

ANALYSIS OF THE LÉVY FLIGHT FORAGING HYPOTHESIS IN \mathbb{R}^n AND UNRELIABILITY OF THE MOST REWARDING STRATEGIES

SERENA DIPIERRO, GIOVANNI GIACOMIN, AND ENRICO VALDINOCI

ABSTRACT. We analyze the searching strategies of a forager diffusing in the whole space via an equation of fractional type. Specifically, the diffusion of the forager is regulated by a Lévy flight whose exponent can be chosen in order to optimize a suitable foraging efficiency functional. Here, the dimension of the space is arbitrary.

On the one hand, we show that the exponent $s = 0$ corresponding to the limit case of heavy-tailed Lévy flights is a pessimizer for the efficiency functional.

On the other hand, we prove that, in situations of biological interest, one finds the most rewarding strategies arbitrarily close to $s = 0$.

The combination of these results give that the most rewarding searching option may turn out to be unfeasible, or at least unreliable, in practice, since small perturbations of the optimal searching exponent lead to pessimal patterns.

The cases analyzed specifically are those of a target located in the proximity of the forager and that of sparse prey modeled by a target infinitely far from the initial position of the seeker.

The efficiency functionals taken into account are either of pointwise type (in which the predator and the prey are modeled by moving points) and of set-dependent type (in which the predator and the prey correspond to regions of space with uniform density, thus modeling also the case of a sight range of the biological individuals involved).

To implement our analysis, we also provide a number of structural results about finiteness, continuity and asymptotic behaviors of the efficiency functionals.

It is suggestive to relate the adoption of the most rewarding searching pattern close to pessimizers to a “high-risk/high-gain” strategy, in which the forager aims at high-energy content prey to mitigate the risk of failure.

This setting is also connected to foraging modes of “ambush” type.

1. INTRODUCTION

The Lévy flight foraging hypothesis conjectures that search efficiency can be optimized by a random relocation induced by a distribution with a fat tail. In this, the theory of Lévy flights challenged the previous paradigm which used Brownian motion as a default template for describing ecological movements. Experimental evidence and empirical results supporting animal diffusion related to Lévy flights have been repeatedly put forth, see e.g. [VAB⁺96, ARMA02, RFMM⁺04, SWR⁺06, SSH⁺08, HQD⁺10, HWQ⁺12, RJB⁺15]. In several occasions, the Lévy flights observed in concrete situations appeared to be independent of the specific distribution of resource [dJBK⁺14] and it has been conjectured that they are a consequence of an adaptive process [VAB⁺96].

A large collection of Lévy flights has been observed also at a microscopical level, in collective movements, in trace fossils, etc., see for instance Table 1 in [Rey18] for an interesting list of ubiquitous evidence of Lévy flights, as well as an account on the known generative mechanisms and selection criteria.

Key words and phrases. Lévy flights, fractional Laplacian, optimal strategies, high risk/high gain.

This research has been supported by the Australian Research Council DECRA DE180100957 “PDEs, free boundaries and applications” and the Australian Laureate Fellowship FL190100081 “Minimal surfaces, free boundaries and partial differential equations”. The authors declare that they have no known competing financial interests that could have appeared to influence the work reported in this paper. It is a pleasure to thank Kurt Williams for interesting discussions.

As every topic of real scientific interest, the Lévy flight foraging hypothesis is also subject to debate and controversy. Typically, issues can arise about the statistical methods used in some of these studies (see e.g. [EPW⁺07]) and on the mathematical framework developed and utilized (see e.g. [LTBV20, BRB⁺21, LTBV21]). Therefore, when dealing with the Lévy flight foraging hypothesis, a very careful, and even pedantic, mathematical setting is absolutely necessary in order to specify exactly the assumptions taken and the models used, and give precise meaning, and possibly limitations, to the results obtained.

In this paper, we consider a forager diffusing in \mathbb{R}^n according to the fractional heat equation. The fractional parameter characterizing this diffusion is denoted by $s \in (0, 1]$. The example of fractional diffusion in the whole space was already considered in a previous work [DGV] when $n = 1$ and here we address the higher dimensional situation (the case in which the anomalous diffusion is confined in a bounded region will be considered in the forthcoming paper [DGV22b]). For completeness, we do not even limit ourselves to the cases $n = 2$ and $n = 3$, and instead we comprise all dimensions $n \in \mathbb{N}$. We recall that the analysis of animal foraging in dimension higher than 1 is not only important for biological purposes, but it also entails a delicate, and possibly controversial, mathematical scrutiny, see e.g. [LTBV20, BRB⁺21, LTBV21].

We will consider an efficiency functional accounting for the success of the search strategy. This efficiency functional is directly proportional to the number of random encounters between predators and targets and inversely proportional to the time of the search process (differently from most of the existing literature, we will look at time averages of successes, rather than at situations at a frozen instant of time).

The results that we provide are both in terms of most rewarding strategy and of pessimal searching patterns.

The main results related to most rewarding strategies (see Theorems 2.2, 2.4, 2.10 and 2.11) are of asymptotic type and establish that *if either the target position converges to the forager initial position, or the target position diverges towards infinity, then the most rewarding search strategy approaches $s = 0$* . Note that these two cases correspond, respectively, to the biological scenarios of *targets adjacent to the seeker* and of *sparse targets*.

The results concerning pessimal seeking options (see Theorems 2.1 and 2.9) are of universal type and prove that, in our framework, *the search strategy given by $s = 0$ is always a pessimizer*.

The combination of these types of main results reveals that *these most rewarding strategies are somewhat unsafe, in the sense that small perturbations of the optimal fractional parameter may produce pessimal searching patterns*.

The main results are complemented by several foundational results, that will be precisely stated in Appendix B, establishing the finiteness and continuity of the efficiency functionals that we take into account (see Propositions B.1, B.2 and B.3).

As a technical remark, we observe that the model presented here deals with a linear diffusive equation governed by the fractional Laplacian, in which individuals have no preexisting knowledge of the environment and no memory of previous events (from the biological point of view, this situation occurs e.g. when targets are mobile, thus providing no useful reference for subsequent foraging patterns).

The scenario considered is that of a non-destructive foraging in which exploited targets are replaced at once. No biological competition is taken into account, nor changes in the numerosness and behaviors of individuals (roughly speaking, this corresponds to the very common biological setting in which the logistic terms play a role only on a significantly longer time scale than the foraging process).

The environment and the time are modeled as continua (see below for a precise mathematical set-up).

The rest of this paper is organized as follows. Section 2 presents the mathematical set-up in which we work and showcases our main results. The proofs of the main results are given in Section 3. The appendices collect some ancillary results needed for the proofs of the main results. The conclusions of this article are summarized in the last section.

2. MATHEMATICAL SETTING AND MAIN RESULTS

In this section, we describe the mathematical framework used in this paper and we state our main results.

2.1. Set-up for fractional diffusion and efficiency functionals. To start with, one needs to model the distribution of predators and targets at a given instant of time t . For this, we let $p(t, x)$ be the target distribution in $(0, +\infty) \times \mathbb{R}^n$.

As for the forager, the anomalous diffusion considered here is that modeled by the fractional heat equation

$$(2.1) \quad \begin{cases} \partial_t u(t, x) = -(-\Delta)^s u(t, x) & \text{for all } (t, x) \in (0, +\infty) \times \mathbb{R}^n, \\ u(0, x) = \delta_y(x). \end{cases}$$

When $s = 1$, the operator $-(-\Delta)^s$ reduces to the standard Laplacian (i.e., the sum of pure second derivatives). Instead, when $s \in (0, 1)$ the fractional Laplacian is defined as

$$-(-\Delta)^s u(x) := \int_{\mathbb{R}^n} \frac{u(x+y) + u(x-y) - 2u(x)}{|y|^{n+2s}} dy,$$

up to a normalizing constant that we disregard for simplicity (see e.g. [AV19] and the references therein for the basics of the fractional Laplace operator¹ and [Law21] for further connection between this operator, the notion of random walks and its link to Lévy flights in biology).

We assume that the forager may tune the fractional parameter $s \in (0, 1]$ in order to maximize the convenience of the corresponding hunting strategy. In this context, we recall that $s = 1$ in (2.1) corresponds to the classical diffusion pattern induced by the Brownian motion, while $s \in (0, 1)$ is related to an anomalous diffusion produced by the corresponding Lévy flight, see e.g. Section 4 in [AV19] and the references therein.

We denote by $G^s(t, x, y)$ the solution to (2.1). In our model, $G^s(t, x, y)$ describes the forager's distribution at time $t \in (0, +\infty)$ with respect to the spatial position $x \in \mathbb{R}^n$. The point $y \in \mathbb{R}^n$ identifies the initial position of the forager (indeed, as customary, the notation δ_y in (2.1) represents the Dirac delta distribution).

Hence, the expected value of encounters between the forager and the target in the time span $(0, T)$ is proportional to the quantity

$$(2.2) \quad \int_0^T \int_{\mathbb{R}^n} G^s(t, \xi, y) p(t, \xi) d\xi dt.$$

This can be considered as a first approximation of a *foraging success functional* (based on the ansatz that the higher the number of encounters between predators and targets, the more rewarding the hunting for the predator).

¹In the literature, in the one-dimensional case, the fractional Laplacian is sometimes called the fractional Riesz derivative, or the fractional Riesz-Weil derivative, see e.g. [DGNBD17]. The name of derivative can however be confusing, since the fractional Laplacian is invariant under reflections, while usually derivatives are not.

See also [MPV13, SV17, DV21] and the references therein for several applications of the fractional Laplacian to problems of biology and ethology.

In particular, if the target is stationary, i.e. it remains at a given point $x \in \mathbb{R}^n$ for all times, we have that $p(t, \xi) = \delta_x(\xi)$ and the foraging success functional in (2.2) reduces to

$$(2.3) \quad \Psi^{x,y}(s, T) := \int_0^T G^s(t, x, y) dt.$$

From this, we introduce an *efficiency functional* to be defined as the ratio between the foraging success functional and the time employed for the search (based on the ansatz that for a searching strategy to be efficient, it is important not only to ensure a foraging success for the predator, but also to do so as quickly as possible). These considerations lead us to the definition

$$(2.4) \quad \mathcal{E}^{x,y}(s, T) := \frac{\Psi^{x,y}(s, T)}{T} = \frac{1}{T} \int_0^T G^s(t, x, y) dt.$$

This type of efficiency functional was introduced in the one-dimensional framework in [DGV].

Furthermore, we will also consider the scenario in which the precise locations of the seeker and the prey are replaced by uniform distributions in subregions of \mathbb{R}^n (this setting is also biologically relevant, since it replaces the exact location of individuals with regions of interests of individuals, e.g. accounting for visual interactions with the environment).

More precisely, given some measurable and bounded sets $\Omega_1, \Omega_2 \subset \mathbb{R}^n$, we will consider the *set-dependent efficiency functional*

$$(2.5) \quad \tilde{\mathcal{E}}^{\Omega_1, \Omega_2}(s, T) := \frac{1}{|\Omega_1||\Omega_2|} \int_{\Omega_1 \times \Omega_2} \mathcal{E}^{x,y}(s, T) dx dy = \frac{1}{T|\Omega_1||\Omega_2|} \int_0^T \int_{\Omega_1 \times \Omega_2} G^s(t, x, y) dx dy dt.$$

As customary, here above we have denoted by $|\Omega_1|$ and $|\Omega_2|$ the Lebesgue measures of the sets Ω_1 and Ω_2 (we have included these quantities in (2.5) for normalizing purposes).

We stress that this set-dependent approach to foraging, which is new in the literature, can serve as a useful technical *simplification of the notion of “direct vision”* which was often utilized to account for the predator directly reaching the target when it lies within a given distance.

Indeed, with this set-dependent approach one still identifies foragers and targets with regions of space, rather than points, but maintaining the advantage of working always with the same diffusion equation, holding at every spatial scale, without the necessity of truncating the Lévy distribution near its singularity, and without introducing an additional parameter in many steps of the intermediate calculations.

2.2. Main results. To study the efficiency functionals introduced in Section 2.1, one needs firstly to understand their basic mathematical properties in terms of *finiteness* and *continuity* with respect to the fractional parameter s (we stress that our goal is to pick s in order to optimize the foraging strategy, hence these properties are indispensable for any further analysis).

In this spirit, we observe that the efficiency functional in (2.4) is finite for $x = y$ only if $n = 1$ and $s \in (\frac{1}{2}, 1]$, as stated in Proposition B.1 in Appendix B.

Since our goal here is to focus on higher dimensional phenomena (i.e., on the case $n \geq 2$, being the case $n = 1$ contained in [DGV]), from now on we will consider the case $x \neq y$ in (2.4). In this setting, in Proposition B.2 we show that $\mathcal{E}^{x,y}(\cdot, T) \in C((0, 1])$ and also that $\mathcal{E}^{x,y}(\cdot, T) \in (0, +\infty)$.

Now we turn our attention to the detection of the *most and least rewarding searching strategies*. We establish that *independently from the starting position of the forager $y \in \mathbb{R}^n$, of the prey location $x \in \mathbb{R}^n$ (as far as $x \neq y$) and of the time span $T \in (0, +\infty)$, the strategy $s = 0$ is a global minimizer for \mathcal{E} . That is, $s = 0$ is a pessimizer for the foraging strategy, as stated in the following result:*

Theorem 2.1. *Let $(x, y, T) \in \mathbb{R}^n \times \mathbb{R}^n \times (0, +\infty)$ such that $x \neq y$.*

Then,

$$(2.6) \quad \inf_{s \in (0, 1)} \mathcal{E}^{x,y}(s, T) = \lim_{s \searrow 0} \mathcal{E}^{x,y}(s, T) = 0.$$

In view of Theorem 2.1 and Proposition B.2, we evince that we can extend by continuity the functional $\mathcal{E}^{x,y}(\cdot, T)$ to the compact interval $[0, 1]$. From now on we will adopt the notation

$$(2.7) \quad \mathcal{E}^{x,y}(0, T) := \lim_{s \searrow 0} \mathcal{E}^{x,y}(s, T) = 0,$$

for all $(x, y, T) \in \mathbb{R}^n \times \mathbb{R}^n \times (0, +\infty)$ with $x \neq y$.

Since Theorem 2.1 addressed the case of the least rewarding strategy, we now focus on finding the optimal pattern. While one can imagine that the best strategy may depend on the specific configuration of targets, some general results can be obtained in terms of asymptotic analysis.

One case of interest is the one in which *the prey is located close to the initial position of the forager*. In this situation we show that *the most rewarding strategy corresponds to values of the fractional parameter close to $s = 0$* . More specifically, we show that as the prey location gets closer and closer to the initial position of the forager the fractional parameter maximizing the efficiency functional approaches the value $s = 0$. The precise statement goes as follows:

Theorem 2.2. *Let $T \in (0, +\infty)$.*

Then, for each $\varepsilon \in (0, 1)$ there exists $r = r_{\varepsilon,n,T} \in (0, +\infty)$ such that for each $x, y \in \mathbb{R}^n$ satisfying $|x - y| \in (0, r)$ it holds that

$$(2.8) \quad \sup_{s \in (0,1)} \mathcal{E}^{x,y}(s, T) = \mathcal{E}^{x,y}(s_{x,y,T}, T) \quad \text{with} \quad s_{x,y,T} \in (0, \varepsilon).$$

Remark 2.3. In Figure 1 we provide some plots of $\mathcal{E}^{x,y}$ for smaller and smaller values of $|x - y|$: we see that the maximizer approaches the value $s = 0$, as stated in Theorem 2.2. Notice also that the case $s = 0$ is however a minimizer, as pointed out in Theorem 2.1. This situation clearly shows that in cases of biological interest *the most rewarding strategy (i.e. the one maximizing an efficiency functional) can result to be unsafe (because dangerously close to strategies that instead pessimize the efficiency functional)*. From the mathematical viewpoint, this phenomenon is described by an efficiency functional which *develops a global maximizer in a small neighborhood of a global minimizer*.

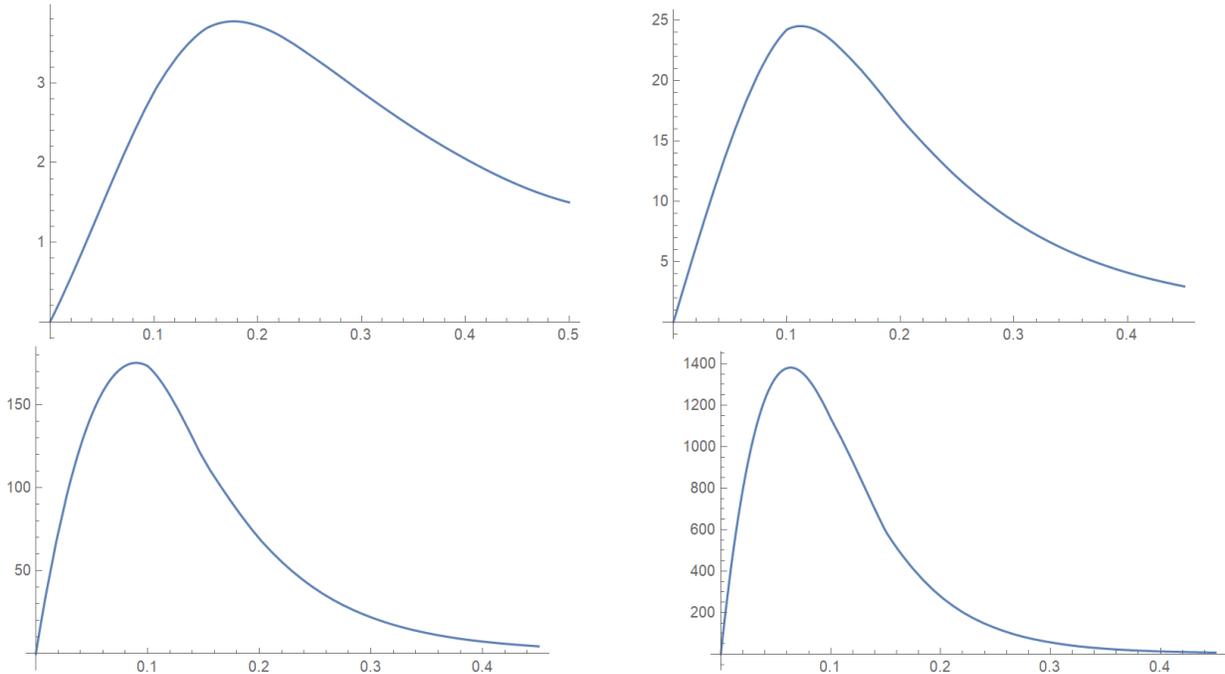


FIGURE 1. Plots of $(0, \frac{1}{2}) \ni s \mapsto \mathcal{E}^{x,y}(s, 1)$ for $(x, y) \in \{(1, 1.01), (1, 1.001), (1, 1.0001), (1, 1.00001)\}$ and $n = 1$.

A natural question in this setting is thereby whether or not this phenomenon of finding *optimizers and pessimizers of efficiency functionals arbitrary close to each other* is induced by the targets being located “too close” to the initial location of the predator. We will see below that this phenomenon is indeed persistent and remains true also for the case of sparse targets.

Indeed, while Theorem 2.2 dealt with targets located close to the forager, we now focus our attention on the case of sparse targets. From the point of view of asymptotic analysis, it is customary (see e.g. [VBH⁺99]) to model sparse prey with a target located arbitrarily far away from the forager initial position. We show that *when the target diverges towards infinity, the most rewarding strategy converges to the value $s = 0$* . More precisely:

Theorem 2.4. *Let $T \in (0, +\infty)$.*

Then, for each $\varepsilon \in (0, 1)$ there exists $R = R_{\varepsilon, n, T} \in (0, +\infty)$ such that for each $x, y \in \mathbb{R}^n$ satisfying $|x - y| > R$ it holds that

$$(2.9) \quad \sup_{s \in (0, 1)} \mathcal{E}^{x, y}(s, T) = \mathcal{E}^{x, y}(s_{x, y, T}, T) \quad \text{with} \quad s_{x, y, T} \in (0, \varepsilon).$$

Remark 2.5. Once again, we can compare Theorem 2.4 with Theorem 2.1: while the latter states that $s = 0$ is a pessimizer for the efficiency functional, from Theorem 2.4 we have that the most rewarding strategy is also arbitrarily close to $s = 0$ as the target diverges from the predator.

In this sense, the phenomenon described in Remark 2.3 for which optimizers and pessimizers of an efficiency functional can get arbitrarily close to each other, making the most rewarding searching strategy somewhat unsafe, persists also in the case of sparse targets.

Remark 2.6. From the mathematical point of view, the diffusion models related to $s = 0$ are related to the new, but rapidly growing, field of integral equations with zero order kernels, see [CW19, KM17, CLS22]. In terms of applications, these equations arise in several fields of science, including probability [ŠSV06, Beg14], relativistic Schrödinger operators [Feu21], and reconstruction of blurred images from the Hubble space telescope [Car09].

Remark 2.7. From the biological point of view, the diffusion models related to $s = 0$ have often appeared in both theoretical analyses² and experimental data.

In terms of theoretical deductions, it was already observed in [VBH⁺99, VRdL08] that the strategy $s = 0$ is optimal in one-dimensional environments when the location of targets, once visited, become depleted and are no longer targets for future searches (this case being named “destructive search”; see also [SRVdL04] for an interpolation of targets that regenerate arbitrarily quickly and those vanishing after a single visit). We point out that the setting considered here is structurally different from that in [VBH⁺99, SRVdL04, VRdL08]. Indeed, the framework here is that of a “non-destructive search” in which the previously-visited sites can be revisited infinitely often and the consumed targets get replaced instantaneously.

Moreover, in [VBH⁺99, SRVdL04, VRdL08] the analysis relied on the notion of a “direct vision”, i.e. an additional distance parameter below which the forager moves on a straight line to the nearest target site (instead, we do not distinguish between small and large scale motions, considering the diffusive pattern as produced by the full fractional Laplacian, not a modified operator³ that avoids singularities).

²As a notational remark, the literature often denotes by $\mu \in [1, 3]$ the exponent of the fat tail of a Lévy distribution. In our notation, $\mu = 1 + 2s$, hence the setting $s = 0$ corresponds to $\mu = 1$. The notation is anyway not uniform over the literature, for instance such μ corresponds to a in [RFMM⁺04], to $2\alpha + 1$ in [DGNBD17], to β in [BRFM⁺06], to $\beta + 1$ in [PBMP], to γ in [BR13], to u in [CSS22], etc. Also, the specific details of each article may be structurally different, therefore comparisons between different works have to be taken with a pinch of salt.

The situation $s = 0$ is also referred with the name of “ballistic”. See also [ZDK15] for a thorough discussion of this case and several applications to physics.

³The specific assumptions on the singularity of the diffusive operator taken into account are important, though sometimes overlooked or not explicitly stressed in the literature, see e.g. [PV21].

In addition, the optimality condition in [VBH⁺99, SRVdL04, VRdL08] had a “monotonic” behavior and no issue of reliability of the most rewarding strategy appeared in that context (compare instead with Remarks 2.3 and 2.5 here).

Furthermore, our efficiency functional considers averages in time over the time span of the search process, rather than single situations at a frozen instant of time.

Besides, our mathematical analysis here is not confined to the one-dimensional case and holds true in every dimension.

The calculation in [VBH⁺99] has also been retaken in [JPE11], where it was shown that the exponent $s = 0$ attains optimality when foragers and targets are located far apart (this is in agreement with our result in Theorem 2.4, though, once again, our setting differs from that in [JPE11] for the reasons explained above).

In terms of experimental evidence, several cases have been recorded of Lévy flights with exponent close to $s = 0$. For instance, just to name a few, black-browed albatrosses and wandering albatrosses (see [HWQ⁺12]) have exhibited a Lévy distribution with $s = 0.135$ and $s = 0.095$, respectively, male spider monkeys exhibit $s = 0.235$ (see [RFMM⁺04]), seals ranged between $s = 0.06$ and $s = 0.15$ (see [ABM04]), etc.

In some cases, the empirical exponents measured corresponded to values of s which are so small that a reanalysis with different plotting methods led to a value of s below zero (not corresponding to a properly defined probability distribution), see e.g. Figure 4c in [SRP07], relating to the value $s = -0.1$. Similarly, when dealing with fruit flies, exponents such as $s = 0.15$ and $s = -0.11$ have been obtained in Figures 3B.1 and 3B.2 of [MHSB07].

In a slightly different setting related to waiting times, it has been suggested in [WMH⁺14] that the case $s = 0$ is closely related to “ambush” strategies by predators and to a high energy content of the targets, as indicated also by the patterns shown by anglers and blonde skates, which present a Lévy exponent very close to the value $s = 0$ and largely specializing on high-energy content prey.

It is quite suggestive to relate this high-energy content of the targets and the high-risk strategy related to $s = 0$: in a sense, one can imagine that a forager may not be discouraged by the fact that the most rewarding strategy can be dangerously close to the pessimizer (as pointed out in Remarks 2.3 and 2.5) precisely when the target is also highly rewarding. The adoption of an optimal strategy close to $s = 0$ would then correspond to a *high-risk/high-gain* situation, in which the expectation of the forager is that the hunt’s gain will be so high to compensate for the risk of failure.

In human cognition, power law behaviors with exponents as low as $s = 0.185$ have been detected in [BR13]. See also [CSS22] for numerical experiments related to the Lévy exponent $s = 0$.

Remark 2.8. Regarding [DGNBD17], it is also interesting to relate the case $s = 0$ to an “ambush” strategy of the forager. Namely, for small values of s , on the one hand, the fat tail of the Lévy distribution “tends to dislocate mass towards infinity”; on the other hand, the diminished regularity effect of the diffusive equation also “leaves more mass near the origin” (see Figure 2 for a sketch of this phenomenon). That is, roughly speaking, a strategy close to $s = 0$ allows the predator to switch rather abruptly from the initial location to a remote region (which, in a sense, provides a good mathematical description of an ambush).

Now we consider the set-dependent efficiency functional introduced in (2.5). As done in Propositions B.1 and B.2, the first step towards the understanding of this situation deals with detecting finiteness and continuity results. To this end, in Proposition B.3 in Appendix B we prove that if $\Omega_1, \Omega_2 \subset \mathbb{R}^n$ are bounded and measurable, then $\tilde{\mathcal{E}}^{\Omega_1, \Omega_2}(s, T) \in (0, +\infty)$ for all $s \in (0, 1]$ and $\tilde{\mathcal{E}}^{\Omega_1, \Omega_2}(\cdot, T) \in C((0, 1])$.

We stress that in Proposition B.3 the sets Ω_1 and Ω_2 are not necessarily disjoint: this detail was perhaps not obvious to start with since, in view of Proposition B.1, the corresponding pointwise

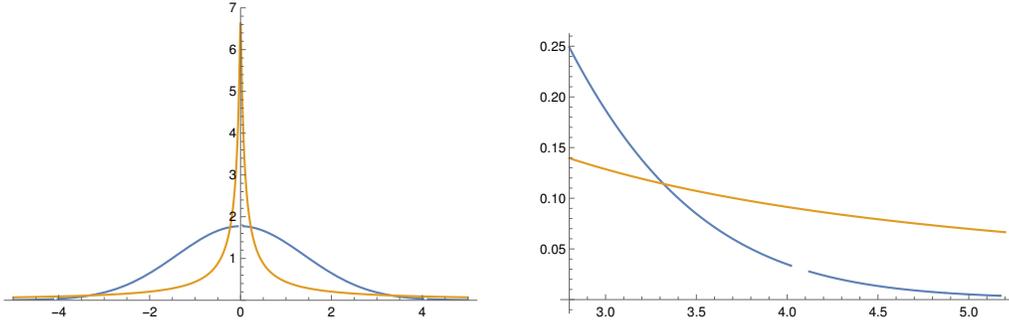


FIGURE 2. Plots of the fractional heat kernel when $s = 0.99$ (in blue) and when $s = 0.2$ (in yellow). Note the displacement of mass both near the origin and at infinity for small values of s (“ambush” foraging mode).

efficiency functional $\mathcal{E}^{x,y}$ fails to be finite when $x = y$ (except in the special case in which $n = 1$ and $s \in (\frac{1}{2}, 1]$). The finiteness of the set-dependent efficiency functional is a consequence of the measure theoretic properties of the Dirac delta distribution (whose singularity produces a finite outcome after integration).

In the case where Ω_1 and Ω_2 are disjoint and regular enough, we observe that a result analogous to the one given in Theorem 2.1 holds true. Namely, we can show that the infimum for $\tilde{\mathcal{E}}^{\Omega_1, \Omega_2}$ is attained at $s = 0$.

Theorem 2.9. *Let $\Omega_1, \Omega_2 \subset \mathbb{R}^n$ be bounded, smooth and disjoint.*

Then, for each $T \in (0, +\infty)$ it holds that

$$(2.10) \quad \inf_{s \in (0,1)} \tilde{\mathcal{E}}^{\Omega_1, \Omega_2}(s, T) = \lim_{s \searrow 0} \tilde{\mathcal{E}}^{\Omega_1, \Omega_2}(s, T) = 0.$$

From Theorem 2.9 and Proposition B.3 we deduce that we can extend by continuity $\tilde{\mathcal{E}}^{\Omega_1, \Omega_2}(\cdot, T)$ in $[0, 1]$. Hence, we will adopt the notation

$$\tilde{\mathcal{E}}^{\Omega_1, \Omega_2}(0, T) := \lim_{s \searrow 0} \tilde{\mathcal{E}}^{\Omega_1, \Omega_2}(s, T),$$

for all $T \in (0, +\infty)$ and $\Omega_1, \Omega_2 \subset \mathbb{R}^n$ bounded, smooth and disjoint.

Now we address the results related to the optimization of the set-dependent efficiency functional. Specifically, we establish results in line with Theorems 2.2 and 2.4 by replacing the pointwise functional in (2.4) with the set-dependent functional defined in (2.5). More precisely, we show that *if $\Omega_1, \Omega_2 \subset \mathbb{R}^n$ are close enough, or sufficiently far away from each other, then the maximizer of $\tilde{\mathcal{E}}^{\Omega_1, \Omega_2}$ is in some small neighborhood of $s = 0$.* The formal statements are the following ones:

Theorem 2.10. *Let $T \in (0, +\infty)$.*

Then, for each $\varepsilon \in (0, 1)$ there exists $r = r_{\varepsilon, n, T} \in (0, +\infty)$ such that if $\Omega_1, \Omega_2 \subset \mathbb{R}^n$ are smooth, disjoint and such that $\Omega_1, \Omega_2 \subset B_r(x_0)$ for some $x_0 \in \mathbb{R}^n$, it holds that

$$(2.11) \quad \sup_{s \in (0,1)} \tilde{\mathcal{E}}^{\Omega_1, \Omega_2}(s, T) = \tilde{\mathcal{E}}^{\Omega_1, \Omega_2}(s_{\Omega_1, \Omega_2, T}, T) \quad \text{with } s_{\Omega_1, \Omega_2, T} \in (0, \varepsilon).$$

Theorem 2.11. *Let $T \in (0, +\infty)$.*

Then, for each $\varepsilon \in (0, 1)$ there exists $R = R_{\varepsilon, n, T} \in (0, +\infty)$ such that if $\Omega_1, \Omega_2 \subset \mathbb{R}^n$ satisfy

$$\inf_{\substack{x \in \Omega_1 \\ y \in \Omega_2}} |x - y| \geq R,$$

then it holds that

$$(2.12) \quad \sup_{s \in (0,1)} \tilde{\mathcal{E}}^{\Omega_1, \Omega_2}(s, T) = \tilde{\mathcal{E}}^{\Omega_1, \Omega_2}(s_{\Omega_1, \Omega_2, T}, T) \quad \text{with } s_{\Omega_1, \Omega_2, T} \in (0, \varepsilon).$$

Remark 2.12. The situation described in Remarks 2.3 and 2.5 carries over for the case of set-dependent efficiency functionals: namely, from Theorems 2.9, 2.10 and 2.11 we can evince that the most rewarding strategy for $\tilde{\mathcal{E}}^{\Omega_1, \Omega_2}$ is unreliable, developing pessimizers and optimizers arbitrarily close to each other, both when the average locations of predators and targets are close and when they are far away.

3. PROOFS OF THE MAIN RESULTS

This section develops the necessary tools to prove the main results and completes the proofs of the theorems.

Some of the intermediate results obtained may have independent interest. For instance, we provide two-sided estimates for the efficiency functional $\Psi^{x,y}(s, T)$ in (2.3) under different environmental scenarios. In particular, we establish a polynomial growth for $\Psi^{x,y}$ when the forager starting position and the target location are close enough. This is the content of Theorem 3.4 below.

Moreover, in Theorems 3.2 and 3.3 we prove some estimates for $\Psi^{x,y}$ when the target is far away from the forager starting point. These results on the behavior of $\Psi^{x,y}$ under these two different configurations will be pivotal in order to show the main results presented in Section 2.2.

3.1. Analysis of the efficiency functionals. In this section we study the efficiency functionals given in (2.4) and (2.5). We will start the section by proving an identity regarding the fractional heat kernel $G^s(t, x, y)$, that is the unique solution to (2.1). We show that, for each $m \in \mathbb{N}$, $G^s(t, x, y)$ can be written as the linear superposition over the time variable of the classical kernel of the m -order heat equation and the density $\mu_t^{s/m}$ of the s/m -stable subordinator (see [DGV22a, Definition 2.4] and the references in [DGV22a] for the basic properties of such a subordinator).

This result will be useful later in order to prove an upper bound for the foraging success functional $\Psi^{x,y}$. The result goes as follows:

Proposition 3.1. *Let $s \in (0, 1)$ and $m \in \mathbb{N}$. Let $G^m(t, x, y)$ be the solution to the higher order parabolic equation*

$$\begin{cases} \partial_t u(t, x) = -(-\Delta)^m u(t, x) & \text{in } (0, +\infty) \times \mathbb{R}^n, \\ u(0, x) = \delta_y(x). \end{cases}$$

Then, it holds that

$$(3.1) \quad G^s(t, x, y) = \int_0^{+\infty} G^m(l, x, y) \mu_t^{\frac{s}{m}}(l) dl \quad \text{for all } (t, x, y) \in (0, +\infty) \times \mathbb{R}^n \times \mathbb{R}^n.$$

Proof. Up to a translation we can assume that $y = 0$. Moreover, we denote by $P^s(t, x)$ the function defined by

$$(3.2) \quad P^s(t, x) := \int_0^{+\infty} G^m(l, x) \mu_t^{\frac{s}{m}}(l) dl \quad \text{with } (t, x) \in (0, +\infty) \times \mathbb{R}^n.$$

For $m > 1$, $G^m(t, x)$ is not positive everywhere, see for instance [GP02] and references therein. Nevertheless, if $m > 1$, we have the existence of two positive constants d, D , depending only on n and m , such that

$$(3.3) \quad |G^m(t, x)| \leq \frac{D}{t^{\frac{n}{2m}}} \exp\left(-\frac{d|x|^\alpha}{t^{\frac{\alpha}{2m}}}\right) \quad \text{for all } (t, x) \in (0, +\infty) \times \mathbb{R}^n,$$

where we defined

$$\alpha := \frac{2m}{2m-1},$$

see Proposition 2.1 and equation (2.8) in [GP02]. In particular, from the inequality in (3.3), we obtain that

$$(3.4) \quad \left| G^m(\cdot, \cdot) \mu_t^{\frac{s}{m}}(\cdot) \right| \in L^1((0, +\infty) \times \mathbb{R}^n).$$

Indeed, using (3.3), we can compute

$$\begin{aligned} \int_0^{+\infty} \int_{\mathbb{R}^n} |G^m(l, x) \mu_t^{\frac{s}{m}}(l)| \, dx \, dl &\leq \int_0^{+\infty} \frac{D\mu_t^{\frac{s}{m}}(l)}{l^{\frac{n}{2m}}} \int_{\mathbb{R}^n} \exp\left(-\frac{d|x|^\alpha}{l^{\frac{\alpha}{2m}}}\right) \, dx \, dl \\ &= \int_0^{+\infty} \frac{D\mu_t^{\frac{s}{m}}(l)}{l^{\frac{n}{2m}}} \int_0^{+\infty} \int_{\partial B_\rho} \exp\left(-\frac{d\rho^\alpha}{l^{\frac{\alpha}{2m}}}\right) \, dH^{n-1} \, d\rho \, dl \\ &= \int_0^{+\infty} \frac{D\mu_t^{\frac{s}{m}}(l)}{l^{\frac{n}{2m}}} \int_0^{+\infty} \omega_{n-1} \rho^{n-1} \exp\left(-\frac{d\rho^\alpha}{l^{\frac{\alpha}{2m}}}\right) \, d\rho \, dl \\ &= \int_0^{+\infty} \frac{\omega_{n-1} D\mu_t^{\frac{s}{m}}(l)}{\alpha d^{\frac{n}{\alpha}}} \int_0^{+\infty} \theta^{\frac{n}{\alpha}-1} e^{-\theta} \, d\theta \, dl \\ &= \int_0^{+\infty} \frac{\omega_{n-1} D\mu_t^{\frac{s}{m}}(l)}{\alpha d^{\frac{n}{\alpha}}} \Gamma\left(\frac{n}{\alpha}\right) \, dl \\ &= \frac{\omega_{n-1} D}{\alpha d^{\frac{n}{\alpha}}} \Gamma\left(\frac{n}{\alpha}\right) \end{aligned}$$

where $\omega_{n-1} := H^{n-1}(\partial B_1)$, and between the third and fourth line we applied the change of variable $\rho = \frac{\theta^{\frac{1}{\alpha}} l^{\frac{1}{2m}}}{d^{\frac{1}{\alpha}}}$. This proves (3.4).

Therefore, taking the Fourier transform in the space variable of both sides of (3.2), we obtain for each $\xi \in \mathbb{R}^n$ that

$$\begin{aligned} \mathcal{F}(P^s(\cdot, t))(\xi) &= \mathcal{F}\left(\int_0^{+\infty} G^m(l, x) \mu_t^{\frac{s}{m}}(l) \, dl\right)(\xi) \\ &= \int_0^{+\infty} \mathcal{F}(G^m(l, \cdot))(\xi) \mu_t^{\frac{s}{m}}(l) \, dl \\ &= \int_0^{+\infty} e^{-l(2\pi|\xi|)^{2m}} \mu_t^{\frac{s}{m}}(l) \, dl \\ &= e^{-(2\pi|\xi|)^{2s}t} \end{aligned}$$

where the second identity is due to equation (3.4) and Fubini's Theorem.

As a consequence, $\mathcal{F}(P^s(\cdot, t))(\xi)$ is the unique solution to the problem

$$\begin{cases} \partial_t u(t, \xi) = -|2\pi\xi|^{2s} u(t, \xi) & \text{in } (0, +\infty) \times \mathbb{R}^n, \\ u(\xi, 0) = 1. \end{cases}$$

Thus, taking the inverse of the Fourier's transform of the last relation we obtain that $P^s(t, x)$ is the unique solution to (2.1) with $y = 0$, see e.g. [AV19], and then it must coincide with $G^s(t, x, 0)$, as desired. \square

It is well-known that the fractional heat kernel has the polynomial growth

$$(3.5) \quad G^s(1, x, y) \leq \frac{C_{n,s}}{1 + |x - y|^{n+2s}} \quad \text{for all } x, y \in \mathbb{R}^n,$$

where $C_{n,s}$ is a constant depending on n and s , see for instance [AV19] and the references therein. By using the scaling property of G^s , see for instance [AV19], and integrating over t in the time

interval $(0, T)$, one then obtains from (3.5) the inequality

$$\Psi^{x,y}(s, T) \leq \frac{C_{n,s,T}}{|x-y|^{n+2s}},$$

where $C_{n,s,T}$ depends on n , s and T .

In the following theorem, using the identity proved in Proposition 3.1, we show that we can choose $C_{n,s,T}$ and some constant $\tilde{C}_{n,T}$, such that

$$C_{n,s,T} \leq s\tilde{C}_{n,T},$$

for each $s \in (0, 1)$. This result will be employed in Section 3.2 in the proof of Theorem 2.4.

Theorem 3.2. *Let $s \in (0, 1)$ and $(x, y, T) \in \mathbb{R}^n \times \mathbb{R}^n \times (0, +\infty)$ satisfying $x \neq y$.*

Then, there exists some constant $\tilde{C}_{n,T} \in (0, +\infty)$ such that

$$(3.6) \quad \Psi^{x,y}(s, T) \leq \frac{s\tilde{C}_{n,T}}{|x-y|^{n+2s}}.$$

Proof. Using Proposition 3.1 with $m = 2$, we can write for each $s \in (0, 1)$ that

$$G^s(t, x, y) = \int_0^{+\infty} G^2(l, x, y) \mu_t^{\frac{s}{2}}(l) dl \quad \text{for all } (t, x, y) \in (0, +\infty) \times \mathbb{R}^n \times \mathbb{R}^n,$$

where G^2 is the kernel of the biharmonic heat equation. Thus, using the inequality in (3.3) with $m = 2$, we obtain that

$$(3.7) \quad G^2(t, x, y) \leq \frac{D_2}{t^{\frac{n}{4}}} \exp\left(-\frac{d_2|x-y|^{\frac{4}{3}}}{t^{\frac{1}{3}}}\right) \quad \text{for all } (t, x, y) \in (0, +\infty) \times \mathbb{R}^n \times \mathbb{R}^n,$$

where d_2 and D_2 depend only on n .

We now set $s_0 := s/2$ and we recall that

$$\mu_t^{s_0}(l) \leq \frac{s_0 t \Gamma(1 + s_0)}{l^{1+s_0}} \quad \text{for all } l \in (0, +\infty),$$

thanks to Theorem 2.3 in [DGV22b].

Accordingly, using this and (3.7) we deduce that

$$(3.8) \quad \begin{aligned} \frac{G^s(t, x, y)}{s} &= \frac{1}{2s_0} \int_0^{+\infty} G^2(l, x, y) \mu_t^{s_0}(l) dl \\ &\leq \frac{1}{2s_0} \int_0^{+\infty} \frac{D_2}{l^{\frac{n}{4}}} \exp\left(-\frac{d_2|x-y|^{\frac{4}{3}}}{l^{\frac{1}{3}}}\right) \mu_t^{s_0}(l) dl \\ &\leq \frac{t\Gamma(1+s_0)}{2} \int_0^{+\infty} \frac{D_2}{l^{\frac{n}{4}+1+s_0}} \exp\left(-\frac{d_2|x-y|^{\frac{4}{3}}}{l^{\frac{1}{3}}}\right) dl \\ &= \frac{3D_2 t \Gamma(1+s_0)}{2d_2^{\frac{3n}{4}+3s_0}} |x-y|^{-n-4s_0} \int_0^{+\infty} a^{\frac{3n}{4}+3s_0-1} e^{-a} da \\ &= \frac{3D_2 t \Gamma(1+s_0)}{2d_2^{\frac{3n}{4}+3s_0}} \frac{\Gamma\left(\frac{3n}{4}+3s_0\right)}{|x-y|^{n+4s_0}} \\ &\leq \frac{2\tilde{C}_n t}{|x-y|^{n+2s}}, \end{aligned}$$

where we applied the change of variable $a = \frac{d_2|x-y|^{\frac{4}{3}}}{l^{\frac{1}{3}}}$ and denoted

$$(3.9) \quad \tilde{C}_n := \frac{1}{4} \sup_{s_0 \in (0, \frac{1}{2})} \frac{3D_2\Gamma(1+s_0)\Gamma\left(\frac{3n}{4} + 3s_0\right)}{2d_2^{\frac{3n}{4}+3s_0}}.$$

Integrating both sides in (3.8) with respect to t in the interval $(0, T)$ and defining

$$(3.10) \quad \tilde{C}_{n,T} := \tilde{C}_n T^2,$$

we obtain the desired result. \square

With the following theorem we establish a lower bound for $\Psi^{x,y}(s, T)$ when $s \in (0, \frac{1}{2})$. This result shows that the functional can be bounded by below by a function with the same polynomial growth at infinity as the one in the right-hand side of (3.6).

Moreover, this result, together with Theorem 3.2, will be necessary in order to prove Theorem 2.4 on the optimality of a highly nonlocal search strategy when the forager is far enough from the prey.

Theorem 3.3. *Let $s \in (0, \frac{1}{2})$ and $T \in (0, +\infty)$.*

Then, for each $\delta \in (0, +\infty)$ there exists a constant $C_{\delta,T} \in (0, +\infty)$, depending only on δ and T , such that for all $x, y \in \mathbb{R}^n$ satisfying $|x - y| > \delta$ it holds that

$$(3.11) \quad \Psi^{x,y}(s, T) \geq \frac{sC_{\delta,T}}{|x - y|^{n+2s}}.$$

Proof. We recall that, if $s \in (0, \frac{1}{2})$ and $t \in (0, 1)$,

$$\mu_t^s(l) \geq \frac{stC_1}{\pi l^{1+s}} \quad \text{for all } l \in \left(t^{\frac{1}{s}}, +\infty\right),$$

for some $C_1 \in (0, +\infty)$, independent of s and l , see Theorem 2.3 in [DGV22b].

Using this and (3.1), for each $s \in (0, \frac{1}{2})$ and $t \in (0, 1)$ we have that

$$(3.12) \quad \begin{aligned} G^s(t, x, y) &= \int_0^{+\infty} \frac{1}{(4\pi l)^{\frac{n}{2}}} \exp\left(-\frac{|x-y|^2}{4l}\right) \mu_t^s(l) dl \\ &\geq \int_{t^{\frac{1}{s}}}^{+\infty} \frac{1}{(4\pi l)^{\frac{n}{2}}} \exp\left(-\frac{|x-y|^2}{4l}\right) \mu_t^s(l) dl \\ &\geq \int_{t^{\frac{1}{s}}}^{+\infty} \frac{1}{(4\pi l)^{\frac{n}{2}}} \exp\left(-\frac{|x-y|^2}{4l}\right) \frac{stC_1}{\pi l^{1+s}} dl \\ &= \frac{stC_1}{4^{\frac{n}{2}}\pi^{\frac{n}{2}+1}} \int_{t^{\frac{1}{s}}}^{+\infty} \frac{1}{l^{\frac{n}{2}+s+1}} \exp\left(-\frac{|x-y|^2}{4l}\right) dl \\ &= \frac{stC_1}{4^{1-s}\pi^{\frac{n}{2}+1}} \int_0^{\frac{|x-y|^2}{4t^{\frac{1}{s}}}} \frac{a^{\frac{n}{2}+s-1}}{|x-y|^{n+2s}} e^{-a} da \\ &\geq \frac{stC_1}{4^{1-s}\pi^{\frac{n}{2}+1}} \int_0^{\frac{\delta^2}{4}} \frac{a^{\frac{n}{2}+s-1}}{|x-y|^{n+2s}} e^{-a} da \\ &\geq \frac{2C_\delta st}{|x-y|^{n+2s}}, \end{aligned}$$

where we defined

$$(3.13) \quad C_\delta := \frac{1}{2} \inf_{s \in (0,1)} \frac{C_1}{4^{1-s}\pi^{\frac{n}{2}+1}} \int_0^{\frac{\delta^2}{4}} a^{\frac{n}{2}+s-1} e^{-a} da.$$

If $T \in (0, 1]$, integrating both sides of (3.12) with respect to t in the interval $(0, T)$, we obtain (3.11) with

$$(3.14) \quad C_{\delta, T} := C_{\delta} T^2$$

If instead $T \in (1, +\infty)$, it is enough to observe that

$$\Psi^{x, y}(s, T) \geq \Psi^{x, y}(s, 1) \geq \frac{s C_{\delta}}{|x - y|^{n+2s}}. \quad \square$$

Now, we focus our attention to the main tools that we will use in order to prove Theorem 2.2. For this, for every $n \in \mathbb{N}$ and $s \in (0, 1)$, we define

$$(3.15) \quad A_{n, s} := \left(0, 1 + \frac{n}{2} - s\right) \cap \left[\frac{n}{2} - s, 1 + \frac{n}{2} - s\right)$$

and we establish the following asymptotic behavior:

Theorem 3.4. *Let $(z, T) \in \mathbb{R}^n \times (0, +\infty)$.*

Then, for all $n \leq 2$, $s \in (0, 1)$ and $\mu \in A_{n, s}$, where $A_{n, s}$ is given in (3.15), there exists a constant $C_{\mu, T, B_1(z)} \in (0, +\infty)$ such that

$$(3.16) \quad \Psi^{x, y}(s, T) \leq \frac{C_{\mu, T, B_1(z)}}{|x - y|^{2\mu}},$$

for all $x, y \in B_1(z)$ with $x \neq y$.

Also, if $n \geq 3$ it holds that

$$(3.17) \quad \Psi^{x, y}(s, T) \leq \frac{C_n}{\Gamma(s)} \frac{1}{|x - y|^{n-2s}},$$

for all $x, y \in \mathbb{R}^n$ satisfying $x \neq y$, where C_n is defined as

$$(3.18) \quad C_n := \sup_{s \in (0, 1)} \frac{4^{-s}}{\pi^{\frac{n}{2}} \Gamma(s)} \Gamma\left(\frac{n}{2} - s\right).$$

Proof. Up to a translation we can assume that $z = 0$ and we consider the classical Dirichlet heat kernel $p_D^{B_2}(t, x, y)$ in the ball B_2 . Thanks to Lemma 2.1 in [Zha02], we have the existence of two constants c_1, c_2 and a time span $T_{B_1, B_2} \in (0, +\infty)$, depending on B_1 and B_2 , such that

$$p_D^{B_2}(t, x, y) \geq \frac{c_1}{t^{\frac{n}{2}}} \exp\left(-\frac{c_2|x - y|^2}{t}\right) \quad \text{for all } (t, x, y) \in (0, T_{B_1, B_2}] \times B_1 \times B_1.$$

Now, we fix two constants $c_0, \eta_0 \in (0, +\infty)$ such that

$$c_0 c_1 \geq \frac{1}{(4\pi)^{\frac{n}{2}}} \quad \text{and} \quad \eta_0^2 c_2 \leq \frac{1}{4}$$

and we obtain that, for each $\eta \in (0, \eta_0)$ and $(t, x, y) \in (0, T_{B_1, B_2}] \times B_1 \times B_1$,

$$(3.19) \quad \begin{aligned} c_0 p_D^{B_2}(t, \eta x, \eta y) - G(t, x, y) &\geq \frac{c_0 c_1}{t^{\frac{n}{2}}} \exp\left(-\frac{c_2|\eta x - \eta y|^2}{t}\right) - \frac{1}{(4\pi t)^{\frac{n}{2}}} \exp\left(-\frac{|x - y|^2}{4t}\right) \\ &= \frac{\exp\left(-\frac{|x - y|^2}{4t}\right)}{(4\pi t)^{\frac{n}{2}}} \left(c_0 c_1 (4\pi)^{\frac{n}{2}} \exp\left(\frac{\left(\frac{1}{4} - c_2 \eta^2\right) |x - y|^2}{t}\right) - 1 \right) \\ &\geq 0. \end{aligned}$$

Let us now prove (3.16). To do so, assume that $n \leq 2$, $s \in (0, 1)$ and $\mu \in A_{n, s}$, where the set $A_{n, s}$ is given in (3.15).

Then, if $x, y \in B_1$, thanks to (3.19) here, (2.54) in [DGV22b] and Proposition 3.1 with $m = 1$, for all $\eta \in (0, \eta_0)$ we have that

$$\begin{aligned}
(3.20) \quad \Psi^{x,y}(s, T) &= \int_0^T \int_0^{+\infty} G(l, x, y) \mu_t^s(l) \, dl \, dt \\
&= \int_0^T \int_0^{T_{B_1, B_2}} G(l, x, y) \mu_t^s(l) \, dl \, dt + \int_0^T \int_{T_{B_1, B_2}}^{+\infty} G(l, x, y) \mu_t^s(l) \, dl \, dt \\
&\leq \int_0^T \int_0^{T_{B_1, B_2}} c_0 p_D^{B_2}(l, \eta x, \eta y) \mu_t^s(l) \, dl \, dt + \int_0^T \int_{T_{B_1, B_2}}^{+\infty} G(l, x, y) \mu_t^s(l) \, dl \, dt \\
&\leq C_{\mu, B_1, B_2} \frac{\eta^{-2\mu}}{|x - y|^{2\mu}} + C_{B_1, B_2} T,
\end{aligned}$$

where we defined

$$(3.21) \quad C_{B_1, B_2} := \max_{l \in (T_{B_1, B_2}, +\infty)} G(l, x, x) \quad \text{and} \quad C_{\mu, B_1, B_2} := C_{D, \mu} c_0,$$

being $C_{D, \mu}$ given in formula (2.55) in [DGV22b].

Accordingly, if we take $\eta_1 \in (0, \eta_0)$ such that

$$\eta_1 \leq \left(\frac{C_{\mu, B_1, B_2}}{C_{B_1, B_2} T} \right)^{-2\mu},$$

depending on B_1, B_2, μ and T , then we obtain from (3.20) that, for all $s \in (0, 1)$ and $x, y \in B_1$,

$$\Psi^{x,y}(s, T) \leq \frac{C_{\mu, T, B_1}}{|x - y|^{2\mu}}.$$

The constant C_{μ, T, B_1} is defined as

$$C_{\mu, T, B_1} := 2C_{\mu, B_1, B_2} \eta_1^{-2\mu}.$$

This concludes the proof of (3.16).

Let us now show (3.17). To do so, if $x, y \in \mathbb{R}^n$ satisfy $x \neq y$, we consider the following identity for the Green function of the fractional Laplacian

$$\int_0^{+\infty} G^s(t, x, y) \, dt = \frac{1}{\Gamma(s)} \int_0^{+\infty} G(t, x, y) t^{s-1} \, dt,$$

which for the convenience of the reader is proved in Proposition A.1.

We also point out that, using the change of variable $a = \frac{|x-y|^2}{4t}$,

$$\begin{aligned}
(3.22) \quad \int_0^{+\infty} G(t, x, y) t^{s-1} \, dt &= \int_0^{+\infty} \frac{t^{s-1}}{(4\pi t)^{\frac{n}{2}}} \exp\left(-\frac{|x-y|^2}{4t}\right) \, dt \\
&= \frac{4^{-s}}{\pi^{\frac{n}{2}} |x-y|^{n-2s}} \int_0^{+\infty} a^{\frac{n}{2}-1-s} e^{-a} \, da \\
&= \frac{4^{-s}}{\pi^{\frac{n}{2}} |x-y|^{n-2s}} \Gamma\left(\frac{n}{2} - s\right).
\end{aligned}$$

Accordingly, we obtain that

$$\begin{aligned}
\Psi^{x,y}(s, T) &:= \int_0^T G^s(t, x, y) dt \\
&\leq \int_0^{+\infty} G^s(t, x, y) dt \\
&= \frac{1}{\Gamma(s)} \int_0^{+\infty} G(t, x, y) t^{s-1} dt \\
&= \frac{1}{\Gamma(s)} \int_0^{+\infty} \frac{t^{s-1}}{(4\pi t)^{\frac{n}{2}}} \exp\left(-\frac{|x-y|^2}{4t}\right) dt \\
&\leq \frac{C_n}{\Gamma(s)} \frac{1}{|x-y|^{n-2s}},
\end{aligned}$$

where C_n is given in (3.18). □

If $\Omega \subset \mathbb{R}^n$ is some smooth, bounded and connected domain, then thanks to the Weak Maximum Principle, one has that

$$(3.23) \quad p_D^\Omega(t, x, y) \leq \frac{1}{(4\pi t)^{\frac{n}{2}}} \exp\left(-\frac{|x-y|^2}{4t}\right) \quad \text{for all } (t, x, y) \in (0, +\infty) \times \Omega \times \Omega.$$

In what follows we will denote by r_D^s the Dirichlet spectral fractional heat kernel in Ω , see for instance [DGV22b]. Also, adopting the same notation of [DGV22b], the success functional associated to r_D^s is denoted by Φ_D , which is defined as

$$\Phi_D^{x,y}(s, T) = \int_0^T r_D^s(t, x, y) dt,$$

see equation (1.5) in [DGV22b].

As a direct consequence of the identity in (3.1) and the estimate in (3.23) we have the following corollary. From this result we can infer a lower bound for $\Psi^{x,y}$ analogous to the one in equation (2.26) of Lemma 2.7 for $\Phi_D^{x,y}$.

Such a lower bound will be employed in order to prove Theorem 2.2.

Corollary 3.5. *Let $s \in (0, 1)$ and $\Omega \subset \mathbb{R}^n$ be a bounded, smooth and connected domain.*

Then, it holds that

$$(3.24) \quad r_D^s(t, x, y) \leq G^s(t, x, y) \quad \text{for all } (t, x, y) \in (0, +\infty) \times \Omega \times \Omega.$$

3.2. Completion of the proof of the main results. This section is devoted to the proofs of all the main results contained in Section 2.

Proof of Theorem 2.1. Let $x, y \in \mathbb{R}^n$ such that $x \neq y$ and $T \in (0, +\infty)$. Then, thanks to equation (3.6) in Theorem 3.2, we have that

$$(3.25) \quad \lim_{s \searrow 0} \mathcal{E}^{x,y}(s, T) \leq \lim_{s \searrow 0} \frac{s \tilde{C}_{n,T}}{s |x-y|^{n+2s}} = 0,$$

where $\tilde{C}_{n,T}$ is given in (3.10). This limit and equation (B.2) lead to (2.6). □

Proof of Theorem 2.2. Let $T \in (0, +\infty)$. From Theorem 2.1 we have that, that for all $x, y \in \mathbb{R}^n$ such that $x \neq y$,

$$(3.26) \quad \sup_{s \in (0,1)} \mathcal{E}^{x,y}(s, T) = \mathcal{E}^{x,y}(s_{x,y,T}, T) \quad \text{with } s_{x,y,T} \in (0, 1].$$

We will first prove Theorem 2.2 for $n \leq 2$. Let $z \in \mathbb{R}^n$ and let r_D^s be the Dirichlet spectral fractional heat kernel in $B_1(z)$. Then, thanks to Lemma 2.7 in [DGV22b], applied to the functional Φ_D in $B_1(z)$,

and thanks to Corollary 3.5 here, we have that for each $s_0 \in (0, \frac{1}{2})$ there exists some $\widehat{\delta} = \widehat{\delta}_{s_0, z, T, B_1(z)}$, given by Lemma 2.7 in [DGV22b] (defined explicitly in formula (2.27) of [DGV22b]), such that

$$(3.27) \quad \Psi^{x,y}(s_0, T) \geq \frac{C_{s_0, z, B_1(z)}}{|x - y|^{n-2s_0}},$$

for all $x, y \in B_{\widehat{\delta}}(z)$ with $x \neq y$, where $C_{s_0, z, B_1(z)}$ is some positive constant, given by formula (2.34) of [DGV22b].

We also recall that in Theorem 3.4 we showed that for all $s_1 \in (0, 1)$ and $\mu \in A_{n, s_1}$, where A_{n, s_1} is the set provided in equation (3.15), there exists some constant C_{μ, T, B_1} such that

$$(3.28) \quad \Psi^{x,y}(s_1, T) \leq \frac{C_{\mu, T, B_1(z)}}{|x - y|^{2\mu}},$$

for all $x, y \in B_1(z)$ such that $x \neq y$. Therefore, using (3.27) and (3.28), for all $s_0 \in (0, \frac{1}{2})$, $s_1 \in (s_0, 1)$ and $\mu \in A_{n, s_1}$, one obtains that

$$(3.29) \quad \frac{\mathcal{E}^{x,y}(s_0, T)}{\mathcal{E}^{x,y}(s_1, T)} = \frac{\Psi^{x,y}(s_0, T)}{\Psi^{x,y}(s_1, T)} \geq \frac{C_{\mu, s_0, z, T, B_1(z)}}{|x - y|^{n-2s_0-2\mu}}$$

for all $x, y \in B_{\widehat{\delta}}(z)$ such that $x \neq y$, where we denoted

$$C_{\mu, s_0, z, T, B_1(z)} := \frac{C_{s_0, z, B_1(z)}}{C_{\mu, T, B_1(z)}}.$$

Therefore, for each $\varepsilon \in (0, 1)$, by choosing $s_0 := \frac{\varepsilon}{4}$, $s_1 \in (\varepsilon, 1)$ and $\mu := (n - \varepsilon)/2$ in (3.29), and using also (3.26), we obtain that there exists some $r = r_{\varepsilon, n, T}$ such that, for all $x, y \in \mathbb{R}^n$ satisfying $|x - y| \in (0, r)$,

$$\sup_{s \in (0, 1)} \mathcal{E}^{x,y}(s, T) = \mathcal{E}^{x,y}(s_{x,y,T}, T) \quad \text{with } s_{x,y,T} \in (0, \varepsilon).$$

This concludes the proof of (2.8) when $n \leq 2$.

In order to complete the proof of Theorem 2.2, it is only left to show (2.8) when $n \geq 3$. In this case, we just have to replace the inequality in (3.28) with the one in (3.17). By doing so, equation (3.29) becomes

$$(3.30) \quad \frac{\mathcal{E}^{x,y}(s_0, T)}{\mathcal{E}^{x,y}(s_1, T)} \geq \frac{\widetilde{C}_{s_0, z, T, B_1(z)}}{|x - y|^{2(s_1 - s_0)}},$$

for all $s_0 \in (0, \frac{1}{2})$, $s_1 \in (s_0, 1)$ and $x, y \in B_{\widehat{\delta}}(z)$ with $x \neq y$, where we defined the constant

$$\widetilde{C}_{s_0, z, T, B_1(z)} := \inf_{s_1 \in (s_0, 1)} \Gamma(s_1) \frac{C_{s_0, z, B_1(z)}}{C_n}.$$

The constant C_n is given in (3.18). Therefore, if for each $\varepsilon \in (0, 1)$ we choose $s_0 := \frac{\varepsilon}{2}$ and $s_1 \in (\varepsilon, 1)$ in (3.30), we can easily show (2.8). \square

Proof of Theorem 2.4. Let $T \in (0, +\infty)$. Thanks to equations (3.6) and (3.11) we have that for all $\delta > 0$ and $x, y \in \mathbb{R}^n$ such that $|x - y| > \delta$, and for all $s_0 \in (0, \frac{1}{2})$ and $s_1 \in (s_0, 1)$,

$$(3.31) \quad \frac{\mathcal{E}^{x,y}(s_0, T)}{\mathcal{E}^{x,y}(s_1, T)} = \frac{\Psi^{x,y}(s_0, T)}{\Psi^{x,y}(s_1, T)} \geq s_0 C_{\delta, n, T} |x - y|^{2(s_1 - s_0)},$$

where we set

$$C_{\delta, n, T} := \frac{C_{\delta, T}}{\widetilde{C}_{n, T}}.$$

The constant $C_{\delta, T}$ is given in (3.13) and (3.14) (for $T \in (1, +\infty)$ and $T \in (0, 1]$ respectively), while $\widetilde{C}_{n, T}$ is provided in (3.10).

As a consequence, by choosing $s_0 := \frac{\varepsilon}{2}$ and $s_1 \in (\varepsilon, 1)$ in (3.31), and making use of (3.26), we obtain the existence of some $R = R_{n,\varepsilon,T}$ such that, for all $x, y \in \mathbb{R}^n$ satisfying $|x - y| > R$,

$$\sup_{s \in (0,1)} \mathcal{E}^{x,y}(s, T) = \mathcal{E}^{x,y}(s_{x,y,T}, T) \quad \text{with } s_{x,y,T} \in (0, \varepsilon). \quad \square$$

Proof of Theorem 2.9. Let $T \in (0, +\infty)$ and $\Omega_1, \Omega_2 \subset \mathbb{R}^n$ be bounded, smooth and disjoint.

We recall that

$$(3.32) \quad \mu_t^s(l) = \frac{1}{\pi} \int_0^{+\infty} e^{-lu - tu^s \cos(\pi s)} \sin(tu^s \sin(\pi s)) du \quad \text{for all } (l, s) \in (0, +\infty) \times (0, 1),$$

see Proposition 3.1 in [KV18].

Therefore, if $s \in (0, \frac{1}{2})$, using also (3.1) with $m = 1$, we obtain that

$$(3.33) \quad \begin{aligned} \Psi^{x,y}(s, T) &= \int_0^T G^s(t, x, y) dt \\ &= \int_0^T \int_0^{+\infty} G(l, x, y) \mu_t^s(l) dl dt \\ &\leq 2 \int_0^T \int_0^{+\infty} G(l, x, y) \int_0^{+\infty} e^{-lu} du dl dt \\ &= (4\pi)^{-\frac{n}{2}} \int_0^T \int_0^{+\infty} \frac{1}{l^{\frac{n}{2}+1}} \exp\left(-\frac{|x-y|^2}{4l}\right) dl dt \\ &= \pi^{-\frac{n}{2}} |x-y|^{-n} \int_0^T \int_0^{+\infty} a^{\frac{n}{2}-1} e^{-a} da dt \\ &= T \pi^{-\frac{n}{2}} \Gamma\left(\frac{n}{2}\right) |x-y|^{-n} \\ &=: h(x, y), \end{aligned}$$

for all $(x, y) \in \Omega_1 \times \Omega_2$. Note that since Ω_1, Ω_2 are bounded, smooth and disjoint, one has that $h \in L^1(\Omega_1 \times \Omega_2)$. Therefore, recalling (3.25), we obtain that

$$(3.34) \quad \lim_{s \searrow 0} \tilde{\mathcal{E}}^{\Omega_1, \Omega_2}(s, T) = 0.$$

Proposition B.3, together with the limit in (3.34), gives the identities in (2.10), thus completing the proof of Theorem 2.9. \square

Proof of Theorem 2.10. Let $T \in (0, +\infty)$ and $\varepsilon \in (0, 1)$. We will begin by proving the result when $n \leq 2$. To do so, we consider equation (3.29) with $s_0 := \frac{\varepsilon}{4}$, $s_1 \in (\varepsilon, 1)$ and $\mu := (n - \varepsilon)/2$. Then, we obtain the existence of some $r = r_{\varepsilon, n, T}$ such that, for all $x, y \in \mathbb{R}^n$ satisfying $|x - y| \in (0, r)$,

$$(3.35) \quad \mathcal{E}^{x,y}\left(\frac{\varepsilon}{4}, T\right) \geq \mathcal{E}^{x,y}(s, T),$$

if $s \in (\varepsilon, 1)$.

Therefore, if for some $x_0 \in \mathbb{R}^n$ we have that $\Omega_1, \Omega_2 \subset B_{\frac{r}{2}}(x_0)$, then integrating both sides of (3.35) in $\Omega_1 \times \Omega_2$ we obtain that

$$\tilde{\mathcal{E}}^{\Omega_1, \Omega_2}\left(\frac{\varepsilon}{4}, T\right) \geq \tilde{\mathcal{E}}^{\Omega_1, \Omega_2}(s, T),$$

for all $s \in (\varepsilon, 1)$. The latter inequality, together with Theorem 2.9, gives (2.11), which concludes the proof of Theorem 2.10 when $n \leq 2$.

In order to deal with the case $n \geq 3$, it is enough to repeat the above reasoning, using equation (3.30) instead of (3.29), with $s_0 := \frac{\varepsilon}{2}$ and $s_1 \in (\varepsilon, 1)$. \square

Proof of Theorem 2.11. Let $T \in (0, +\infty)$ and $\varepsilon \in (0, 1)$. We begin by taking into account equation (3.31) with $s_0 := \frac{\varepsilon}{2}$ and $s_1 \in (\varepsilon, 1)$. Then, we obtain that there exists some $R = R_{n,\varepsilon,T}$ such that, for all $x, y \in \mathbb{R}^n$ satisfying $|x - y| \in (R, +\infty)$,

$$(3.36) \quad \mathcal{E}^{x,y} \left(\frac{\varepsilon}{2}, T \right) \geq \mathcal{E}^{x,y}(s, T),$$

for all $s \in (\varepsilon, 1)$. Therefore, if $\Omega_1, \Omega_2 \subset \mathbb{R}^n$ are disjoint and satisfy

$$\inf_{\substack{x \in \Omega_1 \\ y \in \Omega_2}} |x - y| \geq R,$$

by integrating in $\Omega_1 \times \Omega_2$ both sides of (3.36), we obtain that

$$\tilde{\mathcal{E}}^{\Omega_1, \Omega_2} \left(\frac{\varepsilon}{2}, T \right) \geq \tilde{\mathcal{E}}^{\Omega_1, \Omega_2}(s, T),$$

for all $s \in (\varepsilon, 1)$. This latter inequality, together with Theorem 2.1, entails (2.12), which concludes the proof of Theorem 2.11. \square

APPENDIX A. GREEN FUNCTION FOR THE FRACTIONAL LAPLACIAN

This section is devoted to the proof of an identity for the Green function of the fractional Laplacian. We recall that for all $x, y \in \mathbb{R}^n$ such that $x \neq y$, the Green function $G^s(x, y)$ for the fractional Laplacian is given by

$$G^s(x, y) := \int_0^{+\infty} G^s(t, x, y) dt,$$

where $G^s(t, x, y)$ is the kernel of the spectral fractional heat equation.

Proposition A.1. *Let $n \geq 2$ and $s \in (0, 1)$.*

Then, for all $x, y \in \mathbb{R}^n$ satisfying $x \neq y$ it holds that⁴

$$(A.1) \quad \int_0^{+\infty} G^s(t, x, y) dt = \frac{1}{\Gamma(s)} \int_0^{+\infty} G(t, x, y) t^{s-1} dt.$$

Proof. Let us set the notation

$$\begin{aligned} \mathcal{I}(x, y) &:= \int_0^{+\infty} G^s(t, x, y) dt \\ \text{and} \quad \mathcal{J}(x, y) &:= \frac{1}{\Gamma(s)} \int_0^{+\infty} G(t, x, y) t^{s-1} dt, \end{aligned}$$

for $x, y \in \mathbb{R}^n$ such that $x \neq y$.

It is well-known that $G^s(t, x, y), G(t, x, y) \in C((0, +\infty) \times \mathbb{R}^n \times \mathbb{R}^n)$.

Furthermore, let

$$\mathcal{R}^n := (\mathbb{R}^n \times \mathbb{R}^n) \setminus \{(x, x) \text{ s.t. } x \in \mathbb{R}^n\}.$$

In this way, if $\{(x_k, y_k)\}_k \subset \mathcal{R}^n$ is some sequence such that $(x_k, y_k) \rightarrow (x, y) \in \mathcal{R}^n$ for $k \rightarrow +\infty$, then using (3.1) with $m = 2$, one observes that

$$\begin{aligned} G(t, x_k, y_k) t^{s-1} &\leq \sup_{k \in \mathbb{N}} \frac{t^{s-1}}{(4\pi t)^{\frac{n}{2}}} \exp\left(-\frac{|x_k - y_k|^2}{4t}\right) \in L^1(0, +\infty) \\ \text{and} \quad G^s(t, x_k, y_k) &\leq \int_0^{+\infty} \sup_{k \in \mathbb{N}} |G^2(l, x_k, y_k)| \mu_t^{\frac{s}{2}}(l) dl \in L^1(0, +\infty), \end{aligned}$$

⁴As a side remark, we point out that, differently from the Green function for the Dirichlet spectral fractional Laplacian (see e.g. Proposition A.1 in [DGV22b]), the identity in (A.1) holds true only for $n \geq 2$. This is due to the fact that the heat kernel in this case decays polynomially with respect to the time, while the Dirichlet heat kernel decays exponentially.

for each $s \in (0, 1)$ and $n \geq 2$. Indeed, if $d = \inf_{k \in \mathbb{N}} |x_k - y_k| > 0$, then using inequality (3.7) and recalling (3.32), we obtain that

$$\begin{aligned}
& \int_0^{+\infty} \int_0^{+\infty} \sup_{k \in \mathbb{N}} |G^2(l, x_k, y_k)| \mu_t^{s_0}(l) dl dt \\
& \leq \int_0^{+\infty} \int_0^{+\infty} \frac{D_2}{l^{\frac{n}{4}}} \exp\left(-\frac{d_2 d^{\frac{4}{3}}}{l^{\frac{1}{3}}}\right) \mu_t^{s_0}(l) dl dt \\
\text{(A.2)} \quad & = \int_0^{+\infty} \int_0^{+\infty} \frac{D_2}{l^{\frac{n}{4}}} \exp\left(-\frac{d_2 d^{\frac{4}{3}}}{l^{\frac{1}{3}}}\right) \mu_t^{s_0}(l) dt dl \\
& = \int_0^{+\infty} \int_0^{+\infty} \frac{D_2}{l^{\frac{n}{4}}} \exp\left(-\frac{d_2 d^{\frac{4}{3}}}{l^{\frac{1}{3}}}\right) \left(\frac{1}{\pi} \int_0^{+\infty} e^{-lu - tu^{s_0} \cos(\pi s_0)} \sin(tu^{s_0} \sin(\pi s_0)) du\right) dt dl,
\end{aligned}$$

where we denoted $s_0 := s/2$,

Setting $F(t) := e^{-t\alpha} \sin(t\beta)$, with $\alpha := u^{s_0} \cos(\pi s_0)$ and $\beta := u^{s_0} \sin(\pi s_0)$, for each $T \in (0, +\infty)$ we integrate by parts and see that

$$\begin{aligned}
\int_0^T F(t) dt &= -\frac{1}{\alpha} e^{-t\alpha} \sin(t\beta) \Big|_0^T + \frac{\beta}{\alpha} \int_0^T e^{-t\alpha} \cos(t\beta) dt \\
&= -\frac{1}{\alpha} e^{-T\alpha} \sin(T\beta) - \frac{\beta}{\alpha^2} e^{-t\alpha} \cos(t\beta) \Big|_0^T - \frac{\beta^2}{\alpha^2} \int_0^T e^{-t\alpha} \sin(t\beta) dt \\
&= -\frac{1}{\alpha} e^{-T\alpha} \sin(T\beta) - \frac{\beta}{\alpha^2} e^{-T\alpha} \cos(T\beta) + \frac{\beta}{\alpha^2} - \frac{\beta^2}{\alpha^2} \int_0^T F(t) dt.
\end{aligned}$$

Therefore, by replacing α and β with their corresponding values, one obtains that

$$\begin{aligned}
\text{(A.3)} \quad & \int_0^T F(t) dt \\
&= -\frac{\cos(\pi s_0)}{u^{s_0}} e^{-Tu^{s_0} \cos(\pi s_0)} \sin(Tu^{s_0} \sin(\pi s_0)) - \frac{\sin(\pi s_0)}{u^{s_0}} e^{-Tu^{s_0} \cos(\pi s_0)} \cos(Tu^{s_0} \sin(\pi s_0)) + \frac{\sin(\pi s_0)}{u^{s_0}} \\
&= \frac{1}{u^{s_0}} \left(\sin(\pi s_0) - e^{-Tu^{s_0} \cos(\pi s_0)} \sin(Tu^{s_0} \sin(\pi s_0)) + \pi s_0 \right).
\end{aligned}$$

Taking the limit as $T \rightarrow +\infty$ in (A.3) we thus obtain that

$$\int_0^{+\infty} F(t) dt = \frac{\sin(\pi s_0)}{u^{s_0}}.$$

Plugging this information into (A.2), we conclude that

$$\begin{aligned}
& \int_0^{+\infty} \int_0^{+\infty} \sup_{k \in \mathbb{N}} |G^2(l, x_k, y_k)| \mu_t^{s_0}(l) dl dt \\
& \leq \int_0^{+\infty} \frac{D_2}{\pi l^{\frac{n}{4}}} \exp\left(-\frac{d_2 d^{\frac{4}{3}}}{l^{\frac{1}{3}}}\right) \int_0^{+\infty} e^{-lu} u^{-s_0} \sin(\pi s_0) du dl \\
& = \frac{D_2 \Gamma(1-s_0) \sin(\pi s_0)}{\pi} \int_0^{+\infty} \frac{1}{l^{\frac{n}{4}+1-s_0}} \exp\left(-\frac{d_2 d^{\frac{4}{3}}}{l^{\frac{1}{3}}}\right) dl \\
& = \frac{3D_2 \Gamma(1-s_0) \sin(\pi s_0)}{\pi} d_2^{-\frac{3n}{4}+3s_0} d^{-n+4s_0} \int_0^{+\infty} a^{\frac{3n}{4}-3s_0-1} e^{-a} da \\
& = \frac{3D_2 \Gamma(1-s_0) \sin(\pi s_0)}{\pi} d_2^{-\frac{3n}{4}+3s_0} d^{-n+4s_0} \Gamma\left(\frac{3n}{4}-3s_0\right),
\end{aligned}$$

where we have applied the change of variable $a := \frac{d_2 d^{\frac{4}{3}}}{l^{\frac{1}{3}}}$.

Therefore, by the Dominated Convergence Theorem we obtain that $\mathcal{I}, \mathcal{J} \in C(\mathbb{R}^n)$.

Now let $f \in C_c^\infty(\mathbb{R}^n)$ such that $f \geq 0$. Using the Fourier transform, for each $x \in \mathbb{R}^n$ we compute

$$\begin{aligned}
\int_{\mathbb{R}^n} \mathcal{I}(x, y) f(y) dy &= \int_{\mathbb{R}^n} \int_0^{+\infty} G^s(t, x, y) f(y) dt dy \\
&= \int_0^{+\infty} \int_{\mathbb{R}^n} G^s(t, x, y) f(y) dy dt \\
&= \int_0^{+\infty} \int_{\mathbb{R}^n} \widehat{G}^s(t, x, \xi) \overline{\widehat{f}}(\xi) d\xi dt \\
&= \int_0^{+\infty} \int_{\mathbb{R}^n} e^{-2\pi i x \cdot \xi} e^{-(2\pi|\xi|)^{2s} t} \overline{\widehat{f}}(\xi) d\xi dt \\
&= \int_{\mathbb{R}^n} \int_0^{+\infty} e^{-2\pi i x \cdot \xi} e^{-(2\pi|\xi|)^{2s} t} \overline{\widehat{f}}(\xi) dt d\xi \\
&= \int_{\mathbb{R}^n} e^{-2\pi i x \cdot \xi} \frac{\overline{\widehat{f}}(\xi)}{(2\pi|\xi|)^{2s}} d\xi,
\end{aligned} \tag{A.4}$$

where we denoted by $\overline{\widehat{f}}(\xi)$ the complex conjugate of $\widehat{f}(\xi)$.

Notice that between the fourth and fifth line we change the order of integration since

$$e^{-2\pi i x \cdot \xi} e^{-(2\pi|\xi|)^{2s} t} \overline{\widehat{f}}(\xi) \in L^1((0, +\infty) \times \mathbb{R}^n).$$

Indeed, we have that

$$\begin{aligned}
\int_0^{+\infty} \int_{\mathbb{R}^n} \left| e^{-2\pi i x \cdot \xi} e^{-(2\pi|\xi|)^{2s} t} \overline{\widehat{f}}(\xi) \right| d\xi dt &= \int_0^{+\infty} \int_{\mathbb{R}^n} e^{-(2\pi|\xi|)^{2s} t} \left| \overline{\widehat{f}}(\xi) \right| d\xi dt \\
&= \int_{\mathbb{R}^n} \frac{\left| \overline{\widehat{f}}(\xi) \right|}{(2\pi|\xi|)^{2s}} d\xi,
\end{aligned}$$

and the last integral is finite since $n \geq 2$ and $\overline{\widehat{f}}$ is a Schwarz function.

Using the Fourier transform, we obtain also that

$$\begin{aligned}
\int_{\mathbb{R}^n} \mathcal{J}(x, y) f(y) dy &= \frac{1}{\Gamma(s)} \int_{\mathbb{R}^n} \int_0^{+\infty} G(t, x, y) t^{s-1} f(y) dt dy \\
&= \frac{1}{\Gamma(s)} \int_0^{+\infty} \int_{\mathbb{R}^n} \widehat{G}(t, x, \xi) \widehat{f}(\xi) t^{s-1} d\xi dt \\
\text{(A.5)} \quad &= \frac{1}{\Gamma(s)} \int_0^{+\infty} \int_{\mathbb{R}^n} e^{-2\pi i x \cdot \xi} e^{-(2\pi|\xi|)^2 t} \widehat{f}(\xi) t^{s-1} d\xi dt \\
&= \frac{1}{\Gamma(s)} \int_{\mathbb{R}^n} \int_0^{+\infty} e^{-2\pi i x \cdot \xi} e^{-(2\pi|\xi|)^2 t} \widehat{f}(\xi) t^{s-1} d\xi dt \\
&= \int_{\mathbb{R}^n} e^{-2\pi i x \cdot \xi} \frac{\widehat{f}(\xi)}{(2\pi|\xi|)^{2s}} d\xi.
\end{aligned}$$

Therefore, thanks to (A.4) and (A.5) we conclude that, for each $f \in C_c^\infty(\mathbb{R}^n)$ and $x \in \mathbb{R}^n$,

$$\int_{\mathbb{R}^n} (\mathcal{I}(x, y) - \mathcal{J}(x, y)) f(y) dy = 0.$$

In light of this latter identity and the fact that $\mathcal{J}, \mathcal{I} \in C(\mathcal{R}^n)$ we conclude the proof of (A.1). \square

As a consequence of (A.1), we also obtain the following integral identity:

$$\text{(A.6)} \quad t \int_0^{+\infty} \frac{s}{l^s} \mu_t^s(l) dl = \frac{1}{\Gamma(s)} \quad \text{for all } (s, t) \in (0, 1) \times (0, +\infty).$$

To establish (A.6) we argue as follows. Using the change of variable $\theta = -\frac{|x-y|^2}{4t}$, it is not hard to show that the left-hand side of (A.1) satisfies

$$\text{(A.7)} \quad \frac{1}{\Gamma(s)} \int_0^{+\infty} G(t, x, y) t^{s-1} dt = 4^{-s} \pi^{-\frac{n}{2}} \Gamma\left(\frac{n}{2} - s\right) \frac{1}{\Gamma(s)} |x - y|^{2s-n},$$

see e.g. equation (3.22).

On the other hand, thanks to scaling property of μ_t^s , we have that

$$\text{(A.8)} \quad \mu_t^s(l) = \frac{1}{t^{\frac{1}{s}}} \mu_1^s\left(\frac{l}{t^{\frac{1}{s}}}\right),$$

see e.g. Lemma 2.1 in [DGV22b]. Therefore, using this and the identity (3.1) with $m = 1$, we obtain that

$$\begin{aligned}
\int_0^{+\infty} G^s(t, x, y) dt &= \int_0^{+\infty} \int_0^{+\infty} \frac{1}{(4\pi l)^{\frac{n}{2}}} \exp\left(-\frac{|x-y|^2}{4l}\right) \mu_t^s(l) dl dt \\
\text{(A.9)} \quad &= \int_0^{+\infty} \int_0^{+\infty} \frac{1}{(4\pi l)^{\frac{n}{2}}} \exp\left(-\frac{|x-y|^2}{4l}\right) \frac{1}{t^{\frac{1}{s}}} \mu_1^s\left(\frac{l}{t^{\frac{1}{s}}}\right) dl dt \\
&= \int_0^{+\infty} \int_0^{+\infty} \frac{1}{(4\pi \ell t^{\frac{1}{s}})^{\frac{n}{2}}} \exp\left(-\frac{|x-y|^2}{4\ell t^{\frac{1}{s}}}\right) \mu_1^s(\ell) dt d\ell \\
&= 4^{-s} \pi^{-\frac{n}{2}} |x - y|^{2s-n} \Gamma\left(\frac{n}{2} - s\right) \int_0^{+\infty} \frac{s}{l^s} \mu_1^s(l) dl
\end{aligned}$$

where we have also used the change of variables $\ell = l/t^{1/s}$ and $\theta = -\frac{|x-y|^2}{4\ell t^{\frac{1}{s}}}$ in t .

Accordingly, recalling (A.1), using (A.7) and (A.9) and exploiting again the scaling property in (A.8), we get (A.6), as desired.

APPENDIX B. SOME CONTINUITY RESULTS

Proposition B.1. *Let $(x, T) \in \mathbb{R}^n \times (0, +\infty)$.*

Then, if $n = 1$ and $s \in (\frac{1}{2}, 1]$ we have that

$$\mathcal{E}^{x,x}(s, T) \in (0, +\infty),$$

while if either $n \geq 2$ or $n = 1$ and $s \in (0, \frac{1}{2}]$ it holds that

$$\mathcal{E}^{x,x}(s, T) = +\infty.$$

Proof. Let $(x, T) \in \mathbb{R}^n \times (0, +\infty)$. Then, if $n = 1$ and $s \in (\frac{1}{2}, 1]$, from the first equation in (15) of [DGV], we deduce that $\mathcal{E}^{x,x}(s, T) \in (0, +\infty)$.

If either $n \geq 2$ or $n = 1$ and $s \in (0, \frac{1}{2}]$, as a consequence of Theorem 2.8 in [DGV22b] and Corollary 3.5 here, we obtain that $\mathcal{E}^{x,x}(s, T) = +\infty$. \square

Proposition B.2. *Let $(x, y, T) \in \mathbb{R}^n \times \mathbb{R}^n \times (0, +\infty)$ such that $x \neq y$.*

Then, it holds that $\mathcal{E}^{x,y}(s, T) \in (0, +\infty)$ for all $s \in (0, 1]$ and $\mathcal{E}^{x,y}(\cdot, T) \in C((0, 1])$.

Proof. Let $(x, y, T) \in \mathbb{R}^n \times \mathbb{R}^n \times (0, +\infty)$ with $x \neq y$. Using equation (3.1) with $m = 1$, we see that

$$\begin{aligned} (B.1) \quad G^s(t, x, y) &= \int_0^{+\infty} \frac{1}{(4\pi l)^{\frac{n}{2}}} \exp\left(-\frac{|x-y|^2}{4l}\right) \mu_t^s(l) dl \\ &\leq \sup_{l \in (0, +\infty)} \frac{1}{(4\pi l)^{\frac{n}{2}}} \exp\left(-\frac{|x-y|^2}{4l}\right) \\ &=: C_{x,y}, \end{aligned}$$

where $C_{x,y}$ does not depend on t . Thus, by equation (2.4) and the Maximum Principle for the fractional heat equation, we obtain that

$$(B.2) \quad \mathcal{E}^{x,y}(s, T) \in (0, +\infty),$$

for all $s \in (0, 1]$.

Taking the Fourier Transform of the system of equations (2.1) we obtain that

$$(B.3) \quad G^s(t, x, y) = \int_{\mathbb{R}^n} e^{-|2\pi\xi|^2 s t} e^{2\pi i \xi \cdot (x-y)} d\xi,$$

and we observe that

$$\begin{aligned} \int_{\mathbb{R}^n} \left| e^{-(2\pi|\xi|)^2 s t} e^{2\pi i \xi \cdot (x-y)} \right| d\xi &= \int_{\mathbb{R}^n} e^{-(2\pi|\xi|)^2 s t} d\xi \\ &= \int_0^{+\infty} \int_{\partial B_\rho} e^{-(2\pi\rho)^2 s} dH^{n-1} d\rho \\ &= \int_0^{+\infty} \omega_{n-1} \rho^{n-1} e^{-(2\pi\rho)^2 s} d\rho \\ &= \frac{\omega_{n-1} t^{-\frac{n}{2s}}}{2s(2\pi)^n} \int_0^{+\infty} \theta^{\frac{n}{2s}-1} e^{-\theta} d\theta, \\ &= \frac{\omega_{n-1} t^{-\frac{n}{2s}}}{2s(2\pi)^n} \Gamma\left(\frac{n}{2s}\right). \end{aligned}$$

Thus, for each $\varepsilon \in (0, 1)$ and $t \in (0, +\infty)$, the last term is uniformly bounded in $s \in (\varepsilon, 1]$. Applying the Dominated Convergence Theorem, we obtain the continuity of $G^s(t, x, y)$ in $s \in (0, 1]$. Finally, from this and equation (B.1), we can apply the Dominated Convergence Theorem to prove that $\mathcal{E}^{x,y}(\cdot, T) \in C((0, 1])$. \square

Proposition B.3. *Let $T \in (0, +\infty)$ and $\Omega_1, \Omega_2 \subset \mathbb{R}^n$ be measurable and bounded.*

Then, $\tilde{\mathcal{E}}^{\Omega_1, \Omega_2}(s, T) \in (0, +\infty)$ for all $s \in (0, 1]$ and $\tilde{\mathcal{E}}^{\Omega_1, \Omega_2}(\cdot, T) \in C((0, 1])$.

Proof. For each $(s, t) \in (0, 1) \times (0, +\infty)$ we define the function

$$(B.4) \quad f(s, t) := \int_{\Omega_1 \times \Omega_2} G^s(t, x, y) dx dy.$$

Then, using (3.1) with $m = 1$ and the well-known identity for the solution of the classical heat equation

$$\int_{\mathbb{R}^n} G(t, x, y) dx = 1 \quad \text{for all } (t, y) \in (0, +\infty) \times \mathbb{R}^n,$$

we obtain that

$$(B.5) \quad |f(s, t)| \leq |\Omega_2| \quad \text{for all } (s, t) \in (0, 1] \times (0, +\infty).$$

Thus, as a consequence we obtain that

$$(B.6) \quad \begin{aligned} \tilde{\mathcal{E}}^{\Omega_1, \Omega_2}(s, T) &= \frac{1}{|\Omega_1| |\Omega_2|} \int_{\Omega_1 \times \Omega_2} \mathcal{E}^{x, y}(s, T) dx dy \\ &= \frac{1}{T |\Omega_1| |\Omega_2|} \int_{\Omega_1 \times \Omega_2} \int_0^T G^s(t, x, y) dt dx dy \\ &= \frac{1}{T |\Omega_1| |\Omega_2|} \int_0^T f(s, t) dt \\ &\leq \frac{1}{|\Omega_1|}. \end{aligned}$$

Also, using the positivity of $\mathcal{E}^{x, y}(s, T)$, see Proposition B.2, we obtain that, for all measurable and bounded sets $\Omega_1, \Omega_2 \subset \mathbb{R}^n$ and $T \in (0, +\infty)$,

$$\tilde{\mathcal{E}}^{\Omega_1, \Omega_2}(s, T) \in (0, +\infty).$$

We already proved in Proposition B.2 that $G^s(t, x, y)$ is continuous in $s \in (0, 1]$ for all $(t, x, y) \in (0, +\infty) \times \mathbb{R}^n \times \mathbb{R}^n$. Moreover, in view of equation (B.3) we have that $G^s(t, x, y)$ is uniformly bounded in $(s, x, y) \in (s_0, 1] \times \mathbb{R}^n \times \mathbb{R}^n$ for all $s_0 \in (0, 1)$ and $t \in (0, +\infty)$. Thus, thanks to the Dominated Convergence Theorem we obtain that $f(\cdot, t) \in C((s_0, 1])$ for all $(s_0, t) \in (0, 1) \times (0, +\infty)$.

Finally, using the bound in (B.5) and the continuity of $f(s, t)$ we can conclude by means of the Dominated Convergence Theorem that $\tilde{\mathcal{E}}^{\Omega_1, \Omega_2}(\cdot, T) \in C((0, 1])$ for all $T \in (0, +\infty)$ and all measurable and bounded sets $\Omega_1, \Omega_2 \subset \mathbb{R}^n$. \square

CONCLUSIONS

In this paper we investigated the Lévy flight foraging hypothesis in view of a fractional diffusive equation in \mathbb{R}^n , without any restriction on the number n of dimensions of the ambient space. The anomalous diffusion is regulated by a parameter $s \in (0, 1)$, which the forager can tune in order to maximize the success of its search.

The parameter $s = 0$ corresponds to ballistic motions (i.e., to the limit case of heavy-tailed Lévy flights), while the parameter $s = 1$ corresponds to the classical Brownian motion.

The biological scenarios of a target located in proximity of the forager and of a target arbitrarily far away have been considered in detail. We have also taken into account the case in which predators and targets are distributed uniformly in some regions.

We showed that the exponent $s = 0$ is a pessimizer for the efficiency functional. However, the most rewarding strategy for the forager may also lie arbitrarily close to $s = 0$.

As a result, the most rewarding searching option may turn out to be unreliable, since small perturbations of the optimal searching exponent lead to pessimal patterns.

It is thereby suggestive to link the adoption of the most rewarding strategy, even when close to pessimizers, to a *high-risk/high-gain* program, in which the forager aims at high-energy content prey to mitigate the risk of failure of the search (known experimental evidence on anglers and blonde skates having already related ballistic motions of the forager with the predation of targets with high-energy content).

This foraging mode can also be considered of “ambush” type, due to the tendency of Lévy distributions with fat tails to displace mass both close to the origin and at infinity.

REFERENCES

- [AV19] Nicola Abatangelo and Enrico Valdinoci, *Getting acquainted with the fractional Laplacian*, Contemporary research in elliptic PDEs and related topics, Springer INdAM Ser., vol. 33, Springer, Cham, 2019, pp. 1–105. MR3967804 ↑3, 10
- [ARMA02] R. P. D. Atkinson, C. J. Rhodes, D. W. Macdonald, and R. M. Anderson, *Scale-free dynamics in the movement patterns of jackals*, *Oikos* **98** (2002), no. 1, 134–140. ↑1
- [ABM04] D. Austin, W. D. Bowen, and J. I. McMillan, *Intraspecific variation in movement patterns: modeling individual behaviour in a large marine predator*, *Oikos* **105** (2004), no. 1, 15–30, DOI 10.1111/j.0030-1299.1999.12730.x. ↑7
- [BR13] Andrea Baronchelli and Filippo Radicchi, *Lévy flights in human behavior and cognition*, *Chaos, Solitons Fractals* **56** (2013), 101–105, DOI 10.1016/j.chaos.2013.07.013. Collective Behavior and Evolutionary Games. ↑6, 7
- [Beg14] Luisa Beghin, *Geometric stable processes and related fractional differential equations*, *Electron. Commun. Probab.* **19** (2014), no. 13, 14, DOI 10.1214/ECP.v19-2771. MR3174831 ↑6
- [BRFM⁺06] Denis Boyer, Gabriel Ramos-Fernández, Octavio Miramontes, José L. Mateos, Germinal Cocho, Hernán Larralde, Humberto Ramos, and Fernando Rojas, *Scale-free foraging by primates emerges from their interaction with a complex environment*, *Proc. R. Soc. B.* **273** (2006), 1743–1750, DOI 10.1098/rspb.2005.3462. ↑6
- [BV16] Claudia Bucur and Enrico Valdinoci, *Nonlocal diffusion and applications*, Lecture Notes of the Unione Matematica Italiana, vol. 20, Springer, [Cham]; Unione Matematica Italiana, Bologna, 2016. MR3469920 ↑
- [BRB⁺21] S. V. Buldyrev, E. P. Raposo, F. Bartumeus, S. Havlin, F. R. Rusch, M. G. E. da Luz, and G. M. Viswanathan, *Comment on “Inverse square Lévy walks are not optimal search strategies for $d \geq 2$ ”*, *Phys. Rev. Lett.* **126** (2021), no. 4, Paper No. 048901, 2, DOI 10.1103/physrevlett.126.048901. MR4207853 ↑2
- [CSS22] W. Campeau, A. M. Simons, and B. Stevens, *The evolutionary maintenance of Lévy flight foraging*, *PLoS Comput Biol.*, posted on 2022, DOI 10.1371/journal.pcbi.1009490. ↑6, 7
- [Car09] Alfred S. Carasso, *Bochner subordination, logarithmic diffusion equations, and blind deconvolution of Hubble space telescope imagery and other scientific data*, U.S. Department of Commerce, National Institute of Standards and Technology, 2009. ↑6
- [CLS22] Hector A. Chang-Lara and Alberto Saldaña, *Classical solutions to integral equations with zero order kernels*, arXiv e-prints (2022), available at 2208.12841. ↑6
- [CW19] Huyuan Chen and Tobias Weth, *The Dirichlet problem for the logarithmic Laplacian*, *Comm. Partial Differential Equations* **44** (2019), no. 11, 1100–1139, DOI 10.1080/03605302.2019.1611851. MR3995092 ↑6
- [DGV] Serena Dipierro, Giovanni Giacomini, and Enrico Valdinoci, *Efficiency functionals for the Lévy flight foraging hypothesis*, *J. Math. Biol.* ↑2, 4, 22
- [DGV22a] ———, *Diffusive processes modeled on the spectral fractional Laplacian with Dirichlet and Neumann boundary conditions*, Preprint (2022), available at web.ma.utexas.edu/mp_arc-bin/mpa?yn=22-12. ↑9
- [DGV22b] ———, *The Lévy flight foraging hypothesis in bounded regions: subordinate Brownian motions and high-risk/high-gain strategies*, Preprint (2022), available at https://web.ma.utexas.edu/mp_arc-bin/mpa?yn=22-52. ↑2, 11, 12, 14, 15, 16, 18, 21, 22
- [DV21] Serena Dipierro and Enrico Valdinoci, *Description of an ecological niche for a mixed local/nonlocal dispersal: an evolution equation and a new Neumann condition arising from the superposition of Brownian and Lévy processes*, *Phys. A* **575** (2021), Paper No. 126052, 20, DOI 10.1016/j.physa.2021.126052. MR4249816 ↑3

- [DGNBD17] Bartłomiej Dybiec, Ewa Gudowska-Nowak, Eli Barkai, and Alexander A. Dubkov, *Lévy flights versus Lévy walks in bounded domains*, Phys. Rev. E **95** (2017), 052102.1–052102.13, DOI 10.1103/PhysRevE.95.052102. ↑3, 6, 7
- [EPW⁺07] Andrew M. Edwards, Richard A. Phillips, Nicholas W. Watkins, Mervyn P. Freeman, Eugene J. Murphy, Vsevolod Afanasyev, Sergey V. Buldyrev, M. G. E. da Luz, E. P. Raposo, H. Eugene Stanley, and Gandhimohan M. Viswanathan, *Revisiting Lévy flight search patterns of wandering albatrosses, bumblebees and deer*, Nature **449** (2007), 1044–1048, DOI 10.1038/nature06199. ↑2
- [Feu21] Pierre Aime Feulefack, *The logarithmic Schrödinger operator and associated Dirichlet problems*, posted on 2021, DOI 10.48550/ARXIV.2112.08783. ↑6
- [GP02] V. A. Galaktionov and S. I. Pohozaev, *Existence and blow-up for higher-order semilinear parabolic equations: majorizing order-preserving operators*, Indiana Univ. Math. J. **51** (2002), no. 6, 1321–1338, DOI 10.1512/iumj.2002.51.2131. MR1948452 ↑9, 10
- [HQD⁺10] Nicolas E. Humphries, Nuno Queiroz, Jennifer R. M. Dyer, Nicolas G. Pade, Michael K. Musyl, Kurt M. Schaefer, Daniel W. Fuller, Juerg M. Brunnschweiler, Thomas K. Doyle, and Jonathan D. R. and others Houghton, *Environmental context explains Lévy and Brownian movement patterns of marine predators*, Nature **465** (2010), no. 7301, 1066–1069. ↑1
- [HWQ⁺12] Nicolas E. Humphries, Henri Weimerskirch, Nuno Queiroz, Emily J. Southall, and David W. Sims, *Foraging success of biological Lévy flights recorded in situ*, Proc. Nat. Acad. Sci. **109** (2012), no. 19, 7169–7174. ↑1, 7
- [dJBK⁺14] Monique de Jager, Frederic Bartumeus, Andrea Kölzsch, Franz J. Weissing, Geerten M. Hengeveld, Bart A. Nolet, Peter M. J. Herman, and Johan van de Koppel, *How superdiffusion gets arrested: ecological encounters explain shift from Lévy to Brownian movement*, Proc. R. Soc. B. **281** (2014), 1–7, DOI 10.1098/rspb.2013.2605. ↑1
- [JPE11] Alex James, Michael J. Plank, and Andrew M. Edwards, *Assessing Lévy walks as models of animal foraging*, J. R. Soc. Interface **8** (2011), 1233–1247, DOI 10.1098/rsif.2011.0200. ↑7
- [KM17] Moritz Kassmann and Ante Mimica, *Intrinsic scaling properties for nonlocal operators*, J. Eur. Math. Soc. (JEMS) **19** (2017), no. 4, 983–1011, DOI 10.4171/JEMS/686. MR3626549 ↑6
- [KV18] K. K. Kataria and P. Vellaisamy, *On densities of the product, quotient and power of independent subordinators*, J. Math. Anal. Appl. **462** (2018), no. 2, 1627–1643, DOI 10.1016/j.jmaa.2018.02.059. MR3774308 ↑17
- [Law21] Sean D. Lawley, *Extreme statistics of superdiffusive Lévy flights and every other Lévy subordinate Brownian motion*, Preprint (2021). ↑3
- [LTBV20] Nicolas Levernier, Johannes Textor, Olivier Bénichou, and Raphaël Voituriez, *Inverse square Lévy walks are not optimal search strategies for $d \geq 2$* , Phys. Rev. Lett. **124** (2020), no. 8, 080601, 5, DOI 10.1103/physrevlett.124.080601. MR4071775 ↑2
- [LTBV21] ———, *Reply to “Comment on ‘Inverse square Lévy walks are not optimal search strategies for $d \geq 2$ ’”*, Phys. Rev. Lett. **126** (2021), no. 4, Paper No. 048902, 1, DOI 10.1103/physrevlett.126.048902. MR4207854 ↑2
- [MHSB07] Alexander Maye, Chih-hao Hsieh, George Sugihara, and Björn Brembs, *Order in Spontaneous Behavior*, PLOS ONE **2** (2007), no. 5, 1–14, DOI 10.1371/journal.pone.0000443. ↑7
- [MPV13] Eugenio Montefusco, Benedetta Pellacci, and Gianmaria Verzini, *Fractional diffusion with Neumann boundary conditions: the logistic equation*, Discrete Contin. Dyn. Syst. Ser. B **18** (2013), no. 8, 2175–2202, DOI 10.3934/dcdsb.2013.18.2175. MR3082317 ↑3
- [PV21] Gianni Pagnini and Silvia Vitali, *Should I stay or should I go? Zero-size jumps in random walks for Lévy flights*, Fract. Calc. Appl. Anal. **24** (2021), no. 1, 137–167, DOI 10.1515/fca-2021-0007. MR4225516 ↑6
- [PBMP] Alex Proekt, Jayanth R. Banavar, Amos Maritan, and Donald W. Pfaff, *Scale invariance in the dynamics of spontaneous behavior* **109**, 10564–10569, DOI 10.1073/pnas.1206894109. ↑6
- [Rad73] Hans Rademacher, *Topics in analytic number theory*, Die Grundlehren der mathematischen Wissenschaften, Band 169, Springer-Verlag, New York-Heidelberg, 1973. Edited by E. Grosswald, J. Lehner and M. Newman. MR0364103 ↑
- [RFMM⁺04] Gabriel Ramos-Fernández, José L. Mateos, Octavio Miramontes, Germinal Cocho, Hernán Larralde, and Bárbara Ayala-Orozco, *Lévy walk patterns in the foraging movements of spider monkeys (*Ateles geoffroyi*)*, Behav. Ecol. Sociobiol. **55** (2004), 223–230, DOI 10.1007/s00265-003-0700-6. ↑1, 6, 7
- [RJB⁺15] Julia F. Revere, Jae-Hyung Jeon, Han Bao, Matthias Leippe, Ralf Metzler, and Christine Selhuber-Unkel, *Superdiffusion dominates intracellular particle motion in the supercrowded cytoplasm of pathogenic *Acanthamoeba castellanii** **5** (2015), 12771–12775, DOI 10.1038/srep11690. ↑1

- [Rey18] Andy M. Reynolds, *Current status and future directions of Lévy walk research*, Biol. Open **7** (2018), no. 1, 1–5, DOI 10.1242/bio.030106. ↑1
- [SRVdL04] M. C. Santos, E. P. Raposo, G. M. Viswanathan, and M. G. E. da Luz, *Optimal random searches of revisitable targets: Crossover from superdiffusive to ballistic random walks*, Europhys. Lett, **67** (2004), no. 5, 734–740, DOI 10.1209/epl/i2004-10114-9. ↑6, 7
- [ŠSV06] Hrvoje Šikić, Renming Song, and Zoran Vondraček, *Potential theory of geometric stable processes*, Probab. Theory Related Fields **135** (2006), no. 4, 547–575, DOI 10.1007/s00440-005-0470-3. MR2240700 ↑6
- [SRP07] David W. Sims, David Righton, and Jonathan W. Pitchford, *Minimizing errors in identifying Lévy flight behaviour of organisms*, J. Anim. Ecol. **76** (2007), no. 2, 222–229, DOI 10.1111/j.1365-2656.2006.01208.x. ↑7
- [SSH⁺08] David W. Sims, Emily J. Southall, Nicolas E. Humphries, Graeme C. Hays, Corey J. A. Bradshaw, Jonathan W. Pitchford, Alex James, Mohammed Z. Ahmed, Andrew S. Brierley, and Mark A. and others Hindell, *Scaling laws of marine predator search behaviour*, Nature **451** (2008), no. 7182, 1098–1102. ↑1
- [SWR⁺06] David W. Sims, Matthew J. Witt, Anthony J. Richardson, Emily J. Southall, and Julian D. Metcalfe, *Encounter success of free-ranging marine predator movements across a dynamic prey landscape*, Proc. R. Soc. B. **273** (2006), 1195–1201, DOI 10.1098/rspb.2005.3444. ↑1
- [SV17] Jürgen Sprekels and Enrico Valdinoci, *A new type of identification problems: optimizing the fractional order in a nonlocal evolution equation*, SIAM J. Control Optim. **55** (2017), no. 1, 70–93, DOI 10.1137/16M105575X. MR3590646 ↑3
- [VAB⁺96] G. M. Viswanathan, V. Afanasyev, S. V. Buldyrev, E. J. Murphy, P. A. Prince, and H. E. Stanley, *Lévy flight search patterns of wandering albatrosses*, Nature **381** (1996), 413–415, DOI 10.1038/381413a0. ↑1
- [VBH⁺99] Gandimohan M. Viswanathan, Sergey V. Buldyrev, Shlomo Havlin, M. G. E. Da Luz, E. P. Raposo, and H. Eugene Stanley, *Optimizing the success of random searches*, Nature **401** (1999), no. 6756, 911–914. ↑6, 7
- [VRdL08] G. M. Viswanathan, E. P. Raposo, and M. G. E. da Luz, *Lévy flights and superdiffusion in the context of biological encounters and random searches*, Phys. Life Rev. (2008), 133–150. ↑6, 7
- [WMH⁺14] Victoria J. Wearmouth, Matthew J. McHugh, Nicolas E. Humphries, Aurore Naegelen, Mohammed Z. Ahmed, Emily J. Southall, Andrew M. Reynolds, and David W. Sims, *Scaling laws of ambush predator ‘waiting’ behaviour are tuned to a common ecology*, Proc. R. Soc. B. **281** (2014), 1–9, DOI 10.1098/rspb.2013.2997. ↑7
- [ZDK15] V. Zaburdaev, S. Denisov, and J. Klafter, *Lévy walks*, Rev. Mod. Phys. **87** (2015), 483–530, DOI 10.1103/RevModPhys.87.483. ↑6
- [Zha02] Qi S. Zhang, *The boundary behavior of heat kernels of Dirichlet Laplacians*, J. Differential Equations **182** (2002), no. 2, 416–430, DOI 10.1006/jdeq.2001.4112. MR1900329 ↑13

SERENA DIPIERRO, DEPARTMENT OF MATHEMATICS AND STATISTICS, UNIVERSITY OF WESTERN AUSTRALIA,
35 STIRLING HIGHWAY, WA6009 CRAWLEY, AUSTRALIA
Email address: serena.dipierro@uwa.edu.au

GIOVANNI GIACOMIN, DEPARTMENT OF MATHEMATICS AND STATISTICS, UNIVERSITY OF WESTERN AUSTRALIA,
35 STIRLING HIGHWAY, WA6009 CRAWLEY, AUSTRALIA
Email address: giovanni.giacomin@research.uwa.edu.au

ENRICO VALDINOCI, DEPARTMENT OF MATHEMATICS AND STATISTICS, UNIVERSITY OF WESTERN AUSTRALIA,
35 STIRLING HIGHWAY, WA6009 CRAWLEY, AUSTRALIA
Email address: enrico.valdinoci@uwa.edu.au