ON THE NECESSARY CONDITIONS FOR PRESERVING THE NONNEGATIVE CONE: DOUBLE SCALE ANOMALOUS DIFFUSION

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Abstract: The work deals with the easily verifiable necessary conditions of the preservation of the nonnegativity of the solutions of a system of parabolic equations in the case of the double scale anomalous diffusion when the fractional Laplacian is added to the negative Laplace operator raised to another fractional power in the space of two dimensions. Such necessary conditions are extremely important for the applied analysis society because they impose the necessary form of the system of equations that must be studied mathematically.

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1. Introduction

The solutions of many systems of convection-diffusion-reaction equations arising in biology, physics or engineering describe such quantities as population densities, pressure or concentrations of nutrients and chemicals. Hence, a natural property to require for the solutions is the nonnegativity. Models that do not guarantee the nonnegativity are not valid or break down for small values of the solution. In many situations, demonstrating that a particular model fails to preserve the nonnegativity

yields the better understanding of the model and its limitations. One of the first steps in analyzing ecological or biological or bio-medical models mathematically is to test whether solutions originating from the nonnegative initial data remain nonnegative (as long as they exist). In other words, the model under consideration ensures that the nonnegative cone is positively invariant. Let us recall that if the solutions (of a given evolution PDE) corresponding to the nonnegative initial data remain nonnegative as long as they exist, we say that the system satisfies the nonnegativity property.
For the scalar problems the nonnegativity property follows directly from the maximum principle (see [3] and the references therein). However, for the systems of equations the maximum principle fails to work. In the particular case of the monotone systems the situation is similar to the case of the scalar equations, the sufficient conditions for preserving the nonnegative cone can be found in [3].
The goal of the present article is to establish a simple and easily verifiable criterion, that is, the necessary condition for the nonnegativity of solutions of the systems of nonlinear convection-double scale anomalous diffusion- reaction equations relevant to the modelling of the life sciences. We believe that it could provide a modeler with a tool, which is easy to verify, to approach the issue of the positive invariance of the model.
Our work is devoted to the preservation of the nonnegativity of solutions of the following system of reaction-diffusion equations

$$\frac{\partial u}{\partial t} = -A[(-\Delta_x)^{s_1} + (-\Delta_x)^{s_2}]u + \sum_{l=1}^{m} \Gamma^l \frac{\partial u}{\partial x_l} - F(u). \tag{1.1}$$

Here Δ_x is the standard Laplacian with respect to the x variable, $A,\ \Gamma^l,\ 1 \le l \le m$ are $N \times N$ matrices with constant coefficients, which is relevant to the cell population dynamics in the Mathematical Biology. We call system (1.1) as a (N, m) one. Note that the analogical model can be used to study such branches of science as the Damage Mechanics, the temperature distribution in Thermodynamics. In the present article the space variable x corresponds to the cell genotype, $u_k(x, t)$ stands for the cell density distributions for various groups of cells as functions of their genotype and time,

$$u(x, t) = (u_1(x, t), u_2(x, t), ..., u_N(x, t))^T.$$

The fractional Laplacians in (1.1) describe a particular case of the anomalous diffusion actively used in the context of different applications in plasma physics and turbulence [1], [14], surface diffusion [10], [12], semiconductors [13] and so on. The anomalous diffusion can be viewed as a random process of the particle motion characterized by the probability density distribution of the jump length. The moments of this density distribution are finite in the case of the normal diffusion, but this is not the case for the anomalous diffusion. The asymptotic behavior at infinity of the probability density function determines the value of the power of the Laplacian [11]. The fractional Laplace operators involved in our system (1.1) are defined by means of the spectral calculus. The verification of biomedical processes with anomalous diffusion, transport and interaction of species in the case of the one spatial dimension with a single Laplacian raised to the power $0 < s < \frac{1}{4}$ in the diffusion term was performed in [5] (see also [7]). The similar ideas in the space the of d dimensions, $d \in \mathbb{N},\ d \geq 2$ in the situation when the diffusion term of our system contains the sum of the standard Laplacian acting on the first m variables and the fractional Laplacian with respect to the remaining $d-m$ variables were exploited in [6]. In the article [4] the authors obtain the sufficient and necessary conditions for the positivity of solutions for a large class of quasi-linear parabolic systems. The positivity of solutions of systems of semi-linear parabolic equations under stochastic perturbations was analyzed in [2]. Front propagation problems with anomalous diffusion were studied actively in recent years (see e.g. [15], [16]). The solvability of the single equation containing the Laplacian with drift relevant to the fluid mechanics was treated in [18]. The existence of solutions of the generalized Poisson type equation involving the sum of two distinct fractional powers of a Schrödinger operator with a shallow, short-range potential was discussed in [7]. The nonlocal inverse problem for the space-time fractional equation characterizing the double scale anomalous diffusion was considered in [8]. For the simplicity of presentation we will consider the case of the two spatial dimensions with $0 < s_1 < s_2 < \frac{1}{2}$. Let us assume here that (1.1) involves the square matrices with the entries constant in space and time

$$(A)_{k,j} := a_{k,j}, \quad (\Gamma^l)_{k,j} := \gamma^l_{k,j}, \quad 1 \leq k, j \leq N, \quad l = 1, 2$$

and that the matrix $A + A^* > 0$ for the sake of the global well posedness of system (1.1). Here A^* denotes the adjoint of matrix A. Thus, problem (1.1) can be rewritten in the form

$$\frac{\partial u_k}{\partial t} = -\sum_{j=1}^{N} a_{k,j}[(-\Delta_x)^{s_1} + (-\Delta_x)^{s_2}]u_j + \sum_{l=1}^{2}\sum_{j=1}^{N} \gamma^l_{k,j}\frac{\partial u_j}{\partial x_l} - F_k(u) \qquad (1.2)$$

with $1 \leq k \leq N$ and $0 < s_1 < s_2 < \frac{1}{2}$. Note that in the two dimensional situation discussed in the present work the range of the powers of the fractional Laplacians is broader than in the one dimensional case covered in [5] and [7]. In our article the interaction of species term

$$F(u) = (F_1(u), F_2(u), ..., F_N(u))^T,$$

which can be linear or nonlinear. Let us assume its smoothness in our theorem below for the sake of the well posedness of system (1.1), although, we are not focused on the well posedness issue in the present work. We choose the space dimension $d=2$, which is related to the solvability conditions for the linear Poisson type problem (3.1) stated in Lemma 3.1 below. From the perspective of the applications, the space dimension is not restricted to $d=2$ since the space variable corresponds to cell genotype but not to the usual physical space. Let us denote the inner product as

$$(f(x), g(x))_{L^2(\mathbb{R}^2)} := \int_{\mathbb{R}^2} f(x)\bar{g}(x)dx, \tag{1.3}$$

with a slight abuse of notations when the functions involved in (1.3) do not belong to $L^2(\mathbb{R}^2)$, like for instance the one present in orthogonality condition (3.4) of Lemma 3.1 below. Indeed, if $f(x) \in L^1(\mathbb{R}^2)$ and $g(x) \in L^\infty(\mathbb{R}^2)$, then the integral in the right side of (1.3) is well defined. As for the vector functions, their inner product is defined using their components as

$$(u, v)_{L^2(\mathbb{R}^2, \mathbb{R}^N)} := \sum_{k=1}^{N} (u_k, v_k)_{L^2(\mathbb{R}^2)}. \tag{1.4}$$

Evidently, (1.4) induces the norm

$$\|u\|^2_{L^2(\mathbb{R}^2, \mathbb{R}^N)} = \sum_{k=1}^{N} \|u_k\|^2_{L^2(\mathbb{R}^2)}.$$

Let us use the Sobolev spaces

$$H^{2s}(\mathbb{R}^2) := \left\{u(x) : \mathbb{R}^2 \to \mathbb{R} \mid u(x) \in L^2(\mathbb{R}^2),\ (-\Delta)^s u \in L^2(\mathbb{R}^2)\right\}, \quad 0 < s \leq 1$$

equipped with the norm

$$\|u\|^2_{H^{2s}(\mathbb{R}^2)} := \|u\|^2_{L^2(\mathbb{R}^2)} + \|(-\Delta)^s u\|^2_{L^2(\mathbb{R}^2)}. \tag{1.5}$$

By the nonnegativity of a vector function below we mean the nonnegativity of the each of its components. Our main proposition is as follows.

Theorem 1.1. *Let $F : \mathbb{R}^N \to \mathbb{R}^N$, so that $F \in \mathbb{C}^1$, the initial condition for system (1.1) is $u(x, 0) = u_0(x) \geq 0$ and $u_0(x) \in L^2(\mathbb{R}^2, \mathbb{R}^N)$. We also assume that the off diagonal elements of the matrix A are nonnegative, so that*

$$a_{k,l} \geq 0, \quad 1 \leq k, l \leq N, \quad k \neq l. \tag{1.6}$$

Then the necessary condition for system (1.1) to have a solution $u(x, t) \geq 0$ for all $t \in [0, \infty)$ is that the matrices A and $\Gamma^l,\ l = 1, 2$ are diagonal and for all $1 \leq k \leq N$

$$F_k(s_1, ..., s_{k-1}, 0, s_{k+1}, ..., s_N) \leq 0 \tag{1.7}$$

is valid, where $s_l \geq 0$ *and* $1 \leq l \leq N,\ l \neq k$.

Remark 1.2. *If the interaction of species term is linear, namely when* $F(u) = Lu$, *where* L *is a matrix with elements* $b_{i,j},\ 1 \leq i, j \leq N$ *constant in space and time, our necessary condition above yields the condition that the matrix* L *must be essentially nonpositive, so that* $b_{i,j} \leq 0$ *for* $i \neq j,\ 1 \leq i, j \leq N$.

Remark 1.3. *The proof of our theorem yields that, the necessary condition for preserving the nonnegative cone is carried over from the ODE (the spatially homogeneous case, as described by the ordinary differential equation* $u'(t) = -F(u)$*) to the case of the double scale anomalous diffusion and the convective drift term.*

Remark 1.4. *In the forthcoming papers we intend to treat the following situations:*
a) the necessary and sufficient conditions of the present article,
b) the nonautonomous version of the present work,
c) the density-dependent diffusion matrix,
d) the effect of the delay term in the cases a), b) and c).

We turn our attention to the proof of our main result.

2. The preservation of the nonnegativity of the solution of the system of parabolic equations

Proof of Theorem 1.1. We note that the maximum principle actively exploited for the studies of the solutions of single parabolic equations does not apply to the systems of such equations. Let us consider a time independent, square integrable vector function $v(x)$ and estimate

$$\left(\frac{\partial u}{\partial t}\bigg|_{t=0}, v\right)_{L^2(\mathbb{R}^2,\mathbb{R}^N)} = \left(\lim_{t\to 0^+} \frac{u(x,t)-u_0(x)}{t}, v(x)\right)_{L^2(\mathbb{R}^2,\mathbb{R}^N)}.$$

By virtue of the continuity of the inner product, the right side of the equality above is equal to

$$\lim_{t\to 0^+} \frac{(u(x,t), v(x))_{L^2(\mathbb{R}^2,\mathbb{R}^N)}}{t} - \lim_{t\to 0^+} \frac{(u_0(x), v(x))_{L^2(\mathbb{R}^2,\mathbb{R}^N)}}{t}. \tag{2.1}$$

We choose the initial condition for our system $u_0(x) \geq 0$ and the constant in time vector function $v(x) \geq 0$ to be orthogonal to each other in $L^2(\mathbb{R}^2, \mathbb{R}^N)$. This can be achieved, for example for

$$u_0(x) = (\tilde{u}_1(x), ..., \tilde{u}_{k-1}(x), 0, \tilde{u}_{k+1}(x), ..., \tilde{u}_N(x)), \quad v_j(x) = \tilde{v}(x)\delta_{j,k}. \tag{2.2}$$

Here $1 \leq j \leq N$, $\delta_{j,k}$ stands for the Kronecker symbol and $1 \leq k \leq N$ is fixed. Thus, the second term in (2.1) vanishes and (2.1) is equal to

$$\lim_{t\to 0^+} \frac{\sum_{j=1}^N \int_{\mathbb{R}^2} u_j(x,t)v_j(x)dx}{t} \geq 0$$

because of the nonnegativity of all the components $u_j(x,t)$ and $v_j(x)$ contained in the formula above. Hence, we derive

$$\sum_{j=1}^N \int_{\mathbb{R}^2} \frac{\partial u_j}{\partial t}\Big|_{t=0} v_j(x)dx \geq 0.$$

By means of (2.2), only the k th component of the vector function $v(x)$ does not vanish identically. This gives us

$$\int_{\mathbb{R}^2} \frac{\partial u_k}{\partial t}\Big|_{t=0} \tilde{v}(x)dx \geq 0.$$

By virtue of (1.2), we obtain

$$\int_{\mathbb{R}^2} \Big[-\sum_{j=1,\ j\neq k}^N a_{k,j}[(-\Delta_x)^{s_1} + (-\Delta_x)^{s_2}]\tilde{u}_j(x) + \sum_{l=1}^2 \sum_{j=1,\ j\neq k}^N \gamma_{k,j}^l \frac{\partial \tilde{u}_j}{\partial x_l} -$$
$$-F_k(\tilde{u}_1(x), ..., \tilde{u}_{k-1}(x), 0, \tilde{u}_{k+1}(x), ..., \tilde{u}_N(x))\Big] \tilde{v}(x)dx \geq 0.$$

Because the nonnegative, square integrable function $\tilde{v}(x)$ can be chosen arbitrarily, we have

$$-\sum_{j=1,\ j\neq k}^N a_{k,j}[(-\Delta_x)^{s_1} + (-\Delta_x)^{s_2}]\tilde{u}_j(x) + \sum_{l=1}^2 \sum_{j=1,\ j\neq k}^N \gamma_{k,j}^l \frac{\partial \tilde{u}_j}{\partial x_l} -$$
$$-F_k(\tilde{u}_1(x), ..., \tilde{u}_{k-1}(x), 0, \tilde{u}_{k+1}(x), ..., \tilde{u}_N(x)) \geq 0 \quad a.e. \tag{2.3}$$

For the purpose of the scaling, we replace all the $\tilde{u}_j(x)$ by $\tilde{u}_j\left(\frac{x}{\varepsilon}\right)$ in the inequality above, where $\varepsilon > 0$ is a small parameter. This gives us

$$-\sum_{j=1,\ j\neq k}^N \frac{a_{k,j}}{\varepsilon^{2s_1}}(-\Delta_y)^{s_1}\tilde{u}_j(y) - \sum_{j=1,\ j\neq k}^N \frac{a_{k,j}}{\varepsilon^{2s_2}}(-\Delta_y)^{s_2}\tilde{u}_j(y) + \sum_{l=1}^2 \sum_{j=1,\ j\neq k}^N \frac{\gamma_{k,j}^l}{\varepsilon}\frac{\partial \tilde{u}_j(y)}{\partial y_l}$$
$$-F_k(\tilde{u}_1(y), ..., \tilde{u}_{k-1}(y), 0, \tilde{u}_{k+1}(y), ..., \tilde{u}_N(y)) \geq 0 \quad a.e. \tag{2.4}$$

with $0 < s_1 < s_2 < \frac{1}{2}$. Note that the terms in the left side of inequality (2.4) contain the three scales with respect to our small, positive parameter ε, as distinct from the case of the single fractional Laplacian in the diffusion term discussed in [5]. Obviously, the third term in the left side of (2.4) is the leading one as ε tends to zero. Let us choose

$$\tilde{u}_j(y) = Q_{j,l}(y)e^{-y_l sign \gamma_{k,j}^l}$$

in a neighborhood of the origin, smooth and decaying to zero at the infinities. Here $Q_{j,l}(y)$ is positive and independent of y_l. Then the left side of (2.4) can be made as negative as possible which will violate inequality (2.4). Clearly, the last term in the left side of (2.4) will remain bounded. Hence, for the matrices Γ^l involved in system (1.1), the off diagonal terms should vanish, so that

$$\gamma_{k,j}^l = 0, \quad 1 \le k, j \le N, \quad k \ne j, \quad l = 1, 2.$$

Thus, from (2.4) we derive

$$-\sum_{j=1,\ j\ne k}^{N} \frac{a_{k,j}}{\varepsilon^{2s_1}}(-\Delta_y)^{s_1}\tilde{u}_j(y) - \sum_{j=1,\ j\ne k}^{N} \frac{a_{k,j}}{\varepsilon^{2s_2}}(-\Delta_y)^{s_2}\tilde{u}_j(y) -$$

$$-F_k(\tilde{u}_1(y), ..., \tilde{u}_{k-1}(y), 0, \tilde{u}_{k+1}(y), ..., \tilde{u}_N(y)) \ge 0 \quad a.e. \tag{2.5}$$

Clearly, the second term in the left side of (2.5) is the leading one as $\varepsilon \to 0$. We suppose that some of the $a_{k,j}$ contained in the sums in the left side of (2.5) are strictly positive. Let us choose here all the $\tilde{u}_j(y),\ 1 \le j \le N,\ j \ne k$ to be identical. Consider the equation

$$-(-\Delta_x)^{s_2}\tilde{u}_j(x) = \tilde{v}_j(x), \quad 0 < s_2 < \frac{1}{2}. \tag{2.6}$$

We assume that the right side of (2.6) belongs to $C_c^\infty(\mathbb{R}^2)$. Evidently, $\tilde{v}_j(x) \in L^1(\mathbb{R}^2) \cap L^2(\mathbb{R}^2)$ as well. By virtue of the part 1) of Lemma 3.1 below, (2.6) has a unique solution $\tilde{u}_j(x) \in H^{2s_2}(\mathbb{R}^2)$. Suppose the right side of (2.6) is nonnegative in the whole $\mathbb{R}^2$. Let us use the explicit formula from Section 5.9 of [9], so that

$$\tilde{u}_j(x) = -c_{s_2}\int_{\mathbb{R}^2} |x-y|^{2s_2-2}\tilde{v}_j(y)dy,$$

where $c_{s_2} > 0$ is a constant. Then $\tilde{u}_j(x)$ is negative on $\mathbb{R}^2$, which is a contradiction to our original assumption. Hence, $\tilde{v}_j(x)$ has the points of negativity on the plane. Let us recall that the negativity of the off diagonal elements of the matrix A is ruled out due to assumption (1.6). By making the parameter ε sufficiently small, we can violate the inequality in (2.5). Therefore,

$$a_{k,j} = 0, \quad 1 \le k, j \le N, \quad k \ne j.$$

Thus, by virtue of (2.5) we arrive at

$$F_k(\tilde{u}_1(x), ..., \tilde{u}_{k-1}(x), 0, \tilde{u}_{k+1}(x), ..., \tilde{u}_N(x)) \le 0 \quad a.e.$$

with $\tilde{u}_j(x) \ge 0$ and $\tilde{u}_j(x) \in L^2(\mathbb{R}^2)$ for $1 \le j \le N,\ j \ne k$. ■

3. Auxiliary results

Let us formulate the solvability conditions for the linear Poisson type equation with a square integrable right side

$$(-\Delta)^s u = f(x), \quad x \in \mathbb{R}^2, \quad 0 < s < 1. \tag{3.1}$$

We have the following technical statement. It can be trivially obtained by applying the standard Fourier transform

$$\widehat{\phi}(p) := \frac{1}{2\pi} \int_{\mathbb{R}^2} \phi(x) e^{-ipx} dx, \quad p \in \mathbb{R}^2 \tag{3.2}$$

to both sides of problem (3.1) (see Theorem 1.1 of [19], also [17]). Let us use the obvious upper bound

$$\|\widehat{\phi}(p)\|_{L^\infty(\mathbb{R}^2)} \leq \frac{1}{2\pi} \|\phi(x)\|_{L^1(\mathbb{R}^2)}. \tag{3.3}$$

We will present the proof below for the convenience of the readers.

Lemma 3.1. *Let* $f(x) : \mathbb{R}^2 \to \mathbb{R}, \ f(x) \in L^2(\mathbb{R}^2)$ *and* $s \in (0, 1)$.

1) If $0 < s < \dfrac{1}{2}$ *and in addition* $f(x) \in L^1(\mathbb{R}^2)$*, then equation (3.1) has a unique solution* $u(x) \in H^{2s}(\mathbb{R}^2)$.

2) If $\dfrac{1}{2} \leq s < 1$ *and additionally* $|x| f(x) \in L^1(\mathbb{R}^2)$*, then problem (3.1) admits a unique solution* $u(x) \in H^{2s}(\mathbb{R}^2)$ *if and only if the orthogonality relation*

$$(f(x), 1)_{L^2(\mathbb{R}^2)} = 0 \tag{3.4}$$

is valid.

Proof. Clearly, by means of norm definition (1.5) along with the square integrability of the right side of (3.1), it would be sufficient to demonstrate the solvability of problem (3.1) in $L^2(\mathbb{R}^2)$. Evidently, the solution $u(x) \in L^2(\mathbb{R}^2)$ will belong to $H^{2s}(\mathbb{R}^2), \ 0 < s < 1$ as well.
Let us establish the uniqueness of solutions for equation (3.1). Suppose $u_{1,2}(x) \in H^{2s}(\mathbb{R}^2)$ both satisfy (3.1). Then their difference $w(x) := u_1(x) - u_2(x) \in H^{2s}(\mathbb{R}^2)$ solves the homogeneous equation

$$(-\Delta)^s w = 0.$$

Since the operator $(-\Delta)^s : H^{2s}(\mathbb{R}^2) \to L^2(\mathbb{R}^2)$ does not have any nontrivial zero modes, $w(x)$ vanishes identically in $\mathbb{R}^2$.

Let us apply the standard Fourier transform (3.2) to both sides of equation (3.1). This gives us

$$\widehat{u}(p) = \frac{\widehat{f}(p)}{|p|^{2s}}\chi_{\{p\in\mathbb{R}^2 \ | \ |p|\le 1\}} + \frac{\widehat{f}(p)}{|p|^{2s}}\chi_{\{p\in\mathbb{R}^2 \ | \ |p|>1\}}, \tag{3.5}$$

where χ_A stands for the characteristic function of a set $A\subseteq\mathbb{R}^2$. Obviously, for all $0<s<1$ the second term in the right side of (3.5) belongs to $L^2(\mathbb{R}^2)$ by virtue of the estimate

$$\int_{\mathbb{R}^2}\frac{|\widehat{f}(p)|^2}{|p|^{4s}}\chi_{\{p\in\mathbb{R}^2 \ | \ |p|>1\}}dp \le \|f\|^2_{L^2(\mathbb{R}^2)} < \infty$$

as assumed. To show the square integrability of the first term in the right side of (3.5) for $0<s<\frac{1}{2}$, we use inequality (3.3), which gives us

$$\int_{\mathbb{R}^2}\frac{|\widehat{f}(p)|^2}{|p|^{4s}}\chi_{\{p\in\mathbb{R}^2 \ | \ |p|\le 1\}}dp \le \frac{\|f(x)\|^2_{L^1(\mathbb{R}^2)}}{4\pi(1-2s)} < \infty$$

via the one of our assumptions. This completes the proof of the first part of our lemma. To study the solvability of equation (3.1) for $\frac{1}{2}\le s<1$, we express

$$\widehat{f}(p) = \widehat{f}(0) + \int_0^{|p|}\frac{\partial\widehat{f}(q,\theta)}{\partial q}dq,$$

where θ stands for the angle variable. This allows us to write the first term in the right side of (3.5) as

$$\frac{\widehat{f}(0)}{|p|^{2s}}\chi_{\{p\in\mathbb{R}^2 \ | \ |p|\le 1\}} + \frac{\int_0^{|p|}\frac{\partial\widehat{f}(q,\theta)}{\partial q}dq}{|p|^{2s}}\chi_{\{p\in\mathbb{R}^2 \ | \ |p|\le 1\}}. \tag{3.6}$$

Definition (3.2) yields

$$\left|\frac{\partial\widehat{f}(p)}{\partial|p|}\right| \le \frac{1}{2\pi}\||x|f(x)\|_{L^1(\mathbb{R}^2)},$$

so that

$$\left|\frac{\int_0^{|p|}\frac{\partial\widehat{f}(q,\theta)}{\partial q}dq}{|p|^{2s}}\chi_{\{p\in\mathbb{R}^2 \ | \ |p|\le 1\}}\right| \le \frac{1}{2\pi}\||x|f(x)\|_{L^1(\mathbb{R}^2)}|p|^{1-2s}\chi_{\{p\in\mathbb{R}^2 \ | \ |p|\le 1\}} \in L^2(\mathbb{R}^2).$$

The first term in (3.6) $\frac{\widehat{f}(0)}{|p|^{2s}}\chi_{\{p\in\mathbb{R}^2 \ | \ |p|\le 1\}} \in L^2(\mathbb{R}^2)$ if and only if $\widehat{f}(0)=0$. This gives us orthogonality condition (3.4) in the second case of our lemma. ■

Evidently, the left side of relation (3.4) is well defined under the stated assumptions via Lemma 4.1 of [19]. Note that for the lower values of the power of the fractional Laplacian $0 < s < \frac{1}{2}$ under the given conditions no orthogonality relations are required to solve the linear Poisson type problem (3.1) in $H^{2s}(\mathbb{R}^2)$.

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