Singular continuous phase for Schrödinger operators over circle maps with breaks

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Abstract

We consider Schrödinger operators over a class of circle maps including $C^{2+\epsilon}$-smooth circle maps with finitely many break points. We show that in a region of the Lyapunov exponent — determined by the geometry of the dynamical partitions and α — the spectrum of Schrödinger operators over every such map, is purely singular continuous, for every α-Hölder-continuous potential V. As a corollary, we obtain that for every sufficiently smooth such map, with an invariant measure μ and with rotation number in a set $\mathcal{S}$, and μ-almost all $x \in \mathbb{T}^1$, the corresponding Schrödinger operator has a purely continuous spectrum, for every Hölder-continuous potential V. Set $\mathcal{S}$ includes some Diophantine numbers of class $D(\delta)$, for any $\delta > 1$.

1 Introduction

We consider a class of Schrödinger operators $H = H(T, V, x)$ on a space of square-summable sequences $\ell^2(\mathbb{Z})$, defined by

$$(Hu)_n := u_{n-1} + u_{n+1} + V(T^n x)u_n, \qquad u \in \ell^2(\mathbb{Z}), \tag{1.1}$$

where $V : \mathbb{T}^1 \to \mathbb{R}$ is a potential function, $T : \mathbb{T}^1 \to \mathbb{T}^1$ is an orientation-preserving homeomorphism of the circle $\mathbb{T}^1 = \mathbb{R}/\mathbb{Z}$, and $x \in \mathbb{T}^1$. The study of the spectrum of these operators — referred to as Schrödinger operators over circle maps — initiated in [19]. For an overview of earlier results on spectral theory of Schröedinger operators over dynamically defined potentials the reader is directed, e.g., to [5].

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When the rotation number ρ of T is irrational, this class of operators is a natural generalization of the one-frequency quasiperiodic Schrödinger operators for which $T = R_\rho$, where $R_\rho : x \mapsto x + \rho \mod 1$ is the rigid rotation. When T is transitive, it is topologically conjugate to the rotation, i.e., there is a homeomorphism $\varphi : \mathbb{T}^1 \to \mathbb{T}^1$, such that $T \circ \varphi = \varphi \circ R_\rho$. Hence, in that case, $T^n \circ \varphi = \varphi \circ R_\rho^n$, for every $n \in \mathbb{N}$, and we have $H(T, V, x) = H(R_\rho, V \circ \varphi, y)$, where $x = \varphi(y)$, $y \in \mathbb{T}^1$.

In some cases, the spectral properties of $H(T, V, x)$ can be deduced directly from the spectral properties of the corresponding Schrödinger operator over R_ρ, using this identity. In particular, if T is an analytic circle diffeomorphism with rotation number satisfying Yoccoz's $\mathcal{H}$ arithmetic condition [26], it follows from the theory of Herman [11] and Yoccoz [26] that φ is analytic, and the spectral properties of $H(T, V, x)$, with V analytic [13] follow directly from Avila's global theory of one-frequency quasiperiodic Schrödinger operators over rotations [1]. Although for circle diffeomorphisms T with Liouville rotation numbers the conjugacy to the corresponding rotation can even be singular, certain spectral properties of $H(T, V, x)$, with potentials of the same regularity, are still analogous to those of the one-frequency quasiperiodic Schrödinger operators over rotations with the same rotation numbers [13].

In [19], the study of spectral rigidity properties of Schrödinger operators over more general circle maps was initiated. In particular, in [19], we focus on spectral properties of Schrödinger operators over circle diffeomorphisms with a singularity, i.e., smooth circle diffeomorphisms with a single singular point where the derivative vanishes (critical circle maps) or has a jump discontinuity (circle maps with a break). These maps play an important role in the rigidity theory of circle maps — an extension of Herman's theory on the linearization of circle diffeomorphisms [3, 4, 6, 10, 14, 15, 16, 17, 18]. In this paper, we focus on the spectral properties of Schrödinger operators over a class of circle maps including circle maps with finitely many break points. We are interested in the spectral rigidity properties of these systems, i.e., the spectral properties of these systems that are the same in a large class of them. Here, we are interested in the spectral phase diagram of Schrödinger operators over circle maps and, in particular, the singular continuous phase. Such a phase diagram emerges in one of most studied examples — the almost Mathieu family — which corresponds to $T = R_\rho$ and $V(x) = \lambda \cos(2\pi x)$. It was conjectured by Jitomirskaya [12] (Problem 8 therein), and proved by Avila, You and Zhou [2], that the almost Mathieu operator has a purely singular continuous spectrum in the region $0 < L(E) < \beta$ and that $L(E) = \beta$ is the boundary between continuous and pure point spectrum, for almost all $x \in \mathbb{T}^1$, where $L(E)$ is the Lyapunov exponent and

$$\beta = \beta(\rho) := \limsup_{n\to\infty} \frac{\ln k_{n+1}}{q_n}, \tag{1.2}$$

with k_n and $\frac{p_n}{q_n}$, $n \in \mathbb{N}$, being the partial quotients and rational convergents of $\rho \in (0, 1)\backslash\mathbb{Q}$ (see section 2.2). It was shown in [13] that, in the same region, the spectrum is singular continuous for Schrödinger operators $H(T,V,x)$ with Lipschitz continuous potentials V over C^{1+BV}-smooth circle diffeomorphisms T, for almost all $x\in\mathbb{T}^1$, suggesting that $L(E)=\beta$ could be the boundary between continuous and pure point spectrum, in this case as well. A natural question to ask is if the latter holds for Schrödinger operators over general circle maps, for sufficiently regular potentials. The results of this paper, as well as [19] and [20], for Schrödinger operators over circle diffeomorphisms with singularities, provide a negative answer to that question.

In this paper, we consider Schrödinger operators over C^r-smooth circle maps with k break points, i.e., circle maps that are C^r-smooth diffeomorphisms of a circle outside of k (break) points $x_{\rm br}^{(i)}$, $i=1,\dots,k$, where the *sizes of the breaks* are, respectively,

$$c_i := \sqrt{\frac{T'_-(x_{\rm br}^{(i)})}{T'_+(x_{\rm br}^{(i)})}} \neq 1, \qquad i=1,\dots,k. \tag{1.3}$$

Ergodic Schrödinger operators are intimately related to a family of cocycles — dynamical systems associated with each eigenequation $Hu=Eu$. In the case of a Schrödinger operator over a circle map T with an irrational rotation number ρ, the cocycle is given by

$$(T,A):(x,y)\mapsto(Tx,A(x,E)y), \tag{1.4}$$

where $A\in\mathrm{SL}(2,\mathbb{R})$, $x\in\mathbb{T}^1$, $y\in\mathbb{R}^2$. If $u=(u_n)_{n\in\mathbb{Z}}$ is a sequence satisfying $Hu=Eu$, then

$$\begin{pmatrix}u_{n+1}\\ u_n\end{pmatrix} = A_n(x,E)\begin{pmatrix}u_n\\ u_{n-1}\end{pmatrix}, \quad \text{where} \quad A_n(x,E):=\begin{pmatrix}E-V(T^nx) & -1\\ 1 & 0\end{pmatrix} \tag{1.5}$$

is the transfer matrix. Thus,

$$\begin{pmatrix}u_n\\ u_{n-1}\end{pmatrix} = P_n(x,E)\begin{pmatrix}u_0\\ u_{-1}\end{pmatrix}, \tag{1.6}$$

where $P_n(x,E):=\prod_{i=n-1}^{0}A_i(x,E)=A_{n-1}(x,E)\dots A_0(x,E)$.

Since the rotation number ρ of T is irrational, T is uniquely ergodic [8]. We define the Lyapunov exponent

$$L(E):=\lim_{n\to\infty}\int L_n(x,E)\,d\mu, \quad \text{where} \quad L_n(x,E):=\frac{1}{n}\ln\|P_n(x,E)\|, \tag{1.7}$$

and μ is the unique invariant measure of T. Due to submultiplicativity of $P_n(x,E)$, $L(E)$ exists.

Different components of the spectrum of an operator $H(T,V,x)$ are denoted by σ_{ac} (absolutely continuous), σ_{sc} (singular continuous) and σ_{pp} (pure point). We also denote by $S_{pp}(x)$ the set of eigenvalues of $H(T,V,x)$, with $\sigma_{pp}(x) = \overline{S_{pp}(x)}$. Finally, we set $\mathcal{H} = \ell^2(\mathbb{Z})$, $\mathcal{H}_{sc}(x)$ the corresponding singular continuous subspace, and $P_A(x)$ the operator of spectral projection on a Borel set A, corresponding to $H(T,V,x)$.

The main result of this paper can be formulated as follows. Let

$$\delta_{\max} := \limsup_{n\to\infty} \frac{|\ln \ell_n|}{q_n}, \tag{1.8}$$

where $\ell_n = \min_{I\in\mathcal{P}_{n+1}, I\subset\Delta_0^{(n-1)}\setminus\Delta_0^{(n+1)}} |\tau_n(I)|$ is the length of the smallest renormalized interval of partition $\mathcal{P}_{n+1}$ inside the fundamental interval $\Delta_0^{(n-1)}\setminus\Delta_0^{(n+1)}$ of partition $\mathcal{P}_n$ (see section 2.2).

Theorem 1.1 *Let $T : \mathbb{T}^1 \to \mathbb{T}^1$ be any $C^{2+\epsilon}$-smooth, $\epsilon > 0$, circle map with k breaks of sizes c_i, $i = 1,\dots,k$, with an irrational rotation number $\rho \in (0,1)$ and $\prod_{i=1}^k c_i \neq 1$, and let μ be its unique invariant probability measure. For μ-almost all $x \in \mathbb{T}^1$, and any α-Hölder-continuous potential $V : \mathbb{T}^1 \to \mathbb{R}$, $\alpha \in (0,1]$, we have*

(i) $S_{pp}(x) \cap \{E : 0 \le L(E) < \alpha\delta_{\max}\} = \emptyset$,

(ii) $P_{\{E: 0<L(E)<\alpha\delta_{\max}\}}(x)\mathcal{H} \subset \mathcal{H}_{sc}(x)$.

Remark 1 It was shown in [20] that the theorem can be extended to include Schrödinger operators over C^3-smooth multicritical circle maps.

Remark 2 The claim can also be extended to a class of circle maps with breaks having zero mean nonlinearity, for which the renormalizations converge to piecewise linear (but not linear) maps. Recall that for $T : \mathbb{T}^1 \to \mathbb{T}^1$ such that $D\ln DT \in L^1$, we define the mean nonlinearity as $\mathcal{N}(T) = \int_{x\in\mathbb{T}^1} D\ln DT(x)\, dx$.

Remark 3 The result suggests spectral rigidity. It seems reasonable to expect that for Schrödinger operators over sufficiently smooth circle maps, in a large class of maps including circle diffeomorphisms with singularities, for μ-almost all $x \in \mathbb{T}^1$, and sufficiently regular potentials, the boundary between the singular continuous and pure point spectrum is given by $L(E) = \delta_{\max}$, i.e., that the spectrum is pure point with exponentially decaying eigenfunctions for $L(E) > \delta_{\max}$.

The following theorem is a corollary of our main result. To state the result precisely, we begin with a few more definitions. A number $\rho \in \mathbb{R}\setminus\mathbb{Q}$ is called *Diophantine* of class $D(\delta)$, for some $\delta \ge 0$, if there exists $\mathcal{C} > 0$ such that $|\rho - p/q| > \mathcal{C}/q^{2+\delta}$, for every $p \in \mathbb{Z}$ and $q \in \mathbb{N}$. The set of all Diophantine numbers is denoted by $D := \cup_{\delta\ge 0} D(\delta)$ and the complement of this set in $\mathbb{R}\setminus\mathbb{Q}$ is the set of Liouville numbers. If $\rho \in D(\delta) \cap (0,1)$, then $\limsup_{n\to\infty} \frac{\ln k_{n+1}}{\ln q_n} \le \delta$ and, thus, $\beta(\rho) = 0$. The set of *super Liouville* numbers, characterized by $\beta(\rho) = \infty$, will be denoted by $\mathcal{S}_{\mathcal{L}}$.

We define

$$\beta_e = \beta_e(\rho) := \limsup_{n\to\infty} \frac{k_{2n+1}}{q_{2n}}, \quad \text{and} \quad \beta_o = \beta_o(\rho) := \limsup_{n\to\infty} \frac{k_{2n}}{q_{2n-1}}. \tag{1.9}$$

Let $\mathcal{S} = \mathcal{S}_{\mathrm{br}} \cup \mathcal{S}_{\mathcal{L}}$, where $\mathcal{S}_{\mathrm{br}}$ is the set of $\rho \in (0,1)\backslash\mathbb{Q}$ such that $\beta_{\mathrm{br}} = \infty$, where $\beta_{\mathrm{br}} = \beta_{\mathrm{br}}(\rho) := \beta_e$ if $\prod_{i=1}^{k} c_i < 1$, and $\beta_{\mathrm{br}} = \beta_{\mathrm{br}}(\rho) := \beta_o$ if $\prod_{i=1}^{k} c_i > 1$.

Theorem 1.2 *For every $C^{2+\epsilon}$-smooth, $\epsilon > 0$, circle map with k breaks of sizes c_i, $i = 1, \ldots, k$, with $\prod_{i=1}^{k} c_i \neq 1$, a rotation number $\rho \in \mathcal{S}$, and the invariant measure μ, and μ-almost all $x \in \mathbb{T}^1$, the corresponding Schrödinger operator $H(T, V, x)$ has a purely continuous spectrum, for every Hölder-continuous potential $V : \mathbb{T}^1 \to \mathbb{R}$.*

Remark 4 The set $\mathcal{S} = \mathcal{S}_{\mathrm{br}} \cup \mathcal{S}_{\mathcal{L}}$ of rotation numbers for which the theorem holds contains not only Liouville numbers but also some Diophantine numbers of class $D(\delta)$, for any $\delta > 1$.

The proofs of these results rely on methods of dynamics and renormalization of circle maps. In the next section, we state a sharp version of Gordon's theorem [13] (which is the only input from spectral theory that we need), and introduce dynamical partitions of a circle and renormalizations of circle maps that play an important role in our proofs. In section 3, we define two sets of full invariant measure for circle maps whose renormalizations are piecewise concave and convex, respectively, and prove our main results.

2 Preliminaries

2.1 A criterion for the absence of eigenvalues

In this section, we state an abstract sharp version [13] of a theorem of Gordon [9]. Such a sharp version was first proved in [2] to establish the singular continuous phase for the almost Mathieu operator.

Consider a Schrödinger operator H on $\ell^2(\mathbb{Z})$ given by the action on $u \in \ell^2(\mathbb{Z})$, as

$$(Hu)_n = u_{n+1} + u_{n-1} + V_n u_n. \tag{2.1}$$

As in (1.5), we can define the transfer matrix $A_n(E)$ and, as in (1.6), the n-step transfer matrix $P_n(E) = \prod_{i=n-1}^{0} A_i(E)$. Let also $P_{-n}(E) = \prod_{i=-n}^{-1} (A_i(E))^{-1}$. Let

$$\Lambda(E) := \limsup_{|n|\to\infty} \frac{\ln \|P_n(E)\|}{n}. \tag{2.2}$$

Clearly, for bounded V, $\Lambda(E) < \infty$, for every E. We will use the following sharp Gordon's lemma.

Theorem 2.1 ([13]) *Assume that there exists $\beta > 0$, and an increasing sequence of positive integers q_n diverging to infinity, such that the sequence $\{V_n\}_{n\in Z}$ in* (2.1) *satisfies*

$$\max_{0\le j<q_n} |V_j - V_{j\pm q_n}| \le e^{-\beta q_n}. \tag{2.3}$$

If $\beta > \Lambda(E)$, then E is not an eigenvalue of operator (2.1)*.*

Consider the Schrödinger operator (2.1) with $V_n = V_n(x) = V(T^n x)$ where $V : \mathbb{T}^1 \to \mathbb{R}$ is a bounded real-valued function on the circle and T is an orientation-preserving homeomorphism of a circle with an irrational rotation number ρ. Let the Lyapunov exponent $L(E)$ be defined as in (1.7). Since T is ergodic, by Kingman's ergodic theorem, for almost every x,

$$L(E) = L(x,E) := \lim_{n\to\infty} \frac{1}{n} \ln \|P_n(x,E)\|. \tag{2.4}$$

Theorem 2.2 *Assume that for some $x \in \mathbb{T}^1$, $C > 0$ and $\bar{\beta} > 0$, there is a sequence of positive integers $q_n \to \infty$ such that*

$$\sup_{0\le i<q_n} |V_{i\pm q_n}(x) - V_i(x)| < Ce^{-\bar{\beta} q_n}. \tag{2.5}$$

If $L(E) < \bar{\beta}$, then E is not an eigenvalue of the Schrödinger operator $H(T,V,x)$.

Proof. In order to apply Theorem 2.1, it suffices to prove $\limsup_{|n|\to\infty} \frac{\ln\|P_n(E)\|}{n} \le L(E)$. This follows from a result of Furman [7]. **QED**

For a sequence $q_n \to \infty$, let

$$\hat{\beta} = \hat{\beta}(x) := \liminf_{n\to\infty} \frac{\ln(\sup_{0\le i<q_n} |x_i - x_{i\pm q_n}|)^{-1}}{q_n}, \tag{2.6}$$

where $x_i = T^i x$.

Let $S_{pp}, P_A, \mathcal{H}$, and $\mathcal{H}_{sc}$ be as in Theorem 1.1.

Theorem 2.3 *Let $V : \mathbb{T}^1 \to \mathbb{R}$ be an α-Hölder continuous real-valued function on the circle, with $\alpha \in (0,1)$. Then, we have*

(i) $S_{pp}(x) \cap \{E : 0 \le L(E) < \alpha\hat{\beta}\} = \emptyset$,

(ii) $P_{\{E:0<L(E)<\alpha\hat{\beta}\}}(x)\mathcal{H} \subset \mathcal{H}_{sc}(x)$.

Proof. It suffices to prove part (i) of the claim, i.e., to exclude the point spectrum. Part (ii) of the claim then follows from Kotani's theory [21, 22, 23], the x-independence of the absolutely continuous spectrum [24], and the minimality of T, since the set $\{E : L(E) > 0\}$ does not support any absolutely continuous spectrum.

If $L < \alpha\hat{\beta}$, then $V_i(x) = V(T^i x)$ satisfy the assumption (2.5) of Theorem 2.2 for any $\bar{\beta}$ satisfying $L < \bar{\beta} < \alpha\hat{\beta}$. The claim follows. **QED**

In order to prove Theorem 1.1, we need appropriate bounds on $\hat{\beta}(x)$.

2.2 Dynamical partitions of a circle and renormalizations

The quantity $\hat{\beta}(x)$ involves information about the geometry of the dynamical partitions of a circle. These partitions are obtained using the continued fraction expansion of the rotation number $\rho \in (0,1)$ of the circle map T. Every irrational $\rho \in (0,1)$ can be written uniquely as

$$\rho = \cfrac{1}{k_1 + \cfrac{1}{k_2 + \cfrac{1}{k_3 + \dots}}} =: [k_1, k_2, k_3, \dots], \tag{2.7}$$

with an infinite sequence of partial quotients $k_n \in \mathbb{N}$. Conversely, every infinite sequence of partial quotients defines uniquely an irrational number ρ as the limit of the sequence of rational convergents $p_n/q_n = [k_1, k_2, \dots, k_n]$, obtained by the finite truncations of the continued fraction expansion (2.7). It is well-known that p_n/q_n form a sequence of best rational approximations of an irrational ρ, i.e., there are no rational numbers p/q, with denominators $q \le q_n$, such that $|p - \rho q| < |p_n - \rho q_n|$. The rational convergents can also be defined recursively by $p_n = k_n p_{n-1} + p_{n-2}$ and $q_n = k_n q_{n-1} + q_{n-2}$, starting with $p_0 = 0$, $q_0 = 1$, $p_{-1} = 1$, $q_{-1} = 0$.

To define the dynamical partitions of an orientation-preserving homeomorphism $T : \mathbb{T}^1 \to \mathbb{T}^1$, with an irrational rotation number ρ, we start with a marked point $\chi_0 \in \mathbb{T}^1$, and consider the orbit $\chi_i = T^i \chi_0$, with $i \in \mathbb{Z}$. The subsequence χ_{q_n}, $n \in \mathbb{N}$, indexed by the denominators q_n of the sequence of rational convergents of the rotation number ρ, is called the sequence of dynamical convergents. It follows from the simple arithmetic properties of the rational convergents that the sequence of dynamical convergents χ_{q_n}, $n \in \mathbb{N}$, for the rigid rotation R_ρ has the property that its subsequence with n odd approaches χ_0 from the left and the subsequence with n even approaches χ_0 from the right. Since all circle homeomorphisms with the same irrational rotation number are combinatorially equivalent, the order of the dynamical convergents of T is the same.

The intervals $[\chi_{q_n}, \chi_0]$, for n odd, and $[\chi_0, \chi_{q_n}]$, for n even, will be denoted by $\Delta_0^{(n)}$ or $\Delta_0^{(n)}(\chi_0)$. We also define $\Delta_i^{(n)} = T^i(\Delta_0^{(n)})$, $i \in \mathbb{Z}$. Certain number of images of $\Delta_0^{(n-1)}$ and $\Delta_0^{(n)}$, under the iterations of the map T, cover the whole circle without intersecting each other, except possibly at the end points, and form the n-th dynamical partition of the circle

$$\mathcal{P}_n := \{T^i(\Delta_0^{(n-1)}) : 0 \le i < q_n\} \cup \{T^i(\Delta_0^{(n)}) : 0 \le i < q_{n-1}\}. \tag{2.8}$$

Intervals $\Delta_0^{(n-1)}$ and $\Delta_0^{(n)}$ are called the fundamental intervals of $\mathcal{P}_n$. These partitions are nested, in the sense that intervals of partition $\mathcal{P}_{n+1}$ are obtained by dividing the intervals of partition $\mathcal{P}_n$ into finitely many intervals.

The n-th renormalization of an orientation-preserving homeomorphism $T : \mathbb{T}^1 \to \mathbb{T}^1$, with rotation number ρ, with respect to partition-defining point $\chi_0 \in \mathbb{T}^1$, is a function $f_n : [-1, 0] \to \mathbb{R}$, obtained from the restriction of T^{q_n} to $\Delta_0^{(n-1)}$, by rescaling the coordinates. If τ_n is the affine change of coordinates that maps $\chi_{q_{n-1}}$ to -1 and χ_0 to 0, then

$$f_n := \tau_n \circ T^{q_n} \circ \tau_n^{-1}. \tag{2.9}$$

If we identify χ_0 with zero, then τ_n is just multiplication by $(-1)^n/|\Delta_0^{(n-1)}|$. Here, and in what follows, $|I|$ denotes the length of an interval I on $\mathbb{T}^1$.

For two sequences A_n and B_n, $n \in \mathbb{N}$, we use the notation $A_n = \Theta(B_n)$ to specify that there are two constants $\mathcal{C}_1, \mathcal{C}_2 > 0$ such that $\mathcal{C}_1 B_n \le A_n \le \mathcal{C}_2 B_n$, for all $n \in \mathbb{N}$.

3 Construction of sets of full invariant measure and the proof of the main result

In this section, we construct two sets of full invariant measure for which we have appropriate control on the distances of dynamical convergents, i.e, control of the quantity $\hat{\beta}$ in (2.6), and prove our main results.

3.1 Piecewise concave renormalizations and a full-measure set E

In this section, we assume that there exists an increasing infinite sequence σ_n, $n \in \mathbb{N}$, such that renormalizations f_{σ_n} are piecewise C^2-smooth and concave (downwards) with second derivative bounded and bounded away from zero by a negative constant, on each of the bounded number of intervals.

Let $\ell_{\sigma_n} := \min\limits_{0 \le i \le k_{\sigma_n+1}-1} |\tau_{\sigma_n}(\Delta^{(\sigma_n)}_{q_{\sigma_n-1}+iq_{\sigma_n}})|$. Let $\eta_n \in (0, 1)$, $n \in \mathbb{N}$, be a sequence such that the series $\sum_{n=1}^{\infty}(1-\eta_n)$ diverges to infinity. We also assume that the sequence k_{σ_n+1}, $n \in \mathbb{N}$, diverges to infinity. For $n \in \mathbb{N}$, let

$$\mathcal{I}_{n,0} := \left\{ I \in \mathcal{P}_{\sigma_n+1} \mid I \subset \Delta_0^{(\sigma_n-1)} \backslash \Delta_0^{(\sigma_n+1)},\ |\tau_{\sigma_n}(I)| \le \ell_{\sigma_n}^{\eta_n} \right\}. \tag{3.1}$$

Let

$$E_{n,0} := \bigcup_{I \in \mathcal{I}_{n,0}} I, \quad \text{and} \quad E_{n,i} := T^i(E_{n,0}), \quad \text{for} \quad i = 1, \dots, q_{\sigma_n} - 1. \tag{3.2}$$

We define

$$E_n := \bigcup_{i=0}^{q_{\sigma_n}-1} E_{n,i}, \tag{3.3}$$

and

$$E := \limsup_{n\to\infty} E_n = \bigcap_{n\geq 1}\bigcup_{j\geq n} E_j. \tag{3.4}$$

Proposition 3.1 *There exists $C > 0$ such that*

$$\mu(E_n), \frac{\mu(E_{n,0})}{\mu(\Delta_0^{(\sigma_n-1)} \cup \Delta_0^{(\sigma_n)})} \geq \frac{(1-\eta_n)Ck_{\sigma_n+1}}{k_{\sigma_n+1}+2}, \tag{3.5}$$

for all $n \in \mathbb{N}$.

Proof. If the renormalizations f_{σ_n} are piecewise C^2-smooth and uniformly concave on each of the bounded number of intervals, there exists $\nu \in (0,1)$ and, for each $n \in \mathbb{N}$, there is an interval $\mathcal{I} \subset \Delta_0^{(\sigma_n-1)} \backslash \Delta_0^{(\sigma_n+1)}$ such that the renormalization f_{σ_n} is uniformly concave on $\tau_{\sigma_n}(\mathcal{I})$, and such that the number of intervals of partition $\mathcal{P}_{\sigma_n+1}$ which are subsets of $\mathcal{I}$ is larger than $\nu k_{\sigma_{n+1}}$. It follows that at least one of the subintervals on which f_{σ_n} is monotone contains $\Theta(k_{\sigma_n+1})$ these intervals of partition $\mathcal{P}_{\sigma_n+1}$ and, thus, $\ln \ell_{\sigma_n}^{-1} = \Theta(k_{\sigma_n+1})$. Furthermore, the number N_n of intervals of partition $\mathcal{P}_{\sigma_n+1}$ inside of $\Delta_0^{(\sigma_n-1)} \backslash \Delta_0^{(\sigma_n+1)}$, that are subsets of $E_{n,0}$, satisfies $\ell_{\sigma_n}^{\eta_n-1} \leq b^{N_n}$, for some $b > 1$. Hence, $N_n \geq (1-\eta_n)\ln \ell_{\sigma_n}^{-1}/\ln b \geq (1-\eta_n)Ck_{\sigma_n+1}$, for some $C > 0$.

Recall that the partition $\mathcal{P}_{\sigma_n}$ consists of q_{σ_n} "large" intervals $\Delta_i^{(\sigma_n-1)} = T^i(\Delta_0^{(\sigma_n-1)})$, for $i = 0, \ldots, q_{\sigma_n} - 1$, each of which has invariant measure $\mu(\Delta_0^{(\sigma_n-1)})$ and q_{σ_n-1} "small" intervals $\Delta_i^{(\sigma_n)} = T^i(\Delta_0^{(\sigma_n)})$, for $i = 0, \ldots, q_{\sigma_n-1} - 1$, each of which has invariant measure $\mu(\Delta_0^{(\sigma_n)})$. Since the interval $\Delta_0^{(\sigma_n-1)}$ consists of the union of k_{σ_n+1} disjoint (except at the end points) intervals $\Delta_{q_{\sigma_n-1}+iq_{\sigma_n}}^{(\sigma_n)} \in \mathcal{P}_{\sigma_n+1}$, for $i = 0, \ldots, k_{\sigma_n+1} - 1$, each of which has invariant measure $\mu(\Delta_0^{(\sigma_n)})$, and $\Delta_0^{(\sigma_n+1)} \subset \Delta_{q_{\sigma_n+1}}^{(n)}$, we have that the invariant measure of E_n is

$$\mu(E_n) \geq N_n q_{\sigma_n}\mu(\Delta_0^{(\sigma_n)}) \geq \frac{N_n}{k_{\sigma_n+1}+2}. \tag{3.6}$$

In the last inequality, we have used that

$$k_{\sigma_n+1}q_{\sigma_n}\mu(\Delta_0^{(\sigma_n)}) + q_{\sigma_n}\mu(\Delta_0^{(\sigma_n+1)}) + q_{\sigma_n-1}\mu(\Delta_0^{(\sigma_n)}) = 1. \tag{3.7}$$

We have also used that $q_{\sigma_n-1} \leq q_{\sigma_n}$ and $\mu(\Delta_0^{(\sigma_n+1)}) \leq \mu(\Delta_0^{(\sigma_n)})$.

Similarly,

$$\frac{\mu(E_{n,0})}{\mu(\Delta_0^{(\sigma_n-1)} \cup \Delta_0^{(\sigma_n)})} \geq \frac{N_n}{k_{\sigma_n+1}+2}. \tag{3.8}$$

The claim follows. **QED**

Proposition 3.2 $\mu(E) = 1$.

Proof. For a fixed $m \in \mathbb{N}$, it follows from Proposition 3.1 that

$$\mu(E_m^c) \leq 1 - \frac{(1-\eta_m)Ck_{\sigma_m+1}}{k_{\sigma_m+1}+2}. \tag{3.9}$$

It follows from Proposition 3.1 that

$$\mu(\cap_{j=m}^n E_j^c) \leq \mu(\cap_{j=m}^{n-1} E_j^c)\left(1 - \frac{(1-\eta_n)Ck_{\sigma_n+1}}{k_{\sigma_n+1}+2}\right). \tag{3.10}$$

Applying this estimate recursively, and using that $1 - x < e^{-x}$, for $x > 0$, we have

$$\mu(\cup_{j\geq n} E_j) = 1 - \mu(\cap_{j\geq n} E_j^c) \geq 1 - \exp\left(-\sum_{j\geq n}\frac{(1-\eta_j)Ck_{\sigma_j+1}}{k_{\sigma_j+1}+2}\right). \tag{3.11}$$

If the sequence η_n is such that the series $\sum_{j=1}^{\infty}(1-\eta_j)$ diverges to ∞, $\mu(\cup_{j\geq n} E_j) = 1$, for any $n \in \mathbb{N}$. The claim follows. **QED**

3.2 Piecewise convex renormalizations and a full-measure set $\mathfrak{E}$

In this section, we assume that there is an increasing infinite sequence σ_n, $n \in \mathbb{N}$, such that the renormalizations f_{σ_n} are piecewise $C^{2+\epsilon}$-smooth, $\epsilon > 0$, and convex with second derivative bounded and bounded away from zero by a positive constant, on each of the bounded number of intervals.

We will use the following extension of Yoccoz's lemma that was proved in [19]. Let $k \in \mathbb{N}$ and let $\Delta_1, \Delta_2, \ldots, \Delta_{k+1}$ be consecutive closed intervals on an interval or a circle.

Lemma 3.3 ([19]) *Let $I = \Delta_1 \cup \Delta_2 \cup \cdots \cup \Delta_k$ and let $f : I \to \Delta_2 \cup \Delta_3 \cup \cdots \cup \Delta_{k+1}$ be a $C^{2+\varepsilon}$-smooth diffeomorphism, $\varepsilon \in (0,1)$, satisfying $f(\Delta_i) = \Delta_{i+1}$. Assume that there exist constants $K, K', K'' > 0$ such that*

(i) $\|f\|_{C^2} \leq K$;

(ii) *the set $B_{K'} := \{z \in I : f(z) - z \leq K'\}$ is either an open interval or empty;*

(iii) *$f''(z) \geq K''$, for every $z \in B_{K'}$.*

If $|\Delta_1|, |\Delta_k| \geq \sigma |I|$, for some $\sigma > 0$, then there exists a constant $\mathtt{C} > 1$, such that, for all $i = 1, \ldots, k$,

$$\mathtt{C}^{-1} \frac{1}{(\min\{i, k+1-i\})^2} \leq \frac{|\Delta_i|}{|I|} \leq \mathtt{C} \frac{1}{(\min\{i, k+1-i\})^2}. \tag{3.12}$$

Let $\eta_n \in (0,1)$, $n \in \mathbb{N}$, be a sequence such that the series $\sum_{n=1}^{\infty}(1-\eta_n^{1/2})$ diverges to infinity. We also assume that the sequence k_{σ_n+1}, $n \in \mathbb{N}$, diverges to infinity as well. For each $n \in \mathbb{N}$, let

$$\mathfrak{E}_{n,0} := \bigcup_{I \in J_{n,0}} I, \quad J_{n,0} := \left\{ I \in \mathcal{P}_{\sigma_n+1} | I \subset \Delta_0^{(\sigma_n-1)} \backslash \Delta_0^{(\sigma_n+1)},\ |\tau_{\sigma_n}(I)| \leq \frac{\ell_{\sigma_n}}{\eta_n} \right\}, \tag{3.13}$$

and let

$$\mathfrak{E}_{n,i} := T^i(\mathfrak{E}_{n,0}), \quad \text{for} \quad i = 1, \ldots, q_{\sigma_n} - 1. \tag{3.14}$$

We define

$$\mathfrak{E}_n := \bigcup_{i=0}^{q_{\sigma_n}-1} \mathfrak{E}_{n,i}, \tag{3.15}$$

and

$$\mathfrak{E} := \limsup_{n \to \infty} \mathfrak{E}_n = \bigcap_{n \geq 1} \bigcup_{j \geq n} \mathfrak{E}_j. \tag{3.16}$$

Proposition 3.4 *There exists $\mathfrak{C} > 0$ such that*

$$\mu(\mathfrak{E}_n), \frac{\mu(\mathfrak{E}_{n,0})}{\mu(\Delta_0^{(\sigma_n-1)} \cup \Delta_0^{(\sigma_n)})} \geq \frac{(1-\eta_n^{1/2})\mathfrak{C} k_{\sigma_n+1}}{k_{\sigma_n+1}+2}, \tag{3.17}$$

for all $n \in \mathbb{N}$.

Proof. If the renormalizations f_{σ_n} are piecewise $C^{2+\epsilon}$-smooth, $\epsilon > 0$, and convex, for each $n \in \mathbb{N}$, there exists $\nu \in (0,1)$ and, for each $n \in \mathbb{N}$, there is an interval $\mathcal{J} \subset \Delta_0^{(\sigma_n-1)} \backslash \Delta_0^{(\sigma_n+1)}$ such that f_{σ_n} is convex on $\tau_{\sigma_n}(\mathcal{J})$, with second derivative f''_{σ_n} bounded and bounded away from zero, and such that the number of intervals of partition $\mathcal{P}_{\sigma_n+1}$ that are subsets of $\mathcal{J}$ is larger than $\nu k_{\sigma_{n+1}}$. It follows that there is a subinterval of $\mathcal{J}$ on which f_{σ_n} is smooth and monotone, contains $\Theta(k_{\sigma_n+1})$ of these intervals of partition $\mathcal{P}_{\sigma_n+1}$ and, thus, $\ell_{\sigma_n} = \Theta(k_{\sigma_n+1}^{-2})$. Furthermore, it follows from Lemma 3.3 that the number N_n of intervals of partition $\mathcal{P}_{\sigma_n+1}$ inside of $\Delta_0^{(\sigma_n-1)} \backslash \Delta_0^{(\sigma_n+1)}$, that belong to $\mathfrak{E}_{n,0}$, satisfies $N_n \geq \hat{N}_n$, where $\frac{(\hat{N}_n+M_n)^2}{M_n^2} = \Theta(\eta_n^{-1})$. Here, $\hat{N}_n$ and M_n are the numbers of intervals of partition $\mathcal{P}_{\sigma_n+1}$ on that subinterval of $\mathcal{J}$ which do and do not belong to $\mathfrak{E}_{n,0}$, respectively. Since $M_n + \hat{N}_n = \Theta(k_{\sigma_n+1})$, we have $N_n \geq \Theta(k_{\sigma_n+1})(1-\eta_n^{1/2})$.

Analogously to (3.6) and (3.8), we have

$$\mu(\mathfrak{E}_n), \frac{\mu(\mathfrak{E}_{n,0})}{\mu(\Delta_0^{(\sigma_n-1)} \cup \Delta_0^{(\sigma_n)})} \geq \frac{N_n}{k_{\sigma_n+1}+2}. \tag{3.18}$$

The claim follows. **QED**

Proposition 3.5 $\mu(\mathfrak{E}) = 1$.

Proof. For a fixed $m \in \mathbb{N}$, it follows from Proposition 3.4 that

$$\mu(\mathfrak{E}_m^c) \leq 1 - \frac{(1-\eta_m^{1/2})\mathfrak{C}k_{\sigma_m+1}}{k_{\sigma_m+1}+2}. \tag{3.19}$$

It follows from Proposition 3.4 that

$$\mu(\cap_{j=m}^n \mathfrak{E}_j^c) \leq \mu(\cap_{j=m}^{n-1} \mathfrak{E}_j^c) \left(1 - \frac{(1-\eta_n^{1/2})\mathfrak{C}k_{\sigma_n+1}}{k_{\sigma_n+1}+2}\right). \tag{3.20}$$

Applying this estimate recursively, and using that $1 - x < e^{-x}$, for $x > 0$, we have

$$\mu(\cup_{j\geq n}\mathfrak{E}_j) = 1 - \mu(\cap_{j\geq n}\mathfrak{E}_j^c) \geq 1 - \exp\left(-\sum_{j\geq n} \frac{(1-\eta_j^{1/2})\mathfrak{C}k_{\sigma_j+1}}{k_{\sigma_j+1}+2}\right). \tag{3.21}$$

If the sequence η_n is such that the series $\sum_{j=1}^{\infty}(1-\eta_j^{1/2})$ diverges to ∞, $\mu(\cup_{j\geq n}\mathfrak{E}_j) = 1$, for any $n \in \mathbb{N}$. The claim follows. **QED**

3.3 Distance of dynamical convergents

We consider orientation-preserving homeomorphisms $T : \mathbb{T}^1 \to \mathbb{T}^1$, with an irrational rotation number and logarithm of the derivative of bounded variation $\mathcal{V} := \mathrm{Var}_{\xi\in\mathbb{T}^1} \ln T'(\xi) < \infty$, for which $\ln\,(T^{q_n})'$ is bounded. In particular, the derivative of such a map T has at most finitely many discontinuities that we will denote by $x_{\mathrm{br}}^{(i)}$, $i = 1, \ldots, k$, $k \in \mathbb{N}$. In addition to partitions $\mathcal{P}_n$, defined with a marked point $\chi_0 = x_{\mathrm{br}}^{(1)}$, we consider finer partitions $\mathcal{P}_n^*$, obtained by dividing the interval $\Delta_0^{(n-1)}$ into at most k intervals $\bar{\Delta}_{0,\bar{i}}^{(n-1)}$, $1 \leq \bar{i} \leq k$, on which T^{q_n} is C^1-smooth, by the preimages $T^{-j}(x_{\mathrm{br}}^{(i)})$, $j = 0, \ldots, q_n - 1$, $i = 2, \ldots, k$, and dividing the $q_n - 1$ images $\Delta_i^{(n-1)}$, $i = 1, \ldots, q_n - 1$, into the corresponding images $\bar{\Delta}_{i,\bar{i}}^{(n-1)} = T^i(\bar{\Delta}_{0,\bar{i}}^{(n-1)})$ of these intervals. The following proposition holds for all intervals $I_0 \subset \Delta_0^{(n-1)}$ such that $I_0 \in \mathcal{P}_{n+1}$, and the corresponding intervals $I_i = T^i(I_0)$, $i \in \mathbb{Z}$.

Proposition 3.6 *There exists $C_1 > 0$ such that, for all $n \in \mathbb{N}$ and $i = 0, \ldots, q_n - 1$,*

$$|I_i|, |I_{i-q_n}|, |I_{i+q_n}|, |I_{i+2q_n}| \leq C_1 |\Delta_i^{(n-1)}| \frac{|I_0|}{|\Delta_0^{(n-1)}|}. \tag{3.22}$$

Proof. Let $\bar{I}_0$ be any subinterval of I_0 on which T^{q_n} is C^1-smooth, and $\bar{\Delta}_0^{(n-1)}$ be any subinterval of $\Delta_0^{(n-1)}$ on which T^{q_n} is C^1-smooth, such that $\bar{I}_0 \subset \Delta_0^{(n-1)}$. For $1 = 0, \ldots, q_n$, there exist $\zeta_{i-1} \in \bar{I}_{i-1} \subset \bar{\Delta}_{i-1}^{(n-1)}$ and $\xi_{i-1} \in \bar{\Delta}_{i-1}^{(n-1)}$ such that

$$\frac{|\bar{I}_i|}{|\bar{\Delta}_i^{(n-1)}|} = \frac{|T(\bar{I}_{i-1})|}{|T(\bar{\Delta}_{i-1}^{(n-1)})|} = \frac{T'(\zeta_{i-1})}{T'(\xi_{i-1})} \frac{|\bar{I}_{i-1}|}{|\bar{\Delta}_{i-1}^{(n-1)}|}. \tag{3.23}$$

This implies the estimate

$$\frac{|\bar{I}_i|}{|\bar{\Delta}_i^{(n-1)}|} \leq \left(1 + \frac{|T'(\zeta_{i-1}) - T'(\xi_{i-1})|}{T'(\xi_{i-1})}\right) \frac{|\bar{I}_{i-1}|}{|\bar{\Delta}_{i-1}^{(n-1)}|}. \tag{3.24}$$

By iterating this inequality, we obtain that, for some $\zeta_j, \xi_j \in \bar{\Delta}_j^{(n-1)}$,

$$\frac{|\bar{I}_i|}{|\bar{\Delta}_i^{(n-1)}|} \leq \prod_{j=0}^{i-1} \left(1 + \frac{|T'(\zeta_j) - T'(\xi_j)|}{\min_{\xi \in \mathbb{T}^1} T'(\xi)}\right) \frac{|\bar{I}_0|}{|\bar{\Delta}_0^{(n-1)}|}. \tag{3.25}$$

At points of discontinuity of T', both left and right derivatives are considered. Using the inequality $1 + x \leq e^x$, we obtain

$$\frac{|\bar{I}_i|}{|\bar{\Delta}_i^{(n-1)}|} \leq \exp\left(\sum_{j=0}^{i-1} \frac{|T'(\zeta_j) - T'(\xi_j)|}{\min_{\xi \in \mathbb{T}^1} T'(\xi)}\right) \frac{|\bar{I}_0|}{|\bar{\Delta}_{i-1}^{(n-1)}|}. \tag{3.26}$$

Since, for $i = 0, \ldots, q_n - 1$, the intervals $\bar{\Delta}_i^{(n-1)}$ do not overlap, except possibly at the end points, we have

$$\sum_{j=0}^{q_n-1} |T'(\zeta_j) - T'(\xi_j)| \leq \max_{\xi \in \mathbb{T}^1} T'(\xi) \sum_{j=0}^{q_n-1} |\ln T'(\zeta_j) - \ln T'(\xi_j)| \leq \mathcal{V} \max_{\xi \in \mathbb{T}^1} T'(\xi), \tag{3.27}$$

where $\mathcal{V} = \mathrm{Var}_{\xi \in \mathbb{T}^1} \ln T'(\xi)$. Since the intervals I_i and $\Delta_i^{(n-1)}$ are unions of finitely many intervals of the type $\bar{I}_i$ and $\bar{\Delta}_i^{(n-1)}$, respectively, we obtain

$$\frac{|I_i|}{|\Delta_i^{(n-1)}|} \leq \exp\left(\frac{\mathcal{V} \max_{\xi \in \mathbb{T}^1} T'(\xi)}{\min_{\xi \in \mathbb{T}^1} T'(\xi)}\right) \frac{|I_0|}{|\Delta_0^{(n-1)}|}. \tag{3.28}$$

Since T' is bounded both from below and from above by positive constants, we obtain the desired estimate on $|I_i|$, for $i = 0, \ldots, q_n - 1$. The estimates on $|I_{i-q_n}|$ and $|I_{i+q_n}|$ (with possibly with a different constant C_1) now follow since the logarithm of the derivative of T^{q_n} is bounded. **QED**

Lemma 3.7 *There exists $C_2 > 0$ such that, for all $x \in E$, there are infinitely many $n \in \mathbb{N}$, such that $x \in \Delta_j^{(\sigma_n - 1)}$, for some $0 \le j \le q_{\sigma_n} - 1$, and*

$$|T^{q_{\sigma_n}} x_i - x_i| \le C_2 |\Delta_{j+i}^{(\sigma_n - 1)}| \ell_{\sigma_n}^{\eta_n}, \tag{3.29}$$

where $x_i = T^i x$, $i = -q_{\sigma_n}, \dots, q_{\sigma_n}$.

Proof. For every $x \in E$, there are infinitely many n, such that $x \in E_n$. Furthermore, there exists an element I_j of partition $\mathcal{P}_{\sigma_n + 1}$ inside $E_{n,j} \subset \Delta_j^{(\sigma_n - 1)}$, for some $j = 0, \dots, q_{\sigma_n} - 1$, such that $x \in I_j$, and, thus, the interval $[x, x_{q_{\sigma_n}}]$ if σ_n is even, or $[x_{q_{\sigma_n}}, x]$ if σ_n is odd, is a subset of the union $I_j \cup I_{j + q_{\sigma_n}}$. The claim now follows from Proposition 3.6 as $|T^{q_{\sigma_n}} x - x| \le |I_j| + |T^{q_{\sigma_n}}(I_j)|$. **QED**

Lemma 3.8 *There exists $C_3 > 0$ such that, for all $x \in \mathfrak{E}$, there are infinitely many $n \in \mathbb{N}$, such that $x \in \Delta_j^{(\sigma_n - 1)}$ for some $0 \le j \le q_{\sigma_n} - 1$, and*

$$|T^{q_{\sigma_n}} x_i - x_i| \le C_3 |\Delta_{j+i}^{(\sigma_n - 1)}| \frac{\ell_{\sigma_n}}{\eta_n}, \tag{3.30}$$

where $x_i = T^i x$, $i = -q_{\sigma_n}, \dots, q_{\sigma_n}$.

Proof. For every $x \in \mathfrak{E}$, there are infinitely many n, such that $x \in \mathfrak{E}_n$. Furthermore, there exists an element I_j of partition $\mathcal{P}_{\sigma_n + 1}$ inside $\mathfrak{E}_{n,j} \subset \Delta_j^{(\sigma_n - 1)}$, for some $j = 0, \dots, q_{\sigma_n} - 1$, such that $x \in I_j$, and, thus, the interval $[x, x_{q_{\sigma_n}}]$ if σ_n is even, or $[x_{q_{\sigma_n}}, x]$ if σ_n is odd, is a subset of the union $I_j \cup I_{j + q_{\sigma_n}}$. The claim now follows from Proposition 3.6. **QED**

3.4 Proof of the main theorems

Proof of Theorem 1.1. There is an increasing sequence π_n, $n \in \mathbb{N}$, such that $\delta_{\max} = \lim_{n\to\infty} \frac{\ln \ell_{\pi_n}}{q_{\pi_n}}$. Since $\delta_{\max} > 0$, the sequence $k_{\pi_n + 1}$, $n \in \mathbb{N}$, diverges to infinity. It follows from the properties of the renormalizations of $C^{2+\epsilon}$-smooth circle maps with finitely many ($k \ge 1$) break points with $\prod_{i=1}^k c_i \ne 1$, that there is an increasing infinite subsequence σ_n of π_n, such that the renormalizations f_{σ_n} (defined with one of the break points as a marked point) are piecewise $C^{2+\epsilon}$-smooth and either all uniformly piecewise concave or all uniformly piecewise convex with second derivative bounded and bounded away from zero, on each of the bounded number of at most k (optimal) intervals (i.e., contained in no proper superintervals) on which f_n is smooth. Assume first that there is an increasing, infinite subsequence σ_n of π_n, such that the renormalizations f_{σ_n} are piecewise concave with second derivative negative, piecewise bounded and bounded away from zero. Let $\eta_n \in (0, 1)$, for $n \in \mathbb{N}$, be a sequence converging to 1 such that the series $\sum(1 - \eta_n)$

diverges. We use this sequence to construct set E, as in section 3.1. By Proposition 3.2, $\mu(E) = 1$. For every $x \in E$, by Lemma 3.7, there are infinitely many n, such that estimate (3.29) holds. This implies $\hat{\beta} \geq \delta_{\max}$. Hence, if $L(E) < \alpha\delta_{\max}$, then $L(E) < \alpha\hat{\beta}$, and the claim follows from Theorem 2.3.

Assume now that there is an increasing, infinite subsequence σ_n of π_n, such that the renormalizations f_{σ_n} are piecewise convex with second derivative positive, piecewise bounded and bounded away from zero. Let $\eta_n \in (0,1)$, for $n \in \mathbb{N}$, be a sequence converging to 1 such that the series $\sum(1-\eta_n^{1/2})$ diverges. We use this sequence to construct set $\mathfrak{E}$, as in section 3.2. By Proposition 3.5, $\mu(\mathfrak{E}) = 1$. For every $x \in \mathfrak{E}$, by Lemma 3.8, there are infinitely many n, such that estimate (3.30) holds. This implies $\hat{\beta} \geq \delta_{\max}$. Hence, if $L(E) < \alpha\delta_{\max}$, then $L(E) < \alpha\hat{\beta}$, and the claim follows from Theorem 2.3. **QED**

Theorem 1.2 follows directly from Theorem 1.1, as $\ln \ell_n^{-1} \geq \Theta(k_{n+1})$, for $n \in 2\mathbb{N}$, and $\ell_n \leq \Theta(k_{n+1}^{-2})$, for $n \in 2\mathbb{N}-1$, if $\prod_{i=1}^{k} c_i < 1$; and $\ln \ell_n^{-1} \geq \Theta(k_{n+1})$, for $n \in 2\mathbb{N}-1$, and $\ell_n \leq \Theta(k_{n+1}^{-2})$, for $n \in 2\mathbb{N}$, if $\prod_{i=1}^{k} c_i > 1$, and, thus, $\delta_{\max} \geq \max\{\Theta(\beta_{\mathrm{br}}), \Theta(\beta)\}$.

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References

[1] A. Avila, Global theory of one-frequency Schrödinger operators, *Acta Math.* **215** (2015), 1–54.

[2] A. Avila, J. You, Q. Zhou, Sharp phase transitions for the almost Mathieu operator, *Duke Math. J.* **166** 14 (2017), 2697–2718.

[3] K. Cunha and D. Smania, Renormalization for piecewise smooth homeomorphisms on the circle, *Ann. Inst. H. Poincaré Anal. Non Linéaire* **30** 3 (2013), 441–462.

[4] K. Cunha and D. Smania, Rigidity for piecewise smooth homeomorphisms on the circle, *Adv. Math.* **250** (2014) 193–226.

[5] D. Damanik, Schrödinger operators with dynamically defined potentials, *Ergodic Theory Dynam. Systems* **37** 6 (2017), 1681–1764.

[6] E. de Faria,W. de Melo, Rigidity of critical circle mappings I, *J. Eur. Math. Soc.* **1** 4 (1999), 339–392.

[7] A. Furman, On the multiplicative ergodic theorem for uniquely ergodic systems, *Annales de l'Institut Henri Poincaré (B) Probab. Stat.* **33** 6 (1997), 797–815.

[8] H. Furstenberg, Strict ergodicity and transformation of the torus, *Amer. J. Math.* **83** 4 (1961), 573–601.

[9] A. Y. Gordon, The point spectrum of the one-dimensional Schrödinger operator, *Uspekhi Mat. Nauk* **31** 4 (1976), 257–258.

[10] P. Guarino, M. Martens, W. de Melo, Rigidity of critical circle maps, *Duke Math. J.* **167** 11 (2018), 2125–2188.

[11] M. R. Herman, Sur la conjugasion differentiable des difféomorphismes du cercle a de rotations, *Publ. Math. Inst. Hautes Etudes Sci.* **49** (1979), 5–234.

[12] S. Jitomirskaya, Ergodic Schrödinger operators (on one foot) *Proc. Sympos. Pure Math.* **76** 2 (2007), 613–647.

[13] S. Jitomirskaya, S. Kocić, Spectral theory of Schrödinger operators over circle diffeomorphisms, *Int. Math. Res. Not.* **13** (2022), 9810–9829.

[14] K. Khanin, S. Kocić, Renormalization conjecture and rigidity theory for circle diffeomorphisms with breaks, *Geom. Funct. Anal.* **24** 6 (2014), 2002–2028.

[15] K. Khanin, S. Kocić, E. Mazzeo, C^1-rigidity of circle diffeomorphisms with breaks for almost all rotation numbers, *Ann. Sci. Éc. Norm. Supér.* **50** (2017), 1163–1203.

[16] K. Khanin, A. Teplinsky, Robust rigidity for circle diffeomorphisms with singularities, *Invent. Math.* **169** (2007), 193–218.

[17] K.M. Khanin, E. B. Vul, Circle homeomorphisms with weak discontinuities, *Adv. Sov. Math.* **3** (1991), 57–98.

[18] S. Kocić, Generic rigidity for circle diffeomorphisms with breaks, *Comm. Math. Phys.* **344** 2 (2016), 427–445.

[19] S. Kocić, Singular continuous phase for Schrödinger operators over circle maps, *Preprint:* mp_arc 20-95, (2020).

[20] S. Kocić, Singular continuous phase for Schrödinger operators over multicritical circle maps, *Preprint:* mp_arc 22-46, (2022).

[21] S. Kotani, Ljapunov indices determine absolutely continuous spectra of stationary random one-dimensional Schrödinger operators, *Stochastic Analysis* (Katata/Kyoto, 1982), 225– 247, North-Holland Math. Library, 32. North-Holland, Amsterdam, 1984.

[22] S. Kotani, One-dimensional random Schrödinger operators and Herglotz functions, Probabilistic Methods in Mathematical Physics (Katata/Kyoto, 1985), 219–250, Academic Press, Boston, MA, 1987.

[23] S. Kotani, Generalized Floquet theory for stationary Schrödinger operators in one dimension, Chaos Solitons Fractals **8** 11 (1997), 1817–1854.

[24] Y. Last, B. Simon, Eigenvalues, transfer matrices and absolutely continuous spectrum of one-dimensional Schrödinger operators, *Invent. Math.* **135** 2 (1999), 329–367.

[25] Ya. G. Sinai, K. M. Khanin, Smoothness of conjugacies of diffeomorphisms of the circle with rotations, *Uspekhi Mat. Nauk* **44** 1 (1989), 57–82.

[26] J.-C. Yoccoz, Conjugaison differentiable des difféomorphismes du cercle donc le nombre de rotation vérifie une condition Diophantienne, *Ann. Sci. Éc. Norm. Supér.* **17** (1984), 333–361.