([o,\pi],s) - N\ F_{2N+p}(s)| \le const(s,\varepsilon)\ N^{\varepsilon}$$

for any $\varepsilon > 0$. Similary to the calculations in the section 3 one can show that

$$\sup_{[0,\infty)}\ |\mathbb{E}\ \eta_N([0,\pi],s) - N\ F(s)| = o(N^{1/2\ +\varepsilon})$$

Calculating the variance of $\eta_N([0,\pi],s)$ we note that if

$$x_3,\dots x_{2+m}\ \in [x_1, x_1+s] \sqcup [x_2, x_2+s]$$

then

$$\begin{aligned} & |\mathcal{R}^{(N)}_{2+m,m} - \rho^{(2N+p)}_{2+m,2}| \\ \le\ & \min\Big(\ 4;\ (m+2)!\ 2^{m+2}\cdot(2\ \frac{1}{1+\max(|x_1-x_2|-s;0)}\ \frac{1}{1+2|(x_1+x_2)/2|_1} \\ +\ & (\frac{1}{1+2|(x_1+x_2)/2|_1})^2\ )\Big) \end{aligned}$$

and

$$\begin{aligned} & |\sum_{m=0}^{N-2}\frac{(-1)^m}{m!}\int\limits_{([x_1,x_1+s]\sqcup[x_2,x_2+s])^m} \mathcal{R}^{(N)}_{2+m,2}(x_1,x_2,\dots x_{2+m}) \\ -\ & \rho^{(2N+p)}_{2+m,2}(x_1,x_2,\dots x_{2+m})\ dx_3\dots x_{2+m}| \\ \le\ & const(s,\varepsilon)\ (1/(1+|x_1-x_2|))^{1-\ \varepsilon/2}\ (1/(1+2|(x_1+x_2)/2|_1))^{1-\ \varepsilon/2} \end{aligned}$$

The last inequality leads to the estimate

$$\mathrm{Var}\ \eta_N([0,\pi],s) = b_{2N+p}(s,s)\ N + o(N^{\varepsilon})$$

valid for any $\varepsilon > 0$ and fixed s .
The calculation of higher moments (i.e the proof of lemma 2.2 for $l > 2$) does not

require any alterations. ∎

Since the distribution of the eigenvalues on $[-\pi, 0]$ is the mirror image of that on $[0, \pi]$

$$\eta_N([-\pi,\pi],s) = 2\ \eta_N([0,\pi],s) + (0, 1\ or\ 2)$$

and $(\eta_N([-\pi,\pi],s) - \mathbb{E}\ \eta_N([-\pi,\pi],s))/\ (2N)^{1/2}$ converges in finite-dimensional distributions to $2^{1/2}\ \xi(s)$.

As soon as we proved Theorems 1.1 , 1.2 for SO(2N+p) (U(N)) the same results hold for O(2N+p) , (SU(N)) :

Since $\eta_N([-\pi,\pi],s)$ is invariant under the matrix multiplication by -1 ($\exp(i\theta)$)

$$\int_{SO(2N+p)} \eta([-\pi,\pi],s)\, d\, Haar(SO(2N+p)) = \int_{O(2N+p)} \eta([-\pi,\pi],s)\, d\, Haar(O(2N+p)) ,$$

$$\int_{SU(N)} \eta([-\pi,\pi],s)\, d\, Haar(SU(N)) = \int_{U(N)} \eta([-\pi,\pi],s)\, d\, Haar(U(N)) .$$

Clearly,the analogues of Theorems 1.6,1.7 are valid for the random point field on the semi-axis $[0,\infty)$ with n-point correlation functions given by the formula

$$\rho_n(x_1,\ldots x_n) = \det\left(\frac{\sin\pi(x_i - x_j)}{\pi(x_i - x_j)} + (-1)^{\sigma}\frac{\sin\pi(x_i + x_j)}{\pi(x_i + x_j)}\right)_{i,j=1,\ldots n}$$

$\sigma = 0, 1$

We also want to remark that our methods allow the direct generalization to the case of n-level spacings distribution. Namely, one can define a random variable $\eta_N(l,s)$ as a number of eigenvalues that have exactly l neighbors within the distance $2\pi s/N$ to the right.

(The disribution of $\eta_N(0,s)$ has been studied in our paper). It is absolutely straightforward to prove similar results for a k-dimensional random process

($k = 1, 2, \ldots$)

$$\Big((\eta_N(0,s) - \mathbb{E}\ \eta_N(0,s))/\ N^{1/2}, \ldots (\eta_N(k-1,s) - \mathbb{E}\ \eta_N(k-1,s))/\ N^{1/2}\Big)$$

which in particular would tell us about the global n-level spacings distribution , since the number of n-level spacings greater than $2\pi s/N$ equals to

$$\sum_{l=0}^{n-1} \eta_N(l,s).$$

C.O.E. ($\beta = 1$) and C.S.E. ($\beta = 4$) correspond not to the matrix groups , but to the symmetric spaces ([13]).
C.O.E. gives the distribution of the eigenvalues of

$$M^t M \quad , \ M \in O(N) \backslash U(N)$$

and C.S.E. gives the distribution of the eigenvalues of

$$J^t M^t J M \quad , \ M \in USp(2N) \backslash U(2N) \ ,$$

where J is an antisymmetric matrix : $J_{i,l} = \delta_{i,N+i} \ , i \le l$.
The results similar to Theorems 1.1 and 1.2 hold for $\beta = 1, 4$ as well. The n-point correlation function in these cases are the determinants of some $n \times n$ quaternion matrices (see [4]) ; all combinatorial aspects of the proof remain the same. I was not able to show that

$$\rho_n(x_1, \ldots x_n) \le C^n$$

even with some $C \ge 1$. However for our purposes more trivial and easy to prove estimate

$$\rho_n(x_1, \ldots x_n) \le n^{n/2}$$

is enough.
Again I don't have any simple formulas for the covariance functions of the limiting gaussian processes .
It is also worth to mention that the global distribution of the next-to-nearest spacings in the C.S.E. coincides with the global disribution of the nearest spacings in the C.O.E. ([13]).

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