# SOLVABILITY IN THE SENSE OF SEQUENCES FOR SOME LOGARITHMIC SCHRÖDINGER OPERATORS IN HIGHER DIMENSIONS

Messoud Efendiev[1 2], Vitali Vougalter[3]

[1] Helmholtz Zentrum München, Institut für Computational Biology, Ingolstädter Landstrasse 1
Neuherberg, 85764, Germany
e-mail: messoud.efendiyev@helmholtz-muenchen.de
[2] Department of Mathematics, Marmara University, Istanbul, Turkey
e-mail: m.efendiyev@marmara.edu.tr

[3] Department of Mathematics, University of Toronto, Toronto, Ontario, M5S 2E4, Canada
e-mail: vitali@math.toronto.edu

**Abstract:** We study the solvability of certain linear nonhomogeneous equations containing the logarithm of the sum of the two Schrödinger operators in higher dimensions and demonstrate that under the reasonable technical assumptions the convergence in $L^2(\mathbb{R}^d)$ of the right sides yields the existence and the convergence in $L^2(\mathbb{R}^d)$ of the solutions. The equations involve the operators without the Fredholm property and we use the methods of the spectral and scattering theory for the Schrödinger type operators to generalize the results of our preceding work [19]. As distinct from the many previous articles on the subject, for the operators contained in our equations the essential spectra fill the whole real line.
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## 1. Introduction

Consider the equation

$$-\Delta u + V(x)u - au = f, \tag{1.1}$$

where $u \in E = H^2(\mathbb{R}^d)$ and $f \in F = L^2(\mathbb{R}^d),\ d \in \mathbb{N}$, $a$ is a constant and the scalar potential function $V(x)$ tends to $0$ at infinity. For $a \geq 0$, the essential spectrum of the operator $A : E \to F$, corresponding to the left side of problem (1.1) contains the origin. As a consequence, such operator does not satisfy the Fredholm property.

Its image is not closed, for $d > 1$ the dimension of its kernel and the codimension of its image are not finite. The present article is devoted to the studies of the certain properties of the operators of this kind. We recall that there was a significant amount of work accomplished on the elliptic equations containing the non-Fredholm operators in recent years (see [14], [15], [16], [17], [18], [30], [31], [32], [33], [34], [35], [36], [37], [38], [39], [40], [41], also [4]) along with their potential applications to the theory of the reaction-diffusion problems (see [9], [10]). Fredholm structures, topological invariants and their applications were covered in [11]. The article [12] is devoted to the finite and infinite dimensional attractors for the evolution equations of mathematical physics. The large time behavior of the solutions of a class of fourth-order parabolic equations defined on unbounded domains using the Kolmogorov $\varepsilon$-entropy as a measure was considered in [13]. The attractor for a nonlinear reaction- diffusion system in an unbounded domain in the space of three dimensions was studied in [20]. The works [21] and [27] are important for the understanding of the Fredholm and properness properties of the quasilinear elliptic systems of the second order and of the operators of this kind on $\mathbb{R}^N$. The exponential decay and Fredholm properties in the second-order quasilinear elliptic systems of equations were discussed in [22]. The articles [32] and [38] deal with the solvability conditions for the linearized Cahn-Hilliard equations. The work [37] is devoted to the studies of the Laplacian with transport from the point of view of the non-Fredholm operators. Standing lattice solitons in the discrete NLS equation with saturation were covered in [1]. Particularly, when the constant $a = 0$, our operator $A$ satisfies the Fredholm property in certain properly chosen weighted spaces (see [2], [3], [4], [5], [6]). However, the situation of $a$ nontrivial is significantly different and the approach developed in these articles cannot be used.
One of the important issues concerning the problems with non-Fredholm operators is their solvability. Let us address it in the following setting. Let $f_n$ be a sequence of functions in the image of the operator $A$, so that $f_n \to f$ in $L^2(\mathbb{R}^d)$ as $n \to \infty$. We designate by $u_n$ a sequence of functions from $H^2(\mathbb{R}^d)$, so that

$$Au_n = f_n, \ n \in \mathbb{N}.$$

Since the operator $A$ fails to satisfy the Fredholm property, the sequence $u_n$ may not be convergent. Let us call a sequence $u_n$, so that $Au_n \to f$ a solution in the sense of sequences of equation $Au = f$ (see [30]). If this sequence converges to a function $u_0$ in the norm of the space $E$, then $u_0$ is a solution of this equation. The solution in the sense of sequences is equivalent in this sense to the usual solution. However, in the case of the non-Fredholm operators, this convergence may not hold or it can occur in some weaker sense. In this situation, the solution in the sense of sequences may not imply the existence of the usual solution. In the present article we will find the sufficient conditions of equivalence of the solutions in the sense of sequences and the usual solutions. In the other words, we will find the conditions on the sequences $f_n$ under which the corresponding sequences $u_n$ are strongly convergent. Solvability in the sense of sequences for a linear nonhomogeneous equation involving the logarithmic Laplacian with and without a shallow, short-range scalar potential was covered in [19]. The present work is our modest attempt to generalize these results. In the first part of the article we study the equation

$$\Big[\frac{1}{2}\ln\{-\Delta_x + V(x) - \Delta_y + U(y)\}\Big]u - au = f(x,y), \quad x, y \in \mathbb{R}^3, \quad a \in \mathbb{R} \tag{1.2}$$

with a square integrable right side. The operator in the left side of problem (1.2)

$$H_{U,\,V} := \frac{1}{2}\ln\{-\Delta_x + V(x) - \Delta_y + U(y)\} - a \tag{1.3}$$

is defined by means of the spectral calculus. Here and further down the Laplace operators $\Delta_x$ and $\Delta_y$ are acting on the $x$ and $y$ variables respectively. The sum of the two Schrödinger type operators contained in the right side of (1.3) has the physical meaning of the cumulative hamiltonian of the two non interacting three dimensional quantum particles in external potentials. The logarithmic Laplacian $\ln(-\Delta)$ is the operator with the Fourier symbol $2\ln|p|$. It arises as the formal derivative $\partial_s|_{s=0}(-\Delta)^s$ of the fractional Laplacians at $s = 0$. The operator $(-\Delta)^s$ is extensively used, for instance in the studies of the anomalous diffusion problems (see e.g. [41] and the references therein). Spectral properties of the logarithmic Laplacian in an open set of finite measure with Dirichlet boundary conditions were discussed in [25] (see also [7]). The studies of $\ln(-\Delta)$ are important for the understanding of the asymptotic spectral properties of the family of the fractional Laplacians in the limit $s \to 0^+$. In [23] it was demonstrated that such operator allows to characterize the $s$-dependence of solution to fractional Poisson equations for the full range of exponents $s \in (0,1)$. The scalar potential functions involved in operator $H_{U,\,V}$ are assumed to be shallow and short-range, satisfying the assumptions analogous to the ones of [34] and [35].

**Assumption 1.1.** *The potential functions $V(x), U(y) : \mathbb{R}^3 \to \mathbb{R}$ satisfy the estimates*

$$|V(x)| \le \frac{C}{1 + |x|^{3.5+\varepsilon}}, \quad |U(y)| \le \frac{C}{1 + |y|^{3.5+\varepsilon}}$$

*with a certain $\varepsilon > 0$ and $x, y \in \mathbb{R}^3$ a.e. so that*

$$4^{\frac{1}{9}}\frac{9}{8}(4\pi)^{-\frac{2}{3}}\|V\|_{L^\infty(\mathbb{R}^3)}^{\frac{1}{9}}\|V\|_{L^{\frac{4}{3}}(\mathbb{R}^3)}^{\frac{8}{9}} < 1, \tag{1.4}$$

$$4^{\frac{1}{9}}\frac{9}{8}(4\pi)^{-\frac{2}{3}}\|U\|_{L^\infty(\mathbb{R}^3)}^{\frac{1}{9}}\|U\|_{L^{\frac{4}{3}}(\mathbb{R}^3)}^{\frac{8}{9}} < 1 \tag{1.5}$$

*and*

$$\sqrt{c_{HLS}}\|V\|_{L^{\frac{3}{2}}(\mathbb{R}^3)} < 4\pi, \quad \sqrt{c_{HLS}}\|U\|_{L^{\frac{3}{2}}(\mathbb{R}^3)} < 4\pi.$$

Here and below $C$ will stand for a finite positive constant and $c_{HLS}$ given on p.98 of [26] is the constant in the Hardy-Littlewood-Sobolev inequality

$$\left|\int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\frac{f_1(x)f_1(y)}{|x-y|^2}dxdy\right| \leq c_{HLS}\|f_1\|^2_{L^{\frac{3}{2}}(\mathbb{R}^3)}, \quad f_1 \in L^{\frac{3}{2}}(\mathbb{R}^3).$$

The norm of a function $f_1 \in L^p(\mathbb{R}^d),\ 1 \leq p \leq \infty,\ d \in \mathbb{N}$ is denoted as $\|f_1\|_{L^p(\mathbb{R}^d)}$. By virtue of Lemma 2.3 of [35], under Assumption 1.1 above on the scalar potentials, the operator

$$-\Delta_x + V(x) - \Delta_y + U(y)$$

on $L^2(\mathbb{R}^6)$ is self-adjoint and is unitarily equivalent to $-\Delta_x - \Delta_y$ via the product of the wave operators (see [24], [29])

$$\Omega^{\pm}_V := s - \lim_{t\to\mp\infty} e^{it(-\Delta_x+V(x))}e^{it\Delta_x}, \quad \Omega^{\pm}_U := s - \lim_{t\to\mp\infty} e^{it(-\Delta_y+U(y))}e^{it\Delta_y},$$

where the limits are understood in the strong $L^2$ sense (see e.g. [28] p.34, [8] p.90). Therefore, operator (1.3) has only the essential spectrum, which fills the whole real line and no nontrivial $L^2(\mathbb{R}^6)$ eigenfunctions. Thus, operator (1.3) fails to satisfy the Fredholm property. Note, that in most of the works dealing with the non-Fredholm operators mentioned above except [19] the essential spectra filled only semi-axes. The functions of the continuos spectrum of the first differential operator involved in (1.3) are the solutions to the Schrödinger equation

$$[-\Delta_x + V(x)]\varphi_k(x) = k^2\varphi_k(x), \quad k \in \mathbb{R}^3,$$

in the integral form the Lippmann-Schwinger equation (see e.g. [28] p.98)

$$\varphi_k(x) = \frac{e^{ikx}}{(2\pi)^{\frac{3}{2}}} - \frac{1}{4\pi}\int_{\mathbb{R}^3}\frac{e^{i|k||x-y|}}{|x-y|}(V\varphi_k)(y)dy \tag{1.6}$$

and the orthogonality relations

$$(\varphi_k(x), \varphi_{k_1}(x))_{L^2(\mathbb{R}^3)} = \delta(k-k_1), \quad k,\ k_1 \in \mathbb{R}^3$$

hold. The integral operator contained in (1.6)

$$(Q\varphi)(x) := -\frac{1}{4\pi}\int_{\mathbb{R}^3}\frac{e^{i|k||x-y|}}{|x-y|}(V\varphi)(y)dy, \quad \varphi(x) \in L^\infty(\mathbb{R}^3).$$

We consider $Q : L^\infty(\mathbb{R}^3) \to L^\infty(\mathbb{R}^3)$ and its norm $\|Q\|_\infty < 1$ under our Assumption 1.1 via Lemma 2.1 of [35]. Note that this norm is bounded above by the $k$-independent quantity $I(V)$, which is the left side of inequality (1.4). By virtue of Corollary 2.2 of [35] (see also [34]), under the given conditions for $k \in \mathbb{R}^3$ we have $\varphi_k(x) \in L^\infty(\mathbb{R}^3)$, so that

$$\|\varphi_k(x)\|_{L^\infty(\mathbb{R}^3)} \leq \frac{1}{1-I(V)}\frac{1}{(2\pi)^{\frac{3}{2}}}. \tag{1.7}$$

Similarly, for the second differential operator contained in (1.3) the functions of its continuous spectrum solve

$$[-\Delta_y + U(y)]\eta_q(y) = q^2\eta_q(y), \quad q \in \mathbb{R}^3,$$

in the integral formulation

$$\eta_q(y) = \frac{e^{iqy}}{(2\pi)^{\frac{3}{2}}} - \frac{1}{4\pi}\int_{\mathbb{R}^3} \frac{e^{i|q||y-z|}}{|y-z|}(U\eta_q)(z)dz, \tag{1.8}$$

such that the orthogonality conditions

$$(\eta_q(y), \eta_{q_1}(y))_{L^2(\mathbb{R}^3)} = \delta(q - q_1), \quad q, \ q_1 \in \mathbb{R}^3$$

are valid. The integral operator involved in (1.8) is

$$(P\eta)(y) := -\frac{1}{4\pi}\int_{\mathbb{R}^3} \frac{e^{i|q||y-z|}}{|y-z|}(U\eta)(z)dz, \quad \eta(y) \in L^\infty(\mathbb{R}^3).$$

For $P : L^\infty(\mathbb{R}^3) \to L^\infty(\mathbb{R}^3)$ its norm $\|P\|_\infty < 1$ under Assumption 1.1 by means of Lemma 2.1 of [35]. As above, this norm can be bounded from above by the $q$-independent quantity $I(U)$, which is the left side of (1.5). For $q \in \mathbb{R}^3$, we have $\eta_q(y) \in L^\infty(\mathbb{R}^3)$ and

$$\|\eta_q(y)\|_{L^\infty(\mathbb{R}^3)} \leq \frac{1}{1 - I(U)}\frac{1}{(2\pi)^{\frac{3}{2}}}. \tag{1.9}$$

By means of the spectral theorem,

$$H_{U,\ V}\varphi_k(x)\eta_q(y) = [\ln(\sqrt{k^2 + q^2}) - a]\varphi_k(x)\eta_q(y).$$

We designate by the double tilde sign the generalized Fourier transform with the product of these functions of the continuous spectrum

$$\tilde{\tilde{f}}(k, q) := (f(x, y), \varphi_k(x)\eta_q(y))_{L^2(\mathbb{R}^6)}, \quad k, q \in \mathbb{R}^3. \tag{1.10}$$

(1.10) is a unitary transform on $L^2(\mathbb{R}^6)$. The inner product of two functions is denoted as

$$(f(x), g(x))_{L^2(\mathbb{R}^d)} := \int_{\mathbb{R}^d} f(x)\bar{g}(x)dx, \quad d \in \mathbb{N}, \tag{1.11}$$

with a slight abuse of notations when the functions involved in (1.11) are not square integrable. Indeed, if $f(x) \in L^1(\mathbb{R}^d)$ and $g(x) \in L^\infty(\mathbb{R}^d)$, then the integral in the right side of formula (1.11) makes sense. Let us recall the Fact 2 of [34]. Clearly, under the conditions of Theorem 1.2 below, we have $f(x, y) \in L^1(\mathbb{R}^6)$. The functions of the continuous spectra of our Schrödinger operators $\varphi_k(x)$ and $\eta_q(y)$ are bounded by virtue of the Corollary 2.2 of [35]. Therefore, the left side of formula (1.12) below is well defined. The sphere of radius $r$ in $\mathbb{R}^d, \ d \in \mathbb{N}$ centered at the origin is being designated as $S_r^d$. Our first main proposition is as follows.

**Theorem 1.2.** *Let Assumption 1.1 hold, for the function*

$$f(x,y) \in L^2(\mathbb{R}^6), \quad |x|f(x,y) \in L^1(\mathbb{R}^6), \quad |y|f(x,y) \in L^1(\mathbb{R}^6).$$

*Then problem (1.2) possesses a unique solution $u(x,y) \in L^2(\mathbb{R}^6)$ if and only if the orthogonality relations*

$$(f(x,y), \varphi_k(x)\eta_q(y))_{L^2(\mathbb{R}^6)} = 0 \quad for \quad (k,q) \in S_{e^a}^6 \tag{1.12}$$

*are valid.*

Let us turn our attention to the issue of the solvability in the sense of sequences for our problem. The corresponding sequence of approximate equations with $n \in \mathbb{N}$ is given by

$$\Big[\frac{1}{2}\ln\{-\Delta_x + V(x) - \Delta_y + U(y)\}\Big]u_n - au_n = f_n(x,y), \quad x, \ y \in \mathbb{R}^3 \tag{1.13}$$

with $a \in \mathbb{R}$. The square integrable right sides of (1.13) are converging to the right side of (1.2) in $L^2(\mathbb{R}^6)$ as $n \to \infty$. Our second main result is as follows.

**Theorem 1.3.** *Let Assumption 1.1 hold, $n \in \mathbb{N}$, for the functions*

$$f_n(x,y) \in L^2(\mathbb{R}^6), \quad |x|f_n(x,y) \in L^1(\mathbb{R}^6), \quad |y|f_n(x,y) \in L^1(\mathbb{R}^6),$$

*we have*

$$f_n(x,y) \to f(x,y) \quad in \quad L^2(\mathbb{R}^6), \quad |x|f_n(x,y) \to |x|f(x,y) \quad in \quad L^1(\mathbb{R}^6)$$

*and*

$$|y|f_n(x,y) \to |y|f(x,y) \quad in \quad L^1(\mathbb{R}^6)$$

*as $n \to \infty$. Moreover, the orthogonality conditions*

$$(f_n(x,y), \varphi_k(x)\eta_q(y))_{L^2(\mathbb{R}^6)} = 0 \quad for \quad (k,q) \in S_{e^a}^6 \tag{1.14}$$

*hold for all $n \in \mathbb{N}$. Then equations (1.2) and (1.13) have unique solutions $u(x,y) \in L^2(\mathbb{R}^6)$ and $u_n(x,y) \in L^2(\mathbb{R}^6)$ respectively, so that $u_n(x,y) \to u(x,y)$ in $L^2(\mathbb{R}^6)$ as $n \to \infty$.*

The second part of the work deals with the studies of the equation

$$\Big[\frac{1}{2}\ln\{-\Delta_x - \Delta_y + U(y)\}\Big]u - au = \phi(x,y), \quad x \in \mathbb{R}^d, \quad y \in \mathbb{R}^3 \tag{1.15}$$

with $d \in \mathbb{N}, \ a \in \mathbb{R}$. The scalar potential function contained in (1.15) is shallow and short-range under our Assumption 1.1 and the right side of (1.15) is square integrable. The operator

$$L_U := \frac{1}{2}\ln\{-\Delta_x - \Delta_y + U(y)\} - a \tag{1.16}$$

here is defined by virtue of the spectral calculus. The sum of the free negative Laplacian and the Schrödinger type operator involved in the right side of (1.16) has the physical meaning of the cumulative hamiltonian of a free $d$ dimensional particle and a three dimensional particle in an external potential. These particles do not interact. As above, the operator

$$-\Delta_x - \Delta_y + U(y)$$

on $L^2(\mathbb{R}^{d+3})$ is self-adjoint and is unitarily equivalent to $-\Delta_x - \Delta_y$. Thus, operator (1.16) has only the essential spectrum, which fills the whole real line similarly to the two potential case and no nontrvial $L^2(\mathbb{R}^{d+3})$ eigenfunctions. Therefore, operator (1.16) does not satisfy the Fredholm property. By virtue of the spectral theorem, we have

$$L_U \frac{e^{ikx}}{(2\pi)^{\frac{d}{2}}}\eta_q(y) = [\ln(\sqrt{k^2+q^2}) - a]\frac{e^{ikx}}{(2\pi)^{\frac{d}{2}}}\eta_q(y).$$

We consider another useful generalized Fourier transform with the standard Fourier harmonics and the perturbed plane waves, namely

$$\tilde{\hat{\phi}}(k,q) := \left(\phi(x,y), \frac{e^{ikx}}{(2\pi)^{\frac{d}{2}}}\eta_q(y)\right)_{L^2(\mathbb{R}^{d+3})}, \quad k \in \mathbb{R}^d, \quad q \in \mathbb{R}^3. \tag{1.17}$$

(1.17) is a unitary transform on $L^2(\mathbb{R}^{d+3})$. We have the following proposition.

**Theorem 1.4.** *Let the potential function $U(y)$ satisfy Assumption 1.1, $d \in \mathbb{N}$, for the function*

$$\phi(x,y) \in L^2(\mathbb{R}^{d+3}), \quad |x|\phi(x,y) \in L^1(\mathbb{R}^{d+3}), \quad |y|\phi(x,y) \in L^1(\mathbb{R}^{d+3}).$$

*Then equation (1.15) admits a unique solution $u(x,y) \in L^2(\mathbb{R}^{d+3})$ if and only if the orthogonality conditions*

$$\left(\phi(x,y), \frac{e^{ikx}}{(2\pi)^{\frac{d}{2}}}\eta_q(y)\right)_{L^2(\mathbb{R}^{d+3})} = 0 \quad for \quad (k,q) \in S^{d+3}_{e^a} \tag{1.18}$$

*are valid.*

The final statement of the work is devoted to the issue of the solvability in the sense of sequences for problem (1.15). The corresponding sequence of approximate equations with $n \in \mathbb{N},\ x \in \mathbb{R}^d,\ d \in \mathbb{N},\ y \in \mathbb{R}^3,\ a \in \mathbb{R}$ is given by

$$\Big[\frac{1}{2}\ln\{-\Delta_x - \Delta_y + U(y)\}\Big] u_n - a u_n = \phi_n(x, y). \tag{1.19}$$

The right sides of (1.19) are square integrable. They tend to the right side of (1.15) in $L^2(\mathbb{R}^{d+3})$ as $n \to \infty$.

**Theorem 1.5.** *Let the potential function $U(y)$ satisfy Assumption 1.1, $n \in \mathbb{N},\ d \in \mathbb{N}$, for the functions*

$$\phi_n(x,y) \in L^2(\mathbb{R}^{d+3}), \quad |x|\phi_n(x,y) \in L^1(\mathbb{R}^{d+3}), \quad |y|\phi_n(x,y) \in L^1(\mathbb{R}^{d+3}),$$

*we have*

$$\phi_n(x,y) \to \phi(x,y) \quad in \quad L^2(\mathbb{R}^{d+3}), \quad |x|\phi_n(x,y) \to |x|\phi(x,y) \quad in \quad L^1(\mathbb{R}^{d+3})$$

*and*

$$|y|\phi_n(x,y) \to |y|\phi(x,y) \quad in \quad L^1(\mathbb{R}^{d+3})$$

*as $n \to \infty$. Furthermore, the orthogonality relations*

$$\left(\phi_n(x,y), \frac{e^{ikx}}{(2\pi)^{\frac{d}{2}}}\eta_q(y)\right)_{L^2(\mathbb{R}^{d+3})} = 0 \quad for \quad (k,q) \in S^{d+3}_{e^a} \tag{1.20}$$

*hold for all $n \in \mathbb{N}$. Then problems (1.15) and (1.19) possess unique solutions $u(x,y) \in L^2(\mathbb{R}^{d+3})$ and $u_n(x,y) \in L^2(\mathbb{R}^{d+3})$ respectively, so that $u_n(x,y) \to u(x,y)$ in $L^2(\mathbb{R}^{d+3})$ as $n \to \infty$.*

Let us proceed to the proofs of our propositions.

## 2. Solvability in the sense of sequences with two scalar potentials

*Proof of Theorem 1.2.* First we demonstrate the uniqueness of solutions for problem (1.2). Suppose it has two solutions $u_1(x,y),\ u_2(x,y) \in L^2(\mathbb{R}^6)$. Then their difference $w(x,y) := u_1(x,y) - u_2(x,y) \in L^2(\mathbb{R}^6)$ satisfies the homogeneous equation

$$H_{U,\,V} w = 0.$$

Because operator (1.3) has no nontrivial square integrable zero modes in the whole space as mentioned above, $w(x,y) \equiv 0$ in $\mathbb{R}^6$.
We apply the generalized Fourier transform (1.10) to both sides of equation (1.2) and obtain

$$\tilde{\tilde{u}}(k,q) = \frac{\tilde{\tilde{f}}(k,q)}{\ln\left(\frac{\sqrt{k^2+q^2}}{e^a}\right)}, \quad k,\ q \in \mathbb{R}^3. \tag{2.1}$$

For the technical purposes we introduce the spherical layer

$$A_\delta := \{(k,q) \in \mathbb{R}^6 \mid e^a(1-\delta) \le \sqrt{k^2+q^2} \le e^a(1+\delta)\}, \quad 0<\delta<1, \tag{2.2}$$

which enables us to express

$$\tilde{\tilde{u}}(k,q) = \frac{\tilde{\tilde{f}}(k,q)}{\ln\left(\frac{\sqrt{k^2+q^2}}{e^a}\right)}\chi_{A_\delta} + \frac{\tilde{\tilde{f}}(k,q)}{\ln\left(\frac{\sqrt{k^2+q^2}}{e^a}\right)}\chi_{A_\delta^c}. \tag{2.3}$$

Here and below $A^c$ will denote the complement of a set $A \subseteq \mathbb{R}^d$. The characteristic function of a set $A$ is being designated as $\chi_A$ and $|A|$ will stand for the Lebesgue measure of $A$. Let us define the sets

$$A_\delta^{c+} := \{(k,q) \in \mathbb{R}^6 \mid \sqrt{k^2+q^2} > e^a(1+\delta)\}, \tag{2.4}$$

$$A_\delta^{c-} := \{(k,q) \in \mathbb{R}^6 \mid \sqrt{k^2+q^2} < e^a(1-\delta)\}, \tag{2.5}$$

so that

$$A_\delta^c = A_\delta^{c+} \cup A_\delta^{c-}.$$

Clearly, the second term in the right side of (2.3) can be written as

$$\frac{\tilde{\tilde{f}}(k,q)}{\ln\left(\frac{\sqrt{k^2+q^2}}{e^a}\right)}\chi_{A_\delta^{c+}} + \frac{\tilde{\tilde{f}}(k,q)}{\ln\left(\frac{\sqrt{k^2+q^2}}{e^a}\right)}\chi_{A_\delta^{c-}}. \tag{2.6}$$

We have the elementary upper bounds

$$\frac{|\tilde{\tilde{f}}(k,q)|}{\left|\ln\left(\frac{\sqrt{k^2+q^2}}{e^a}\right)\right|}\chi_{A_\delta^{c+}} \le \frac{|\tilde{\tilde{f}}(k,q)|}{\ln(1+\delta)} \in L^2(\mathbb{R}^6),$$

$$\frac{|\tilde{\tilde{f}}(k,q)|}{\left|\ln\left(\frac{\sqrt{k^2+q^2}}{e^a}\right)\right|}\chi_{A_\delta^{c-}} \le \frac{|\tilde{\tilde{f}}(k,q)|}{-\ln(1-\delta)} \in L^2(\mathbb{R}^6)$$

via the one of our assumptions. Obviously, we can write

$$\tilde{\tilde{f}}(k,q) = \tilde{\tilde{f}}(e^a,\sigma) + \int_{e^a}^{\sqrt{k^2+q^2}} \frac{\partial \tilde{\tilde{f}}(s,\sigma)}{\partial s} ds. \tag{2.7}$$

Here and further down $\sigma$ will denote the angle variables on the sphere. Hence, we can express the first term in the right side of (2.3) as

$$\frac{\tilde{\tilde{f}}(e^a,\sigma)}{\ln\left(\frac{\sqrt{k^2+q^2}}{e^a}\right)}\chi_{A_\delta} + \frac{\int_{e^a}^{\sqrt{k^2+q^2}} \frac{\partial \tilde{\tilde{f}}(s,\sigma)}{\partial s} ds}{\ln\left(\frac{\sqrt{k^2+q^2}}{e^a}\right)}\chi_{A_\delta}. \tag{2.8}$$

Let us recall the result of Lemma 11 of [34]. Thus, under the stated assumptions we have $(\nabla_k + \nabla_q)\tilde{\tilde{f}}(k,q) \in L^\infty(\mathbb{R}^6)$. We estimate the second term in sum (2.8) from above in the absolute value as

$$\left|\frac{\int_{e^a}^{\sqrt{k^2+q^2}} \frac{\partial \tilde{\tilde{f}}(s,\sigma)}{\partial s} ds}{\ln\left(\frac{\sqrt{k^2+q^2}}{e^a}\right)}\chi_{A_\delta}\right| \leq \|(\nabla_k + \nabla_q)\tilde{\tilde{f}}(k,q)\|_{L^\infty(\mathbb{R}^6)} \left|\frac{\sqrt{k^2+q^2} - e^a}{\ln\left(\frac{\sqrt{k^2+q^2}}{e^a}\right)}\right| \chi_{A_\delta} \leq$$

$$\leq C\|(\nabla_k + \nabla_q)\tilde{\tilde{f}}(k,q)\|_{L^\infty(\mathbb{R}^6)}\chi_{A_\delta} \in L^2(\mathbb{R}^6).$$

Therefore, it remains to consider the term

$$\frac{\tilde{\tilde{f}}(e^a,\sigma)}{\ln\left(\frac{\sqrt{k^2+q^2}}{e^a}\right)}\chi_{A_\delta}. \tag{2.9}$$

It can be easily verified that (2.9) belongs to $L^2(\mathbb{R}^6)$ if and only if $\tilde{\tilde{f}}(e^a,\sigma)$ is trivial. This is equivalent to orthogonality conditions (1.12). ■

We turn our attention to the establishing of the solvability in the sense of sequences for our problem in the situation with two scalar potentials.

*Proof of Theorem 1.3.* Evidently, each equation (1.13) admits a unique solution $u_n(x,y) \in L^2(\mathbb{R}^6)$, $n \in \mathbb{N}$ by means of the result of Theorem 1.2 above. Let us verify that the limiting orthogonality conditions

$$(f(x,y), \varphi_k(x)\eta_q(y))_{L^2(\mathbb{R}^6)} = 0 \quad for \quad (k,q) \in S^6_{e^a} \tag{2.10}$$

are valid. Recall the Fact 2 of [34] and the proof of Theorem 2 of [40]. Hence, under the stated assumptions we have $f_n(x,y) \in L^1(\mathbb{R}^6)$, $n \in \mathbb{N}$, so that

$$f_n(x,y) \to f(x,y) \quad in \quad L^1(\mathbb{R}^6) \quad as \quad n \to \infty. \tag{2.11}$$

Using (1.14), (1.7), (1.9) and (2.11), we derive for $(k,q) \in S^6_{e^a}$ that

$$|(f(x,y), \varphi_k(x)\eta_q(y))_{L^2(\mathbb{R}^6)}| = |(f(x,y) - f_n(x,y), \varphi_k(x)\eta_q(y))_{L^2(\mathbb{R}^6)}| \leq$$

$$\leq \frac{1}{(2\pi)^3}\frac{1}{1-I(V)}\frac{1}{1-I(U)}\|f_n(x,y) - f(x,y)\|_{L^1(\mathbb{R}^6)} \to 0, \quad n \to \infty,$$

such that formula (2.10) holds. By virtue of the result of Theorem 1.2, equation (1.2) has a unique solution $u(x,y) \in L^2(\mathbb{R}^6)$.

Let us apply the generalized Fourier transform (1.10) to both sides of problems (1.2) and (1.13). This gives us

$$\tilde{\tilde{u}}(k,q) = \frac{\tilde{\tilde{f}}(k,q)}{\ln\left(\frac{\sqrt{k^2+q^2}}{e^a}\right)}, \quad \tilde{\tilde{u}}_n(k,q) = \frac{\tilde{\tilde{f}}_n(k,q)}{\ln\left(\frac{\sqrt{k^2+q^2}}{e^a}\right)}$$

with $k,\ q\in\mathbb{R}^3$ and $n\in\mathbb{N}$. Hence, $\tilde{\tilde{u}}_n(k,q)-\tilde{\tilde{u}}(k,q)$ can be expressed as

$$\frac{\tilde{\tilde{f}}_n(k,q)-\tilde{\tilde{f}}(k,q)}{\ln\left(\frac{\sqrt{k^2+q^2}}{e^a}\right)}\chi_{A_\delta}+\frac{\tilde{\tilde{f}}_n(k,q)-\tilde{\tilde{f}}(k,q)}{\ln\left(\frac{\sqrt{k^2+q^2}}{e^a}\right)}\chi_{A_\delta^c}. \tag{2.12}$$

Obviously, the second term in (2.12) is given by

$$\frac{\tilde{\tilde{f}}_n(k,q)-\tilde{\tilde{f}}(k,q)}{\ln\left(\frac{\sqrt{k^2+q^2}}{e^a}\right)}\chi_{A_\delta^{c+}}+\frac{\tilde{\tilde{f}}_n(k,q)-\tilde{\tilde{f}}(k,q)}{\ln\left(\frac{\sqrt{k^2+q^2}}{e^a}\right)}\chi_{A_\delta^{c-}}. \tag{2.13}$$

We have the trivial estimates from above

$$\frac{|\tilde{\tilde{f}}_n(k,q)-\tilde{\tilde{f}}(k,q)|}{\left|\ln\left(\frac{\sqrt{k^2+q^2}}{e^a}\right)\right|}\chi_{A_\delta^{c+}}\le\frac{|\tilde{\tilde{f}}_n(k,q)-\tilde{\tilde{f}}(k,q)|}{\ln(1+\delta)},$$

$$\frac{|\tilde{\tilde{f}}_n(k,q)-\tilde{\tilde{f}}(k,q)|}{\left|\ln\left(\frac{\sqrt{k^2+q^2}}{e^a}\right)\right|}\chi_{A_\delta^{c-}}\le\frac{|\tilde{\tilde{f}}_n(k,q)-\tilde{\tilde{f}}(k,q)|}{-\ln(1-\delta)},$$

so that

$$\left\|\frac{\tilde{\tilde{f}}_n(k,q)-\tilde{\tilde{f}}(k,q)}{\ln\left(\frac{\sqrt{k^2+q^2}}{e^a}\right)}\chi_{A_\delta^{c+}}\right\|_{L^2(\mathbb{R}^6)}\le\frac{\|f_n(x,y)-f(x,y)\|_{L^2(\mathbb{R}^6)}}{\ln(1+\delta)}\to 0,\quad n\to\infty,$$

$$\left\|\frac{\tilde{\tilde{f}}_n(k,q)-\tilde{\tilde{f}}(k,q)}{\ln\left(\frac{\sqrt{k^2+q^2}}{e^a}\right)}\chi_{A_\delta^{c-}}\right\|_{L^2(\mathbb{R}^6)}\le\frac{\|f_n(x,y)-f(x,y)\|_{L^2(\mathbb{R}^6)}}{-\ln(1-\delta)}\to 0,\quad n\to\infty$$

as assumed. Let us recall orthogonality conditions (2.10) and (1.14). They yield that

$$\tilde{\tilde{f}}(e^a,\sigma)=0,\quad \tilde{\tilde{f}}_n(e^a,\sigma)=0,\quad n\in\mathbb{N},$$

such that

$$\tilde{\tilde{f}}(k,q)=\int_{e^a}^{\sqrt{k^2+q^2}}\frac{\partial\tilde{\tilde{f}}(s,\sigma)}{\partial s}ds,\quad \tilde{\tilde{f}}_n(k,q)=\int_{e^a}^{\sqrt{k^2+q^2}}\frac{\partial\tilde{\tilde{f}}_n(s,\sigma)}{\partial s}ds,\quad n\in\mathbb{N}.$$

Then the first term in (2.12) can be written as

$$\frac{\int_{e^a}^{\sqrt{k^2+q^2}}\left[\frac{\partial\tilde{\tilde{f}}_n(s,\sigma)}{\partial s}-\frac{\partial\tilde{\tilde{f}}(s,\sigma)}{\partial s}\right]ds}{\ln\left(\frac{\sqrt{k^2+q^2}}{e^a}\right)}\chi_{A_\delta}. \tag{2.14}$$

Clearly, (2.14) can be bounded from above in the absolute value by

$$\|(\nabla_k+\nabla_q)[\tilde{\tilde{f}}_n(k,q)-\tilde{\tilde{f}}(k,q)]\|_{L^\infty(\mathbb{R}^6)}\left|\frac{\sqrt{k^2+q^2}-e^a}{\ln\left(\frac{\sqrt{k^2+q^2}}{e^a}\right)}\right|\chi_{A_\delta}\leq$$

$$\leq C\|(\nabla_k+\nabla_q)[\tilde{\tilde{f}}_n(k,q)-\tilde{\tilde{f}}(k,q)]\|_{L^\infty(\mathbb{R}^6)}\chi_{A_\delta}.$$

Note that under the given conditions, by means of Lemma 11 of [34], we have $(\nabla_k+\nabla_q)\tilde{\tilde{f}}_n(k,q),\ (\nabla_k+\nabla_q)\tilde{\tilde{f}}(k,q)\in L^\infty(\mathbb{R}^6)$. By virtue of Lemma 5 of [40],

$$\|(\nabla_k+\nabla_q)[\tilde{\tilde{f}}_n(k,q)-\tilde{\tilde{f}}(k,q)]\|_{L^\infty(\mathbb{R}^6)}\to 0,\quad n\to\infty. \tag{2.15}$$

Evidently, we have the estimate for the norm

$$\left\|\frac{\int_{e^a}^{\sqrt{k^2+q^2}}\left[\frac{\partial\tilde{\tilde{f}}_n(s,\sigma)}{\partial s}-\frac{\partial\tilde{\tilde{f}}(s,\sigma)}{\partial s}\right]ds}{\ln\left(\frac{\sqrt{k^2+q^2}}{e^a}\right)}\chi_{A_\delta}\right\|_{L^2(\mathbb{R}^6)}\leq$$

$$\leq C\|(\nabla_k+\nabla_q)[\tilde{\tilde{f}}_n(k,q)-\tilde{\tilde{f}}(k,q)]\|_{L^\infty(\mathbb{R}^6)}|A_\delta|^{\frac{1}{2}}\to 0,\quad n\to\infty$$

due to (2.15). Therefore, $u_n(x,y)\to u(x,y)$ in $L^2(\mathbb{R}^6)$ as $n\to\infty$, which completes the proof of the theorem. ∎

In the final section of our article we consider the case of the logarithmic Schrödinger operator involving the free Laplacian added to the three dimensional Schrödinger operator.

## 3. Solvability in the sense of sequences with Laplacian and a single potential

*Proof of Theorem 1.4.* To establish the uniqueness of solutions for our problem, we suppose that (1.15) possesses two solutions $u_1(x,y),\ u_2(x,y)\in L^2(\mathbb{R}^{d+3})$. Then their difference $w(x,y):=u_1(x,y)-u_2(x,y)\in L^2(\mathbb{R}^{d+3})$ solves the homogeneous equation

$$L_U w=0.$$

Since operator (1.16) considered in the whole space does not have any nontrivial square integrable zero modes as stated above, $w(x,y)\equiv 0$ in $\mathbb{R}^{d+3}$.
Let us apply the generalized Fourier transform (1.17) to both sides of problem (1.15). This gives us

$$\tilde{\tilde{u}}(k,q)=\frac{\tilde{\tilde{\phi}}(k,q)}{\ln\left(\frac{\sqrt{k^2+q^2}}{e^a}\right)},\quad k\in\mathbb{R}^d,\quad q\in\mathbb{R}^3. \tag{3.1}$$

We use the spherical layer

$$B_\delta := \{(k,q) \in \mathbb{R}^{d+3} \mid e^a(1-\delta) \le \sqrt{k^2+q^2} \le e^a(1+\delta)\}, \quad 0<\delta<1. \tag{3.2}$$

This allows us to write

$$\tilde{\hat{u}}(k,q) = \frac{\tilde{\hat{\phi}}(k,q)}{\ln\left(\frac{\sqrt{k^2+q^2}}{e^a}\right)}\chi_{B_\delta} + \frac{\tilde{\hat{\phi}}(k,q)}{\ln\left(\frac{\sqrt{k^2+q^2}}{e^a}\right)}\chi_{B_\delta^c}. \tag{3.3}$$

Let us introduce the sets

$$B_\delta^{c+} := \{(k,q) \in \mathbb{R}^{d+3} \mid \sqrt{k^2+q^2} > e^a(1+\delta)\}, \tag{3.4}$$

$$B_\delta^{c-} := \{(k,q) \in \mathbb{R}^{d+3} \mid \sqrt{k^2+q^2} < e^a(1-\delta)\}, \tag{3.5}$$

such that

$$B_\delta^c = B_\delta^{c+} \cup B_\delta^{c-}.$$

Obviously, the second term in the right side of (3.3) can be expressed as

$$\frac{\tilde{\hat{\phi}}(k,q)}{\ln\left(\frac{\sqrt{k^2+q^2}}{e^a}\right)}\chi_{B_\delta^{c+}} + \frac{\tilde{\hat{\phi}}(k,q)}{\ln\left(\frac{\sqrt{k^2+q^2}}{e^a}\right)}\chi_{B_\delta^{c-}}. \tag{3.6}$$

We have the trivial estimates from above

$$\frac{|\tilde{\hat{\phi}}(k,q)|}{\left|\ln\left(\frac{\sqrt{k^2+q^2}}{e^a}\right)\right|}\chi_{B_\delta^{c+}} \le \frac{|\tilde{\hat{\phi}}(k,q)|}{\ln(1+\delta)} \in L^2(\mathbb{R}^{d+3}),$$

$$\frac{|\tilde{\hat{\phi}}(k,q)|}{\left|\ln\left(\frac{\sqrt{k^2+q^2}}{e^a}\right)\right|}\chi_{B_\delta^{c-}} \le \frac{|\tilde{\hat{\phi}}(k,q)|}{-\ln(1-\delta)} \in L^2(\mathbb{R}^{d+3})$$

due to the one of our assumptions. Evidently, we have the formula

$$\tilde{\hat{\phi}}(k,q) = \tilde{\hat{\phi}}(e^a,\sigma) + \int_{e^a}^{\sqrt{k^2+q^2}} \frac{\partial \tilde{\hat{\phi}}(s,\sigma)}{\partial s} ds \tag{3.7}$$

Thus, the first term in the right side of (3.3) can be written as

$$\frac{\tilde{\hat{\phi}}(e^a,\sigma)}{\ln\left(\frac{\sqrt{k^2+q^2}}{e^a}\right)}\chi_{B_\delta} + \frac{\int_{e^a}^{\sqrt{k^2+q^2}} \frac{\partial \tilde{\hat{\phi}}(s,\sigma)}{\partial s} ds}{\ln\left(\frac{\sqrt{k^2+q^2}}{e^a}\right)}\chi_{B_\delta}. \tag{3.8}$$

We recall the result of Lemma 12 of [34]. Hence, under the given conditions we have $(\nabla_k + \nabla_q)\tilde{\hat{\phi}}(k,q) \in L^\infty(\mathbb{R}^{d+3})$. Let us obtain the upper bound in the absolute value on the second term in sum (3.8) as

$$\left| \frac{\int_{e^a}^{\sqrt{k^2+q^2}} \frac{\partial \tilde{\hat{\phi}}(s,\sigma)}{\partial s} ds}{\ln\left(\frac{\sqrt{k^2+q^2}}{e^a}\right)} \chi_{B_\delta} \right| \leq \|(\nabla_k + \nabla_q)\tilde{\hat{\phi}}(k,q)\|_{L^\infty(\mathbb{R}^{d+3})} \left| \frac{\sqrt{k^2+q^2} - e^a}{\ln\left(\frac{\sqrt{k^2+q^2}}{e^a}\right)} \right| \chi_{B_\delta} \leq$$

$$\leq C\|(\nabla_k + \nabla_q)\tilde{\hat{\phi}}(k,q)\|_{L^\infty(\mathbb{R}^{d+3})} \chi_{B_\delta} \in L^2(\mathbb{R}^{d+3}),$$

so that it remains to analyze the term

$$\frac{\tilde{\hat{\phi}}(e^a, \sigma)}{\ln\left(\frac{\sqrt{k^2+q^2}}{e^a}\right)} \chi_{B_\delta}. \tag{3.9}$$

It can be trivially checked that (3.9) is contained in $L^2(\mathbb{R}^{d+3})$ if and only if $\tilde{\hat{\phi}}(e^a, \sigma)$ vanishes. This is equivalent to orthogonality relations (1.18). ■

Let us conclude the article with demonstrating the solvability in the sense of sequences for our equation containing the logarithmic Schrödinger operator when the free Laplacian is added to a three dimensional Schrödinger operator.

*Proof of Theorem 1.5.* Each equation (1.19) has a unique solution $u_n(x,y) \in L^2(\mathbb{R}^{d+3}), \ n \in \mathbb{N}$ due to the result of Theorem 1.4 above. Let us demonstrate that the limiting orthogonality relations

$$\left(\phi(x,y), \frac{e^{ikx}}{(2\pi)^{\frac{d}{2}}}\eta_q(y)\right)_{L^2(\mathbb{R}^{d+3})} = 0 \quad for \quad (k,q) \in S_{e^a}^{d+3} \tag{3.10}$$

hold. By means of Fact 2 of [34], under the given conditions we have $\phi_n(x,y) \in L^1(\mathbb{R}^{d+3}), \ n \in \mathbb{N}$. We recall the argument of the proof of Theorem 3 of [40]. Hence,

$$\phi_n(x,y) \to \phi(x,y) \quad in \quad L^1(\mathbb{R}^{d+3}), \quad n \to \infty. \tag{3.11}$$

Let us use formulas (1.20), (1.9), (3.11) to obtain for $(k,q) \in S_{e^a}^{d+3}$ that

$$\left| \left(\phi(x,y), \frac{e^{ikx}}{(2\pi)^{\frac{d}{2}}}\eta_q(y)\right)_{L^2(\mathbb{R}^{d+3})} \right| = \left| \left(\phi(x,y) - \phi_n(x,y), \frac{e^{ikx}}{(2\pi)^{\frac{d}{2}}}\eta_q(y)\right)_{L^2(\mathbb{R}^{d+3})} \right| \leq$$

$$\leq \frac{1}{(2\pi)^{\frac{d+3}{2}}} \frac{1}{1 - I(U)} \|\phi_n(x,y) - \phi(x,y)\|_{L^1(\mathbb{R}^{d+3})} \to 0, \quad n \to \infty,$$

so that (3.10) is valid. By virtue of Theorem 1.4 above, problem (1.15) possesses a unique solution $u(x,y) \in L^2(\mathbb{R}^{d+3})$.

We apply the generalized Fourier transform (1.17) to both sides of equations (1.15) and (1.19) and obtain that

$$\tilde{\hat{u}}(k,q) = \frac{\tilde{\hat{\phi}}(k,q)}{\ln\left(\frac{\sqrt{k^2+q^2}}{e^a}\right)}, \quad \tilde{\hat{u}}_n(k,q) = \frac{\tilde{\hat{\phi}}_n(k,q)}{\ln\left(\frac{\sqrt{k^2+q^2}}{e^a}\right)}$$

with $k \in \mathbb{R}^d,\ q \in \mathbb{R}^3$ and $n \in \mathbb{N}$. Clearly, $\tilde{\hat{u}}_n(k,q) - \tilde{\hat{u}}(k,q)$ can be written as

$$\frac{\tilde{\hat{\phi}}_n(k,q) - \tilde{\hat{\phi}}(k,q)}{\ln\left(\frac{\sqrt{k^2+q^2}}{e^a}\right)}\chi_{B_\delta} + \frac{\tilde{\hat{\phi}}_n(k,q) - \tilde{\hat{\phi}}(k,q)}{\ln\left(\frac{\sqrt{k^2+q^2}}{e^a}\right)}\chi_{B_\delta^c}. \tag{3.12}$$

Evidently, the second term in (3.12) equals to

$$\frac{\tilde{\hat{\phi}}_n(k,q) - \tilde{\hat{\phi}}(k,q)}{\ln\left(\frac{\sqrt{k^2+q^2}}{e^a}\right)}\chi_{B_\delta^{c+}} + \frac{\tilde{\hat{\phi}}_n(k,q) - \tilde{\hat{\phi}}(k,q)}{\ln\left(\frac{\sqrt{k^2+q^2}}{e^a}\right)}\chi_{B_\delta^{c-}}. \tag{3.13}$$

Obviously, the inequalities

$$\frac{|\tilde{\hat{\phi}}_n(k,q) - \tilde{\hat{\phi}}(k,q)|}{\left|\ln\left(\frac{\sqrt{k^2+q^2}}{e^a}\right)\right|}\chi_{B_\delta^{c+}} \le \frac{|\tilde{\hat{\phi}}_n(k,q) - \tilde{\hat{\phi}}(k,q)|}{\ln(1+\delta)},$$

$$\frac{|\tilde{\hat{\phi}}_n(k,q) - \tilde{\hat{\phi}}(k,q)|}{\left|\ln\left(\frac{\sqrt{k^2+q^2}}{e^a}\right)\right|}\chi_{B_\delta^{c-}} \le \frac{|\tilde{\hat{\phi}}_n(k,q) - \tilde{\hat{\phi}}(k,q)|}{-\ln(1-\delta)}$$

hold. Hence, by means of the one of our assumptions

$$\left\|\frac{\tilde{\hat{\phi}}_n(k,q) - \tilde{\hat{\phi}}(k,q)}{\ln\left(\frac{\sqrt{k^2+q^2}}{e^a}\right)}\chi_{B_\delta^{c+}}\right\|_{L^2(\mathbb{R}^{d+3})} \le \frac{\|\phi_n(x,y) - \phi(x,y)\|_{L^2(\mathbb{R}^{d+3})}}{\ln(1+\delta)} \to 0,$$

$$\left\|\frac{\tilde{\hat{\phi}}_n(k,q) - \tilde{\hat{\phi}}(k,q)}{\ln\left(\frac{\sqrt{k^2+q^2}}{e^a}\right)}\chi_{B_\delta^{c-}}\right\|_{L^2(\mathbb{R}^{d+3})} \le \frac{\|\phi_n(x,y) - \phi(x,y)\|_{L^2(\mathbb{R}^{d+3})}}{-\ln(1-\delta)} \to 0$$

as $n \to \infty$. Let us use orthogonality relations (3.10) and (1.20). Thus,

$$\tilde{\hat{\phi}}(e^a,\sigma) = 0, \quad \tilde{\hat{\phi}}_n(e^a,\sigma) = 0, \quad n \in \mathbb{N},$$

so that

$$\tilde{\hat{\phi}}(k,q) = \int_{e^a}^{\sqrt{k^2+q^2}} \frac{\partial\tilde{\hat{\phi}}(s,\sigma)}{\partial s}ds, \quad \tilde{\hat{\phi}}_n(k,q) = \int_{e^a}^{\sqrt{k^2+q^2}} \frac{\partial\tilde{\hat{\phi}}_n(s,\sigma)}{\partial s}ds, \quad n \in \mathbb{N}.$$

Then the first term in (3.12) is given by

$$\frac{\int_{e^a}^{\sqrt{k^2+q^2}}\left[\frac{\partial\tilde{\hat{\phi}}_n(s,\sigma)}{\partial s}-\frac{\partial\tilde{\hat{\phi}}(s,\sigma)}{\partial s}\right]ds}{\ln\left(\frac{\sqrt{k^2+q^2}}{e^a}\right)}\chi_{B_\delta}. \tag{3.14}$$

Clearly, (3.14) can be estimated from above in the absolute value by

$$\|(\nabla_k+\nabla_q)[\tilde{\hat{\phi}}_n(k,q)-\tilde{\hat{\phi}}(k,q)]\|_{L^\infty(\mathbb{R}^{d+3})}\left|\frac{\sqrt{k^2+q^2}-e^a}{\ln\left(\frac{\sqrt{k^2+q^2}}{e^a}\right)}\right|\chi_{B_\delta}\le$$

$$\le C\|(\nabla_k+\nabla_q)[\tilde{\hat{\phi}}_n(k,q)-\tilde{\hat{\phi}}(k,q)]\|_{L^\infty(\mathbb{R}^{d+3})}\chi_{B_\delta}.$$

Let us recall Lemma 12 of [34]. Hence, under the stated assumptions, we have $(\nabla_k+\nabla_q)\tilde{\hat{\phi}}_n(k,q),\ (\nabla_k+\nabla_q)\tilde{\hat{\phi}}(k,q)\in L^\infty(\mathbb{R}^{d+3})$. According to Lemma 5 of [40],

$$\|(\nabla_k+\nabla_q)[\tilde{\hat{\phi}}_n(k,q)-\tilde{\hat{\phi}}(k,q)]\|_{L^\infty(\mathbb{R}^{d+3})}\to 0,\quad n\to\infty. \tag{3.15}$$

Obviously, we have the upper bound for the norm

$$\left\|\frac{\int_{e^a}^{\sqrt{k^2+q^2}}\left[\frac{\partial\tilde{\hat{\phi}}_n(s,\sigma)}{\partial s}-\frac{\partial\tilde{\hat{\phi}}(s,\sigma)}{\partial s}\right]ds}{\ln\left(\frac{\sqrt{k^2+q^2}}{e^a}\right)}\chi_{B_\delta}\right\|_{L^2(\mathbb{R}^{d+3})}\le$$

$$\le C\|(\nabla_k+\nabla_q)[\tilde{\hat{\phi}}_n(k,q)-\tilde{\hat{\phi}}(k,q)]\|_{L^\infty(\mathbb{R}^{d+3})}|B_\delta|^{\frac{1}{2}}\to 0,\quad n\to\infty$$

via (3.15). This implies that $u_n(x,y)\to u(x,y)$ in $L^2(\mathbb{R}^{d+3})$ as $n\to\infty$, which completes the proof of the theorem. ■

# References


[1] G.L. Alfimov, A.S. Korobeinikov, C.J. Lustri, D.E. Pelinovsky, *Standing lattice solitons in the discrete NLS equation with saturation*, Nonlinearity, **32** (2019), no. 9, 3445–3484.

[2] C. Amrouche, V. Girault, J. Giroire, *Dirichlet and Neumann exterior problems for the $n$-dimensional Laplace operator: an approach in weighted Sobolev spaces*, J. Math. Pures Appl. (9), **76** (1997), no. 1, 55–81.

[3] C. Amrouche, F. Bonzom, *Mixed exterior Laplace's problem*, J. Math. Anal. Appl., **338** (2008), no. 1, 124–140.

[4] N. Benkirane, *Propriétés d'indice en théorie höldérienne pour des opérateurs elliptiques dans $R^n$*, C. R. Acad. Sci. Paris Sér. I Math., **307** (1988), no. 11, 577–580.

[5] P. Bolley, T.L. Pham, *Propriétés d'indice en théorie höldérienne pour des opérateurs différentiels elliptiques dans $R^n$*, J. Math. Pures Appl. (9), **72** (1993), no. 1, 105–119.

[6] P. Bolley, T.L. Pham, *Propriété d'indice en théorie Höldérienne pour le problème extérieur de Dirichlet*, Comm. Partial Differential Equations, **26** (2001), no. 1-2, 315–334.

[7] H. Chen, T. Weth, *The Dirichlet problem for the logarithmic Laplacian*, Comm. Partial Differential Equations, **44** (2019), no. 11, 1100–1139.

[8] H.L. Cycon, R.G. Froese, W. Kirsch, B. Simon, *Schrödinger operators with application to quantum mechanics and global geometry*. Texts and Monographs in Physics. Springer Study Edition. Springer-Verlag, Berlin (1987), 319 pp.

[9] A. Ducrot, M. Marion, V. Volpert, *Systemes de réaction-diffusion sans propriété de Fredholm*, C. R. Math. Acad. Sci. Paris, **340** (2005), no. 9, 659–664.

[10] A. Ducrot, M. Marion, V. Volpert, *Reaction-diffusion problems with non-Fredholm operators*, Adv. Differential Equations, **13** (2008), no. 11-12, 1151–1192.

[11] M. Efendiev, *Fredholm structures, topological invariants and applications*. AIMS Series on Differential Equations & Dynamical Systems, **3**. American Institute of Mathematical Sciences (AIMS), Springfield, MO (2009), 205 pp.

[12] M. Efendiev, *Finite and infinite dimensional attractors for evolution equations of mathematical physics*. GAKUTO International Series. Mathematical Sciences and Applications, **33**. $Gakk\bar{o}tosho\ Co.,\ Ltd., Tokyo$ (2010), 239 pp.

[13] M.A. Efendiev, L.A. Peletier, *On the large time behavior of solutions of fourth order parabolic equations and $\varepsilon$-entropy of their attractors*, C. R. Math. Acad. Sci. Paris, **344** (2007), no. 2, 93–96.

[14] M. Efendiev, V. Vougalter, *Solvability of some integro-differential equations with drift*, Osaka J. Math., **57** (2020), no. 2, 247–265.

[15] M. Efendiev, V. Vougalter, *Solvability in the sense of sequences for some fourth order non-Fredholm operators*, J. Differential Equations, **271** (2021), 280–300.

[16] M. Efendiev, V. Vougalter, *Existence of solutions for some non-Fredholm integro-differential equations with mixed diffusion*, J. Differential Equations, **284** (2021), 83–101.

[17] M. Efendiev, V. Vougalter, *Linear and nonlinear non-Fredholm operators and their applications*, Electron. Res. Arch., **30** (2022), no. 2, 515–534.

[18] M. Efendiev, V. Vougalter, *Existence of solutions for some systems of integro-differential equations with transport and superdiffusion*, Anal. Math. Phys., **12** (2022), no. 5, Paper No. 110, 29 pp.

[19] M. Efendiev, V. Vougalter, *Solvability in the sense of sequences for some non-Fredholm operators with the logarithmic Laplacian*, preprint (2022).

[20] M.A. Efendiev, S.V. Zelik, *The attractor for a nonlinear reaction-diffusion system in an unbounded domain*, Comm. Pure Appl. Math., **54** (2001), no. 6, 625–688.

[21] H.G. Gebran, C.A. Stuart, *Fredholm and properness properties of quasilinear elliptic systems of second order*, Proc. Edinb. Math. Soc. (2), **48** (2005), no. 1, 91–124.

[22] H.G. Gebran, C.A. Stuart, *Exponential decay and Fredholm properties in second-order quasilinear elliptic systems*, J. Differential Equations, **249** (2010), no. 1, 94–117.

[23] S. Jarohs, A. Saldana, T. Weth, *A new look at the fractional Poisson problem via the logarithmic Laplacian*, J. Funct. Anal., **279** (2020), no. 11, 108732, 50 pp.

[24] T. Kato, *Wave operators and similarity for some non-selfadjoint operators*, Math. Ann., **162** (1965/66), 258–279.

[25] A. Laptev, T. Weth, *Spectral properties of the logarithmic Laplacian*, Anal. Math. Phys., **11** (2021), no. 3, Paper No. 133, 24 pp.

[26] E.H. Lieb, M. Loss, *Analysis*. Graduate Studies in Mathematics, **14**, American Mathematical Society, Providence, RI (1997), 278 pp.

[27] P.J. Rabier, C.A. Stuart, *Fredholm and properness properties of quasilinear elliptic operators on* $\mathbb{R}^N$, Math. Nachr., **231** (2001), 129–168.

[28] M. Reed, B. Simon, *Methods of modern mathematical physics. III. Scattering theory*, Academic Press, New York-London (1979), 463 pp.

[29] I. Rodnianski, W. Schlag, *Time decay for solutions of Schrödinger equations with rough and time-dependent potentials*, Invent. Math., **155** (2004), no. 3, 451–513.

[30] V. Volpert, *Elliptic partial differential equations*. Volume 1: Fredholm theory of elliptic problems in unbounded domains. Monographs in Mathematics, **101**. Birkhäuser/Springer Basel AG, Basel (2011), 639 pp.

[31] V. Volpert, B. Kazmierczak, M. Massot, Z. Peradzynski, *Solvability conditions for elliptic problems with non-Fredholm operators,* Appl. Math. (Warsaw), **29** (2002), no. 2, 219–238.

[32] V. Volpert, V. Vougalter, *On the solvability conditions for a linearized Cahn-Hilliard equation,* Rend. Istit. Mat. Univ. Trieste, **43** (2011), 1–9.

[33] V. Vougalter, *On solvability in the sense of sequences for some non-Fredholm operators in higher dimensions,* J. Math. Sci. (N.Y.), **247** (2020), no. 6, Problems in mathematical analysis. No. 102, 850–864.

[34] V. Vougalter, V. Volpert, *On the solvability conditions for some non Fredholm operators,* Int. J. Pure Appl. Math., **60** (2010), no. 2, 169–191.

[35] V. Vougalter, V. Volpert, *Solvability conditions for some non-Fredholm operators*, Proc. Edinb. Math. Soc. (2), **54** (2011), no. 1, 249–271.

[36] V. Vougalter, V. Volpert, *On the existence of stationary solutions for some non-Fredholm integro-differential equations,* Doc. Math., **16** (2011), 561–580.

[37] V. Vougalter, V. Volpert, *On the solvability conditions for the diffusion equation with convection terms,* Commun. Pure Appl. Anal., **11** (2012), no. 1, 365–373.

[38] V. Vougalter, V. Volpert, *Solvability conditions for a linearized Cahn-Hilliard equation of sixth order,* Math. Model. Nat. Phenom., **7** (2012), no. 2, 146–154.

[39] V. Vougalter, V. Volpert, *Solvability conditions for some linear and nonlinear non-Fredholm elliptic problems,* Anal. Math. Phys., **2** (2012), no. 4, 473–496.

[40] V. Vougalter, V. Volpert, *On the solvability in the sense of sequences for some non-Fredholm operators,* Dyn. Partial Differ. Equ., **11** (2014), no. 2, 109–124.

[41] V. Vougalter, V. Volpert, *On the solvability in the sense of sequences for some non-Fredholm operators related to the anomalous diffusion.* Analysis of pseudo-differential operators, Trends Math., Birkhäuser/Springer, Cham (2019), 229–257.