SOLVABILITY IN THE SENSE OF SEQUENCES FOR SOME NON-FREDHOLM OPERATORS WITH THE LOGARITHMIC LAPLACIAN

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Abstract: We establish the solvability of certain linear nonhomogeneous equations and demonstrate that under reasonable technical conditions the convergence in $L^2(\mathbb{R}^d)$ of their right sides implies the existence and the convergence in $L^2(\mathbb{R}^d)$ of the solutions. In the first part of the work the equation involves the logarithmic Laplacian. In the second part we generalize the results derived by incorporating a shallow, short-range scalar potential into the problem. The argument relies on the methods of the spectral and scattering theory for the non-Fredholm Schrödinger type operators. As distinct from the preceding articles on the subject, for the operators involved in the equations the essential spectra fill the whole real line.

AMS Subject Classification: 35P05
Key words: solvability conditions, non-Fredholm operators, logarithmic Laplacian

1. Introduction

Let us consider the problem

$$(-\Delta + V(x))u - au = f, \tag{1.1}$$

with $u \in E = H^2(\mathbb{R}^d)$ and $f \in F = L^2(\mathbb{R}^d),\ d \in \mathbb{N}$, a is a constant and $V(x)$ is a function tending to 0 at infinity. If $a \geq 0$, then the essential spectrum of the operator $A : E \to F$, which corresponds to the left side of equation (1.1) contains the origin. Consequently, such operator does not satisfy the Fredholm property. Its image is

not closed, for $d > 1$ the dimension of its kernel and the codimension of its image are not finite. The present work is devoted to the studies of the certain properties of the logarithms of the operators of this kind. We recall that elliptic problems with non-Fredholm operators were treated extensively in recent years (see [14], [15], [16], [17], [18], [31], [32], [33], [34], [35], [36], [37], [38], [39], [40], [42], also [6]) along with their potential applications to the theory of reaction-diffusion problems (see [9], [10]). Fredholm structures, topological invariants and their application were discussed in [11]. The work [12] deals with the finite and infinite dimensional attractors for the evolution equations of mathematical physics. The large time behavior of the solutions of a class of fourth-order parabolic equations defined on unbounded domains using the Kolmogorov ε-entropy as a measure was treated in [13]. The attractor for a nonlinear reaction-diffusion system in an unbounded domain in the space of three dimensions was considered in [19]. The articles [20] and [28] are devoted to the understanding of the Fredholm and properness properties of the quasilinear elliptic systems of the second order and of the operators of this kind on $\mathbb{R}^N$. The exponential decay and Fredholm properties in the second-order quasilinear elliptic systems of equations were considered in [21]. The article [37] is dedicated to the studies of the Laplace operator with drift from the point of view of the non-Fredholm operators. The linearized Cahn-Hilliard equations were covered in [33] and [38]. Standing lattice solitons in the discrete NLS equation with saturation were discussed in [1]. In the particular case when a is trivial, our operator A mentioned above satisfies the Fredholm property in some properly chosen weighted spaces (see [2], [3], [4], [5], [6]). But the situation when $a \neq 0$ is significantly different and the method developed in these articles cannot be used.
One of the important questions concerning the equations with non-Fredholm operators is their solvability. We address it in the following setting. Let f_n be a sequence of functions in the image of the operator A, so that $f_n \to f$ in $L^2(\mathbb{R}^d)$ as $n \to \infty$. We denote by u_n a sequence of functions from $H^2(\mathbb{R}^d)$, so that

$$Au_n = f_n, \ n \in \mathbb{N}.$$

Because the operator A does not satisfy the Fredholm property, the sequence u_n may not be convergent. We call a sequence u_n such that $Au_n \to f$ a solution in the sense of sequences of problem $Au = f$ (see [31]). If such sequence converges to a function u_0 in the norm of the space E, then u_0 is a solution of this equation. The solution in the sense of sequences is equivalent in this sense to the usual solution. However, in the case of the non-Fredholm operators, this convergence may not hold or it can occur in some weaker sense. In such case, the solution in the sense of sequences may not imply the existence of the usual solution. In the present article we will find sufficient conditions of equivalence of solutions in the sense of sequences and the usual solutions. In the other words, we will determine the conditions on the sequences f_n under which the corresponding sequences u_n are strongly convergent. The solvability in the sense of sequences for the problems involving the

Schrödinger type non-Fredholm operators was considered in [15], [17], [34], [40], [41], [42]. The current work is our attempt to generalize these results by dealing with the solvability of the linear equations involving in their left sides the logarithm of such second order differential operators without the Fredholm property, which can be defined using the spectral calculus.
Let us first consider the equation

$$\Big[\frac{1}{2}\ln(-\Delta)\Big]u - au = f(x), \quad x \in \mathbb{R}^d, \quad d \in \mathbb{N}, \quad a \in \mathbb{R} \tag{1.2}$$

with a square integrable right side. The logarithmic Laplacian $\ln(-\Delta)$ is the operator with Fourier symbol $2\ln|p|$. It arises as formal derivative $\partial_s|_{s=0}(-\Delta)^s$ of fractional Laplacians at $s = 0$. The operator $(-\Delta)^s$ is actively used, for example in the studies of the anomalous diffusion problems (see e.g. [42] and the references therein). Spectral properties of the logarithmic Laplacian in an open set of finite measure with Dirichlet boundary conditions were discussed in [26] (see also [7]). The studies of $\ln(-\Delta)$ are relevant for the understanding of the asymptotic spectral properties of the family of fractional Laplacians in the limit $s \to 0^+$. In [23] it has been shown that this operator enables to characterize the s-dependence of solution to fractional Poisson problems for the full range of exponents $s \in (0, 1)$. The problem analogical to (1.2) but with the standard Laplace operator in the context of the solvability in the sense of sequences was discussed in [34]. The operator in the left side of our equation (1.2) is given by

$$l_a := \frac{1}{2}\ln(-\Delta) - a, \quad a \in \mathbb{R} \tag{1.3}$$

and is considered on $L^2(\mathbb{R}^d),\ d \in \mathbb{N}$. By means of the standard Fourier transform, it can be easily derived that the essential spectrum of (1.3) is given by

$$\lambda_a(p) = \ln|p| - a, \quad a \in \mathbb{R}. \tag{1.4}$$

Note that as distinct from the preceding works dealing with the non-Fredholm operators mentioned above, (1.4) fills not a semi-axis but the whole real line. Thus, the inverse of (1.3) is not bounded.
Let us write down the corresponding sequence of the approximate equations with $n \in \mathbb{N}$ as

$$\Big[\frac{1}{2}\ln(-\Delta)\Big]u_n - au_n = f_n(x), \quad x \in \mathbb{R}^d, \quad d \in \mathbb{N}. \tag{1.5}$$

The right sides of (1.5) are assumed to be square integrable and converging to the right side of (1.2) in $L^2(\mathbb{R}^d)$ as $n \to \infty$. The inner product of two functions is defined as

$$(f(x), g(x))_{L^2(\mathbb{R}^d)} := \int_{\mathbb{R}^d} f(x)\bar{g}(x)dx, \quad d \in \mathbb{N}, \tag{1.6}$$

with a slight abuse of notations when these functions do not belong to $L^2(\mathbb{R}^d)$. Indeed, if $f(x) \in L^1(\mathbb{R}^d)$ and $g(x)$ is bounded, then it is clear that the integral in the right side of (1.6) makes sense, like for instance in the cases of the functions involved in the orthogonality relations of our theorems below. Throughout the article, the sphere of radius $r > 0$ in $\mathbb{R}^d$ centered at the origin will be denoted as S_r^d. Let us first state the solvability relations for problem (1.2).

Theorem 1.1. *Let $f(x) \in L^2(\mathbb{R}^d),\ xf(x) \in L^1(\mathbb{R}^d)$, where $d \in \mathbb{N}$ and $a \in \mathbb{R}$.*

a) If $d = 1$ then equation (1.2) admits a unique solution $u(x) \in L^2(\mathbb{R})$ if and only if the orthogonality conditions

$$\Big(f(x), \frac{e^{\pm ie^a x}}{\sqrt{2\pi}}\Big)_{L^2(\mathbb{R})} = 0 \tag{1.7}$$

hold.

b) If $d \geq 2$ then problem (1.2) possesses a unique solution $u(x) \in L^2(\mathbb{R}^d)$ if and only if the orthogonality relations

$$\Big(f(x), \frac{e^{ipx}}{(2\pi)^{\frac{d}{2}}}\Big)_{L^2(\mathbb{R}^d)} = 0, \quad p \in S_{e^a}^d \tag{1.8}$$

are valid.

Our second main proposition deals with the issue of the solvability in the sense of sequences for our problem.

Theorem 1.2. *Let $n \in \mathbb{N},\ d \in \mathbb{N},\ f_n(x) \in L^2(\mathbb{R}^d),\ xf_n(x) \in L^1(\mathbb{R}^d)$, so that $f_n(x) \to f(x)$ in $L^2(\mathbb{R}^d)$ and $xf_n(x) \to xf(x)$ in $L^1(\mathbb{R}^d)$ as $n \to \infty$.*

a) If $d = 1$, let the orthogonality conditions

$$\Big(f_n(x), \frac{e^{\pm ie^a x}}{\sqrt{2\pi}}\Big)_{L^2(\mathbb{R})} = 0 \tag{1.9}$$

hold for all $n \in \mathbb{N}$. Then equations (1.2) and (1.5) have unique solutions $u(x) \in L^2(\mathbb{R})$ and $u_n(x) \in L^2(\mathbb{R})$ respectively, so that $u_n(x) \to u(x)$ in $L^2(\mathbb{R})$ as $n \to \infty$.

b) If $d \geq 2$, let the orthogonality relations

$$\Big(f_n(x), \frac{e^{ipx}}{(2\pi)^{\frac{d}{2}}}\Big)_{L^2(\mathbb{R}^d)} = 0, \quad p \in S_{e^a}^d \tag{1.10}$$

hold for all $n \in \mathbb{N}$. Then problems (1.2) and (1.5) admit unique solutions $u(x) \in L^2(\mathbb{R}^d)$ and $u_n(x) \in L^2(\mathbb{R}^d)$ respectively, such that $u_n(x) \to u(x)$ in $L^2(\mathbb{R}^d)$ as $n \to \infty$. Let us demonstrate that the limiting orthogonality conditions

Throughout the article we use the hat symbol to designate the standard Fourier transform

$$\widehat{f}(p) := \frac{1}{(2\pi)^{\frac{d}{2}}} \int_{\mathbb{R}^d} f(x)e^{-ipx}dx, \quad p \in \mathbb{R}^d, \quad d \in \mathbb{N}. \tag{1.11}$$

The second part of our work deals with the studies of the problem

$$\Big[\frac{1}{2}\ln(-\Delta+V(x))\Big]u-au=f(x), \quad x\in\mathbb{R}^3, \quad a\in\mathbb{R} \tag{1.12}$$

with a square integrable right side as before. The corresponding sequence of approximate equations for $n\in\mathbb{N}$ is given by

$$\Big[\frac{1}{2}\ln(-\Delta+V(x))\Big]u_n-au_n=f_n(x), \quad x\in\mathbb{R}^3. \tag{1.13}$$

The square integrable right sides in (1.13) tend to the right side of (1.12) in $L^2(\mathbb{R}^3)$ as $n\to\infty$. Let us make the following technical assumptions on the scalar potential contained in the problems above. Note that the conditions on $V(x)$, which is shallow and short-range will be analogical to those given in Assumption 1.1 of [36] (see also [35], [37]). The essential spectrum of such a Schrödinger operator $-\Delta+V(x)$ fills the nonnegative semi-axis (see e.g. [24]).

Assumption 1.3. *The potential function* $V(x):\mathbb{R}^3\to\mathbb{R}$ *satisfies the bound*

$$|V(x)|\le\frac{C}{1+|x|^{3.5+\delta}}$$

with some $\delta>0$ *and* $x=(x_1,x_2,x_3)\in\mathbb{R}^3$ *a.e. and it is such that*

$$4^{\frac{1}{9}}\frac{9}{8}(4\pi)^{-\frac{2}{3}}\|V\|_{L^\infty(\mathbb{R}^3)}^{\frac{1}{9}}\|V\|_{L^{\frac{4}{3}}(\mathbb{R}^3)}^{\frac{8}{9}}<1 \quad \textit{and} \quad \sqrt{c_{HLS}}\|V\|_{L^{\frac{3}{2}}(\mathbb{R}^3)}<4\pi. \tag{1.14}$$

Here and below C stands for a finite positive constant and c_{HLS} given on p.98 of [27] is the constant in the Hardy-Littlewood-Sobolev inequality

$$\Big|\int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\frac{f_1(x)f_1(y)}{|x-y|^2}dxdy\Big|\le c_{HLS}\|f_1\|^2_{L^{\frac{3}{2}}(\mathbb{R}^3)}, \quad f_1\in L^{\frac{3}{2}}(\mathbb{R}^3).$$

By virtue of Lemma 2.3 of [36], under Assumption 1.3 above on the scal potential, the operator $-\Delta+V(x)$ on $L^2(\mathbb{R}^3)$ is self-adjoint and unitarily equivalent to $-\Delta$ via the wave operators (see [25], [30])

$$\Omega^{\pm}:=s-\lim_{t\to\mp\infty}e^{it(-\Delta+V)}e^{it\Delta},$$

where the limit is understood in the strong L^2 sense (see e.g. [29] p.34, [8] p.90). Thus, the operator

$$L_a:=\frac{1}{2}\ln(-\Delta+V(x))-a, \quad a\in\mathbb{R} \tag{1.15}$$

involved in the left sides of problems (1.12) and (1.13) considered on $L^2(\mathbb{R}^3)$ and defined via the spectral calculus has only the essential spectrum

$$\tilde{\lambda}_a(k)=\ln|k|-a, \quad a\in\mathbb{R} \tag{1.16}$$

and no nontrivial $L^2(\mathbb{R}^3)$ eigenfunctions. Note that (1.16) fills the whole real line similarly to the no potential case. By means of the spectral theorem, the functions of the continuous spectrum of (1.15) satisfy

$$L_a\varphi_k(x) = (\ln|k| - a)\varphi_k(x), \quad k \in \mathbb{R}^3, \quad a \in \mathbb{R} \tag{1.17}$$

in the integral formulation the Lippmann-Schwinger equation for the perturbed plane waves (see e.g. [29] p.98)

$$\varphi_k(x) = \frac{e^{ikx}}{(2\pi)^{\frac{3}{2}}} - \frac{1}{4\pi}\int_{\mathbb{R}^3} \frac{e^{i|k||x-y|}}{|x-y|}(V\varphi_k)(y)dy \tag{1.18}$$

and the orthogonality relations

$$(\varphi_k(x), \varphi_q(x))_{L^2(\mathbb{R}^3)} = \delta(k-q), \quad k, q \in \mathbb{R}^3. \tag{1.19}$$

Particularly, when the vector $k = 0$, we have $\varphi_0(x)$. Let us designate the generalized Fourier transform with respect to these functions using the tilde symbol as

$$\tilde{f}(k) := (f(x), \varphi_k(x))_{L^2(\mathbb{R}^3)}, \ k \in \mathbb{R}^3. \tag{1.20}$$

(1.20) is a unitary transform on $L^2(\mathbb{R}^3)$. The integral operator contained in (1.18) is being denoted as

$$(Q\varphi)(x) := -\frac{1}{4\pi}\int_{\mathbb{R}^3} \frac{e^{i|k||x-y|}}{|x-y|}(V\varphi)(y)dy, \quad \varphi \in L^\infty(\mathbb{R}^3).$$

Let us consider $Q : L^\infty(\mathbb{R}^3) \to L^\infty(\mathbb{R}^3)$. Under Assumption 1.3, via Lemma 2.1 of [36] the operator norm $\|Q\|_\infty$ is bounded above by the expression $I(V)$, which is the left side of the first inequality in (1.14), such that $I(V) < 1$. Corollary 2.2 of [36] under our conditions gives us the estimate

$$|\tilde{f}(k)| \leq \frac{1}{(2\pi)^{\frac{3}{2}}} \frac{1}{1 - I(V)} \|f(x)\|_{L^1(\mathbb{R}^3)}. \tag{1.21}$$

Our statement on the solvability of equation (1.12) is as follows.

Theorem 1.4. *Let Assumption 1.3 hold, $f(x) \in L^2(\mathbb{R}^3)$ and $xf(x) \in L^1(\mathbb{R}^3)$. Then equation (1.12) admits a unique solution $u(x) \in L^2(\mathbb{R}^3)$ if and only if the orthogonality conditions*

$$(f(x), \varphi_k(x))_{L^2(\mathbb{R}^3)} = 0, \quad k \in S^3_{e^a} \tag{1.22}$$

are valid.

The final main proposition of our work deals with the solvability in the sense of sequences for problem (1.12).

Theorem 1.5. *Let Assumption 1.3 hold,* $n \in \mathbb{N}$, $f_n(x) \in L^2(\mathbb{R}^3)$, $xf_n(x) \in L^1(\mathbb{R}^3)$, *so that* $f_n(x) \to f(x)$ *in* $L^2(\mathbb{R}^3)$ *and* $xf_n(x) \to xf(x)$ *in* $L^1(\mathbb{R}^3)$ *as* $n \to \infty$. *Let in addition*

$$(f_n(x), \varphi_k(x))_{L^2(\mathbb{R}^3)} = 0, \quad k \in S^3_{e^a} \tag{1.23}$$

hold for all $n \in \mathbb{N}$. *Then problems (1.12) and (1.13) possess unique solutions* $u(x) \in L^2(\mathbb{R}^3)$ *and* $u_n(x) \in L^2(\mathbb{R}^3)$ *respectively, so that* $u_n(x) \to u(x)$ *in* $L^2(\mathbb{R}^3)$ *as* $n \to \infty$.

Let us note that (1.22) and (1.23) are the orthogonality relations to the function of the continuous spectrum of our Schrödinger type operator, as distinct from the Limiting Absorption Principle in which one needs to orthogonalize to the standard Fourier harmonics (see e.g. Lemma 2.3 and Proposition 2.4 of [22]).

2. Solvability in the sense of sequences in the no potential case

Proof of Theorem 1.1. To demonstrate the uniqueness of solutions for our equation, we suppose that (1.2) has two solutions $u_1(x),\ u_2(x) \in L^2(\mathbb{R}^d),\ d \in \mathbb{N}$. Clearly, their difference $w(x) := u_1(x) - u_2(x) \in L^2(\mathbb{R}^d)$ as well and it solves the homogeneous problem

$$\Big[\frac{1}{2}\ln(-\Delta)\Big]w - aw = 0.$$

Since the operator l_a on $L^2(\mathbb{R}^d)$ given by (1.3) has only the essential spectrum and no nontrivial zero modes, $w(x)$ vanishes in $\mathbb{R}^d$.
We apply the standard Fourier transform (1.11) to both sides of problem (1.2) and arrive at

$$\widehat{u}(p) = \frac{\widehat{f}(p)}{\ln\Big(\frac{|p|}{e^a}\Big)}. \tag{2.1}$$

Let us first consider the case b) of our theorem when the dimension of the problem $d \geq 2$. For the technical purposes we introduce the spherical layer

$$A_\delta := \{p \in \mathbb{R}^d \mid e^a(1-\delta) \leq |p| \leq e^a(1+\delta)\}, \quad 0 < \delta < 1, \quad d \geq 2, \tag{2.2}$$

so that

$$\widehat{u}(p) = \frac{\widehat{f}(p)}{\ln\Big(\frac{|p|}{e^a}\Big)}\chi_{A_\delta} + \frac{\widehat{f}(p)}{\ln\Big(\frac{|p|}{e^a}\Big)}\chi_{A_\delta^c}. \tag{2.3}$$

Here and further down A^c will stand for the complement of a set $A \subseteq \mathbb{R}^d$. The characteristic function of a set A is being denoted as χ_A and $|A|$ will designate the Lebesgue measure of a set A. We will use the sets

$$A_\delta^{c+} := \{p \in \mathbb{R}^d \mid |p| > e^a(1+\delta)\}, \tag{2.4}$$

$$A_\delta^{c-} := \{p \in \mathbb{R}^d \mid |p| < e^a(1-\delta)\}, \tag{2.5}$$

such that

$$A_\delta^c = A_\delta^{c+} \cup A_\delta^{c-}.$$

Obviously, the second term in the right side of (2.3) can be expressed as

$$\frac{\widehat{f}(p)}{\ln\left(\frac{|p|}{e^a}\right)}\chi_{A_\delta^{c+}} + \frac{\widehat{f}(p)}{\ln\left(\frac{|p|}{e^a}\right)}\chi_{A_\delta^{c-}}. \tag{2.6}$$

We have the trivial upper bounds

$$\frac{|\widehat{f}(p)|}{\left|\ln\left(\frac{|p|}{e^a}\right)\right|}\chi_{A_\delta^{c+}} \le \frac{|\widehat{f}(p)|}{\ln(1+\delta)} \in L^2(\mathbb{R}^d),$$

$$\frac{|\widehat{f}(p)|}{\left|\ln\left(\frac{|p|}{e^a}\right)\right|}\chi_{A_\delta^{c-}} \le \frac{|\widehat{f}(p)|}{-\ln(1-\delta)} \in L^2(\mathbb{R}^d),$$

as assumed. Let us write

$$\widehat{f}(p) = \widehat{f}(e^a, \sigma) + \int_{e^a}^{|p|} \frac{\partial \widehat{f}(s,\sigma)}{\partial s} ds.$$

Here and below σ will stand for the angle variables on the sphere. This enables us to express the first term in the right side of (2.3) as

$$\frac{\widehat{f}(e^a,\sigma)}{\ln\left(\frac{|p|}{e^a}\right)}\chi_{A_\delta} + \frac{\int_{e^a}^{|p|} \frac{\partial \widehat{f}(s,\sigma)}{\partial s} ds}{\ln\left(\frac{|p|}{e^a}\right)}\chi_{A_\delta}. \tag{2.7}$$

By means of the definition of the standard Fourier transform (1.11), we easily obtain that

$$\left|\frac{\partial \widehat{f}(p)}{\partial |p|}\right| \le \frac{1}{(2\pi)^{\frac{d}{2}}}\|xf(x)\|_{L^1(\mathbb{R}^d)}, \quad p \in \mathbb{R}^d, \quad d \ge 2. \tag{2.8}$$

Thus, the second term in (2.7) can be bounded from above in the absolute value by

$$\frac{1}{(2\pi)^{\frac{d}{2}}}\|xf(x)\|_{L^1(\mathbb{R}^d)}\left|\frac{|p|-e^a}{\ln\left(\frac{|p|}{e^a}\right)}\right|\chi_{A_\delta} \le C\|xf(x)\|_{L^1(\mathbb{R}^d)}\chi_{A_\delta} \in L^2(\mathbb{R}^d).$$

Hence, it remains to analyze the term

$$\frac{\widehat{f}(e^a,\sigma)}{\ln\left(\frac{|p|}{e^a}\right)}\chi_{A_\delta}. \tag{2.9}$$

It can be easily verified that (2.9) is square integrable if and only if $\widehat{f}(e^a, \sigma)$ is trivial. This is equivalent to orthogonality conditions (1.8).
Let us complete the proof of our theorem by treating the case a) when the dimension of the problem $d = 1$. We define the intervals on the real line

$$I_\delta^+ := [e^a(1-\delta),\ e^a(1+\delta)], \quad I_\delta^- := [-e^a(1+\delta),\ -e^a(1-\delta)], \tag{2.10}$$

where $0 < \delta < 1$, so that

$$I_\delta := I_\delta^+ \cup I_\delta^-.$$

Moreover,

$$I_\delta^{c+} := (-\infty,\ -e^a(1+\delta)) \cup (e^a(1+\delta),\ +\infty), \tag{2.11}$$

$$I_\delta^{c-} := (-e^a(1-\delta),\ e^a(1-\delta)), \tag{2.12}$$

such that

$$I_\delta^c = I_\delta^{c+} \cup I_\delta^{c-}.$$

Let us express

$$\widehat{u}(p) = \frac{\widehat{f}(p)}{\ln\left(\frac{|p|}{e^a}\right)}\chi_{I_\delta^+} + \frac{\widehat{f}(p)}{\ln\left(\frac{|p|}{e^a}\right)}\chi_{I_\delta^{c+}} + \frac{\widehat{f}(p)}{\ln\left(\frac{|p|}{e^a}\right)}\chi_{I_\delta^-} + \frac{\widehat{f}(p)}{\ln\left(\frac{|p|}{e^a}\right)}\chi_{I_\delta^{c-}}. \tag{2.13}$$

Clearly, the second term in the right side of (2.13) can be estimated in the absolute value as

$$\frac{|\widehat{f}(p)|}{\left|\ln\left(\frac{|p|}{e^a}\right)\right|}\chi_{I_\delta^{c+}} \le \frac{|\widehat{f}(p)|}{\ln(1+\delta)} \in L^2(\mathbb{R})$$

as assumed. Similarly, for the fourth term in the right side of (2.13) we have

$$\frac{|\widehat{f}(p)|}{\left|\ln\left(\frac{|p|}{e^a}\right)\right|}\chi_{I_\delta^{c-}} \le \frac{|\widehat{f}(p)|}{-\ln(1-\delta)} \in L^2(\mathbb{R})$$

as well. Obviously,

$$\widehat{f}(p) = \widehat{f}(e^a) + \int_{e^a}^{p} \frac{d\widehat{f}(s)}{ds} ds,$$

which enables us to write the first term in the right side of (2.13) as

$$\frac{\widehat{f}(e^a)}{\ln\left(\frac{|p|}{e^a}\right)}\chi_{I_\delta^+} + \frac{\int_{e^a}^{p} \frac{d\widehat{f}(s)}{ds} ds}{\ln\left(\frac{|p|}{e^a}\right)}\chi_{I_\delta^+}. \tag{2.14}$$

By virtue of the definition of the standard Fourier transform (1.11), we have

$$\left|\frac{d\widehat{f}(p)}{dp}\right| \leq \frac{1}{\sqrt{2\pi}}\|xf(x)\|_{L^1(\mathbb{R})}. \tag{2.15}$$

This is the analog of formula (2.8) in one dimension. The second term in (2.14) can be bounded from above in the absolute value by

$$\frac{1}{\sqrt{2\pi}}\|xf(x)\|_{L^1(\mathbb{R})}\left|\frac{p-e^a}{\ln\left(\frac{|p|}{e^a}\right)}\right|\chi_{I_\delta^+} \leq C\|xf(x)\|_{L^1(\mathbb{R})}\chi_{I_\delta^+},$$

which is square integrable on the real line. Let us consider the first term in (2.14). It can be trivially checked that

$$\frac{\widehat{f}(e^a)}{\ln\left(\frac{|p|}{e^a}\right)}\chi_{I_\delta^+} \in L^2(\mathbb{R})$$

if and only if $\widehat{f}(e^a)$ vanishes. This is equivalent to the orthogonality relation

$$\left(f(x), \frac{e^{ie^a x}}{\sqrt{2\pi}}\right)_{L^2(\mathbb{R})} = 0.$$

Clearly,

$$\widehat{f}(p) = \widehat{f}(-e^a) + \int_{-e^a}^{p} \frac{d\widehat{f}(s)}{ds}ds,$$

which allows us to express the third term in the right side of (2.13) as

$$\frac{\widehat{f}(-e^a)}{\ln\left(\frac{|p|}{e^a}\right)}\chi_{I_\delta^-} + \frac{\int_{-e^a}^{p}\frac{d\widehat{f}(s)}{ds}ds}{\ln\left(\frac{|p|}{e^a}\right)}\chi_{I_\delta^-}. \tag{2.16}$$

The second term in (2.16) can be estimated from above in the absolutely value by

$$\frac{1}{\sqrt{2\pi}}\|xf(x)\|_{L^1(\mathbb{R})}\left|\frac{p+e^a}{\ln\left(\frac{-p}{e^a}\right)}\right|\chi_{I_\delta^-} \leq C\|xf(x)\|_{L^1(\mathbb{R})}\chi_{I_\delta^-},$$

which belongs to $L^2(\mathbb{R})$. Finally, we analyze the first term in (2.16). It can be easily verified that

$$\frac{\widehat{f}(-e^a)}{\ln\left(\frac{|p|}{e^a}\right)}\chi_{I_\delta^-} \in L^2(\mathbb{R})$$

if and only if $\widehat{f}(-e^a) = 0$. This is equivalent to the orthogonality condition

$$\left(f(x), \frac{e^{-ie^a x}}{\sqrt{2\pi}}\right)_{L^2(\mathbb{R})} = 0,$$

which completes the proof of our theorem. ■

Let us proceed to establishing the solvability in the sense of sequences for our equation in the no potential case.

Proof of Theorem 1.2. We recall the result of Lemma 4.1 of [42]. Under the stated assumptions, we have $f_n(x) \in L^1(\mathbb{R}^d),\ n \in \mathbb{N},\ d \in \mathbb{N}$, so that

$$f_n(x) \to f(x) \quad in \quad L^1(\mathbb{R}^d) \quad as \quad n \to \infty. \tag{2.17}$$

Let us first treat the case b) of the theorem when the dimension of the problem $d \geq 2$. By means of Theorem 1.1, each equation (1.5) admits a unique solution $u_n(x) \in L^2(\mathbb{R}^d),\ n \in \mathbb{N}$. We demonstrate that the limiting orthogonality conditions

$$\left(f(x), \frac{e^{ipx}}{(2\pi)^{\frac{d}{2}}}\right)_{L^2(\mathbb{R}^d)} = 0, \quad p \in S^d_{e^a} \tag{2.18}$$

are valid. Using (1.10) along with (2.17), we easily obtain that for $p \in S^d_{e^a}$

$$\left|\left(f(x), \frac{e^{ipx}}{(2\pi)^{\frac{d}{2}}}\right)_{L^2(\mathbb{R}^d)}\right| = \left|\left(f(x) - f_n(x), \frac{e^{ipx}}{(2\pi)^{\frac{d}{2}}}\right)_{L^2(\mathbb{R}^d)}\right| \leq$$

$$\leq \frac{1}{(2\pi)^{\frac{d}{2}}}\|f_n(x) - f(x)\|_{L^1(\mathbb{R}^d)} \to 0, \quad n \to \infty.$$

Therefore, by virtue of Theorem 1.1 equation (1.2) possesses a unique solution $u(x) \in L^2(\mathbb{R}^d),\ d \geq 2$. Let us apply the standard Fourier transform (1.11) to both sides of problems (1.5). This yields

$$\widehat{u}_n(p) = \frac{\widehat{f_n}(p)}{\ln\left(\frac{|p|}{e^a}\right)}, \quad n \in \mathbb{N}, \tag{2.19}$$

such that

$$\widehat{u}_n(p) - \widehat{u}(p) = \frac{\widehat{f_n}(p) - \widehat{f}(p)}{\ln\left(\frac{|p|}{e^a}\right)}\chi_{A_\delta} + \frac{\widehat{f_n}(p) - \widehat{f}(p)}{\ln\left(\frac{|p|}{e^a}\right)}\chi_{A_\delta^c}. \tag{2.20}$$

Evidently, the second term in the right side of (2.20) can be written as

$$\frac{\widehat{f_n}(p) - \widehat{f}(p)}{\ln\left(\frac{|p|}{e^a}\right)}\chi_{A_\delta^{c+}} + \frac{\widehat{f_n}(p) - \widehat{f}(p)}{\ln\left(\frac{|p|}{e^a}\right)}\chi_{A_\delta^{c-}}. \tag{2.21}$$

The first term in (2.21) can be easily bounded from above in the absolutely value by $\dfrac{|\widehat{f}_n(p) - \widehat{f}(p)|}{\ln(1+\delta)}$, such that

$$\Bigg\| \frac{\widehat{f}_n(p) - \widehat{f}(p)}{\ln\left(\frac{|p|}{e^a}\right)} \chi_{A_\delta^{c+}} \Bigg\|_{L^2(\mathbb{R}^d)} \leq \frac{1}{\ln(1+\delta)} \|f_n(x) - f(x)\|_{L^2(\mathbb{R}^d)} \to 0$$

as $n \to \infty$ as assumed. Similarly, the second term in (2.21) can be trivially estimated from above in the absolutely value by $\dfrac{|\widehat{f}_n(p) - \widehat{f}(p)|}{-\ln(1-\delta)}$, so that

$$\Bigg\| \frac{\widehat{f}_n(p) - \widehat{f}(p)}{\ln\left(\frac{|p|}{e^a}\right)} \chi_{A_\delta^{c-}} \Bigg\|_{L^2(\mathbb{R}^d)} \leq \frac{1}{-\ln(1-\delta)} \|f_n(x) - f(x)\|_{L^2(\mathbb{R}^d)} \to 0$$

as $n \to \infty$ as well. Let us recall our orthogonality conditions (2.18) and (1.10). They give us that

$$\widehat{f}(e^a, \sigma) = 0, \quad \widehat{f}_n(e^a, \sigma) = 0, \quad n \in \mathbb{N}.$$

Then

$$\widehat{f}(p) = \int_{e^a}^{|p|} \frac{\partial \widehat{f}(s,\sigma)}{\partial s} ds, \quad \widehat{f}_n(p) = \int_{e^a}^{|p|} \frac{\partial \widehat{f}_n(s,\sigma)}{\partial s} ds, \quad n \in \mathbb{N},$$

which enables us to express the first term in the right side of (2.20) as

$$\frac{\int_{e^a}^{|p|} \left[\frac{\partial \widehat{f}_n(s,\sigma)}{\partial s} - \frac{\partial \widehat{f}(s,\sigma)}{\partial s} \right] ds}{\ln\left(\frac{|p|}{e^a}\right)} \chi_{A_\delta}. \tag{2.22}$$

From the definition of the standard Fourier transform (1.11) we easily derive that

$$\left| \frac{\partial \widehat{f}_n(p)}{\partial |p|} - \frac{\partial \widehat{f}(p)}{\partial |p|} \right| \leq \frac{1}{(2\pi)^{\frac{d}{2}}} \|x f_n(x) - x f(x)\|_{L^1(\mathbb{R}^d)}, \quad d \geq 2. \tag{2.23}$$

Hence, expression (2.22) can be bounded from above in the absolute value by

$$\frac{1}{(2\pi)^{\frac{d}{2}}} \|x f_n(x) - x f(x)\|_{L^1(\mathbb{R}^d)} \left| \frac{|p| - e^a}{\ln\left(\frac{|p|}{e^a}\right)} \right| \chi_{A_\delta} \leq C \|x f_n(x) - x f(x)\|_{L^1(\mathbb{R}^d)} \chi_{A_\delta},$$

such that for the norm we have

$$\Bigg\| \frac{\int_{e^a}^{|p|} \left[\frac{\partial \widehat{f}_n(s,\sigma)}{\partial s} - \frac{\partial \widehat{f}(s,\sigma)}{\partial s} \right] ds}{\ln\left(\frac{|p|}{e^a}\right)} \chi_{A_\delta} \Bigg\|_{L^2(\mathbb{R}^d)} \leq C \|x f_n(x) - x f(x)\|_{L^1(\mathbb{R}^d)} |A_\delta|^{\frac{1}{2}} \to 0$$

as $n \to \infty$ due to the one of our assumptions. Therefore,

$$u_n(x) \to u(x) \quad in \quad L^2(\mathbb{R}^d), \quad n \to \infty, \quad d \geq 2$$

in the situation b) of our theorem. We conclude the proof by considering the case a) when the dimension of the problem $d = 1$. By virtue of the result of Theorem 1.1, each equation (1.5) possesses a unique solution $u_n(x) \in L^2(\mathbb{R}),\ n \in \mathbb{N}$. Let us establish that the limiting orthogonality relations

$$\Big(f(x), \frac{e^{\pm ie^a x}}{\sqrt{2\pi}}\Big)_{L^2(\mathbb{R})} = 0 \tag{2.24}$$

hold. We use (1.9) along with (2.17) to obtain that

$$\Big|\Big(f(x), \frac{e^{\pm ie^a x}}{\sqrt{2\pi}}\Big)_{L^2(\mathbb{R})}\Big| = \Big|\Big(f(x) - f_n(x), \frac{e^{\pm ie^a x}}{\sqrt{2\pi}}\Big)_{L^2(\mathbb{R})}\Big| \leq$$

$$\leq \frac{1}{\sqrt{2\pi}} \|f_n(x) - f(x)\|_{L^1(\mathbb{R})} \to 0, \quad n \to \infty.$$

Thus, by means of Theorem 1.1 above, problem (1.2) admits a unique solution $u(x) \in L^2(\mathbb{R})$. By applying the standard Fourier transform (1.11) to both sides of equations (1.5), we obtain the analog of formula (2.19) in one dimension, so that

$$\widehat{u}_n(p) - \widehat{u}(p) = \frac{\widehat{f}_n(p) - \widehat{f}(p)}{\ln\Big(\frac{|p|}{e^a}\Big)} \chi_{I_\delta^+} + \frac{\widehat{f}_n(p) - \widehat{f}(p)}{\ln\Big(\frac{|p|}{e^a}\Big)} \chi_{I_\delta^{c+}} +$$

$$+ \frac{\widehat{f}_n(p) - \widehat{f}(p)}{\ln\Big(\frac{|p|}{e^a}\Big)} \chi_{I_\delta^-} + \frac{\widehat{f}_n(p) - \widehat{f}(p)}{\ln\Big(\frac{|p|}{e^a}\Big)} \chi_{I_\delta^{c-}}. \tag{2.25}$$

The second term in the right side of (2.25) can be bounded from above in the absolute value by $\dfrac{|\widehat{f}_n(p) - \widehat{f}(p)|}{\ln(1+\delta)}$, such that

$$\left\| \frac{\widehat{f}_n(p) - \widehat{f}(p)}{\ln\Big(\frac{|p|}{e^a}\Big)} \chi_{I_\delta^{c+}} \right\|_{L^2(\mathbb{R})} \leq \frac{1}{\ln(1+\delta)} \|f_n(x) - f(x)\|_{L^2(\mathbb{R})} \to 0, \quad n \to \infty$$

as assumed. The fourth term in the right side of (2.25) can be estimated from above in the absolute value by $\dfrac{|\widehat{f}_n(p) - \widehat{f}(p)|}{-\ln(1-\delta)}$, so that

$$\left\| \frac{\widehat{f}_n(p) - \widehat{f}(p)}{\ln\Big(\frac{|p|}{e^a}\Big)} \chi_{I_\delta^{c-}} \right\|_{L^2(\mathbb{R})} \leq \frac{1}{-\ln(1-\delta)} \|f_n(x) - f(x)\|_{L^2(\mathbb{R})} \to 0, \quad n \to \infty$$

as well. Let us recall orthogonality conditions (2.24) and (1.9). They imply that

$$\widehat{f}(e^a)=0, \quad \widehat{f}_n(e^a)=0, \quad n\in\mathbb{N},$$

such that

$$\widehat{f}(p)=\int_{e^a}^{p}\frac{d\widehat{f}(s)}{ds}ds, \quad \widehat{f}_n(p)=\int_{e^a}^{p}\frac{d\widehat{f}_n(s)}{ds}ds, \quad n\in\mathbb{N}.$$

Thus, the first term in the right side of (2.25) can be written as

$$\frac{\int_{e^a}^{p}\left[\frac{d\widehat{f}_n(s)}{ds}-\frac{d\widehat{f}(s)}{ds}\right]ds}{\ln\left(\frac{|p|}{e^a}\right)}\chi_{I_\delta^+}. \tag{2.26}$$

By virtue of the definition of the standard Fourier transform (1.11), we easily obtain that

$$\left|\frac{d\widehat{f}_n(p)}{dp}-\frac{d\widehat{f}(p)}{dp}\right|\le\frac{1}{\sqrt{2\pi}}\|xf_n(x)-xf(x)\|_{L^1(\mathbb{R})}. \tag{2.27}$$

Clearly, formula (2.27) is the one dimensional analog of (2.23). Hence, expression (2.26) can be bounded from above in the absolute value by

$$\frac{1}{\sqrt{2\pi}}\|xf_n(x)-xf(x)\|_{L^1(\mathbb{R})}\left|\frac{p-e^a}{\ln\left(\frac{|p|}{e^a}\right)}\right|\chi_{I_\delta^+}\le C\|xf_n(x)-xf(x)\|_{L^1(\mathbb{R})}\chi_{I_\delta^+},$$

so that for the norm we have

$$\left\|\frac{\int_{e^a}^{p}\left[\frac{d\widehat{f}_n(s)}{ds}-\frac{d\widehat{f}(s)}{ds}\right]ds}{\ln\left(\frac{|p|}{e^a}\right)}\chi_{I_\delta^+}\right\|_{L^2(\mathbb{R})}\le C\sqrt{2\delta e^a}\|xf_n(x)-xf(x)\|_{L^1(\mathbb{R})}\to 0$$

as $n\to\infty$ due to the one of our assumptions. Orthogonality relations (2.24) and (1.9) give us that

$$\widehat{f}(-e^a)=0, \quad \widehat{f}_n(-e^a)=0, \quad n\in\mathbb{N}.$$

Hence,

$$\widehat{f}(p)=\int_{-e^a}^{p}\frac{d\widehat{f}(s)}{ds}ds, \quad \widehat{f}_n(p)=\int_{-e^a}^{p}\frac{d\widehat{f}_n(s)}{ds}ds, \quad n\in\mathbb{N},$$

such that the third term in the right side of (2.25) can be expressed as

$$\frac{\int_{-e^a}^{p}\left[\frac{d\widehat{f}_n(s)}{ds}-\frac{d\widehat{f}(s)}{ds}\right]ds}{\ln\left(\frac{|p|}{e^a}\right)}\chi_{I_\delta^-}. \tag{2.28}$$

Using formula (2.27), we estimate (2.28) from above in the absolute value by

$$\frac{1}{\sqrt{2\pi}}\|xf_n(x)-xf(x)\|_{L^1(\mathbb{R})}\left|\frac{p+e^a}{\ln\left(\frac{-p}{e^a}\right)}\right|\chi_{I_\delta^-}\leq C\|xf_n(x)-xf(x)\|_{L^1(\mathbb{R})}\chi_{I_\delta^-}.$$

Then for the norm we have

$$\left\|\frac{\int_{-e^a}^{p}\left[\frac{d\widehat{f_n}(s)}{ds}-\frac{d\widehat{f}(s)}{ds}\right]ds}{\ln\left(\frac{|p|}{e^a}\right)}\chi_{I_\delta^-}\right\|_{L^2(\mathbb{R})}\leq C\sqrt{2\delta e^a}\|xf_n(x)-xf(x)\|_{L^1(\mathbb{R})}\to 0$$

as $n\to\infty$ as assumed. Therefore,

$$u_n(x)\to u(x)\quad in\quad L^2(\mathbb{R}),\quad n\to\infty$$

in the situation a) of our theorem as well. ■

3. Solvability in the sense of sequences with a scalar potential

Proof of Theorem 1.4. Let us first establish the uniqueness of solutions for our problem. We suppose that there exist $u_1(x), u_2(x)\in L^2(\mathbb{R}^3)$ satisfying (1.12). Then their square integrable difference $w(x):=u_1(x)-u_2(x)$ solves the homogeneous equation

$$L_a w=0.$$

Since the operator L_a defined in (1.15) has no nontrivial zero modes belonging to $L^2(\mathbb{R}^3)$ as discussed above, $w(x)$ vanishes identically in $\mathbb{R}^3$.
For the technical purposes we introduce the spherical layer in our space of three dimensions as

$$B_\delta:=\{k\in\mathbb{R}^3\mid e^a(1-\delta)\leq|k|\leq e^a(1+\delta)\},\quad 0<\delta<1. \tag{3.1}$$

$|B_\delta|$ will stand for its Lebesgue measure. Let us apply the generalized Fourier transform (1.20) with the functions of the continuous spectrum of our Schrödinger operator to both sides of problem (1.12). This yields

$$\tilde{u}(k)=\frac{\tilde{f}(k)}{\ln\left(\frac{|k|}{e^a}\right)}\chi_{B_\delta}+\frac{\tilde{f}(k)}{\ln\left(\frac{|k|}{e^a}\right)}\chi_{B_\delta^c}. \tag{3.2}$$

We define the sets

$$B_\delta^{c+}:=\{k\in\mathbb{R}^3\mid |k|>e^a(1+\delta)\}, \tag{3.3}$$

$$B_\delta^{c-}:=\{k\in\mathbb{R}^3\mid |k|<e^a(1-\delta)\}, \tag{3.4}$$

so that

$$B_\delta^c=B_\delta^{c+}\cup B_\delta^{c-}.$$

The second term in the right side of (3.2) can be easily written as

$$\frac{\tilde{f}(k)}{\ln\left(\frac{|k|}{e^a}\right)}\chi_{B_\delta^{c+}} + \frac{\tilde{f}(k)}{\ln\left(\frac{|k|}{e^a}\right)}\chi_{B_\delta^{c-}} \tag{3.5}$$

Evidently, we have the upper bound

$$\frac{|\tilde{f}(k)|}{\left|\ln\left(\frac{|k|}{e^a}\right)\right|}\chi_{B_\delta^{c+}} \leq \frac{|\tilde{f}(k)|}{\ln(1+\delta)} \in L^2(\mathbb{R}^3)$$

via the one of our assumptions. Similarly,

$$\frac{|\tilde{f}(k)|}{\left|\ln\left(\frac{|k|}{e^a}\right)\right|}\chi_{B_\delta^{c-}} \leq \frac{|\tilde{f}(k)|}{-\ln(1-\delta)} \in L^2(\mathbb{R}^3).$$

Clearly, we can write

$$\tilde{f}(k) = \tilde{f}(e^a, \sigma) + \int_{e^a}^{|k|} \frac{\partial \tilde{f}(s,\sigma)}{\partial s} ds. \tag{3.6}$$

This enables us to express the first term in the right side of (3.2) as

$$\frac{\tilde{f}(e^a,\sigma)}{\ln\left(\frac{|k|}{e^a}\right)}\chi_{B_\delta} + \frac{\int_{e^a}^{|k|} \frac{\partial \tilde{f}(s,\sigma)}{\partial s} ds}{\ln\left(\frac{|k|}{e^a}\right)}\chi_{B_\delta}. \tag{3.7}$$

Let us recall Lemma 2.4 of [36]. Hence, under the given conditions we have $\nabla_q \tilde{f}(q) \in L^\infty(\mathbb{R}^3)$. We estimate the second term in sum (3.7) in the absolute value as

$$\left|\frac{\int_{e^a}^{|k|} \frac{\partial \tilde{f}(s,\sigma)}{\partial s} ds}{\ln\left(\frac{|k|}{e^a}\right)}\chi_{B_\delta}\right| \leq \|\nabla_q \tilde{f}(q)\|_{L^\infty(\mathbb{R}^3)} \left| |k| - e^a \ln\left(\frac{|k|}{e^a}\right)\right| \chi_{B_\delta} \leq$$

$$\leq C \|\nabla_q \tilde{f}(q)\|_{L^\infty(\mathbb{R}^3)} \chi_{B_\delta} \in L^2(\mathbb{R}^3).$$

Therefore, it remains to analyze the term

$$\frac{\tilde{f}(e^a,\sigma)}{\ln\left(\frac{|k|}{e^a}\right)}\chi_{B_\delta}. \tag{3.8}$$

It can trivially checked that (3.8) is square integrable if and only if $\tilde{f}(e^a, \sigma)$ vanishes. This is equivalent to orthogonality relations (1.22). ■

We turn our attention to the demonstration of the validity of the result of our final main proposition, which deals with the solvability in the sense of sequences.

Proof of Theorem 1.5. Clearly, each equation (1.13) has a unique solution $u_n(x) \in L^2(\mathbb{R}^3),\ n \in \mathbb{N}$ by means of the result of Theorem 1.4 above. It can be easily verified that the limiting orthogonality relations

$$(f(x), \varphi_k(x))_{L^2(\mathbb{R}^3)} = 0, \quad k \in S^3_{e^a} \tag{3.9}$$

hold. Let us recall the result of Lemma 4.1 of [42]. Under the given conditions, we have $f_n(x) \in L^1(\mathbb{R}^3), n \in \mathbb{N}$, so that $f_n(x) \to f(x)$ in $L^1(\mathbb{R}^3)$ as $n \to \infty$. By virtue of (1.23) along with upper bound (1.21)

$$|(f(x), \varphi_k(x))_{L^2(\mathbb{R}^3)}| = |(f(x) - f_n(x), \varphi_k(x))_{L^2(\mathbb{R}^3)}| \leq$$
$$\leq \frac{1}{(2\pi)^{\frac{3}{2}}} \frac{1}{1 - I(V)} \|f_n(x) - f(x)\|_{L^1(\mathbb{R}^3)} \to 0, \quad n \to \infty$$

with $k \in S^3_{e^a}$. Therefore, limiting equation (1.12) possesses a unique solution $u(x) \in L^2(\mathbb{R}^3)$ via the result of Theorem 1.4. Let us apply the generalized Fourier transform (1.20) to both sides of problem (1.13). This gives us

$$\tilde{u}_n(k) = \frac{\tilde{f}_n(k)}{\ln\left(\frac{|k|}{e^a}\right)}, \quad n \in \mathbb{N},$$

so that

$$\tilde{u}_n(k) - \tilde{u}(k) = \frac{\tilde{f}_n(k) - \tilde{f}(k)}{\ln\left(\frac{|k|}{e^a}\right)} \chi_{B_\delta} + \frac{\tilde{f}_n(k) - \tilde{f}(k)}{\ln\left(\frac{|k|}{e^a}\right)} \chi_{B^c_\delta}. \tag{3.10}$$

Obviously, the second term in the right side of (3.10) can be trivially written as

$$\frac{\tilde{f}_n(k) - \tilde{f}(k)}{\ln\left(\frac{|k|}{e^a}\right)} \chi_{B^{c+}_\delta} + \frac{\tilde{f}_n(k) - \tilde{f}(k)}{\ln\left(\frac{|k|}{e^a}\right)} \chi_{B^{c-}_\delta}. \tag{3.11}$$

Evidently, the first term in (3.11) can be easily estimated from above in the absolute value by $\dfrac{|\tilde{f}_n(k) - \tilde{f}(k)|}{\ln(1+\delta)}$. Hence,

$$\left\| \frac{\tilde{f}_n(k) - \tilde{f}(k)}{\ln\left(\frac{|k|}{e^a}\right)} \chi_{B^{c+}_\delta} \right\|_{L^2(\mathbb{R}^3)} \leq \frac{1}{\ln(1+\delta)} \|f_n(x) - f(x)\|_{L^2(\mathbb{R}^3)} \to 0, \quad n \to \infty$$

as assumed. Analogously, the second term in (3.11) can be bounded from above in the absolute value by $\dfrac{|\tilde{f}_n(k) - \tilde{f}(k)|}{-\ln(1-\delta)}$, so that

$$\left\| \frac{\tilde{f}_n(k) - \tilde{f}(k)}{\ln\left(\frac{|k|}{e^a}\right)} \chi_{B^{c-}_\delta} \right\|_{L^2(\mathbb{R}^3)} \leq \frac{1}{-\ln(1-\delta)} \|f_n(x) - f(x)\|_{L^2(\mathbb{R}^3)} \to 0, \quad n \to \infty$$

via the one of our assumptions. By virtue of the orthogonality relations (3.9) and (1.23), we have

$$\tilde{f}(e^a,\sigma)=0, \quad \tilde{f}_n(e^a,\sigma)=0, \quad n\in\mathbb{N},$$

such that

$$\tilde{f}(k)=\int_{e^a}^{|k|}\frac{\partial \tilde{f}(s,\sigma)}{\partial s}ds, \quad \tilde{f}_n(k)=\int_{e^a}^{|k|}\frac{\partial \tilde{f}_n(s,\sigma)}{\partial s}ds, \quad n\in\mathbb{N}.$$

This enables us to express the first term in the right side of (3.10) as

$$\frac{\int_{e^a}^{|k|}\left[\frac{\partial \tilde{f}_n(s,\sigma)}{\partial s}-\frac{\partial \tilde{f}(s,\sigma)}{\partial s}\right]ds}{\ln\left(\frac{|k|}{e^a}\right)}\chi_{B_\delta},$$

which can be trivially estimated from above in the absolute value by

$$\|\nabla_q[\tilde{f}_n(q)-\tilde{f}(q)]\|_{L^\infty(\mathbb{R}^3)}\left|\frac{|k|-e^a}{\ln\left(\frac{|k|}{e^a}\right)}\right|\chi_{B_\delta}\le C\|\nabla_q[\tilde{f}_n(q)-\tilde{f}(q)]\|_{L^\infty(\mathbb{R}^3)}\chi_{B_\delta}.$$

Thus,

$$\left\|\frac{\tilde{f}_n(k)-\tilde{f}(k)}{\ln\left(\frac{|k|}{e^a}\right)}\chi_{B_\delta}\right\|_{L^2(\mathbb{R}^3)}\le C\|\nabla_q[\tilde{f}_n(q)-\tilde{f}(q)]\|_{L^\infty(\mathbb{R}^3)}\sqrt{|B_\delta|}.$$

Let us recall the result of Lemma 3.4 of [34]. Under the given conditions, we have

$$\|\nabla_q[\tilde{f}_n(q)-\tilde{f}(q)]\|_{L^\infty(\mathbb{R}^3)}\to 0, \quad n\to\infty.$$

Therefore,

$$u_n(x)\to u(x) \quad in \quad L^2(\mathbb{R}^3), \quad n\to\infty$$

which completes the proof of our theorem. ∎

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