Monotone ODE with Discontinuous Vector Fields in Sequence Spaces

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Abstract. We consider a system of ODE in a Fréchet space with unconditional Schauder basis. The right side of the ODE is a discontinuous function. Under certain monotonicity conditions we prove an existence theorem for the corresponding initial value problem.

1. Introduction

Analysis of ODE with non Lipschitz right hand side has long history. Without any claims on a completeness of exposition we just note some principle points of this history. A detailed discussion of further developments in anyone of these points requires a separate survey.

The first result belongs to G. Peano (1890). G. Peano considered an initial value problem

$$\dot{x} = f(t, x), \quad x(t_0) = x_0 \tag{1.1}$$

where f is a continuous mapping of some domain

$$D \subset \mathbb{R}^{m+1} = \{(t, x)\}, \quad x = (x_1, \ldots, x_m)$$

with values in $\mathbb{R}^m$.

G. Peano stated that this problem has a solution that is defined locally for small $|t - t_0|$. This solution may not be unique.

C. Carathéodory relaxed the conditions of this theorem up to measurability of the function f in t.

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All these results are essentially based on the fact that a closed ball in $\mathbb{R}^m$ is compact. In an infinite dimensional Banach or more generally locally convex space they are in general invalid.

The corresponding example was first constructed by J. Dieudonné [**1**]. His example was as follows:

$$\dot{x}_k = \sqrt{|x_k|} + \frac{1}{k}, \quad x_k(0) = 0, \quad k \in \mathbb{N}, \quad t \geq 0.$$

It is easy to check that this IVP does not have solutions $x(t) = \{x_k(t)\} \in c_0$.

To recover an existence in the infinite dimensional case one must impose some extra compactness conditions on f [**7**] or consider the measure of non compactness [**8**].

Observe that all the existence results mentioned above follow in one way or another from the Schauder-Tychonoff fixed point theorem.

The next stop in this journey is the concept differential inclusions. If the right side of equation (1.1) is just a measurable function then even for continuous $x(t)$ a mapping $t \mapsto f(t, x(t))$ is not obliged to be measurable [**4**].

The corresponding transformation of the notion of a solution was proposed A. Filippov [**5**]. According to him an absolutely continuous function $x(t)$ is a solution to (1.1) if for almost all t the following inclusion holds

$$\dot{x}(t) \in \bigcap_{r>0} \bigcap_{N} \operatorname{conv} f\big(t, B_r(x(t)) \backslash N\big).$$

Here $B_r(x) \subset \mathbb{R}^m$ is an open ball of the radius r and the center at x. The intersection $\bigcap_N$ is taken over all measure-null sets N; and conv stands for the closed convex hull of a set.

It is important to stress that once we have denied the classical concept of a solution then there are a lot of reasonable generalizations arise. Filippov's concept is good for control and for dry friction mechanics [**10**]. A very different approach by DiPerna and Lions is good for PDE and fluid mechanics [**2**].

2. The Main Theorems

Let E stand for a Fréchet space. Its topology is defined by a collection of seminorms $\{\|\cdot\|_n\}_{n\in\mathbb{N}}$.

Recall that such a space is completely metrizable by the following metrics

$$\rho(x, y) = \sum_{k=1}^{\infty} \frac{1}{2^k} \min\{1, \|x - y\|_k\}.$$

Assume that the space E possesses an unconditional Schauder basis $\{e_k\}_{k\in\mathbb{N}}$. Recall several definitions.

Definition 1. *A sequence $\{e_k\}_{k\in\mathbb{N}} \subset E$ is called a Schauder basis if for every $x \in E$ there is a unique sequence of scalars $\{x_k\}_{k\in\mathbb{N}}$ such that*

$$x = \sum_{k=1}^{\infty} x_k e_k. \tag{2.1}$$

This series is convergent in the topology of E.

We shall say that $\{e_k\}_{k\in\mathbb{N}}$ *is an unconditional basis if for any* $x \in E$ *and for any permutation* $\pi : \mathbb{N} \to \mathbb{N}$ *the sum*

$$\sum_{k=1}^{\infty} x_{\pi(k)} e_{\pi(k)}$$

is convergent.

Introduce a notation $I_T = [0, T], \quad T > 0$.

Equip the space E with a partial order $\ll$ as follows

$$x = \sum_{k=1}^{\infty} x_k e_k \ll y = \sum_{k=1}^{\infty} y_k e_k \Longleftrightarrow x_i \le y_i, \quad i \in \mathbb{N}.$$

DEFINITION 2. *We shall say that a function* $g : E \to \mathbb{R}$ *is left continuous if for all* $\tilde{x} \in E$ *and for all sequences*

$$x_k \to \tilde{x}, \quad x_k \ll x_{k+1}, \quad k \in \mathbb{N}$$

one has

$$\lim_{k\to\infty} g(x_k) = g(\tilde{x}).$$

The main object of our study is the following initial value problem

$$\dot{x}(t) = f\big(t, x(t)\big) = \sum_{k=1}^{\infty} f_k\big(t, x(t)\big) e_k, \quad x(0) = \hat{x} \in E. \tag{2.2}$$

Here the function $f : I_T \times E \to E$ is such that all the functions $f_k : I_T \times E \to \mathbb{R}$ are left continuous in the second argument when the first one is fixed.

For any fixed x the function $t \mapsto f(t, x)$ is integrable on I_T. For details on the Lebesgue integrable functions with values in locally convex spaces see [**7**].

Assume that there exists an element $C = \sum_{k=1}^{\infty} C_k e_k \in E$ such that for any $x \in E$ the following inequality holds

$$f(t, x) \ll C. \tag{2.3}$$

Assume that there exists an element $x_* \in E$ such that the following inequality holds

$$x_* \ll \hat{x} + \int_0^t f(s, x_*) ds, \quad t \in I_T. \tag{2.4}$$

DEFINITION 3. *We shall say that a function* $x \in C(I_T, E)$ *is a solution to IVP (2.2) if a function* $t \mapsto f(t, x(t))$ *is integrable in* I_T *and the following equation*

$$x(t) = \hat{x} + \int_0^t f\big(s, x(s)\big) ds, \quad t \in I_T$$

is satisfied.

THEOREM 1. *In addition to the hypotheses above assume that* f *is monotone:*

$$x \ll y \Longrightarrow f(t, x) \ll f(t, y), \quad \forall x, y \in E, \quad \forall t \in I_T. \tag{2.5}$$

Then problem (2.2) has a solution $x(t)$.

Theorem 1 is proved in section 3.2.

Theorem 2. *Assume in addition that E is a reflexive space. Then for almost all t the solution $x(t)$ from theorem 1 is weakly differentiable that is there exists $\dot{x}(t)\in E$ such that for almost all t one has*

$$\Big(\psi,\frac{x(t+h)-x(t)}{h}\Big)\to(\psi,\dot{x}(t)),\quad h\to 0,\quad h\ne 0,\quad \forall\psi\in E'$$

and

$$\dot{x}(t)=f(t,x(t)).$$

Theorem 2 is proved in section 3.3.

Remark 1. *The assertions of theorems 1, 2 remain valid if the functions f_k are right continuous in the second argument.*

Moreover if in formulas (2.3), (2.4) one replaces "$\ll$" with "$\gg$" and "$f(t,x)\ll f(t,y)$" with "$f(t,x)\gg f(t,y)$" in (2.5) then both theorems hold.

3. Proofs of the Theorems

We denote all inessential positive constants by the same letter c.

3.1. Auxiliary Facts. The following theorem is essentially based on the assumption that the Schauder basis is unconditional. This theorem generalizes the corresponding result for Banach spaces [**6**].

Theorem 3 ([**9**]). *Fix a sequence $\lambda=\{\lambda_j\}_{j\in\mathbb{N}}\in\ell_\infty$. Then*

$$\mathcal{M}_\lambda x=\sum_{k=1}^\infty\lambda_k x_k e_k,\quad x=\sum_{k=1}^\infty x_k e_k\in E$$

is a bounded linear operator of E to E and for any number i' there exists a number i and a positive constant c both independent on λ such that

$$\|\mathcal{M}_\lambda x\|_{i'}\le c\|\lambda\|_{\ell_\infty}\cdot\|x\|_i,\quad \forall x\in E.$$

Particularly this theorem implies

$$y=\sum_{k=1}^\infty y_k e_k\in E\Longrightarrow |y|:=\sum_{k=1}^\infty |y_k|e_k\in E.$$

Other consequence of this theorem is as follows.

Lemma 1. *Assume that constant vectors*

$$a=\sum_{k=1}^\infty a_k e_k,\quad b=\sum_{k=1}^\infty b_k e_k\in E$$

are such that $a\ll b$. Then for any sequence of reals y_k, $\quad a_k\le y_k\le b_k$ an element

$$y=\sum_{k=1}^\infty y_k e_k\in E$$

is well defined.

Indeed,

$$y=\mathcal{M}_\alpha a+\mathcal{M}_\beta b,$$

where

$$\alpha=\{\alpha_k\},\quad \beta=\{\beta_k\}\in\ell_\infty,\quad \alpha_k+\beta_k=1,\quad \alpha_k,\beta_k\ge 0.$$

Lemma 2. *Assume that constant vectors*

$$a=\sum_{k=1}^{\infty}a_ke_k,\quad b=\sum_{k=1}^{\infty}b_ke_k\in E$$

are such that $a\ll b$. *Then the interval*

$$[a,b]:=\{x\in E\mid a\ll x\ll b\}$$

is a compact set.

Moreover for any $i'\in\mathbb{N}$ *there exists* $i\in\mathbb{N}$ *and a positive constant* $c>0$ *such that*

$$x\in[a,b]\Longrightarrow\|x\|_{i'}\le c(\|a+b\|_{i'}+\|a-b\|_i). \tag{3.1}$$

The constant c *does not depend on* a,b.

Proof of lemma 2. Let us shift the set $[a,b]$ and consider a set

$$J=[a-s,b-s],\quad s=\frac{a+b}{2}.$$

The set $[a,b]$ is compact iff the set J is compact.

Consider a projection $P_ny=\sum_{k=1}^{n}y_ke_k$. Each set

$$K_n=P_n(J)\subset J$$

is compact since it is a closed and bounded subset of $\mathbb{R}^n$.

Show that the sets $\{K_n\}$ form ε-nets in J.

Indeed, take any element $y\in J$ and present it as follows

$$y=P_ny+q_n,\quad q_n=\sum_{k=n+1}^{\infty}y_ke_k,\quad |y_k|\le r_k=\frac{b_k-a_k}{2}.$$

A series $R_n=\sum_{k=n+1}^{\infty}r_ke_k\in E$ is a tail of the expansion of the element $(b-a)/2$ and thus for all i it follows that $\|R_n\|_i\to 0$. Observe that

$$q_n=\mathcal{M}_\lambda R_n,$$

where $\lambda=\{\lambda_j\},\quad \lambda_j=y_j/r_j$ provided $r_j\ne 0$ and $\lambda_j=0$ otherwise.

Theorem 3 implies that for any i' there exists i such that

$$\|q_n\|_{i'}=\|\mathcal{M}_\lambda R_n\|_{i'}\le c\|R_n\|_i\to 0. \tag{3.2}$$

The limit in the last part of formula (3.2) is uniform in $y\in J$. This proves the lemma in the part of compactness.

From the formulas above it also follows that $y=\mathcal{M}_\lambda R_0,\quad y\in J,$

$$\|y\|_{i'}\le c\|R_0\|_i=c\Big\|\frac{a-b}{2}\Big\|_i.$$

This readily implies estimate (3.1).

Lemma 2 is proved.

Introduce linear functions $e^j:E\to\mathbb{R}$ by the formula $e^j(y)=y_j$. These functions are continuous [**3**].

Lemma 3. *Let* $W\subset E$ *be a chain (linearly ordered set) with an upper bound* $\overline{w}\in E$.

Then the element

$$\sup W:=\sum_{k=1}^{\infty}\sup\{e^k(W)\}e_k\in E$$

is well defined and $\sup W \ll \overline{w}$.

Indeed, the assertion follows from lemma 1: take any $x \in W$; then $\sup W \in [x, \overline{w}]$.

LEMMA 4. *Let a function* $F : I_T \times E \to \mathbb{R}$ *be left continuous in the second argument and a function* $t \mapsto F(t,x)$ *be integrable on* I_T *for each* $x \in E$.

Suppose also that F *is monotone:*

$$x \ll y \Longrightarrow F(t,x) \le F(t,y), \quad \forall t \in I_T.$$

Assume that a function $u : I_T \to E$,

$$u(t) = \sum_{k=1}^{\infty} u_k(t) e_k$$

is such that all the functions $u_k : I_T \to \mathbb{R}$ *are integrable and for some* $a, b \in E, \quad a \ll b$ *and for all* $t \in I_T$ *one has*

$$u(t) \in [a, b].$$

Then a mapping $t \mapsto F(t, u(t))$ *is integrable on* I_T.

Proof of lemma 4. From [**4**] we know that for each k there exists a sequence $\varphi_{k,j}(t)$ of simple functions such that

$$a_k \le \varphi_{k,j} \le \varphi_{k,j+1} \le b_k, \quad a = \sum_{r=1}^{\infty} a_r e_r, \quad b = \sum_{r=1}^{\infty} b_r e_r$$

and $\varphi_{k,j} \to u_k$ pointwise for each $t \in I_T$ as $j \to \infty$.

Introduce the following functions

$$[a,b] \ni U_j(t) = \sum_{k=1}^{j} \varphi_{k,j}(t) e_k + \sum_{r=j+1}^{\infty} a_r e_r, \quad F_j(t) = F(t, U_j(t)).$$

Each function U_j has a finite set of values in E and $U_j \ll U_{j+1}$. Thus $F_j : I_T \to E$ is integrable.

Let us show that $U_j \to u$ pointwise in E. Indeed, consider an estimate:

$$\|U_j(t) - u(t)\|_i \le \Big\| \sum_{k=1}^{N} (\varphi_{k,j}(t) - u_k(t)) e_k \Big\|_i$$
$$+ \Big\| \sum_{k=N+1}^{j} (\varphi_{k,j}(t) - u_k(t)) e_k + \sum_{k=j+1}^{\infty} (a_k - u_k(t)) e_k \Big\|_i, \quad j > N.$$

The first summand in the right hand side of this formula vanishes as $j \to \infty$. By lemma 2 the second summand is estimated from above in terms of

$$\Big\| \sum_{k=N+1}^{j} a_k e_k \Big\|_{i_1}, \quad \Big\| \sum_{k=N+1}^{j} a_k e_k \Big\|_{i_2}, \quad \Big\| \sum_{k=N+1}^{j} b_k e_k \Big\|_{i_1}, \quad \Big\| \sum_{k=N+1}^{j} b_k e_k \Big\|_{i_2}$$

and

$$\Big\| \sum_{k=j+1}^{\infty} a_k e_k \Big\|_{i_1}, \quad \Big\| \sum_{k=j+1}^{\infty} a_k e_k \Big\|_{i_2}, \quad \Big\| \sum_{k=j+1}^{\infty} b_k e_k \Big\|_{i_1}, \quad \Big\| \sum_{k=j+1}^{\infty} b_k e_k \Big\|_{i_2}.$$

These terms vanish as $N \to \infty$.

So that $F_j(t) \to F(t, u(t))$ pointwise. On the other hand

$$F(t, a) \le F_j(t) \le F(t, b).$$

Therefore the assertion of the lemma follows from the Dominated convergence theorem.

Lemma 4 is proved.

LEMMA 5. *Take a function* $u : I_T \to [a, b] \subset E$,

$$u(t) = \sum_{k=1}^{\infty} u_k(t) e_k$$

with integrable u_k. *Then a function* $t \mapsto f(t, u(t))$ *is integrable in* I_T.

Indeed, from lemma 4 we know that the functions $f_k(\cdot, u(\cdot)) : I_T \to \mathbb{R}$ are integrable and $f_k(t, a) \le f_k(t, u(t)) \le f_k(t, b)$.

Introduce functions

$$\phi_n(\cdot) = \sum_{k=1}^{n} f_k(\cdot, u(\cdot)) e_k$$

and observe that $\phi_n(\cdot) \to f(\cdot, u(\cdot))$ pointwise in E.

The functions ϕ_n are integrable. Moreover,

$$\mathcal{M}_{\lambda_n} f(t, a) \ll \phi_n(t) \ll \mathcal{M}_{\lambda_n} f(t, b), \quad \lambda_n = (\underbrace{1, \dots, 1}_{n \text{ times}}, 0, 0, \dots)$$

and from lemma 2 and theorem 3 it follows that for any $i \in \mathbb{N}$ there are $i', i'' \in \mathbb{N}$ and a constant $c > 0$ such that for all t one has

$$\|\phi_n(t)\|_i \le c(\|f(t, a)\|_{i'} + \|f(t, b)\|_{i'} + \|f(t, a)\|_{i''} + \|f(t, b)\|_{i''}).$$

The function in the right side of this inequality is integrable by the statement of the problem.

The Dominated convergence theorem concludes the proof.

3.2. Proof of Theorem 1. Here we employ a version of N. Bourbaki's famous idea.

Introduce a set

$$S = \Big\{ u(t) = \sum_{k=1}^{\infty} u_k(t) e_k \in E \,\Big|\, u_k \quad \text{are lower semicontinuous,}$$
$$x_* \ll u(t) \ll \Phi(u)(t), \quad t \in I_T \Big\},$$

where

$$\Phi(u)(t) = \hat{x} + \int_0^t f(s, u(s)) ds.$$

The set S is not empty since $x_* \in S$.

By formula (2.3) if $u \in S$ then one has

$$\Phi(u)(t) \in [x_*, \hat{x} + T|C|], \quad u(t) \in [x_*, \hat{x} + T|C|], \quad t \in I_T. \tag{3.3}$$

Observe also that $\Phi(S) \subset S$. Indeed, this follows by lemma 5 from the first inclusion of (3.3) and monotonicity of the mapping f.

The set S is partially ordered by the following binary relation. For any $u, v \in S$ by definition put

$$u \prec v \Longleftrightarrow u(t) \ll v(t) \quad \forall t \in I_T.$$

Let $W \subset S$ be a chain.

By lemma 3 a function $w^*(t) = (\sup W)(t)$ is correctly defined. We then obtain

$$w^*(t) = \sum_{k=1}^{\infty} w_k^*(t) e_k \in [x_*, \hat{x} + T|C|].$$

From [**3**] we know that all the functions w_k^* are lower semicontinuous.

Furthermore

$$\begin{aligned} W \ni w(t) \ll w^*(t) &\Longrightarrow f(t, w(t)) \ll f(t, w^*(t)) \\ &\Longrightarrow \int_0^t f(s, w(s))ds \ll \int_0^t f(s, w^*(s))ds. \end{aligned}$$

And thus

$$w(t) \ll \hat{x} + \int_0^t f(s, w(s))ds \ll \hat{x} + \int_0^t f(s, w^*(s))ds.$$

The last estimate holds for all $w \in W$. This implies

$$w^*(t) \ll \hat{x} + \int_0^t f(s, w^*(s))ds$$

and $w^* \in S$. Therefore w^* is an upper bound for W. Thus each chain of S has an upper bound. By the Zorn lemma S has a maximal element, say $x(t)$. Therefore one obtains

$$x(\cdot) \prec \Phi(x(\cdot)) \in S \Longrightarrow x(\cdot) = \Phi(x(\cdot))$$

and this maximal element is the desired solution to problem (2.2).

The theorem is proved.

3.3. Proof of Theorem 2. Recall that a function $w : I_T \to \mathbb{R}$ is absolutely continuous iff it can be presented in the form

$$w(t) = \int_0^t p(s)ds, \quad p \in L^1(I_T).$$

In this case the classical derivative $\dot{w}$ exists for almost all t and

$$\dot{w}(t) = p(t) \quad \text{almost everywhere.}$$

From lemma 5 we know that $f(\cdot, x(\cdot)) \in L^1(I_T, E)$. Thus

$$(\psi, x(t)) = (\psi, \hat{x}) + \Big(\psi, \int_0^t f(s, x(s))ds\Big) = (\psi, \hat{x}) + \int_0^t \big(\psi, f(s, x(s))\big)ds$$

is an absolutely continuous function and

$$\frac{(\psi, x(t+h)) - (\psi, x(t))}{h} = \Big(\psi, \frac{x(t+h) - x(t)}{h}\Big), \quad h \neq 0. \tag{3.4}$$

For almost all t one has

$$\frac{d}{dt}(\psi, x(t)) = \big(\psi, f(t, x(t))\big). \tag{3.5}$$

The space E is separable so that E'' is also separable and thus E' is separable. Let $\Psi \subset E'$ be a countable dense set.

Repeating the argument above we obtain $f(t,x(t))\in[f(t,x_*),C]$ and for any i and for almost all t we have

$$\Big\|\frac{1}{h}\int_t^{t+h}f(s,x(s))ds\Big\|_i\le\frac{1}{|h|}\Big|\int_t^{t+h}\|f(s,x(s))\|_i ds\Big|$$
$$\le\frac{1}{|h|}\Big|\int_t^{t+h}\tilde c_i\big(\|f(s,x_*)+C\|_i+\|f(s,x_*)-C\|_{i'}\big)ds\Big|\le c_i(t). \qquad (3.6)$$

Here $\tilde c_i$ is a positive constant.

Let Θ_ψ be a set of values $t\in I_T$ for those formula (3.5) does not hold. The measure of Θ_ψ is equal to zero so that the measure of a set

$$\Theta=\bigcup_{\psi\in\Psi}\Theta_\psi$$

equals zero as well.

By the same reason a set

$$\tilde\Theta=\Theta\cup\{t\text{ for those formula (3.6) does not hold}\}$$

is of measure zero too.

Introduce a set $I'_T=I_T\backslash\tilde\Theta$. Consider an element

$$X_{t,h}=\frac{x(t+h)-x(t)}{h}$$

as a linear function on E'.

For each $t\in I'$ one therefore obtains $\|X_{t,h}\|_i\le c_i(t)$.

Fix $t\in I'_T$. From formula (3.4) for any $\psi\in\Psi$ one gets

$$(\psi,X_{t,h})\to\frac{d}{dt}(\psi,x(t)),\quad h\to 0.$$

From the Banach-Steinhaus theorem [**3**] it follows that the limit

$$(q(t),\psi)=\lim_{h\to 0}(\psi,X_{t,h})$$

exists for all $\psi\in E'$. For each $t\in I'_T$ the function $q(t)\in E''$. By reflexivity of E we can regard this as $q(t)\in E$ and put $\dot x=q$.

The theorem is proved.

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