SOLVABILITY OF SOME INTEGRO-DIFFERENTIAL EQUATIONS WITH TRANSPORT AND CONCENTRATED SOURCES

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Abstract: The work deals with the existence of solutions of an integro-differential equation in the case of the normal diffusion and the influx/efflux term proportional to the Dirac delta function in the presence of the drift term. The proof of the existence of solutions relies on a fixed point technique. We use the solvability conditions for the non-Fredholm elliptic operators in unbounded domains and discuss how the introduction of the transport term influences the regularity of the solutions.

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1. Introduction

In the present article we establish the existence of stationary solutions of the following nonlocal reaction-diffusion equation

$$\frac{\partial u}{\partial t} = D\frac{\partial^2 u}{\partial x^2} + b\frac{\partial u}{\partial x} + \int_{-\infty}^{\infty} K(x-y)g(w(y)u(y,t))dy + \alpha\delta(x), \tag{1.1}$$

where the constants $b, \ \alpha \in \mathbb{R}$ are nontrivial and $w(x)$ is the cut-off function. The conditions on it will be stated further down. The equations of this kind are used in the cell population dynamics. The solvability of the problem analogical to (1.1)

without the transport term was studied in [35]. Emergence and propagation of patterns in nonlocal reaction-diffusion equations arising in the theory of speciation and containing the transport term were discussed in [33]. The space variable x here corresponds to the cell genotype, $u(x, t)$ stands for the cell density as a function of their genotype and time. The right side of (1.1) describes the evolution of the cell density via the cell proliferation, mutations and the cell influx/efflux. The diffusion term here corresponds to the change of genotype by means of the small random mutations, and the nonlocal term describes large mutations. The function $g(w(x)u(x))$ denotes the rate of cell birth which depends on u, w (density dependent proliferation), and the kernel $K(x-y)$ gives the proportion of newly born cells, which change their genotype from y to x. Let us assume that it depends on the distance between the genotypes. Finally, the last term in the right side of our equation, which is proportional to the Dirac delta function is the influx/efflux of cells for different genotypes. A similar equation on the real line in the case of the standard negative Laplace operator raised to the power $0 < s < \frac{1}{4}$ in the diffusion term was discussed recently in [42]. But in the article [42] it was assumed that the influx/efflux term $f(x) \in L^1(\mathbb{R}) \cap L^2(\mathbb{R})$. Thus, in the present work we address the more singular situation. In neuroscience, the integro-differential problems describe the nonlocal interaction of neurons (see [9] and the references therein).
We set $D = 1$ and demonstrate the existence of solutions of the equation

$$\frac{d^2u}{dx^2} + b\frac{du}{dx} + \int_{-\infty}^{\infty} K(x-y)g(w(y)u(y))dy + \alpha\delta(x) = 0. \tag{1.2}$$

Let us discuss the situation when the linear part of such operator does not satisfy the Fredholm property. As a consequence, the conventional methods of the nonlinear analysis may not be applicable. We use the solvability conditions for the non-Fredholm operators along with the method of contraction mappings.
Consider the problem

$$-\Delta u + V(x)u - au = f, \tag{1.3}$$

where $u \in E = H^2(\mathbb{R}^d)$ and $f \in F = L^2(\mathbb{R}^d)$, $d \in \mathbb{N}$, a is a constant and the scalar potential function $V(x)$ is either trivial or tends to 0 at infinity. For $a \geq 0$, the essential spectrum of the operator $A : E \to F$, which corresponds to the left side of equation (1.3) contains the origin. As a consequence, such operator does not satisfy the Fredholm property. Its image is not closed, for $d > 1$ the dimension of its kernel and the codimension of its image are not finite. The present article is deals with the studies of the certain properties of the operators of this kind. Note that the elliptic equations involving the non-Fredholm operators were treated actively in recent years. Approaches in weighted Sobolev and Hölder spaces were developed in [4], [5], [7], [8], [6]. The non-Fredholm Schrödinger type operators were studied with the methods of the spectral and the scattering theory in [17], [20], [30], [32], [36], [37]. Fredholm structures, topological invariants and their applications were covered in [13]. The article [14] deals with the finite and infinite dimensional attractors for the evolution problems of the mathematical physics. The large time behavior of the solutions of a class of fourth-order parabolic equations defined on unbounded domains via the Kolmogorov ε-entropy as a measure was investigated in [15]. The attractor for a nonlinear reaction-diffusion system in an unbounded domain in $\mathbb{R}^3$ was studied in [22]. The works [24] and [29] are important for the understanding of the Fredholm and properness properties of quasilinear elliptic systems of the second order and of the operators of this kind on $\mathbb{R}^N$. The exponential decay and Fredholm properties in the second-order quasilinear elliptic systems were covered in [25]. The Laplace operator with drift from the point of view of the non-Fredholm operators was considered in [39] and [29] and the linearized Cahn-Hilliard equations in [32] and [40]. The nonlinear non-Fredholm elliptic problems were considered in [16], [18], [19], [20], [21], [31], [38], [41], [42]. The interesting applications to the theory of the reaction-diffusion equations were developed in [11], [12]. The non-Fredholm operators arise also when studying wave systems with an infinite number of localized traveling waves (see [2]). The standing lattice solitons in the discrete NLS problem with saturation were considered in [3]. In particular, when $a = 0$ our operator A is Fredholm in certain properly chosen weighted spaces (see [4], [5], [7], [8], [6]). However, the case of $a \neq 0$ is considerably different and the method developed in these works cannot be used. The existence, stability and bifurcations of the solutions of the nonlinear partial differential equations involving the Dirac delta function potentials were treated actively in [1], [23], [26], [27].

Let us set $K(x) = \varepsilon\mathcal{K}(x)$ with $\varepsilon \geq 0$. When our nonnegative parameter ε is trivial, we arrive at the linear Poisson type equation with drift, namely

$$-\frac{d^2u}{dx^2} - b\frac{du}{dx} = \alpha\delta(x), \tag{1.4}$$

where $b, \ \alpha \in \mathbb{R}$ and $b, \ \alpha \neq 0$ are the constants. It can be trivially checked that problem (1.4) admits a continuous solution, which vanishes on the negative semi-axis. It is given by

$$u_0(x) := \begin{cases} \frac{\alpha}{b}(e^{-bx} - 1), & x \geq 0 \\ 0, & x < 0 \end{cases} \tag{1.5}$$

Clearly, $u_0(x)$ does not belong to $H^1(\mathbb{R})$. It is bounded for $b > 0$ and it is unbounded if $b < 0$. Let us recall the analogous situation described in [35]. The solution of the Poisson equation without the drift term considered there was proportional to the ramp function. It was unbounded and it did not belong to $H^1(\mathbb{R})$. In [42] the authors were dealing with the Poisson type equation involving the fractional Laplacian and the transport term. Its bounded solution was contained in $H^1(\mathbb{R})$. Let us suppose that the assumption below is fulfilled.

Assumption 1.1. *Let $\mathcal{K}(x) : \mathbb{R} \to \mathbb{R}$ be nontrivial, so that $\mathcal{K}(x), \ x\mathcal{K}(x) \in L^1(\mathbb{R})$ and orthogonality relation (4.2) is valid. We also assume that the cut-off function*

$w(x) : \mathbb{R} \to \mathbb{R}$ *is such that* $w(x)u_0(x)$ *is nontrivial and* $w(x)u_0(x) \in H^1(\mathbb{R})$. *Furthermore,* $w(x) \in H^1(\mathbb{R})$ *and for* $b,\ \alpha \in \mathbb{R},\ b,\ \alpha \neq 0$ *the inequality*

$$\|w(x)u_0(x)\|_{H^1(\mathbb{R})} \le 1 \tag{1.6}$$

holds.

It can be trivially checked that $w(x) = e^{-2|b||x|},\ x \in \mathbb{R}$ satisfies the conditions above. Thus, it can be used as our cut-off function. Note that in the argument of [42] such cut-off function was not needed due to the more regular behaviour of the solution of the Poisson type equation. In our work we choose the space dimension $d = 1$, which is related to the solvability of the linear Poisson type equation (1.4) discussed above. From the point of view of the applications, the space dimension is not restricted to $d = 1$ because the space variable corresponds to the cell genotype but not to the usual physical space. We use the Sobolev space

$$H^1(\mathbb{R}) := \Big\{u(x) : \mathbb{R} \to \mathbb{R} \mid u(x) \in L^2(\mathbb{R}),\ \frac{du}{dx} \in L^2(\mathbb{R})\Big\}.$$

It is equipped with the norm

$$\|u\|^2_{H^1(\mathbb{R})} := \|u\|^2_{L^2(\mathbb{R})} + \Big\|\frac{du}{dx}\Big\|^2_{L^2(\mathbb{R})}. \tag{1.7}$$

Obviously, by means of the standard Fourier transform (2.1), this norm can be expressed as

$$\|u\|^2_{H^1(\mathbb{R})} = \|\widehat{u}(p)\|^2_{L^2(\mathbb{R})} + \|p\widehat{u}(p)\|^2_{L^2(\mathbb{R})}. \tag{1.8}$$

By virtue of the Sobolev inequality in one dimension (see e.g. Sect 8.5 of [28]), the upper bound

$$\|u(x)\|_{L^\infty(\mathbb{R})} \le \frac{1}{\sqrt{2}}\|u(x)\|_{H^1(\mathbb{R})}. \tag{1.9}$$

holds. We seek the resulting solution of nonlinear equation (1.2) as

$$u(x) = u_0(x) + u_p(x). \tag{1.10}$$

Evidently, we arrive at the perturbative equation

$$-\frac{d^2u_p(x)}{dx^2} - b\frac{du_p(x)}{dx} = \varepsilon \int_{-\infty}^{\infty} \mathcal{K}(x-y)g(w(y)[u_0(y) + u_p(y)])dy. \tag{1.11}$$

Let us use a closed ball in our Sobolev space

$$B_\rho := \{u(x) \in H^1(\mathbb{R}) \mid \|u\|_{H^1(\mathbb{R})} \le \rho\}, \quad 0 < \rho \le 1. \tag{1.12}$$

We look for the solution of equation (1.11) as the fixed point of the auxiliary nonlinear problem

$$-\frac{d^2u(x)}{dx^2} - b\frac{du(x)}{dx} = \varepsilon \int_{-\infty}^{\infty} \mathcal{K}(x-y)g(w(y)[u_0(y) + v(y)])dy \tag{1.13}$$

in ball (1.12). For a given function $v(y)$ this is an equation with respect to $u(x)$. The left side of (1.13) contains the operator

$$L_b := -\frac{d^2}{dx^2} - b\frac{d}{dx} \tag{1.14}$$

acting on $L^2(\mathbb{R})$. By means of the standard Fourier transform, it can be trivially checked that the essential spectrum of L_b is given by

$$\lambda_b(p) := p^2 - ibp, \quad p \in \mathbb{R}. \tag{1.15}$$

Since (1.15) contains the origin, L_b does not satisfy the Fredholm property, such operator has no bounded inverse. The similar situation in the context of the integro-differential equations occurred also in works [38] and [41]. The problems studied there also required the application of the orthogonality relations. The contraction argument was used in [34] to estimate the perturbation to the standing solitary wave of the Nonlinear Schrödinger (NLS) equation when either the external potential or the nonlinear term in the NLS were perturbed but the Schrödinger operator involved in the nonlinear equation there satisfied the Fredholm property (see Assumption 1 of [34], also [10]). Let us introduce the interval on the real line

$$I := \Big[-\frac{1}{\sqrt{2}} - \frac{1}{2}\|w(x)\|_{H^1(\mathbb{R})}, \ \frac{1}{\sqrt{2}} + \frac{1}{2}\|w(x)\|_{H^1(\mathbb{R})}\Big] \tag{1.16}$$

along with the closed ball in the space of $C^1(I)$ functions, namely

$$D_M := \{g(z) \in C^1(I) \mid \|g\|_{C^1(I)} \le M\}, \quad M > 0. \tag{1.17}$$

In this context the norm

$$\|g\|_{C^1(I)} := \|g\|_{C(I)} + \|g'\|_{C(I)}, \tag{1.18}$$

where $\|g\|_{C(I)} := \max_{z\in I}|g(z)|$. From the biological point of view, the rate of cell birth function is nonlinear and is trivial at the origin.

Assumption 1.2. *Let $g(z) : \mathbb{R} \to \mathbb{R}$, such that $g(0) = 0$. It is also assumed that $g(z) \in D_M$ and it does not vanish identically on the interval I.*

We recall the article [42]. The function $g(z)$ there was assumed to be twice continuously differentiable on the corresponding interval I. Let us use the following positive auxiliary expression

$$Q := \max\left\{\left\|\frac{\widehat{\mathcal{K}}(p)}{p^2 - ibp}\right\|_{L^\infty(\mathbb{R})}, \ \left\|\frac{\widehat{\mathcal{K}}(p)}{p - ib}\right\|_{L^\infty(\mathbb{R})}\right\}. \tag{1.19}$$

We introduce the operator T_g, so that $u = T_g v$, where u is a solution of equation (1.13). Our first main statement is as follows.

Theorem 1.3. *Let Assumptions 1.1 and 1.2 hold. Then equation (1.13) defines the map $T_g : B_\rho \to B_\rho$, which is a strict contraction for all*

$$0 < \varepsilon \leq \frac{\rho}{2\sqrt{\pi}QM\Big(1 + \frac{1}{\sqrt{2}}\|w(x)\|_{H^1(\mathbb{R})}\Big)}. \tag{1.20}$$

The unique fixed point $u_p(x)$ of this map T_g is the only solution of problem (1.11) in B_ρ.

Clearly, the resulting solution of equation (1.2) given by formula (1.10) will not vanish identically on the real line, because $g(0) = 0$ and $\alpha \neq 0$ due to our assumptions.

Our second main proposition is about the continuity of the cumulative solution of problem (1.2) given by (1.10) with respect to the nonlinear function g. We introduce the following positive, auxiliary quantity

$$\sigma := \sqrt{2\pi}QM\|w(x)\|_{H^1(\mathbb{R})}. \tag{1.21}$$

Theorem 1.4. *Let $j = 1, 2$, suppose that the assumptions of Theorem 1.3 hold, such that $u_{p,j}(x)$ is the unique fixed point of the map $T_{g_j} : B_\rho \to B_\rho$, which is a strict contraction for all the values of ε satisfying (1.20) and the resulting solution of problem (1.2) with $g(z) = g_j(z)$ is given by*

$$u_j(x) = u_0(x) + u_{p,j}(x). \tag{1.22}$$

Then for all values of ε, which satisfy inequality (1.20), the estimate

$$\|u_1(x) - u_2(x)\|_{H^1(\mathbb{R})} \leq$$

$$\leq \frac{2\sqrt{\pi}\varepsilon Q\Big(1 + \frac{1}{\sqrt{2}}\|w(x)\|_{H^1(\mathbb{R})}\Big)}{1 - \varepsilon\sigma}\|g_1(z) - g_2(z)\|_{C^1(I)} \tag{1.23}$$

is valid.

We proceed to the proof of our first main result.

2. The existence of the perturbed solution

Proof of Theorem 1.3. Let us choose arbitrarily $v(x) \in B_\rho$ and denote the term contained in the integral expression in the right side of problem (1.13) as

$$G(x) := g(w(x)[u_0(x) + v(x)]).$$

The standard Fourier transform is defined as

$$\widehat{\phi}(p) := \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \phi(x) e^{-ipx} dx, \quad p \in \mathbb{R}. \tag{2.1}$$

Evidently, the inequality

$$\|\widehat{\phi}(p)\|_{L^\infty(\mathbb{R})} \leq \frac{1}{\sqrt{2\pi}} \|\phi(x)\|_{L^1(\mathbb{R})} \tag{2.2}$$

is valid. We apply (2.1) to both sides of problem (1.13). This gives us

$$\widehat{u}(p) = \varepsilon\sqrt{2\pi} \frac{\widehat{\mathcal{K}}(p)\widehat{G}(p)}{p^2 - ibp}, \quad p\widehat{u}(p) = \varepsilon\sqrt{2\pi} \frac{\widehat{\mathcal{K}}(p)\widehat{G}(p)}{p - ib},$$

so that

$$|\widehat{u}(p)| \leq \varepsilon\sqrt{2\pi} Q |\widehat{G}(p)|, \quad |p\widehat{u}(p)| \leq \varepsilon\sqrt{2\pi} Q |\widehat{G}(p)|, \tag{2.3}$$

where Q is defined in (1.19). Note that under the stated assumptions $Q < \infty$ by means of Lemma 4.1 below. By virtue of (1.8) along with (2.3) we easily estimate the norm as

$$\|u(x)\|^2_{H^1(\mathbb{R})} \leq 4\pi\varepsilon^2 Q^2 \|G(x)\|^2_{L^2(\mathbb{R})}. \tag{2.4}$$

It can be trivially checked that for $v(x) \in B_\rho$, we have

$$|w(x)[u_0(x) + v(x)]| \leq \frac{1}{\sqrt{2}} + \frac{1}{2} \|w(x)\|_{H^1(\mathbb{R})}. \tag{2.5}$$

Indeed, the left side of (2.5) can be bounded from above using inequalities (1.6) and (1.9) by

$$\|w(x)u_0(x)\|_{L^\infty(\mathbb{R})} + \|w(x)\|_{L^\infty(\mathbb{R})} \|v(x)\|_{L^\infty(\mathbb{R})} \leq$$

$$\leq \frac{1}{\sqrt{2}} \|w(x)u_0(x)\|_{H^1(\mathbb{R})} + \frac{1}{\sqrt{2}} \|w(x)\|_{H^1(\mathbb{R})} \frac{1}{\sqrt{2}} \|v(x)\|_{H^1(\mathbb{R})} \leq$$

$$\leq \frac{1}{\sqrt{2}} + \frac{1}{2} \|w(x)\|_{H^1(\mathbb{R})}.$$

Similarly, for $v(x) \in B_\rho$ the estimate

$$\|w(x)[u_0(x) + v(x)]\|_{L^2(\mathbb{R})} \leq 1 + \frac{1}{\sqrt{2}} \|w(x)\|_{H^1(\mathbb{R})} \tag{2.6}$$

holds. Clearly, the left side of (2.6) can be estimated from above by virtue of (1.6) and (1.9) by

$$\|w(x)u_0(x)\|_{L^2(\mathbb{R})} + \|w(x)v(x)\|_{L^2(\mathbb{R})} \leq \|w(x)u_0(x)\|_{H^1(\mathbb{R})} +$$

$$+ \|w(x)\|_{L^\infty(\mathbb{R})} \|v(x)\|_{L^2(\mathbb{R})} \leq 1 + \frac{1}{\sqrt{2}} \|w(x)\|_{H^1(\mathbb{R})}.$$

Obviously,

$$G(x) = \int_0^{w(x)[u_0(x)+v(x)]} g'(z)dz.$$

Hence,

$$|G(x)| \leq \max_{z \in I} |g'(z)| |w(x)[u_0(x)+v(x)]| \leq M|w(x)[u_0(x)+v(x)]|, \quad (2.7)$$

with the interval I is defined in (1.16). By means of (2.7) along with (2.6) we arrive at

$$\|G(x)\|_{L^2(\mathbb{R})} \leq M\|w(x)[u_0(x)+v(x)]\|_{L^2(\mathbb{R})} \leq M\Big(1 + \frac{1}{\sqrt{2}}\|w(x)\|_{H^1(\mathbb{R})}\Big). \quad (2.8)$$

Upper bounds (2.4) and (2.8) give us

$$\|u(x)\|_{H^1(\mathbb{R})} \leq 2\sqrt{\pi}\varepsilon QM\Big(1 + \frac{1}{\sqrt{2}}\|w(x)\|_{H^1(\mathbb{R})}\Big) \leq \rho \quad (2.9)$$

for all the values of the parameter ε, which satisfy (1.20). Thus, $u(x) \in B_\rho$ as well. Let us suppose that for a certain $v(x) \in B_\rho$ there exist two solutions $u_{1,2}(x) \in B_\rho$ of problem (1.13). Their difference $w(x) := u_1(x) - u_2(x) \in L^2(\mathbb{R})$ solves the homogeneous equation

$$-\frac{d^2w(x)}{dx^2} - b\frac{dw(x)}{dx} = 0.$$

The operator L_b defined in (1.14) and considered on the whole real line does not have any nontrivial square integrable zero modes, such that $w(x)$ vanishes identically on $\mathbb{R}$. Therefore, problem (1.13) defines a map $T_g : B_\rho \to B_\rho$ for all the values of ε, which satisfy inequality (1.20).

Let us demonstrate that under the stated assumptions this map is a strict contraction. We choose arbitrarily $v_{1,2}(x) \in B_\rho$. By virtue of the argument above $u_{1,2} := T_g v_{1,2} \in B_\rho$ as well for ε satisfying inequality (1.20). By means of (1.13), we have precisely

$$-\frac{d^2u_1(x)}{dx^2} - b\frac{du_1(x)}{dx} = \varepsilon\int_{-\infty}^{\infty} \mathcal{K}(x-y)g(w(y)[u_0(y)+v_1(y)])dy, \quad (2.10)$$

$$-\frac{d^2u_2(x)}{dx^2} - b\frac{du_2(x)}{dx} = \varepsilon\int_{-\infty}^{\infty} \mathcal{K}(x-y)g(w(y)[u_0(y)+v_2(y)])dy. \quad (2.11)$$

Let us introduce

$$G_1(x) := g(w(x)[u_0(x)+v_1(x)]), \quad G_2(x) := g(w(x)[u_0(x)+v_2(x)])$$

and apply the standard Fourier transform (2.1) to both sides of problems (2.10) and (2.11). This gives us

$$\widehat{u_1}(p) = \varepsilon\sqrt{2\pi}\frac{\widehat{\mathcal{K}}(p)\widehat{G_1}(p)}{p^2 - ibp}, \quad \widehat{u_2}(p) = \varepsilon\sqrt{2\pi}\frac{\widehat{\mathcal{K}}(p)\widehat{G_2}(p)}{p^2 - ibp},$$

so that

$$\widehat{u_1}(p) - \widehat{u_2}(p) = \varepsilon\sqrt{2\pi}\frac{\widehat{\mathcal{K}}(p)[\widehat{G_1}(p) - \widehat{G_2}(p)]}{p^2 - ibp},$$

$$p[\widehat{u_1}(p) - \widehat{u_2}(p)] = \varepsilon\sqrt{2\pi}\frac{\widehat{\mathcal{K}}(p)[\widehat{G_1}(p) - \widehat{G_2}(p)]}{p - ib}.$$

Thus, the upper bounds

$$|\widehat{u_1}(p) - \widehat{u_2}(p)| \leq \varepsilon\sqrt{2\pi}Q|\widehat{G_1}(p) - \widehat{G_2}(p)|,$$

$$|p[\widehat{u_1}(p) - \widehat{u_2}(p)]| \leq \varepsilon\sqrt{2\pi}Q|\widehat{G_1}(p) - \widehat{G_2}(p)|$$

hold. This enables us to estimate the norm via (1.8) as

$$\|u_1(x) - u_2(x)\|^2_{H^1(\mathbb{R})} = \int_{-\infty}^{\infty} |\widehat{u_1}(p) - \widehat{u_2}(p)|^2 dp + \int_{-\infty}^{\infty} |p(\widehat{u_1}(p) - \widehat{u_2}(p))|^2 dp \leq$$

$$\leq 4\pi\varepsilon^2 Q^2 \|G_1(x) - G_2(x)\|^2_{L^2(\mathbb{R})},$$

so that

$$\|u_1(x) - u_2(x)\|_{H^1(\mathbb{R})} \leq 2\sqrt{\pi}\varepsilon Q\|G_1(x) - G_2(x)\|_{L^2(\mathbb{R})}. \tag{2.12}$$

Clearly, we have the equality

$$G_1(x) - G_2(x) = \int_{w(x)[u_0(x)+v_2(x)]}^{w(x)[u_0(x)+v_1(x)]} g'(z)dz,$$

such that $|G_1(x) - G_2(x)| \leq$

$$\leq \max_{z\in I}|g'(z)||w(x)(v_1(x) - v_2(x))| \leq M|w(x)(v_1(x) - v_2(x))|. \tag{2.13}$$

Let us obtain the upper bound on the right side of (2.13) using (1.9) as

$$M\|w(x)\|_{L^\infty(\mathbb{R})}|v_1(x) - v_2(x)| \leq \frac{M}{\sqrt{2}}\|w(x)\|_{H^1(\mathbb{R})}|v_1(x) - v_2(x)|.$$

This allows us to estimate the norm as

$$\|G_1(x) - G_2(x)\|_{L^2(\mathbb{R})} \leq \frac{M}{\sqrt{2}}\|w(x)\|_{H^1(\mathbb{R})}\|v_1(x) - v_2(x)\|_{L^2(\mathbb{R})} \leq$$

$$\leq \frac{M}{\sqrt{2}}\|w(x)\|_{H^1(\mathbb{R})}\|v_1(x) - v_2(x)\|_{H^1(\mathbb{R})}. \tag{2.14}$$

By means of (2.12) along with (2.14) we derive

$$\|u_1(x) - u_2(x)\|_{H^1(\mathbb{R})} \leq \sqrt{2\pi}\varepsilon QM\|w(x)\|_{H^1(\mathbb{R})}\|v_1(x) - v_2(x)\|_{H^1(\mathbb{R})}. \tag{2.15}$$

Evidently,

$$\frac{\rho}{2\sqrt{\pi}QM\Big(1+\frac{1}{\sqrt{2}}\|w(x)\|_{H^1(\mathbb{R})}\Big)} < \frac{1}{\sqrt{2\pi}QM\|w(x)\|_{H^1(\mathbb{R})}}.$$

By virtue of inequality (1.20) for our parameter ε we have

$$0<\varepsilon<\frac{1}{\sqrt{2\pi}QM\|w(x)\|_{H^1(\mathbb{R})}},$$

so that the constant in the right side of upper bound (2.15) is less than one. This implies that our map $T_g : B_\rho \to B_\rho$ defined by problem (1.13) is a strict contraction for all the values of ε satisfying (1.20). Its unique fixed point $u_p(x)$ is the only solution of equation (1.11) in the ball B_ρ. We easily deduce from (2.9) that $\|u_p(x)\|_{H^1(\mathbb{R})} \to 0$ as $\varepsilon \to 0$. The resulting $u(x)$ given by formula (1.10) is a solution of problem (1.2). ■

We turn our attention to the demonstration of the validity of the second main proposition of our work.

3. The continuity of the cumulative solution

Proof of Theorem 1.4. Clearly, for all the values of ε, which satisfy inequality (1.20), we have

$$u_{p,1} = T_{g_1}u_{p,1}, \quad u_{p,2} = T_{g_2}u_{p,2}.$$

Thus,

$$u_{p,1}-u_{p,2} = T_{g_1}u_{p,1} - T_{g_1}u_{p,2} + T_{g_1}u_{p,2} - T_{g_2}u_{p,2}.$$

We obtain

$$\|u_{p,1}-u_{p,2}\|_{H^1(\mathbb{R})} \le \|T_{g_1}u_{p,1} - T_{g_1}u_{p,2}\|_{H^1(\mathbb{R})} + \|T_{g_1}u_{p,2} - T_{g_2}u_{p,2}\|_{H^1(\mathbb{R})}.$$

By means of bound (2.15), we have

$$\|T_{g_1}u_{p,1} - T_{g_1}u_{p,2}\|_{H^1(\mathbb{R})} \le \varepsilon\sigma\|u_{p,1}-u_{p,2}\|_{H^1(\mathbb{R})},$$

where σ is defined in (1.21). Evidently, $\varepsilon\sigma < 1$, since the map $T_{g_1} : B_\rho \to B_\rho$ is a strict contraction under the stated assumptions. Hence, we arrive at

$$(1-\varepsilon\sigma)\|u_{p,1}-u_{p,2}\|_{H^1(\mathbb{R})} \le \|T_{g_1}u_{p,2} - T_{g_2}u_{p,2}\|_{H^1(\mathbb{R})}. \tag{3.1}$$

Evidently, for our fixed point $T_{g_2}u_{p,2} = u_{p,2}$. We introduce $\xi(x) := T_{g_1}u_{p,2}$. Therefore,

$$-\frac{d^2\xi(x)}{dx^2} - b\frac{d\xi(x)}{dx} = \varepsilon\int_{-\infty}^{\infty}\mathcal{K}(x-y)g_1(w(y)[u_0(y)+u_{p,2}(y)])dy, \tag{3.2}$$

$$-\frac{d^2u_{p,2}(x)}{dx^2} - b\frac{du_{p,2}(x)}{dx} = \varepsilon \int_{-\infty}^{\infty} \mathcal{K}(x-y)g_2(w(y)[u_0(y)+u_{p,2}(y)])dy. \quad (3.3)$$

Let us designate

$$G_{1,2}(x) := g_1(w(x)[u_0(x)+u_{p,2}(x)]),\ G_{2,2}(x) := g_2(w(x)[u_0(x)+u_{p,2}(x)]).$$

We apply the standard Fourier transform (2.1) to both sides of equations (3.2) and (3.3) above. This gives us

$$\widehat{\xi}(p) = \varepsilon\sqrt{2\pi}\frac{\widehat{\mathcal{K}}(p)\widehat{G_{1,2}}(p)}{p^2-ibp}, \quad \widehat{u_{p,2}}(p) = \varepsilon\sqrt{2\pi}\frac{\widehat{\mathcal{K}}(p)\widehat{G_{2,2}}(p)}{p^2-ibp},$$

so that

$$\widehat{\xi}(p) - \widehat{u_{p,2}}(p) = \varepsilon\sqrt{2\pi}\frac{\widehat{\mathcal{K}}(p)}{p^2-ibp}[\widehat{G_{1,2}}(p) - \widehat{G_{2,2}}(p)],$$

$$p[\widehat{\xi}(p) - \widehat{u_{p,2}}(p)] = \varepsilon\sqrt{2\pi}\frac{\widehat{\mathcal{K}}(p)}{p-ib}[\widehat{G_{1,2}}(p) - \widehat{G_{2,2}}(p)].$$

This allows us to derive the estimates from above

$$|\widehat{\xi}(p) - \widehat{u_{p,2}}(p)| \le \varepsilon\sqrt{2\pi}Q|\widehat{G_{1,2}}(p) - \widehat{G_{2,2}}(p)|, \quad (3.4)$$

$$|p[\widehat{\xi}(p) - \widehat{u_{p,2}}(p)]| \le \varepsilon\sqrt{2\pi}Q|\widehat{G_{1,2}}(p) - \widehat{G_{2,2}}(p)|. \quad (3.5)$$

By means of (3.4), we have

$$\|\widehat{\xi}(p) - \widehat{u_{p,2}}(p)\|^2_{L^2(\mathbb{R})} = \int_{-\infty}^{\infty} |\widehat{\xi}(p) - \widehat{u_{p,2}}(p)|^2 dp \le$$

$$\le 2\pi\varepsilon^2 Q^2 \|G_{1,2}(x) - G_{2,2}(x)\|^2_{L^2(\mathbb{R})}. \quad (3.6)$$

Similarly, using (3.5) we obtain

$$\|p[\widehat{\xi}(p) - \widehat{u_{p,2}}(p)]\|^2_{L^2(\mathbb{R})} = \int_{-\infty}^{\infty} |p[\widehat{\xi}(p) - \widehat{u_{p,2}}(p)]|^2 dp \le$$

$$\le 2\pi\varepsilon^2 Q^2 \|G_{1,2}(x) - G_{2,2}(x)\|^2_{L^2(\mathbb{R})}. \quad (3.7)$$

By virtue of (1.8) along with inequalities (3.6) and (3.7), the norm can be easily bounded above as

$$\|\xi(x) - u_{p,2}(x)\|^2_{H^1(\mathbb{R})} = \|\widehat{\xi}(p) - \widehat{u_{p,2}}(p)\|^2_{L^2(\mathbb{R})} + \|p[\widehat{\xi}(p) - \widehat{u_{p,2}}(p)]\|^2_{L^2(\mathbb{R})} \le$$

$$\le 4\pi\varepsilon^2 Q^2 \|G_{1,2}(x) - G_{2,2}(x)\|^2_{L^2(\mathbb{R})},$$

so that

$$\|\xi(x) - u_{p,2}(x)\|_{H^1(\mathbb{R})} \le 2\sqrt{\pi}\varepsilon Q \|G_{1,2}(x) - G_{2,2}(x)\|_{L^2(\mathbb{R})}. \quad (3.8)$$

Clearly,

$$G_{1,2}(x) - G_{2,2}(x) = \int_0^{w(x)[u_0(x)+u_{p,2}(x)]} [g_1'(z) - g_2'(z)]dz.$$

Hence,

$$|G_{1,2}(x) - G_{2,2}(x)| \leq \max_{z\in I}|g_1'(z) - g_2'(z)||w(x)[u_0(x) + u_{p,2}(x)]| \leq$$

$$\leq \|g_1(z) - g_2(z)\|_{C^1(I)}|w(x)[u_0(x) + u_{p,2}(x)]|.$$

This enables us to estimate the norm by means of (2.6) as

$$\|G_{1,2}(x) - G_{2,2}(x)\|_{L^2(\mathbb{R})} \leq \|g_1(z) - g_2(z)\|_{C^1(I)}\|w(x)[u_0(x) + u_{p,2}(x)]\|_{L^2(\mathbb{R})} \leq$$

$$\leq \|g_1(z) - g_2(z)\|_{C^1(I)}\Big(1 + \frac{1}{\sqrt{2}}\|w(x)\|_{H^1(\mathbb{R})}\Big). \tag{3.9}$$

Using (3.8) along with (3.9) we obtain $\|\xi(x) - u_{p,2}(x)\|_{H^1(\mathbb{R})} \leq$

$$\leq 2\sqrt{\pi}\varepsilon Q\|g_1(z) - g_2(z)\|_{C^1(I)}\Big(1 + \frac{1}{\sqrt{2}}\|w(x)\|_{H^1(\mathbb{R})}\Big). \tag{3.10}$$

By virtue of (3.1) and (3.10) we arrive at $\|u_{p,1}(x) - u_{p,2}(x)\|_{H^1(\mathbb{R})} \leq$

$$\leq \frac{2\sqrt{\pi}\varepsilon Q\Big(1 + \frac{1}{\sqrt{2}}\|w(x)\|_{H^1(\mathbb{R})}\Big)}{1 - \varepsilon\sigma}\|g_1(z) - g_2(z)\|_{C^1(I)}. \tag{3.11}$$

Equalities (1.22) along with inequality (3.11) imply the validity of (1.23). ■

4. Auxiliary results

Let us obtain the conditions under which the expression Q introduced in (1.19) is finite. We denote the inner product as

$$(f(x), g(x))_{L^2(\mathbb{R})} := \int_{-\infty}^{\infty} f(x)\bar{g}(x)dx, \tag{4.1}$$

with a slight abuse of notations when the functions involved in (4.1) do not belong to $L^2(\mathbb{R})$, like for instance the ones present in orthogonality relation (4.2) of Lemma 4.1 below. Indeed, if $f(x) \in L^1(\mathbb{R})$ and $g(x) \in L^\infty(\mathbb{R})$, then the integral in the right side of (4.1) is well defined. The proof of Lemma 4.1 was partially presented in the part b) of Lemma A1 of [16]. Let us present it here for the convenience of the readers.

Lemma 4.1. *Let $\mathcal{K}(x) : \mathbb{R} \to \mathbb{R}$ be nontrivial, so that $\mathcal{K}(x),\ x\mathcal{K}(x) \in L^1(\mathbb{R})$. Then $Q < \infty$ if and only if the orthogonality condition*

$$(\mathcal{K}(x), 1)_{L^2(\mathbb{R})} = 0 \tag{4.2}$$

is valid.

Proof. It can be trivially checked using (2.2) that $\dfrac{\widehat{\mathcal{K}}(p)}{p - ib} \in L^\infty(\mathbb{R})$. Indeed, we have

$$\left|\frac{\widehat{\mathcal{K}}(p)}{p - ib}\right| = \frac{|\widehat{\mathcal{K}}(p)|}{\sqrt{p^2 + b^2}} \leq \frac{1}{\sqrt{2\pi}b}\|\mathcal{K}(x)\|_{L^1(\mathbb{R})} < \infty$$

as assumed. Note that when the drift constant b vanishes, the situation here becomes more singular. Clearly, we can write

$$\widehat{\mathcal{K}}(p) = \widehat{\mathcal{K}}(0) + \int_0^p \frac{d\widehat{\mathcal{K}}(q)}{dq}dq,$$

so that

$$\frac{\widehat{\mathcal{K}}(p)}{p^2 - ibp} = \frac{\widehat{\mathcal{K}}(0)}{p^2 - ibp} + \frac{\int_0^p \frac{d\widehat{\mathcal{K}}(q)}{dq}dq}{p^2 - ibp}. \tag{4.3}$$

From the definition of the standard Fourier transform (2.1) it can be easily derived that

$$\left|\frac{d\widehat{\mathcal{K}}(p)}{dp}\right| \leq \frac{1}{\sqrt{2\pi}}\|x\mathcal{K}(x)\|_{L^1(\mathbb{R})}.$$

Hence,

$$\left|\frac{\int_0^p \frac{d\widehat{\mathcal{K}}(q)}{dq}dq}{p^2 - ibp}\right| \leq \frac{1}{\sqrt{2\pi}|b|}\|x\mathcal{K}(x)\|_{L^1(\mathbb{R})} < \infty$$

via the one of our assumptions. By virtue of definition (2.1), we have

$$\widehat{\mathcal{K}}(0) = \frac{1}{\sqrt{2\pi}}(\mathcal{K}(x), 1)_{L^2(\mathbb{R})}.$$

Therefore, the first term in the right side of (4.3) is given by

$$\frac{(\mathcal{K}(x), 1)_{L^2(\mathbb{R})}}{\sqrt{2\pi}(p^2 - ibp)}. \tag{4.4}$$

Obviously, expression (4.4) is bounded if and only if orthogonality relation (4.2) is valid. ∎

Note that as distinct from the similar proposition in the situation without a drift term discussed in [35], the statement of Lemma 4.1 above relies only on a single orthogonality condition (4.2) and the argument of the proof is less cumbersome. The argument of [42] does not use the orthogonality relations at all.

References

[1] R. Adami, D. Noja, *Exactly solvable models and bifurcations: the case of the cubic NLS with a δ or a δ' interaction in dimension one*, Math. Model. Nat. Phenom., **9** (2014), no. 5, 1–16.

[2] G.L. Alfimov, E.V. Medvedeva, D.E. Pelinovsky, *Wave systems with an infinite number of localized traveling waves*, Phys. Rev. Lett., **112** (2014), 054103, 5 pp.

[3] G.L. Alfimov, A.S. Korobeinikov, C.J. Lustri, D.E. Pelinovsky, *Standing lattice solitons in the discrete NLS equation with saturation*, Nonlinearity, **32** (2019), no. 9, 3445–3484.

[4] C. Amrouche, V. Girault, J. Giroire, *Dirichlet and Neumann exterior problems for the n-dimensional Laplace operator: an approach in weighted Sobolev spaces*, J. Math. Pures Appl. (9), **76** (1997), no. 1, 55–81.

[5] C. Amrouche, F. Bonzom, *Mixed exterior Laplace's problem*, J. Math. Anal. Appl., **338** (2008), no. 1, 124–140.

[6] N. Benkirane, *Propriétés d'indice en théorie höldérienne pour des opérateurs elliptiques dans R^n*, C. R. Acad. Sci. Paris Sér. I Math., **307** (1988), no. 11, 577–580.

[7] P. Bolley, T.L. Pham, *Propriétés d'indice en théorie höldérienne pour des opérateurs différentiels elliptiques dans R^n*, J. Math. Pures Appl. (9), **72** (1993), no. 1, 105–119.

[8] P. Bolley, T.L. Pham, *Propriété d'indice en théorie Höldérienne pour le problème extérieur de Dirichlet*, Comm. Partial Differential Equations, **26** (2001), no. 1-2, 315–334.

[9] S. Coombes, P. beim Graben, R. Potthast, *Tutorial on neural field theory*. Neural fields, Springer, Heidelberg (2014), 1–43.

[10] S. Cuccagna, D. Pelinovsky, V. Vougalter, *Spectra of positive and negative energies in the linearized NLS problem*, Comm. Pure Appl. Math., **58** (2005), no. 1, 1–29.

[11] A. Ducrot, M. Marion, V. Volpert, *Systemes de réaction-diffusion sans propriété de Fredholm*, C. R. Math. Acad. Sci. Paris, **340** (2005), no. 9, 659–664.

[12] A. Ducrot, M. Marion, V. Volpert, *Reaction-diffusion problems with non-Fredholm operators*, Adv. Differential Equations, **13** (2008), no. 11-12, 1151–1192.

[13] M. Efendiev, *Fredholm structures, topological invariants and applications*. AIMS Series on Differential Equations & Dynamical Systems, **3**. American Institute of Mathematical Sciences (AIMS), Springfield, MO (2009), 205 pp.

[14] M. Efendiev, *Finite and infinite dimensional attractors for evolution equations of mathematical physics*. GAKUTO International Series. Mathematical Sciences and Applications, **33**. $Gakk\bar{o}tosho\ Co.,\ Ltd., Tokyo$ (2010), 239 pp.

[15] M.A. Efendiev, L.A. Peletier, *On the large time behavior of solutions of fourth order parabolic equations and ε-entropy of their attractors*, C. R. Math. Acad. Sci. Paris, **344** (2007), no. 2, 93–96.

[16] M. Efendiev, V. Vougalter, *Solvability of some integro-differential equations with drift*, Osaka J. Math., **57** (2020), no. 2, 247–265.

[17] M. Efendiev, V. Vougalter, *Solvability in the sense of sequences for some fourth order non-Fredholm operators*, J. Differential Equations, **271** (2021), 280–300.

[18] M. Efendiev, V. Vougalter, *Existence of solutions for some non-Fredholm integro-differential equations with mixed diffusion*, J. Differential Equations, **284** (2021), 83–101.

[19] M. Efendiev, V. Vougalter, *On the solvability of some systems of integro-differential equations with drift*. Mathematical methods in modern complexity science, Nonlinear Syst. Complex., **33** (2022), Springer, Cham, 141–167.

[20] M. Efendiev, V. Vougalter, *Linear and nonlinear non-Fredholm operators and their applications*, Electron. Res. Arch., **30** (2022), no. 2, 515–534.

[21] M. Efendiev, V. Vougalter, *Existence of solutions for some systems of integro-differential equations with transport and superdiffusion*, Anal. Math. Phys., **12** (2022), no. 5, Paper No. 110.

[22] M.A. Efendiev, S.V. Zelik, *The attractor for a nonlinear reaction-diffusion system in an unbounded domain*, Comm. Pure Appl. Math., **54** (2001), no. 6, 625–688.

[23] R. Fukuizumi, L. Jeanjean, *Stability of standing waves for a nonlinear Schrödinger equation with a repulsive Dirac delta potential*, Discrete Contin. Dyn. Syst., **21** (2008), no. 1, 121–136.

[24] H.G. Gebran, C.A. Stuart, *Fredholm and properness properties of quasilinear elliptic systems of second order*, Proc. Edinb. Math. Soc. (2), **48** (2005), no. 1, 91–124.

[25] H.G. Gebran, C.A. Stuart, *Exponential decay and Fredholm properties in second order quasilinear elliptic systems*, J. Differential Equations, **249** (2010), no. 1, 94–117.

[26] J. Holmer, J. Marzuola, M. Zworski, *Fast soliton scattering by delta impurities,* Comm. Math. Phys, **274** (2007), no. 1, 187–216.

[27] S. Le Coz, R. Fukuizumi, G. Fibich, B. Ksherim, Y. Sivan, *Instability of bound states of a nonlinear Schrödinger equation with a Dirac potential*, Phys. D., **237** (2008), no. 8, 1103–1128.

[28] E.H. Lieb, M. Loss, *Analysis. Graduate Studies in Mathematics*, **14**, American Mathematical Society, Providence, RI (1997), 278 pp.

[29] P.J. Rabier, C.A. Stuart, *Fredholm and properness properties of quasilinear elliptic operators on* $\mathbb{R}^N$, Math. Nachr., **231** (2001), 129–168.

[30] V. Volpert, Elliptic partial differential equations. Volume 1: Fredholm theory of elliptic problems in unbounded domains. Monographs in Mathematics, **101**. Birkhäuser/Springer Basel AG, Basel (2011), 639 pp.

[31] V. Volpert, B. Kazmierczak, M. Massot, Z. Peradzynski, *Solvability conditions for elliptic problems with non-Fredholm operators*, Appl. Math. (Warsaw), **29** (2002), no. 2, 219–238.

[32] V. Volpert, V. Vougalter, *On the solvability conditions for a linearized Cahn-Hilliard equation*, Rend. Istit. Mat. Univ. Trieste, **43** (2011), 1–9.

[33] V. Volpert, V. Vougalter, *Emergence and propagation of patterns in nonlocal reaction-diffusion equations arising in the theory of speciation*. Dispersal, individual movement and spatial ecology, Lecture Notes in Math., **2071**, Springer, Heidelberg (2013), 331–353.

[34] V. Vougalter, *On threshold eigenvalues and resonances for the linearized NLS equation*, Math. Model. Nat. Phenom., **5** (2010), no. 4, 448–469.

[35] V. Vougalter, *Solvability of some integro-differential equations with concentrated sources,* Complex Var. Elliptic Equ., **67** (2022), no. 4, 975–987.

[36] V. Vougalter, V. Volpert, *On the solvability conditions for some non Fredholm operators*, Int. J. Pure Appl. Math., **60** (2010), no. 2, 169–191.

[37] V. Vougalter, V. Volpert, *Solvability conditions for some non-Fredholm operators*, Proc. Edinb. Math. Soc. (2), **54** (2011), no. 1, 249–271.

[38] V. Vougalter, V. Volpert, *On the existence of stationary solutions for some non-Fredholm integro-differential equations,* Doc. Math., **16** (2011), 561–580.

[39] V. Vougalter, V. Volpert, *On the solvability conditions for the diffusion equation with convection terms*, Commun. Pure Appl. Anal., **11** (2012), no. 1, 365–373.

[40] V. Vougalter, V. Volpert, *Solvability conditions for a linearized Cahn-Hilliard equation of sixth order*, Math. Model. Nat. Phenom., **7** (2012), no. 2, 146–154.

[41] V. Vougalter, V. Volpert, *Solvability conditions for some linear and nonlinear non-Fredholm elliptic problems*, Anal. Math. Phys., **2** (2012), no. 4, 473–496.

[42] V. Vougalter, V. Volpert, *Solvability of some integro-differential equations with anomalous diffusion and transport*, Anal. Math. Phys., **11** (2021), no. 3, Paper No. 135, 26 pp.