SOLVABILITY OF SOME QUADRATIC INTEGRAL EQUATIONS

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Abstract: The work deals with the existence of solutions of a certain quadratic integral equation in $H^1(\mathbb{R})$. The theory of quadratic integral equations has many useful applications in the mathematical physics, economics, biology, as well as in describing the real world problems. The proof of the existence of solutions is based on a fixed point technique in our Sobolev space on the real line.

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1. Introduction

The present article is devoted to the existence of solutions of the following integral equation

$$u(x) = u_0(x) + [Tu(x)] \int_{-\infty}^{\infty} K(x-y)g(u(y))dy, \quad x \in \mathbb{R}. \tag{1.1}$$

The precise conditions on the functions $u_0(x),\ g(u)$, the linear operator T and the kernel $K(x)$ will be specified further down. The theory of integral equations has many useful applications in describing the numerous events and problems of the real world. It is caused by the fact that this theory is frequently applicable in various branches of mathematics and in mathematical physics, economics, biology as well as in dealing with the real world problems. The quadratic integral equations arise in the theories of the radiative transfer, neutron transport, in the kinetic theory of gases, in the design of the bandlimited signals for the binary communication using the simple memoryless correlation detection, when the signals are disturbed by the additive white Gaussian noise (see e.g. [1], [5], [11] and the references therein). The article [1] deals with the solvability of a nonlinear quadratic integral equation in the Banach space of the real functions being defined and continuous on a bounded and closed interval using the fixed point result. The works [2] and

[4] are devoted to the studies of the existence of solutions for quadratic integral equations on unbounded intervals. The existence of solutions for quadratic integral inclusions was treated in [3]. The paper [10] deals with the nondecreasing solutions of a quadratic integral equation of Urysohn-Stieltjes type. The solvability of the quadratic integral equations in Orlicz spaces was covered in [7], [8], [9]. The reduction of dimension in multi-dimensional integral equations was discussed in [15]. The integro-differential equations, which may involve either Fredholm or non Fredholm operators arise in the mathematical biology when studying the systems with the nonlocal consumption of resources and the intra-specific competition (see [12], [13], [17], [18] and the references therein). The contraction argument was used in [16] to estimate the perturbation to the standing solitary wave of the Nonlinear Schrödinger (NLS) equation when either the external potential or the nonlinear term were perturbed. The similar ideas were exploited to show the persistence of pulses for certain reaction-diffusion type equations (see [6]). Suppose that the assumption below is fulfilled.

Assumption 1.1. *Let the kernel $K(x) : \mathbb{R} \to \mathbb{R}$ be nontrivial, such that $K(x) \in W^{1,1}(\mathbb{R})$. The function $u_0(x) : \mathbb{R} \to \mathbb{R}$ does not vanish identically on the real line and $u_0(x) \in H^1(\mathbb{R})$. Suppose also that the linear operator $T : H^1(\mathbb{R}) \to H^1(\mathbb{R})$ is bounded, such that its norm $0 < \|T\| < \infty$.*

Let the function $V(x) : \mathbb{R} \to \mathbb{R}$ be nontrivial and $V(x) \in W^{1,\infty}(\mathbb{R})$, such that $V(x)$ and its derivative $\frac{dV}{dx}$ are bounded on the whole real line. Then it can be easily verified that the multiplication operator

$$Tu(x) = V(x)u(x), \quad u(x) \in H^1(\mathbb{R}) \tag{1.2}$$

satisfies the assumption above. We will use the Sobolev space

$$H^1(\mathbb{R}) := \Big\{u(x) : \mathbb{R} \to \mathbb{R} \mid u(x) \in L^2(\mathbb{R}), \ \frac{du}{dx} \in L^2(\mathbb{R})\Big\}. \tag{1.3}$$

It is equipped with the norm

$$\|u\|^2_{H^1(\mathbb{R})} := \|u\|^2_{L^2(\mathbb{R})} + \Big\|\frac{du}{dx}\Big\|^2_{L^2(\mathbb{R})}. \tag{1.4}$$

Another norm relevant to our argument is given by

$$\|K\|_{W^{1,1}(\mathbb{R})} := \|K\|_{L^1(\mathbb{R})} + \Big\|\frac{dK}{dx}\Big\|_{L^1(\mathbb{R})}. \tag{1.5}$$

By means of the Sobolev inequality in one dimension (see e.g. Sect 8.5 of [14]), we have

$$\|u(x)\|_{L^\infty(\mathbb{R})} \le \frac{1}{\sqrt{2}}\|u(x)\|_{H^1(\mathbb{R})}. \tag{1.6}$$

Let us recall the algebraic property of our Sobolev space, namely that for any $u(x), v(x) \in H^1(\mathbb{R})$

$$\|u(x)v(x)\|_{H^1(\mathbb{R})} \le c_a \|u(x)\|_{H^1(\mathbb{R})} \|v(x)\|_{H^1(\mathbb{R})}, \tag{1.7}$$

where $c_a > 0$ is a constant. Estimate from above (1.7) can be easily derived, for instance via (1.6). The Young's inequality (see e.g. Section 4.2 of [14]) enables us to estimate the norm of the convolution as

$$\|u * v\|_{L^2(\mathbb{R})} \le \|u\|_{L^1(\mathbb{R})} \|v\|_{L^2(\mathbb{R})}. \tag{1.8}$$

Clearly, inequality (1.8) yields the upper bound

$$\Big\| \frac{d}{dx} \int_{-\infty}^{\infty} u(x-y)v(y)dy \Big\|_{L^2(\mathbb{R})} \le \Big\| \frac{du}{dx} \Big\|_{L^1(\mathbb{R})} \|v\|_{L^2(\mathbb{R})}. \tag{1.9}$$

We seek the resulting solution of nonlinear equation (1.1) as

$$u(x) = u_0(x) + u_p(x). \tag{1.10}$$

Evidently, we arrive at the perturbative equation

$$u_p(x) = [T(u_0(x) + u_p(x))] \int_{-\infty}^{\infty} K(x-y) g(u_0(y) + u_p(y)) dy. \tag{1.11}$$

Let us introduce a closed ball in our Sobolev space

$$B_\rho := \{u(x) \in H^1(\mathbb{R}) \mid \|u\|_{H^1(\mathbb{R})} \le \rho\}, \quad 0 < \rho \le 1. \tag{1.12}$$

We seek the solution of equation (1.11) as the fixed point of the auxiliary nonlinear problem

$$u(x) = [T(u_0(x) + v(x))] \int_{-\infty}^{\infty} K(x-y) g(u_0(y) + v(y)) dy \tag{1.13}$$

in ball (1.12). Let us introduce the interval on the real line

$$I := \Big[-\frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}} \|u_0\|_{H^1(\mathbb{R})}, \ \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}} \|u_0\|_{H^1(\mathbb{R})} \Big] \tag{1.14}$$

along with the closed ball in the space of $C_1(I)$ functions, namely

$$D_M := \{g(z) \in C_1(I) \mid \|g\|_{C_1(I)} \le M\}, \quad M > 0. \tag{1.15}$$

In this context the norm

$$\|g\|_{C_1(I)} := \|g\|_{C(I)} + \|g'\|_{C(I)}, \tag{1.16}$$

where $\|g\|_{C(I)} := \max_{z\in I}|g(z)|$.

Assumption 1.2. *Let $g(z) : \mathbb{R} \to \mathbb{R}$, such that $g(0) = 0$. It is also assumed that $g(z) \in D_M$ and it does not vanish identically on the interval I.*

Let us introduce the operator t_g, such that $u = t_g v$, where u is a solution of equation (1.13). Our first main result is as follows.

Theorem 1.3. *Let Assumptions 1.1 and 1.2 hold and*

$$c_a\|T\|(\|u_0\|_{H^1(\mathbb{R})} + 1)^2\|K\|_{W^{1,1}(\mathbb{R})}M \le \frac{\rho}{2}. \tag{1.17}$$

Then equation (1.13) defines the map $t_g : B_\rho \to B_\rho$, which is a strict contraction. The unique fixed point $u_p(x)$ of this map t_g is the only solution of problem (1.11) in B_ρ.

Obviously, the resulting solution of equation (1.1) given by (1.10) will not vanish identically on the real line because $g(0) = 0$, the operator T is linear and the function $u_0(x)$ is nontrivial according to our assumptions.

For the technical purposes we define

$$\sigma := 2c_a(\|u_0\|_{H^1(\mathbb{R})} + 1)\|T\|M\|K\|_{W^{1,1}(\mathbb{R})} > 0. \tag{1.18}$$

Our second major statement is about the continuity of the cumulative solution of problem (1.1) given by formula (1.10) with respect to the function g.

Theorem 1.4. *Let $j = 1, 2$, the assumptions of Theorem 1.3 are valid, such that $u_{p,j}(x)$ is the unique fixed point of the map $t_{g_j} : B_\rho \to B_\rho$, which is a strict contraction since inequality (1.17) holds and the resulting solution of problem (1.1) with $g(z) = g_j(z)$ is given by*

$$u_j(x) = u_0(x) + u_{p,j}(x). \tag{1.19}$$

Then the estimate from above

$$\|u_1(x) - u_2(x)\|_{H^1(\mathbb{R})} \le \frac{\sigma}{2M(1-\sigma)}(\|u_0\|_{H^1(\mathbb{R})} + 1)\|g_1(z) - g_2(z)\|_{C_1(I)} \tag{1.20}$$

is valid.

Let us proceed to the proof of our first main proposition.

2. The existence of the perturbed solution

Proof of Theorem 1.3. Let us choose arbitrarily $v(x) \in B_\rho$. By means of (1.13) along with (1.7) we obtain the upper bound $\|u\|_{H^1(\mathbb{R})} \leq$

$$\leq c_a \|T(u_0(x)+v(x))\|_{H^1(\mathbb{R})} \Big\| \int_{-\infty}^{\infty} K(x-y)g(u_0(y)+v(y))dy \Big\|_{H^1(\mathbb{R})}. \quad (2.1)$$

Let us estimate the right side of (2.1). Clearly, we have

$$\|T(u_0(x)+v(x))\|_{H^1(\mathbb{R})} \leq \|T\|(\|u_0(x)\|_{H^1(\mathbb{R})}+1). \quad (2.2)$$

By means of inequality (1.8), we obtain

$$\Big\| \int_{-\infty}^{\infty} K(x-y)g(u_0(y)+v(y))dy \Big\|_{L^2(\mathbb{R})} \leq \|K\|_{L^1(\mathbb{R})} \|g(u_0(x)+v(x))\|_{L^2(\mathbb{R})}. \quad (2.3)$$

Similarly, (1.9) yields

$$\Big\| \frac{d}{dx} \int_{-\infty}^{\infty} K(x-y)g(u_0(y)+v(y))dy \Big\|_{L^2(\mathbb{R})} \leq$$

$$\leq \Big\| \frac{dK}{dx} \Big\|_{L^1(\mathbb{R})} \|g(u_0(x)+v(x))\|_{L^2(\mathbb{R})}. \quad (2.4)$$

Estimates (2.3) and (2.4) give us

$$\Big\| \int_{-\infty}^{\infty} K(x-y)g(u_0(y)+v(y))dy \Big\|_{H^1(\mathbb{R})} \leq$$

$$\leq \|K\|_{W^{1,1}(\mathbb{R})} \|g(u_0(x)+v(x))\|_{L^2(\mathbb{R})}. \quad (2.5)$$

Let us express

$$g(u_0(x)+v(x)) = \int_0^{u_0(x)+v(x)} g'(z)dz. \quad (2.6)$$

For $v(x) \in B_\rho$ using inequality (1.6) we easily derive

$$|u_0+v| \leq \frac{1}{\sqrt{2}}(\|u_0\|_{H^1(\mathbb{R})}+1). \quad (2.7)$$

Hence,

$$|g(u_0(x)+v(x))| \leq \max_{z\in I}|g'(z)||u_0(x)+v(x)| \leq M|u_0(x)+v(x)|, \quad (2.8)$$

where the interval I is defined in (1.14). This yields

$$\|g(u_0(x)+v(x))\|_{L^2(\mathbb{R})} \leq M(\|u_0\|_{H^1(\mathbb{R})}+1). \quad (2.9)$$

Therefore, we arrive at

$$\|u(x)\|_{H^1(\mathbb{R})} \leq c_a\|T\|(\|u_0\|_{H^1(\mathbb{R})}+1)^2\|K\|_{W^{1,1}(\mathbb{R})}M. \quad (2.10)$$

By virtue of (1.17), we have $\|u(x)\|_{H^1(\mathbb{R})} \leq \rho$. Thus, the function $u(x)$, which is uniquely determined by (1.13) belongs to B_ρ as well. This means that equation (1.13) defines a map $t_g : B_\rho \to B_\rho$ under the given conditions.
Let us establish that under the stated assumptions this map is a strict contraction. We choose arbitrarily $v_{1,2}(x) \in B_\rho$. The argument above yields that $u_{1,2} := t_g v_{1,2} \in B_\rho$. By virtue of (1.13) we have

$$u_1(x) = [T(u_0(x) + v_1(x))] \int_{-\infty}^{\infty} K(x-y) g(u_0(y) + v_1(y)) dy, \tag{2.11}$$

$$u_2(x) = [T(u_0(x) + v_2(x))] \int_{-\infty}^{\infty} K(x-y) g(u_0(y) + v_2(y)) dy. \tag{2.12}$$

From (2.11) and (2.12) we easily deduce that

$$u_1(x) - u_2(x) = [Tv_1(x) - Tv_2(x)] \int_{-\infty}^{\infty} K(x-y) g(u_0(y) + v_1(y)) dy +$$

$$+ [T(u_0(x) + v_2(x))] \int_{-\infty}^{\infty} K(x-y)[g(u_0(y) + v_1(y)) - g(u_0(y) + v_2(y))] dy. \tag{2.13}$$

By means of (2.13) along with (1.7) we derive

$$\|u_1(x) - u_2(x)\|_{H^1(\mathbb{R})} \leq c_a \|Tv_1(x) - Tv_2(x)\|_{H^1(\mathbb{R})} \times$$

$$\times \Big\| \int_{-\infty}^{\infty} K(x-y) g(u_0(y) + v_1(y)) dy \Big\|_{H^1(\mathbb{R})} + c_a \|T(u_0(x) + v_2(x))\|_{H^1(\mathbb{R})} \times$$

$$\times \Big\| \int_{-\infty}^{\infty} K(x-y)[g(u_0(y) + v_1(y)) - g(u_0(y) + v_2(y))] dy \Big\|_{H^1(\mathbb{R})}. \tag{2.14}$$

Let us obtain the upper bound on the right side of (2.14). Obviously,

$$\|Tv_1(x) - Tv_2(x)\|_{H^1(\mathbb{R})} \leq \|T\| \|v_1(x) - v_2(x)\|_{H^1(\mathbb{R})}. \tag{2.15}$$

Using inequality (1.8), we arrive at

$$\Big\| \int_{-\infty}^{\infty} K(x-y) g(u_0(y) + v_1(y)) dy \Big\|_{L^2(\mathbb{R})} \leq$$

$$\leq \|K\|_{L^1(\mathbb{R})} \|g(u_0(x) + v_1(x))\|_{L^2(\mathbb{R})}. \tag{2.16}$$

By applying (1.9), we have

$$\Big\| \frac{d}{dx} \int_{-\infty}^{\infty} K(x-y) g(u_0(y) + v_1(y)) dy \Big\|_{L^2(\mathbb{R})} \leq$$

$$\leq \Big\| \frac{dK}{dx} \Big\|_{L^1(\mathbb{R})} \|g(u_0(x) + v_1(x))\|_{L^2(\mathbb{R})}. \tag{2.17}$$

Estimates from above (2.16) and (2.17) give us

$$\Big\| \int_{-\infty}^{\infty} K(x-y) g(u_0(y)+v_1(y)) dy \Big\|_{H^1(\mathbb{R})} \leq$$

$$\leq \|K\|_{W^{1,1}(\mathbb{R})} \|g(u_0(x)+v_1(x))\|_{L^2(\mathbb{R})}. \tag{2.18}$$

Clearly,

$$g(u_0(x)+v_1(x)) = \int_0^{u_0(x)+v_1(x)} g'(z) dz. \tag{2.19}$$

From (2.19) we easily deduce that

$$|g(u_0(x)+v_1(x))| \leq max_{z \in I} |g'(z)| |u_0(x)+v_1(x)| \leq M |u_0(x)+v_1(x)|, \tag{2.20}$$

such that

$$\|g(u_0(x)+v_1(x))\|_{L^2(\mathbb{R})} \leq M(\|u_0\|_{H^1(\mathbb{R})} + 1). \tag{2.21}$$

Therefore, the first term in the right side of inequality (2.14) can be bounded from above by

$$c_a \|T\| \|v_1(x) - v_2(x)\|_{H^1(\mathbb{R})} \|K\|_{W^{1,1}(\mathbb{R})} M(\|u_0\|_{H^1(\mathbb{R})} + 1). \tag{2.22}$$

Hence, it remains to estimate the second term in the right side of (2.14). Evidently,

$$\|T(u_0(x)+v_2(x))\|_{H^1(\mathbb{R})} \leq \|T\|(\|u_0\|_{H^1(\mathbb{R})} + 1). \tag{2.23}$$

By means of inequality (1.8), we easily derive

$$\Big\| \int_{-\infty}^{\infty} K(x-y)[g(u_0(y)+v_1(y)) - g(u_0(y)+v_2(y))] dy \Big\|_{L^2(\mathbb{R})} \leq$$

$$\leq \|K\|_{L^1(\mathbb{R})} \|g(u_0(x)+v_1(x)) - g(u_0(x)+v_2(x))\|_{L^2(\mathbb{R})}. \tag{2.24}$$

Upper bound (1.9) yields

$$\Big\| \frac{d}{dx} \int_{-\infty}^{\infty} K(x-y)[g(u_0(y)+v_1(y)) - g(u_0(y)+v_2(y))] dy \Big\|_{L^2(\mathbb{R})} \leq$$

$$\leq \Big\| \frac{dK}{dx} \Big\|_{L^1(\mathbb{R})} \|g(u_0(x)+v_1(x)) - g(u_0(x)+v_2(x))\|_{L^2(\mathbb{R})}. \tag{2.25}$$

Using (2.24) and (2.25), we arrive at

$$\Big\| \int_{-\infty}^{\infty} K(x-y)[g(u_0(y)+v_1(y)) - g(u_0(y)+v_2(y))] dy \Big\|_{H^1(\mathbb{R})} \leq$$

$$\leq \|K\|_{W^{1,1}(\mathbb{R})} \|g(u_0(x)+v_1(x)) - g(u_0(x)+v_2(x))\|_{L^2(\mathbb{R})}. \tag{2.26}$$

We easily express

$$g(u_0(x)+v_1(x))-g(u_0(x)+v_2(x))=\int_{u_0(x)+v_2(x)}^{u_0(x)+v_1(x)}g'(z)dz. \tag{2.27}$$

Formula (2.27) gives us

$$|g(u_0(x)+v_1(x))-g(u_0(x)+v_2(x))|\le max_{z\in I}|g'(z)||v_1(x)-v_1(x)|\le$$

$$\le M|v_1(x)-v_1(x)|, \tag{2.28}$$

such that

$$\|g(u_0(x)+v_1(x))-g(u_0(x)+v_2(x))\|_{L^2(\mathbb{R})}\le M\|v_1(x)-v_2(x)\|_{H^1(\mathbb{R})}. \tag{2.29}$$

Thus, the second term in the right side of inequality (2.14) can be estimated from above by expression (2.22) as well. Hence, we obtain $\|u_1(x)-u_2(x)\|_{H^1(\mathbb{R})}\le$

$$\le 2c_a(\|u_0\|_{H^1(\mathbb{R})}+1)\|T\|M\|K\|_{W^{1,1}(\mathbb{R})}\|v_1(x)-v_2(x)\|_{H^1(\mathbb{R})}. \tag{2.30}$$

By virtue of (2.30) along with definition (1.18), we have

$$\|t_gv_1(x)-t_gv_2(x)\|_{H^1(\mathbb{R})}\le\sigma\|v_1(x)-v_2(x)\|_{H^1(\mathbb{R})}. \tag{2.31}$$

It can be easily verified using (1.17) that the constant in the right side of inequality above

$$\sigma<1. \tag{2.32}$$

This implies that our map $t_g : B_\rho \to B_\rho$ defined by equation (1.13) is a strict contraction under the given conditions. Its unique fixed point $u_p(x)$ is the only solution of problem (1.11) in the ball B_ρ. The resulting $u(x)$ given by (1.10) is a solution of equation (1.1). ∎

Let us conclude the article by establishing our second main result.

3. The continuity of the resulting solution

Proof of Theorem 1.4. Obviously, under the stated assumptions, we have

$$u_{p,1}=t_{g_1}u_{p,1}, \quad u_{p,2}=t_{g_2}u_{p,2}. \tag{3.1}$$

Thus,

$$u_{p,1}-u_{p,2}=t_{g_1}u_{p,1}-t_{g_1}u_{p,2}+t_{g_1}u_{p,2}-t_{g_2}u_{p,2}. \tag{3.2}$$

Therefore,

$$\|u_{p,1}-u_{p,2}\|_{H^1(\mathbb{R})}\le\|t_{g_1}u_{p,1}-t_{g_1}u_{p,2}\|_{H^1(\mathbb{R})}+\|t_{g_1}u_{p,2}-t_{g_2}u_{p,2}\|_{H^1(\mathbb{R})}. \tag{3.3}$$

By means of estimate (2.31), we have

$$\|t_{g_1}u_{p,1} - t_{g_1}u_{p,2}\|_{H^1(\mathbb{R})} \le \sigma \|u_{p,1} - u_{p,2}\|_{H^1(\mathbb{R})} \tag{3.4}$$

with σ given by (1.18), such that (2.32) holds. Hence, we obtain

$$(1-\sigma)\|u_{p,1} - u_{p,2}\|_{H^1(\mathbb{R})} \le \|t_{g_1}u_{p,2} - t_{g_2}u_{p,2}\|_{H^1(\mathbb{R})}. \tag{3.5}$$

Evidently, for our fixed point $t_{g_2}u_{p,2} = u_{p,2}$. We denote $r(x) := t_{g_1}u_{p,2}$ and arrive at

$$r(x) = [T(u_0(x) + u_{p,2}(x))] \int_{-\infty}^{\infty} K(x-y) g_1(u_0(y) + u_{p,2}(y)) dy, \tag{3.6}$$

$$u_{p,2}(x) = [T(u_0(x) + u_{p,2}(x))] \int_{-\infty}^{\infty} K(x-y) g_2(u_0(y) + u_{p,2}(y)) dy. \tag{3.7}$$

Formulas (3.6) and (3.7) yield

$$r(x) - u_{p,2}(x) = [T(u_0(x) + u_{p,2}(x))]\times$$

$$\times \int_{-\infty}^{\infty} K(x-y)[g_1(u_0(y) + u_{p,2}(y)) - g_2(u_0(y) + u_{p,2}(y))] dy. \tag{3.8}$$

By virtue of (1.7), we derive

$$\|r(x) - u_{p,2}(x)\|_{H^1(\mathbb{R})} \le c_a \|T(u_0(x) + u_{p,2}(x))\|_{H^1(\mathbb{R})}\times$$

$$\times \Big\| \int_{-\infty}^{\infty} K(x-y)[g_1(u_0(y) + u_{p,2}(y)) - g_2(u_0(y) + u_{p,2}(y))] dy \Big\|_{H^1(\mathbb{R})}. \tag{3.9}$$

Clearly, we have the upper bound

$$\|T(u_0(x) + u_{p,2}(x))\|_{H^1(\mathbb{R})} \le \|T\|(\|u_0\|_{H^1(\mathbb{R})} + 1). \tag{3.10}$$

By means of inequality (1.8),

$$\Big\| \int_{-\infty}^{\infty} K(x-y)[g_1(u_0(y) + u_{p,2}(y)) - g_2(u_0(y) + u_{p,2}(y))] dy \Big\|_{L^2(\mathbb{R})} \le$$

$$\le \|K\|_{L^1(\mathbb{R})} \|g_1(u_0(x) + u_{p,2}(x)) - g_2(u_0(x) + u_{p,2}(x))\|_{L^2(\mathbb{R})}. \tag{3.11}$$

Similarly, (1.9) gives us

$$\Big\| \frac{d}{dx} \int_{-\infty}^{\infty} K(x-y)[g_1(u_0(y) + u_{p,2}(y)) - g_2(u_0(y) + u_{p,2}(y))] dy \Big\|_{L^2(\mathbb{R})} \le$$

$$\le \Big\| \frac{dK}{dx} \Big\|_{L^1(\mathbb{R})} \|g_1(u_0(x) + u_{p,2}(x)) - g_2(u_0(x) + u_{p,2}(x))\|_{L^2(\mathbb{R})}. \tag{3.12}$$

Estimates (3.11) and (3.12) easily imply

$$\Big\| \int_{-\infty}^{\infty} K(x-y)[g_1(u_0(y)+u_{p,2}(y)) - g_2(u_0(y)+u_{p,2}(y))]dy \Big\|_{H^1(\mathbb{R})} \leq$$

$$\leq \|K\|_{W^{1,1}(\mathbb{R})} \|g_1(u_0(x)+u_{p,2}(x)) - g_2(u_0(x)+u_{p,2}(x))\|_{L^2(\mathbb{R})}. \tag{3.13}$$

Evidently,

$$g_1(u_0(x)+u_{p,2}(x)) - g_2(u_0(x)+u_{p,2}(x)) = \int_0^{u_0(x)+u_{p,2}(x)} [g_1'(z) - g_2'(z)]dz. \tag{3.14}$$

From (3.14) we deduce

$$|g_1(u_0(x)+u_{p,2}(x)) - g_2(u_0(x)+u_{p,2}(x))| \leq max_{z\in I}|g_1'(z) - g_2'(z)||u_0(x)+u_{p,2}(x)|$$

$$\leq \|g_1(z) - g_2(z)\|_{C_1(I)} |u_0(x)+u_{p,2}(x)|, \tag{3.15}$$

so that

$$\|g_1(u_0(x)+u_{p,2}(x)) - g_2(u_0(x)+u_{p,2}(x))\|_{L^2(\mathbb{R})} \leq$$

$$\leq \|g_1(z) - g_2(z)\|_{C_1(I)} (\|u_0\|_{H^1(\mathbb{R})} + 1). \tag{3.16}$$

By virtue of upper bounds (3.9), (3.10), (3.13), (3.16) obtained above, we derive

$$\|r(x) - u_{p,2}(x)\|_{H^1(\mathbb{R})} \leq$$

$$\leq c_a \|T\| (\|u_0\|_{H^1(\mathbb{R})} + 1)^2 \|K\|_{W^{1,1}(\mathbb{R})} \|g_1(z) - g_2(z)\|_{C_1(I)}. \tag{3.17}$$

Inequalities (3.5) and (3.17) give us

$$\|u_{p,1}(x) - u_{p,2}(x)\|_{H^1(\mathbb{R})} \leq$$

$$\leq \frac{c_a}{1-\sigma} \|T\| (\|u_0\|_{H^1(\mathbb{R})} + 1)^2 \|K\|_{W^{1,1}(\mathbb{R})} \|g_1(z) - g_2(z)\|_{C_1(I)}. \tag{3.18}$$

By means of (1.19) along with (3.18) and definition (1.18) estimate (1.20) holds. ■

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