Solvability of some integro-differential equations with drift and superdiffusion

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Abstract. We establish the existence in the sense of sequences of solutions for some integro-differential type equations containing the drift term and the square root of the one dimensional negative Laplacian, on the whole real line or on a finite interval with periodic boundary conditions in the corresponding H^2 spaces. The argument relies on the fixed point technique when the elliptic equations involve first order differential operators with and without Fredholm property. It is proven that, under the reasonable technical assumptions, the convergence in L^1 of the integral kernels implies the existence and convergence in H^2 of solutions.

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1 Introduction

Let us recall that a linear operator L acting from a Banach space E into another Banach space F satisfies the Fredholm property if its image is closed, the dimension of its kernel and the codimension of its image are finite. As a consequence, the problem $Lu = f$ is solvable if and only if $\phi_i(f) = 0$ for a finite number of functionals ϕ_i from the dual space F^*. These properties of Fredholm operators are widely used in many methods of linear and nonlinear analysis.

Elliptic problems in bounded domains with a sufficiently smooth boundary satisfy the Fredholm property if the ellipticity condition, proper ellipticity and Lopatinskii conditions are fulfilled (see e.g. [1], [7], [20], [25]). This is the main result of the theory of linear elliptic equations. In the case of unbounded domains, these conditions may not be sufficient and the Fredholm property may not be satisfied. For example, the Laplace operator, $Lu = \Delta u$, in $\mathbb{R}^d$

does not satisfy the Fredholm property when considered in Hölder spaces, $L : C^{2+\alpha}(\mathbb{R}^d) \to C^{\alpha}(\mathbb{R}^d)$, or in Sobolev spaces, $L : H^2(\mathbb{R}^d) \to L^2(\mathbb{R}^d)$.

Linear elliptic problems in unbounded domains satisfy the Fredholm property if and only if, in addition to the conditions mentioned above, limiting operators are invertible (see [26]). In certain trivial cases, the limiting operators can be constructed explicitly. For instance, if

$$Lu = a(x)u'' + b(x)u' + c(x)u, \quad x \in \mathbb{R},$$

where the coefficients of the operator have limits at infinity,

$$a_{\pm} = \lim_{x\to\pm\infty} a(x), \quad b_{\pm} = \lim_{x\to\pm\infty} b(x), \quad c_{\pm} = \lim_{x\to\pm\infty} c(x),$$

the limiting operators are given by:

$$L_{\pm}u = a_{\pm}u'' + b_{\pm}u' + c_{\pm}u.$$

Because the coefficients are constants, the essential spectrum of the operator, that is the set of complex numbers λ for which the operator $L - \lambda$ fails to satisfy the Fredholm property, can be explicitly found by virtue of the Fourier transform:

$$\lambda_{\pm}(\xi) = -a_{\pm}\xi^2 + b_{\pm}i\xi + c_{\pm}, \quad \xi \in \mathbb{R}.$$

Invertibility of limiting operators is equivalent to the condition that the origin does not belong to the essential spectrum.

In the case of general elliptic problems, the same assertions hold true. The Fredholm property is satisfied if the essential spectrum does not contain the origin or if the limiting operators are invertible. However, such conditions may not be explicitly written.

In the case of non-Fredholm operators the usual solvability conditions may not be applicable and solvability conditions are, in general, unknown. There are certain classes of operators for which solvability relations are derived. Let us illustrate them with the following example. Consider the problem

$$Lu \equiv \Delta u + au = f \tag{1.1}$$

in $\mathbb{R}^d$, where a is a positive constant. The operator L coincides with its limiting operators. The homogeneous equation has a nontrivial bounded solution. Hence the Fredholm property is not satisfied. However, since the operator has constant coefficients, we can apply the Fourier transform and find the solution explicitly. Solvability relations can be formulated as follows. If $f \in L^2(\mathbb{R}^d)$ and $xf \in L^1(\mathbb{R}^d)$, then there exists a solution of this equation in $H^2(\mathbb{R}^d)$ if and only if

$$\left(f(x), \frac{e^{ipx}}{(2\pi)^{\frac{d}{2}}} \right)_{L^2(\mathbb{R}^d)} = 0, \quad p \in S^d_{\sqrt{a}} \quad a.e.$$

(see [33]). Here $S^d_{\sqrt{a}}$ stands for the sphere in $\mathbb{R}^d$ of radius $\sqrt{a}$ centered at the origin. Hence, though the operator does not satisfy the Fredholm property, solvability conditions are formulated similarly. However, this similarity is only formal since the range of the operator is not closed.

In the case of the operator with a scalar potential,

$$Lu \equiv \Delta u + a(x)u = f,$$

the Fourier transform is not directly applicable. Nevertheless, the solvability relations in $\mathbb{R}^3$ can be derived by a rather sophisticated application of the theory of self-adjoint operators (see [30]). As before, the solvability conditions are formulated in terms of the orthogonality to the solutions of the homogeneous adjoint problem. There are several other examples of linear elliptic non Fredholm operators for which solvability conditions can be derived (see [12], [26], [27], [30], [32], [33]).

Solvability relations play a significant role in the analysis of nonlinear elliptic problems. In the case of non-Fredholm operators, in spite of some progress in the understanding of linear equations, there exist only few examples where nonlinear non-Fredholm operators are analyzed (see [6], [11], [14], [31], [33], [35]). The article [8] is devoted to the studies of the finite and infinite dimensional attractors for evolution equations of mathematical physics. The large time behavior of solutions of a class of fourth-order parabolic equations defined on unbounded domains using the Kolmogorov ε-entropy as a measure was investigated in [10]. In [15] the authors consider the attractor for a nonlinear reaction-diffusion system in an unbounded domain in $\mathbb{R}^3$. The works [16] and [24] are devoted to the understanding of the Fredholm and properness properties of quasilinear elliptic systems of second order and of the operators of this kind on $\mathbb{R}^N$. In [17] the authors establish the exponential decay and study the Fredholm properties in second order quasilinear elliptic systems. [9] deals with a systematic study of a dynamical systems approach to investigating the symmetrization and stabilization properties of nonnegative solutions of nonlinear elliptic problems in asymptotically symmetric unbounded domains. In the present article we consider another class of stationary nonlinear problems, for which the Fredholm property may not be satisfied:

$$-\sqrt{-\frac{d^2}{dx^2}}u + b\frac{du}{dx} + au + \int_\Omega G(x-y)F(u(y),y)dy = 0, \quad x \in \Omega, \tag{1.2}$$

where Ω is a domain on the real line, $a \geq 0$, $b \in \mathbb{R}$, $b \neq 0$ are the constants. The operator $\sqrt{-\frac{d^2}{dx^2}}$ here is defined by virtue of the spectral calculus and is actively used, for instance in the studies of the superdiffusion problems (see e.g. [34], [36] and the references therein). Superdiffusion can be described as a random process of particle motion characterized by the probability density distribution of the jump length. The moments of this density distribution are finite in the case of normal diffusion, but this is not the case for superdiffusion. Asymptotic behavior at the infinity of the probability density function determines the value of the power of the negative Laplacian (see [23]). For the simplicity of presentation we restrict ourselves to the one dimensional case (the multidimensional situation is more technical). The solvability of the integro-differential equation with the transport and influx/efflux terms on the real line was studied recently in [37]. The method used there worked for the powers of the fractional negative Laplacian $0 < s < \frac{1}{4}$ but it did not cover the situation when $s = \frac{1}{2}$. In the population dynamics the integro-differential equations describe the models with the intra-specific competition and the nonlocal consumption of resources (see e.g. [2], [3]). We use the explicit form of the solvability relations and study the existence of solutions of this nonlinear problem. The studies of the solutions of the integro-differential equations with the drift term are relevant to the understanding of the emergence and propagation of patterns in the theory of speciation (see [28]). The solvability of the linear problem involving the Laplace operator with the transport term was addressed in [32], see also [4]. In the case when the drift term vanishes, namely when $b = 0$, the equation analogous to (1.2) was considered in [36] (see also [34]). Verification of biomedical processes with anomalous diffusion, transport and interaction of species was performed in [13]. Weak solutions of the Dirichlet and Neumann problems for the elliptic operator involving the transport term were studied in [21]. In [22] the authors established the imbedding theorems and treated the spectrum of a certain pseudodifferential operator.

2 Formulation of the results

The nonlinear part of problem (1.2) will satisfy the following regularity conditions.

Assumption 1. *Function $F(u, x) : \mathbb{R} \times \Omega \to \mathbb{R}$ is satisfying the Caratheodory condition (see [19]), so that*

$$|F(u, x)| \le k|u| + h(x) \quad for \quad u \in \mathbb{R}, \ x \in \Omega \tag{2.1}$$

with a constant $k > 0$ and $h(x) : \Omega \to \mathbb{R}^+$, $h(x) \in L^2(\Omega)$. Furthermore, it is a Lipschitz continuous function, so that

$$|F(u_1, x) - F(u_2, x)| \le l|u_1 - u_2| \quad for \quad any \quad u_{1,2} \in \mathbb{R}, \quad x \in \Omega \tag{2.2}$$

with a constant $l > 0$.

The article [5] was devoted to the solvability of a local elliptic equation in a bounded domain in $\mathbb{R}^N$. The nonlinear function there was allowed to have a sublinear growth. In order to demonstrate the solvabity of equation (1.2), we will use the auxiliary problem

$$\sqrt{-\frac{d^2}{dx^2}}u - b\frac{du}{dx} - au = \int_\Omega G(x - y)F(v(y), y)dy. \tag{2.3}$$

Let us introduce

$$(f_1(x), f_2(x))_{L^2(\Omega)} := \int_\Omega f_1(x)\bar{f}_2(x)dx, \tag{2.4}$$

with a slight abuse of notations when these functions are not square integrable, like for example those involved in orthogonality condition (5.5) further down. Indeed, if $f_1(x) \in L^1(\Omega)$ and $f_2(x)$ is bounded, then the integral in the right side of (2.4) makes sense. In the first part of the article we discuss the situation on the whole real line, $\Omega = \mathbb{R}$, such that the appropriate Sobolev space is equipped with the norm

$$\|u\|^2_{H^2(\mathbb{R})} := \|u\|^2_{L^2(\mathbb{R})} + \left\|\frac{d^2u}{dx^2}\right\|^2_{L^2(\mathbb{R})}. \tag{2.5}$$

The main issue for our equation above is that in the absence of the drift term we were dealing with the self-adjoint, non Fredholm operator

$$\sqrt{-\frac{d^2}{dx^2}} - a : H^1(\mathbb{R}) \to L^2(\mathbb{R}), \ a \geq 0,$$

which was the obstacle to solve our problem (see [34], [36]). The similar situations but in linear problems, both self- adjoint and non self-adjoint containing the differential operators without the Fredholm property have been studied extensively in recent years (see [26], [27], [30], [32], [33]). However, the situation is different when the constant in the drift term $b \neq 0$. The operator

$$L_{a,\ b} := \sqrt{-\frac{d^2}{dx^2}} - b\frac{d}{dx} - a : \quad H^1(\mathbb{R}) \to L^2(\mathbb{R}), \tag{2.6}$$

where $a \geq 0$ and $b \in \mathbb{R},\ b \neq 0$ involved in the left side of equation (2.3) is non-selfadjoint. By virtue of the standard Fourier transform, it can be easily obtained that the essential spectrum of such operator $L_{a,\ b}$ is given by

$$\lambda_{a,\ b}(p) = |p| - a - ibp, \quad p \in \mathbb{R}.$$

Evidently, when the constant $a > 0$ the operator $L_{a,\ b}$ is satisfies the Fredholm property, because its essential spectrum does not contain the origin. But when a vanishes, our operator $L_{a,\ b}$ does not satisfy the Fredholm property since the origin belongs to its essential spectrum. Let us demonstrate that under the reasonable technical assumptions equation (2.3) defines a map $T_{a,\ b} : H^2(\mathbb{R}) \to H^2(\mathbb{R})$ with the constants $a \geq 0,\ b \in \mathbb{R},\ b \neq 0$, which is a strict contraction.

Theorem 1. *Let $\Omega = \mathbb{R},\ G(x) : \mathbb{R} \to \mathbb{R},\ G(x) \in W^{1,1}(\mathbb{R})$ and Assumption 1 holds.*

I) When $a > 0,\ b \in \mathbb{R},\ b \neq 0$ we assume that $2\sqrt{\pi}N_{a,\ b}l < 1$, where $N_{a,\ b}$ is introduced in (5.4). Then the map $v \mapsto T_{a,b}v = u$ on $H^2(\mathbb{R})$ defined by problem (2.3) has a unique fixed point $v_{a,b}$, which is the only solution of equation (1.2) in $H^2(\mathbb{R})$.

II) When $a = 0,\ b \in \mathbb{R},\ b \neq 0$ we assume that $xG(x) \in L^1(\mathbb{R})$, orthogonality condition (5.5) holds and $2\sqrt{\pi}N_{0,\ b}l < 1$. Then the map $T_{0,b}v = u$ on $H^2(\mathbb{R})$ defined by problem (2.3) possesses a unique fixed point $v_{0,b}$, which is the only solution of equation (1.2) in $H^2(\mathbb{R})$.

In both cases I and II the fixed point $v_{a,b},\ a \geq 0,\ \ b \in \mathbb{R},\ b \neq 0$ is nontrivial provided the intersection of supports of the Fourier transforms of functions $supp\widehat{F(0,x)} \cap supp\widehat{G}$ is a set of nonzero Lebesgue measure in $\mathbb{R}$.

Let us note that in the situation when $a > 0$ of the theorem above, as distinct from part I) of Theorem 1 of [36] describing the equation without the drift term, the orthogonality relations are not needed. We introduce the sequence of approximate equations related to problem (1.2) on the whole real line, namely

$$-\sqrt{-\frac{d^2}{dx^2}}u_m + b\frac{du_m}{dx} + au_m + \int_{-\infty}^{\infty} G_m(x-y)F(u_m(y), y)dy = 0 \tag{2.7}$$

with the constants $a \geq 0,\ b \in \mathbb{R},\ b \neq 0$ and $m \in \mathbb{N}$. The sequence of kernels $\{G_m(x)\}_{m=1}^{\infty}$ converges to $G(x)$ as $m \to \infty$ in the appropriate function spaces discussed further down. Let us demonstrate that, under the certain technical assumptions, each of equations (2.7) possesses a unique solution $u_m(x) \in H^2(\mathbb{R})$, the limiting problem (1.2) admits a unique solution $u(x) \in H^2(\mathbb{R})$, and $u_m(x) \to u(x)$ in $H^2(\mathbb{R})$ as $m \to \infty$. This is the so-called *existence of solutions in the sense of sequences*. In such case, the solvability relations can be formulated for the iterated kernels G_m. They imply the convergence of the kernels in terms of the Fourier transforms (see the Appendix) and, as a consequence, the convergence of the solutions (Theorems 2, 4). The analogous ideas in the sense of the standard Schrödinger type operators were exploited in [12], [29]. Our second main proposition is as follows.

Theorem 2. *Let $\Omega = \mathbb{R},\ m \in \mathbb{N},\ G_m(x) : \mathbb{R} \to \mathbb{R},\ G_m(x) \in W^{1,1}(\mathbb{R})$ are such that $G_m(x) \to G(x)$ in $W^{1,1}(\mathbb{R})$ as $m \to \infty$. Let Assumption 1 hold.*

I) If $a > 0,\ b \in \mathbb{R},\ b \neq 0$, assume that

$$2\sqrt{\pi}N_{a,\ b,\ m}l \leq 1 - \varepsilon$$

for all $m \in \mathbb{N}$ with a certain fixed $0 < \varepsilon < 1$ and $N_{a,\ b,\ m}$ defined in (5.8). Then each problem (2.7) has a unique solution $u_m(x) \in H^2(\mathbb{R})$, and limiting equation (1.2) admits a unique solution $u(x) \in H^2(\mathbb{R})$.

II) If $a = 0,\ b \in \mathbb{R},\ b \neq 0$, assume that $xG_m(x) \in L^1(\mathbb{R}),\ xG_m(x) \to xG(x)$ in $L^1(\mathbb{R})$ as $m \to \infty$, orthogonality relation (5.10) is valid and

$$2\sqrt{\pi}N_{0,\ b,\ m}l \leq 1 - \varepsilon$$

for all $m \in \mathbb{N}$ with some fixed $0 < \varepsilon < 1$. Then each problem (2.7) admits a unique solution $u_m(x) \in H^2(\mathbb{R})$, and limiting equation (1.2) possesses a unique solution $u(x) \in H^2(\mathbb{R})$.

In both cases I and II, we have $u_m(x) \to u(x)$ in $H^2(\mathbb{R})$ as $m \to \infty$.

The unique solution $u_m(x)$ of each problem (2.7) does not vanish identically on the whole real line provided that the intersection of supports of the Fourier images of functions $supp\widehat{F(0,x)} \cap supp\widehat{G}_m$ is a set of nonzero Lebesgue measure in $\mathbb{R}$. Similarly, the unique solution $u(x)$ of limiting equation (1.2) is nontrivial if $supp\widehat{F(0,x)} \cap supp\widehat{G}$ is a set of nonzero Lebesgue measure in $\mathbb{R}$.

In the second part of the article we consider the analogous problem on the finite interval $\Omega = I := [0,\ 2\pi]$ with periodic boundary conditions, such that the appropriate functional space is given by

$$H^2(I) = \{u(x) : I \to \mathbb{R} \mid u(x), u''(x) \in L^2(I), \quad u(0) = u(2\pi), \quad u'(0) = u'(2\pi)\}.$$

For the technical purposes, we will use the following auxiliary constrained subspace

$$H_0^2(I) = \{u(x) \in H^2(I) \mid (u(x), 1)_{L^2(I)} = 0\}, \tag{2.8}$$

which is a Hilbert space as well (see e.g. Chapter 2.1 of [18]). Similarly,

$$H_0^1(I) = \{u(x) \in H^1(I) \mid (u(x), 1)_{L^2(I)} = 0\}.$$

Let us demonstrate that equation (2.3) in this case defines a map $\tau_{a,b}$ on the above mentioned spaces with the constants $a \geq 0,\ b \in \mathbb{R},\ b \neq 0$. This map will be a strict contraction under the stated technical assumptions.

Theorem 3. *Let*

$$\Omega = I,\ G(x) : I \to \mathbb{R},\ G(x) \in C(I),\ \frac{dG(x)}{dx} \in L^1(I),\ G(0) = G(2\pi),\ F(u, 0) = F(u, 2\pi)$$

for $u \in \mathbb{R}$ and Assumption 1 holds.

I) If $a > 0,\ b \in \mathbb{R},\ b \neq 0$ we assume that $2\sqrt{\pi}\mathcal{N}_{a,\ b}l < 1$ with $\mathcal{N}_{a,\ b}$ given by (5.25). Then the map $v \mapsto \tau_{a,b}v = u$ on $H^2(I)$ defined by problem (2.3) possesses a unique fixed point $v_{a,b}$, the only solution of equation (1.2) in $H^2(I)$.

II) If $a = 0,\ b \in \mathbb{R},\ b \neq 0$, let us assume that orthogonality condition (5.26) holds and $2\sqrt{\pi}\mathcal{N}_{0,\ b}l < 1$. Then the map $\tau_{0,b}v = u$ on $H_0^2(I)$ defined by problem (2.3) has a unique fixed point $v_{0,b}$, the only solution of equation (1.2) in $H_0^2(I)$.

In both cases I and II the fixed point $v_{a,b},\ a \geq 0,\ b \in \mathbb{R},\ b \neq 0$ does vanish identically on the interval I provided the Fourier coefficients $G_n F(0, x)_n \neq 0$ for a certain $n \in \mathbb{Z}$.

Remark 1. *We use the constrained subspace $H_0^2(I)$ in case II) of our theorem, so that the Fredholm operator $\sqrt{-\frac{d^2}{dx^2}} - b\frac{d}{dx} : H_0^1(I) \to L^2(I)$ has the trivial kernel.*

To establish the existence in the sense of sequences of solutions for our integro-differential equation on the interval I, we consider the sequence of approximate equations, analogously to the situation on the whole real line with $m \in \mathbb{N}$, namely

$$-\sqrt{-\frac{d^2}{dx^2}}u_m + b\frac{du_m}{dx} + au_m + \int_0^{2\pi} G_m(x-y)F(u_m(y), y)dy = 0, \tag{2.9}$$

where $a \geq 0,\ b \in \mathbb{R},\ b \neq 0$ are the constants. Our final main proposition is as follows.

Theorem 4. *Let* $\Omega = I,\ m \in \mathbb{N},\ G_m(x) : I \to \mathbb{R},\ G_m(x) \in C(I),\ \dfrac{dG_m(x)}{dx} \in L^1(I)$, *so that*

$$G_m(x) \to G(x) \quad in \quad C(I), \quad \frac{dG_m(x)}{dx} \to \frac{dG(x)}{dx} \quad in \quad L^1(I) \quad as \quad m \to \infty,$$

$G_m(0) = G_m(2\pi),\ F(u,0) = F(u,2\pi)$ *for* $u \in \mathbb{R}$. *Let Assumption 1 hold.*

I) If $a > 0,\ b \in \mathbb{R},\ b \neq 0$, *assume that*

$$2\sqrt{\pi}\mathcal{N}_{a,\ b,\ m} l \leq 1 - \varepsilon$$

for all $m \in \mathbb{N}$ *with a certain fixed* $0 < \varepsilon < 1$ *and* $\mathcal{N}_{a,\ b,\ m}$ *defined in (5.30). Then each problem (2.9) admits a unique solution* $u_m(x) \in H^2(I)$ *and limiting equation (1.2) has a unique solution* $u(x) \in H^2(I)$.

II) If $a = 0,\ b \in \mathbb{R},\ b \neq 0$, *assume that the orthogonality condition (5.32) is valid and*

$$2\sqrt{\pi}\mathcal{N}_{0,\ b,\ m} l \leq 1 - \varepsilon$$

for all $m \in \mathbb{N}$ *with some fixed* $0 < \varepsilon < 1$. *Then each problem (2.9) has a unique solution* $u_m(x) \in H_0^2(I)$ *and limiting equation (1.2) possesses a unique solution* $u(x) \in H_0^2(I)$.

In both cases I and II we have $u_m(x) \to u(x)$ *as* $m \to \infty$ *in the norms in* $H^2(I)$ *and* $H_0^2(I)$ *respectively.*

The unique solution $u_m(x)$ *of each problem (2.9) does not vanish identically on the interval* I *provided that the Fourier coefficients* $G_{m,n}F(0,x)_n \neq 0$ *for some* $n \in \mathbb{Z}$. *Similarly, the unique solution* $u(x)$ *of limiting equation (1.2) is nontrivial if* $G_nF(0,x)_n \neq 0$ *for a certain* $n \in \mathbb{Z}$.

Remark 2. *In the article we deal with the real valued functions by virtue of the assumptions on* $F(u,x),\ G_m(x)$ *and* $G(x)$ *involved in the integral terms of the iterated and limiting problems discussed above.*

Remark 3. *The importance of Theorems 2 and 4 above is the continuous dependence of the solutions with respect to the integral kernels.*

3 The Whole Real Line Case

Proof of Theorem 1. Let us first suppose that in the situation of $\Omega = \mathbb{R}$ for a certain $v \in H^2(\mathbb{R})$ there exist two solutions $u_{1,2} \in H^2(\mathbb{R})$ of problem (2.3). Then their difference $w(x) := u_1(x) - u_2(x) \in H^2(\mathbb{R})$ will solve the homogeneous equation

$$\sqrt{-\frac{d^2}{dx^2}}w - b\frac{dw}{dx} - aw = 0.$$

Because the operator $L_{a,\ b} : H^1(\mathbb{R}) \to L^2(\mathbb{R})$ introduced in (2.6) does not possess any nontrivial zero modes, $w(x) \equiv 0$ on $\mathbb{R}$.

We choose an arbitrary $v(x) \in H^2(\mathbb{R})$ and apply the standard Fourier transform (5.1) to both sides of (2.3). This yields

$$\widehat{u}(p) = \sqrt{2\pi}\frac{\widehat{G}(p)\widehat{f}(p)}{|p| - a - ibp}, \quad p^2\widehat{u}(p) = \sqrt{2\pi}\frac{p^2\widehat{G}(p)\widehat{f}(p)}{|p| - a - ibp}, \tag{3.1}$$

where $\widehat{f}(p)$ denotes the Fourier image of $F(v(x), x)$. Clearly, we have the estimates from above

$$|\widehat{u}(p)| \leq \sqrt{2\pi}N_{a,\ b}|\widehat{f}(p)| \quad and \quad |p^2\widehat{u}(p)| \leq \sqrt{2\pi}N_{a,\ b}|\widehat{f}(p)|.$$

Note that $N_{a,\ b} < \infty$ by means of Lemma A1 of the Appendix without any orthogonality relations for $a > 0$ and under orthogonality condition (5.5) when $a = 0$. This enables us to obtain the upper bound on the norm

$$\|u\|^2_{H^2(\mathbb{R})} = \|\widehat{u}(p)\|^2_{L^2(\mathbb{R})} + \|p^2\widehat{u}(p)\|^2_{L^2(\mathbb{R})} \leq 4\pi N^2_{a,\ b}\|F(v(x), x)\|^2_{L^2(\mathbb{R})} < \infty$$

due to inequality (2.1) of Assumption 1 above with $v(x) \in L^2(\mathbb{R})$. Thus, for an arbitrary $v(x) \in H^2(\mathbb{R})$ there exists a unique solution $u(x) \in H^2(\mathbb{R})$ of problem (2.3) with its Fourier transform given by (3.1) and the map $T_{a,b} : H^2(\mathbb{R}) \to H^2(\mathbb{R})$ is well defined. This enables us to choose arbitrarily $v_{1,2}(x) \in H^2(\mathbb{R})$ such that their images $u_{1,2} = T_{a,b}v_{1,2} \in H^2(\mathbb{R})$. According to (2.3), we have

$$\sqrt{-\frac{d^2}{dx^2}}u_1 - b\frac{du_1}{dx} - au_1 = \int_{-\infty}^{\infty} G(x-y)F(v_1(y), y)dy, \tag{3.2}$$

$$\sqrt{-\frac{d^2}{dx^2}}u_2 - b\frac{du_2}{dx} - au_2 = \int_{-\infty}^{\infty} G(x-y)F(v_2(y), y)dy. \tag{3.3}$$

Let us apply the standard Fourier transform (5.1) to both sides of equations (3.2) and (3.3), which gives us

$$\widehat{u}_1(p) = \sqrt{2\pi}\frac{\widehat{G}(p)\widehat{f}_1(p)}{|p| - a - ibp}, \quad p^2\widehat{u}_1(p) = \sqrt{2\pi}\frac{p^2\widehat{G}(p)\widehat{f}_1(p)}{|p| - a - ibp},$$

$$\widehat{u}_2(p) = \sqrt{2\pi}\frac{\widehat{G}(p)\widehat{f}_2(p)}{|p| - a - ibp}, \quad p^2\widehat{u}_2(p) = \sqrt{2\pi}\frac{p^2\widehat{G}(p)\widehat{f}_2(p)}{|p| - a - ibp},$$

where $\widehat{f}_1(p)$ and $\widehat{f}_2(p)$ stand for the Fourier images of $F(v_1(x), x)$ and $F(v_2(x), x)$ respectively. Apparently, we have the estimates from above

$$|\widehat{u}_1(p) - \widehat{u}_2(p)| \leq \sqrt{2\pi}N_{a,\ b}|\widehat{f}_1(p) - \widehat{f}_2(p)|, \quad |p^2\widehat{u}_1(p) - p^2\widehat{u}_2(p)| \leq \sqrt{2\pi}N_{a,\ b}|\widehat{f}_1(p) - \widehat{f}_2(p)|.$$

This enables us to obtain the inequality for the norms

$$\|u_1 - u_2\|^2_{H^2(\mathbb{R})} \leq 4\pi N^2_{a,\ b}\|F(v_1(x), x) - F(v_2(x), x)\|^2_{L^2(\mathbb{R})}.$$

Clearly, $v_{1,2}(x) \in H^2(\mathbb{R}) \subset L^\infty(\mathbb{R})$ via the Sobolev embedding. By virtue of upper bound (2.2) we easily arrive at

$$\|T_{a,b}v_1 - T_{a,b}v_2\|_{H^2(\mathbb{R})} \le 2\sqrt{\pi} N_{a,\ b} l \|v_1 - v_2\|_{H^2(\mathbb{R})}. \tag{3.4}$$

The constant in the right side of (3.4) is less than one as assumed. By means of the Fixed Point Theorem, there exists a unique function $v_{a,b} \in H^2(\mathbb{R})$ with the property $T_{a,b}v_{a,b} = v_{a,b}$. This is the only solution of problem (1.2) in $H^2(\mathbb{R})$. Suppose $v_{a,b}(x)$ is trivial on the real line. This will contradict to our assumption that the Fourier transforms of $G(x)$ and $F(0,x)$ do not vanish on a set of nonzero Lebesgue measure in $\mathbb{R}$. ■

Let us proceed to establishing the existence in the sense of sequences of the solutions for our integro-differential equation on the real line.

Proof of Theorem 2. By virtue of the result of Theorem 1 above, each equation (2.7) has a unique solution $u_m(x) \in H^2(\mathbb{R}),\ m \in \mathbb{N}$. Limiting problem (1.2) possesses a unique solution $u(x) \in H^2(\mathbb{R})$ by means of Lemma A2 below along with Theorem 1. Let us apply the standard Fourier transform (5.1) to both sides of (1.2) and (2.7). This yields

$$\widehat{u}(p) = \sqrt{2\pi}\frac{\widehat{G}(p)\widehat{\varphi}(p)}{|p| - a - ibp}, \quad \widehat{u}_m(p) = \sqrt{2\pi}\frac{\widehat{G}_m(p)\widehat{\varphi}_m(p)}{|p| - a - ibp}, \quad m \in \mathbb{N}, \tag{3.5}$$

where $\widehat{\varphi}(p)$ and $\widehat{\varphi}_m(p)$ stand for the Fourier images of $F(u(x),x)$ and $F(u_m(x),x)$ respectively. Obviously,

$$|\widehat{u}_m(p) - \widehat{u}(p)| \le \sqrt{2\pi}\left\|\frac{\widehat{G}_m(p)}{|p| - a - ibp} - \frac{\widehat{G}(p)}{|p| - a - ibp}\right\|_{L^\infty(\mathbb{R})} |\widehat{\varphi}(p)| +$$

$$+\sqrt{2\pi}\left\|\frac{\widehat{G}_m(p)}{|p| - a - ibp}\right\|_{L^\infty(\mathbb{R})} |\widehat{\varphi}_m(p) - \widehat{\varphi}(p)|,$$

so that

$$\|u_m - u\|_{L^2(\mathbb{R})} \le \sqrt{2\pi}\left\|\frac{\widehat{G}_m(p)}{|p| - a - ibp} - \frac{\widehat{G}(p)}{|p| - a - ibp}\right\|_{L^\infty(\mathbb{R})} \|F(u(x),x)\|_{L^2(\mathbb{R})} +$$

$$+\sqrt{2\pi}\left\|\frac{\widehat{G}_m(p)}{|p| - a - ibp}\right\|_{L^\infty(\mathbb{R})} \|F(u_m(x),x) - F(u(x),x)\|_{L^2(\mathbb{R})}.$$

Inequality (2.2) of Assumption 1 above gives us

$$\|F(u_m(x),x) - F(u(x),x)\|_{L^2(\mathbb{R})} \le l\|u_m(x) - u(x)\|_{L^2(\mathbb{R})}. \tag{3.6}$$

Evidently, $u_m(x),\ u(x) \in H^2(\mathbb{R}) \subset L^\infty(\mathbb{R})$ due to the Sobolev embedding. Thus, we arrive at

$$\|u_m(x) - u(x)\|_{L^2(\mathbb{R})}\left\{1 - \sqrt{2\pi}\left\|\frac{\widehat{G}_m(p)}{|p| - a - ibp}\right\|_{L^\infty(\mathbb{R})} l\right\} \le$$

$$\leq \sqrt{2\pi} \left\| \frac{\widehat{G}_m(p)}{|p| - a - ibp} - \frac{\widehat{G}(p)}{|p| - a - ibp} \right\|_{L^\infty(\mathbb{R})} \|F(u(x), x)\|_{L^2(\mathbb{R})}.$$

By virtue of (5.9) when $a > 0$ and of (5.11) for $a = 0$, we derive

$$\|u_m(x) - u(x)\|_{L^2(\mathbb{R})} \leq \frac{\sqrt{2\pi}}{\varepsilon} \left\| \frac{\widehat{G}_m(p)}{|p| - a - ibp} - \frac{\widehat{G}(p)}{|p| - a - ibp} \right\|_{L^\infty(\mathbb{R})} \|F(u(x), x)\|_{L^2(\mathbb{R})}.$$

Inequality (2.1) of Assumption 1 above implies that $F(u(x), x) \in L^2(\mathbb{R})$ for $u(x) \in H^2(\mathbb{R})$. Therefore,

$$u_m(x) \to u(x), \quad m \to \infty \tag{3.7}$$

in $L^2(\mathbb{R})$ due to the result of Lemma A2 of the Appendix. Evidently,

$$p^2\widehat{u}(p) = \sqrt{2\pi} \frac{p^2\widehat{G}(p)\widehat{\varphi}(p)}{|p| - a - ibp}, \quad p^2\widehat{u}_m(p) = \sqrt{2\pi} \frac{p^2\widehat{G}_m(p)\widehat{\varphi}_m(p)}{|p| - a - ibp}, \quad m \in \mathbb{N}.$$

Hence

$$|p^2\widehat{u}_m(p) - p^2\widehat{u}(p)| \leq \sqrt{2\pi} \left\| \frac{p^2\widehat{G}_m(p)}{|p| - a - ibp} - \frac{p^2\widehat{G}(p)}{|p| - a - ibp} \right\|_{L^\infty(\mathbb{R})} |\widehat{\varphi}(p)| +$$

$$+ \sqrt{2\pi} \left\| \frac{p^2\widehat{G}_m(p)}{|p| - a - ibp} \right\|_{L^\infty(\mathbb{R})} |\widehat{\varphi}_m(p) - \widehat{\varphi}(p)|.$$

Using (3.6), we obtain

$$\left\| \frac{d^2u_m}{dx^2} - \frac{d^2u}{dx^2} \right\|_{L^2(\mathbb{R})} \leq \sqrt{2\pi} \left\| \frac{p^2\widehat{G}_m(p)}{|p| - a - ibp} - \frac{p^2\widehat{G}(p)}{|p| - a - ibp} \right\|_{L^\infty(\mathbb{R})} \|F(u(x), x)\|_{L^2(\mathbb{R})} +$$

$$+ \sqrt{2\pi} \left\| \frac{p^2\widehat{G}_m(p)}{|p| - a - ibp} \right\|_{L^\infty(\mathbb{R})} l \|u_m(x) - u(x)\|_{L^2(\mathbb{R})}.$$

By means of the result of Lemma A2 of the Appendix along with (3.7), we derive $\frac{d^2u_m}{dx^2} \to \frac{d^2u}{dx^2}$ in $L^2(\mathbb{R})$ as $m \to \infty$. Definition (2.5) of the norm gives us $u_m(x) \to u(x)$ in $H^2(\mathbb{R})$ as $m \to \infty$.

Let us suppose that the unique solution $u_m(x)$ of equation (2.7) discussed above is trivial on the whole real line for a certain $m \in \mathbb{N}$. This will contradict to the assumption that the Fourier transforms of $G_m(x)$ and $F(0, x)$ do not vanish identically on a set of nonzero Lebesgue measure in $\mathbb{R}$. The similar argument is valid for the unique solution $u(x)$ of limiting problem (1.2). ■

4 The Problem on the Finite Interval

Proof of Theorem 3. Let us present the proof of our theorem in the situation when $a > 0$. If the constant a vanishes, the ideas will be analogous. When $a = 0$, we will need to use the constrained subspace (2.8) instead of $H^2(I)$. The non-selfadjoint operator in the left side of problem (2.3)

$$\mathcal{L}_{a,\ b} := \sqrt{-\frac{d^2}{dx^2}} - b\frac{d}{dx} - a : \quad H^1(I) \to L^2(I) \tag{4.1}$$

satisfies the Fredholm property. Its set of eigenvalues is given by

$$\lambda_{a,\ b}(n) = |n| - a - ibn, \quad n \in \mathbb{Z} \tag{4.2}$$

and its eigenfunctions are the standard Fourier harmonics $\frac{e^{inx}}{\sqrt{2\pi}}$, $n \in \mathbb{Z}$. Note that the eigenvalues of the the operator $\mathcal{L}_{a,\ b}$ are simple, as distinct from the similar situation without the drift term, when the eigenvalues corresponding to $n \neq 0$ are two-fold degenerate (see [36]).

First suppose that for some $v(x) \in H^2(I)$ there exist two solutions $u_{1,2}(x) \in H^2(I)$ of problem (2.3) with $\Omega = I$. Then the function $w(x) := u_1(x) - u_2(x) \in H^2(I)$ will be a solution of the homogeneous equation

$$\sqrt{-\frac{d^2}{dx^2}}w - b\frac{dw}{dx} - aw = 0.$$

Since the operator $\mathcal{L}_{a,\ b} :\ H^1(I) \to L^2(I)$ discussed above does not possess nontrivial zero modes, we have $w(x) \equiv 0$ in I.

We choose arbitrarily $v(x) \in H^2(I)$ and apply the Fourier transform (5.21) to problem (2.3) studied on the interval I. This yields

$$u_n = \sqrt{2\pi}\frac{G_n f_n}{|n| - a - ibn}, \quad n^2 u_n = \sqrt{2\pi}\frac{n^2 G_n f_n}{|n| - a - ibn}, \quad n \in \mathbb{Z} \tag{4.3}$$

with $f_n := F(v(x), x)_n$. This enables us to derive the upper bounds

$$|u_n| \le \sqrt{2\pi}\mathcal{N}_{a,\ b}|f_n|, \quad |n^2 u_n| \le \sqrt{2\pi}\mathcal{N}_{a,\ b}|f_n|.$$

Note that $\mathcal{N}_{a,\ b} < \infty$ under our assumptions by virtue of Lemma A3 of the Appendix. Thus, we arrive at

$$\|u\|^2_{H^2(I)} = \sum_{n=-\infty}^{\infty} |u_n|^2 + \sum_{n=-\infty}^{\infty} |n^2 u_n|^2 \le 4\pi\mathcal{N}^2_{a,\ b}\|F(v(x), x)\|^2_{L^2(I)} < \infty$$

due to inequality (2.1) of Assumption 1 for $v(x) \in H^2(I)$. Hence, for an arbitrary $v(x) \in H^2(I)$ there exists a unique $u(x) \in H^2(I)$, which solves equation (2.3) with its Fourier transform given by (4.3), such the map $\tau_{a,b} : H^2(I) \to H^2(I)$ in the first case of our the theorem is well defined.

Let us consider any $v_{1,2}(x) \in H^2(I)$, such their images under the map mentioned above $u_{1,2} = \tau_{a,b} v_{1,2} \in H^2(I)$. By virtue of (2.3), we have

$$\sqrt{-\frac{d^2}{dx^2}}u_1 - b\frac{du_1}{dx} - au_1 = \int_0^{2\pi} G(x-y)F(v_1(y), y)dy, \tag{4.4}$$

$$\sqrt{-\frac{d^2}{dx^2}}u_2 - b\frac{du_2}{dx} - au_2 = \int_0^{2\pi} G(x-y)F(v_2(y), y)dy. \tag{4.5}$$

By means of Fourier transform (5.21) applied to both sides of (4.4) and (4.5), we easily obtain

$$u_{1,n} = \sqrt{2\pi}\frac{G_n f_{1,n}}{|n| - a - ibn}, \quad u_{2,n} = \sqrt{2\pi}\frac{G_n f_{2,n}}{|n| - a - ibn},$$

$$n^2 u_{1,n} = \sqrt{2\pi}\frac{n^2 G_n f_{1,n}}{|n| - a - ibn}, \quad n^2 u_{2,n} = \sqrt{2\pi}\frac{n^2 G_n f_{2,n}}{|n| - a - ibn}, \quad n \in \mathbb{Z}$$

with $f_{j,n} := F(v_j(x), x)_n, \ j = 1, 2$. Hence,

$$|u_{1,n} - u_{2,n}| \le \sqrt{2\pi}\mathcal{N}_{a,\ b}|f_{1,n} - f_{2,n}|, \quad |n^2(u_{1,n} - u_{2,n})| \le \sqrt{2\pi}\mathcal{N}_{a,\ b}|f_{1,n} - f_{2,n}|,$$

so that

$$\|u_1 - u_2\|^2_{H^2(I)} = \sum_{n=-\infty}^{\infty} |u_{1,n} - u_{2,n}|^2 + \sum_{n=-\infty}^{\infty} |n^2(u_{1,n} - u_{2,n})|^2 \le$$

$$\le 4\pi\mathcal{N}^2_{a,\ b}\|F(v_1(x), x) - F(v_2(x), x)\|^2_{L^2(I)}.$$

Obviously, $v_{1,2}(x) \in H^2(I) \subset L^\infty(I)$ due to the Sobolev embedding. By virtue of inequality (2.2) we easily arrive at

$$\|\tau_{a,b}v_1 - \tau_{a,b}v_2\|_{H^2(I)} \le 2\sqrt{\pi}\mathcal{N}_{a,\ b}l\|v_1 - v_2\|_{H^2(I)}. \tag{4.6}$$

The constant in the right side of (4.6) is less than one via the one of our assumptions. Then the Fixed Point Theorem implies the existence and uniqueness of a function $v_{a,b} \in H^2(I)$ which satisfies $\tau_{a,b}v_{a,b} = v_{a,b}$. This is the only solution of equation (1.2) in $H^2(I)$ in the first case of our theorem. Let us suppose that $v_{a,b}(x)$ is trivial in I. This will contradict to the assumption that $G_n F(0, x)_n \neq 0$ for a certain $n \in \mathbb{Z}$. Note that in the situation of the theorem when $a > 0$ our argument does not rely on any orthogonality conditions. ■

We turn our attention to establishing the final main result of the article.

Proof of Theorem 4. Evidently, the limiting kernel $G(x)$ is also a periodic function on our interval I (see the argument of Lemma A4 of the Appendix). Each problem (2.9) has a unique solution $u_m(x), \ m \in \mathbb{N}$, which belongs to $H^2(I)$ in the case if $a > 0$ and to $H^2_0(I)$

in the situation when $a = 0$ by virtue of Theorem 3. Limiting equation (1.2) possesses a unique solution $u(x)$ belonging to $H^2(I)$ in the situation when $a > 0$ and to $H_0^2(I)$ in the case when a vanishes via Lemma A4 below along with Theorem 3.

Let us apply Fourier transform (5.21) to both sides of equations (1.2) and (2.9). This yields

$$u_n = \sqrt{2\pi}\frac{G_n\varphi_n}{|n| - a - ibn}, \quad u_{m,n} = \sqrt{2\pi}\frac{G_{m,n}\varphi_{m,n}}{|n| - a - ibn}, \quad n \in \mathbb{Z}, \quad m \in \mathbb{N}. \tag{4.7}$$

Here φ_n and $\varphi_{m,n}$ are the Fourier images of $F(u(x), x)$ and $F(u_m(x), x)$ respectively under transform (5.21). We easily estimate obtain the upper bound

$$|u_{m,n} - u_n| \le \sqrt{2\pi}\left\|\frac{G_{m,n}}{|n| - a - ibn} - \frac{G_n}{|n| - a - ibn}\right\|_{l^\infty} |\varphi_n| + \sqrt{2\pi}\left\|\frac{G_{m,n}}{|n| - a - ibn}\right\|_{l^\infty} |\varphi_{m,n} - \varphi_n|.$$

Hence,

$$\|u_m - u\|_{L^2(I)} \le \sqrt{2\pi}\left\|\frac{G_{m,n}}{|n| - a - ibn} - \frac{G_n}{|n| - a - ibn}\right\|_{l^\infty} \|F(u(x), x)\|_{L^2(I)} +$$

$$+\sqrt{2\pi}\left\|\frac{G_{m,n}}{|n| - a - ibn}\right\|_{l^\infty} \|F(u_m(x), x) - F(u(x), x)\|_{L^2(I)}.$$

By virtue of inequality (2.2) of Assumption 1, we derive

$$\|F(u_m(x), x) - F(u(x), x)\|_{L^2(I)} \le l\|u_m(x) - u(x)\|_{L^2(I)}. \tag{4.8}$$

Obviously, $u_m(x), u(x) \in H^2(I) \subset L^\infty(I)$ via the Sobolev embedding. Clearly,

$$\|u_m - u\|_{L^2(I)} \left\{1 - \sqrt{2\pi}l\left\|\frac{G_{m,n}}{|n| - a - ibn}\right\|_{l^\infty}\right\} \le$$

$$\le \sqrt{2\pi}\left\|\frac{G_{m,n}}{|n| - a - ibn} - \frac{G_n}{|n| - a - ibn}\right\|_{l^\infty} \|F(u(x), x)\|_{L^2(I)}.$$

By means of inequalities (5.31) in the case when $a > 0$ and (5.33) in the situation when a vanishes, we arrive at

$$\|u_m - u\|_{L^2(I)} \le \frac{\sqrt{2\pi}}{\varepsilon}\left\|\frac{G_{m,n}}{|n| - a - ibn} - \frac{G_n}{|n| - a - ibn}\right\|_{l^\infty} \|F(u(x), x)\|_{L^2(I)}.$$

Apparently, $F(u(x), x) \in L^2(I)$ for $u(x) \in H^2(I)$ due to upper bound (2.1) of Assumption 1. The result of Lemma A4 of the Appendix implies that

$$u_m(x) \to u(x), \quad m \to \infty \tag{4.9}$$

in $L^2(I)$. Evidently,

$$|n^2u_{m,n} - n^2u_n| \le \sqrt{2\pi}\left\|\frac{n^2G_{m,n}}{|n|-a-ibn} - \frac{n^2G_n}{|n|-a-ibn}\right\|_{l^\infty}|\varphi_n| +$$

$$+\sqrt{2\pi}\left\|\frac{n^2G_{m,n}}{|n|-a-ibn}\right\|_{l^\infty}|\varphi_{m,n}-\varphi_n|.$$

Let us use (4.8) to obtain

$$\left\|\frac{d^2u_m}{dx^2} - \frac{d^2u_m}{dx^2}\right\|_{L^2(I)} \le \sqrt{2\pi}\left\|\frac{n^2G_{m,n}}{|n|-a-ibn} - \frac{n^2G_n}{|n|-a-ibn}\right\|_{l^\infty}\|F(u(x),x)\|_{L^2(I)} +$$

$$+\sqrt{2\pi}\left\|\frac{n^2G_{m,n}}{|n|-a-ibn}\right\|_{l^\infty} l\|u_m(x)-u(x)\|_{L^2(I)}.$$

By means of Lemma A4 along with (4.9), we derive $\frac{d^2u_m}{dx^2} \to \frac{d^2u}{dx^2}$ as $m \to \infty$ in $L^2(I)$. Thus, $u_m(x) \to u(x)$ in the $H^2(I)$ norm as $m \to \infty$.

If we suppose that $u_m(x) \equiv 0$ in the interval I for some $m \in \mathbb{N}$, then we will obtain a contradiction to the assumption that $G_{m,n}F(0,x)_n \neq 0$ for a certain $n \in \mathbb{Z}$. The similar reasoning is valid for the solution $u(x)$ of limiting equation (1.2). ■

5 Appendix

Let $G(x)$ be a function, $G(x) : \mathbb{R} \to \mathbb{R}$, for which we denote its standard Fourier transform using the hat symbol as

$$\widehat{G}(p) := \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} G(x)e^{-ipx}dx, \quad p \in \mathbb{R}, \tag{5.1}$$

so that

$$\|\widehat{G}(p)\|_{L^\infty(\mathbb{R})} \le \frac{1}{\sqrt{2\pi}}\|G(x)\|_{L^1(\mathbb{R})} \tag{5.2}$$

and $G(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\widehat{G}(q)e^{iqx}dq,\ x \in \mathbb{R}$. Clearly, (5.2) yields

$$\|p\widehat{G}(p)\|_{L^\infty(\mathbb{R})} \le \frac{1}{\sqrt{2\pi}}\left\|\frac{dG(x)}{dx}\right\|_{L^1(\mathbb{R})}. \tag{5.3}$$

For the technical purposes we introduce the auxiliary quantities

$$N_{a,\ b} := max\left\{\left\|\frac{\widehat{G}(p)}{|p|-a-ibp}\right\|_{L^\infty(\mathbb{R})}, \quad \left\|\frac{p^2\widehat{G}(p)}{|p|-a-ibp}\right\|_{L^\infty(\mathbb{R})}\right\}, \tag{5.4}$$

where $a \geq 0, \; b \in \mathbb{R}, \; b \neq 0$ are the constants.

Lemma A1. *Let* $G(x) : \mathbb{R} \to \mathbb{R}, \; G(x) \in W^{1,1}(\mathbb{R})$.

a) If $a > 0, \; b \in \mathbb{R}, \; b \neq 0$ *then* $N_{a, \, b} < \infty$.

b) If $a = 0, \; b \in \mathbb{R}, \; b \neq 0$ *and additionally* $xG(x) \in L^1(\mathbb{R})$ *then* $N_{0, \, b} < \infty$ *if and only if the orthogonality condition*

$$(G(x), 1)_{L^2(\mathbb{R})} = 0 \tag{5.5}$$

is valid.

Proof. First of all, let us observe that in both cases a) and b) of our lemma, under the given conditions the expression

$$\frac{p^2 \widehat{G}(p)}{|p| - a - ibp} \in L^\infty(\mathbb{R}).$$

Indeed, it can be easily verified that the function $\dfrac{p}{|p| - a - ibp}$ is bounded on the whole real line and $p\widehat{G}(p) \in L^\infty(\mathbb{R})$ due to inequality (5.3) above. Let us turn our attention to establishing the result of the part a) of the lemma. We need to estimate the expression

$$\frac{|\widehat{G}(p)|}{\sqrt{(|p| - a)^2 + b^2 p^2}}. \tag{5.6}$$

Apparently, the numerator of (5.6) can be bounded from above by virtue of (5.2) and the denominator in (5.6) can be trivially estimated below by a finite, positive constant, such that

$$\left| \frac{\widehat{G}(p)}{|p| - a - ibp} \right| \leq C \|G(x)\|_{L^1(\mathbb{R})} < \infty$$

due to the one of our assumptions. Here and further down C will denote a finite, positive constant. This yields that under the given assumptions, for $a > 0$ we have $N_{a, \, b} < \infty$. In the situation when $a = 0$, we will use the identity

$$\widehat{G}(p) = \widehat{G}(0) + \int_0^p \frac{d\widehat{G}(s)}{ds} ds.$$

Hence

$$\frac{\widehat{G}(p)}{|p| - ibp} = \frac{\widehat{G}(0)}{|p| - ibp} + \frac{\int_0^p \frac{d\widehat{G}(s)}{ds} ds}{|p| - ibp}. \tag{5.7}$$

By means of definition (5.1) of the standard Fourier transform, we easily arrive at

$$\left| \frac{d\widehat{G}(p)}{dp} \right| \leq \frac{1}{\sqrt{2\pi}} \|xG(x)\|_{L^1(\mathbb{R})}.$$

Therefore,

$$\left|\frac{\int_0^p \frac{d\widehat{G}(s)}{ds}ds}{|p|-ibp}\right| \le \frac{\|xG(x)\|_{L^1(\mathbb{R})}}{\sqrt{2\pi(1+b^2)}} < \infty$$

as assumed. Thus, the expression in the left side of (5.7) is bounded if and only if $\widehat{G}(0)=0$. This is equivalent to orthogonality condition (5.5). ■

Let us define the following technical expressions, which will help to study problems (2.7).

$$N_{a,\ b,\ m} := max\left\{\left\|\frac{\widehat{G}_m(p)}{|p|-a-ibp}\right\|_{L^\infty(\mathbb{R})}, \quad \left\|\frac{p^2\widehat{G}_m(p)}{|p|-a-ibp}\right\|_{L^\infty(\mathbb{R})}\right\} \tag{5.8}$$

with constants $a \ge 0,\ b\in\mathbb{R},\ b\neq 0$ and $m\in\mathbb{N}$. We have the following auxiliary statement.

Lemma A2. *Let $m\in\mathbb{N}$, $G_m(x):\mathbb{R}\to\mathbb{R}$, $G_m(x)\in W^{1,1}(\mathbb{R})$ and $G_m(x)\to G(x)$ in $W^{1,1}(\mathbb{R})$ as $m\to\infty$.*

a) If $a>0,\ b\in\mathbb{R},\ b\neq 0$, let

$$2\sqrt{\pi}N_{a,\ b,\ m}l \le 1-\varepsilon \tag{5.9}$$

for all $m\in\mathbb{N}$ with some fixed $0<\varepsilon<1$.

b) If $a=0,\ b\in\mathbb{R},\ b\neq 0$, let $xG_m(x)\in L^1(\mathbb{R})$, $xG_m(x)\to xG(x)$ in $L^1(\mathbb{R})$ as $m\to\infty$, the orthogonality relation

$$(G_m(x),1)_{L^2(\mathbb{R})}=0, \quad m\in\mathbb{N} \tag{5.10}$$

holds. Let in addition

$$2\sqrt{\pi}N_{0,\ b,\ m}l \le 1-\varepsilon \tag{5.11}$$

for all $m\in\mathbb{N}$ with a certain fixed $0<\varepsilon<1$.

Then

$$\frac{\widehat{G}_m(p)}{|p|-a-ibp} \to \frac{\widehat{G}(p)}{|p|-a-ibp}, \quad m\to\infty, \tag{5.12}$$

$$\frac{p^2\widehat{G}_m(p)}{|p|-a-ibp} \to \frac{p^2\widehat{G}(p)}{|p|-a-ibp}, \quad m\to\infty \tag{5.13}$$

in $L^\infty(\mathbb{R})$, so that

$$\left\|\frac{\widehat{G}_m(p)}{|p|-a-ibp}\right\|_{L^\infty(\mathbb{R})} \to \left\|\frac{\widehat{G}(p)}{|p|-a-ibp}\right\|_{L^\infty(\mathbb{R})}, \quad m\to\infty, \tag{5.14}$$

$$\left\|\frac{p^2\widehat{G}_m(p)}{|p|-a-ibp}\right\|_{L^\infty(\mathbb{R})} \to \left\|\frac{p^2\widehat{G}(p)}{|p|-a-ibp}\right\|_{L^\infty(\mathbb{R})}, \quad m\to\infty. \tag{5.15}$$

Moreover,

$$2\sqrt{\pi}N_{a,\ b}l \le 1-\varepsilon. \tag{5.16}$$

Proof. Using inequality (5.2), we easily obtain

$$\|\widehat{G}_m(p) - \widehat{G}(p)\|_{L^\infty(\mathbb{R})} \leq \frac{1}{\sqrt{2\pi}}\|G_m(x) - G(x)\|_{L^1(\mathbb{R})} \to 0, \quad m \to \infty \tag{5.17}$$

via the one of our assumptions. Obviously, (5.14) and (5.15) will easily follow from (5.12) and (5.13) respectively by virtue of the standard triangle inequality.

Let us use the fact that the function $\dfrac{p}{|p| - a - ibp} \in L^\infty(\mathbb{R})$ along with the analog of estimate (5.3) to derive

$$\left|\frac{p^2\widehat{G}_m(p)}{|p| - a - ibp} - \frac{p^2\widehat{G}(p)}{|p| - a - ibp}\right| \leq C\|p[\widehat{G}_m(p) - \widehat{G}(p)]\|_{L^\infty(\mathbb{R})} \leq \frac{C}{\sqrt{2\pi}}\left\|\frac{dG_m(x)}{dx} - \frac{dG(x)}{dx}\right\|_{L^1(\mathbb{R})}.$$

Hence

$$\left\|\frac{p^2\widehat{G}_m(p)}{|p| - a - ibp} - \frac{p^2\widehat{G}(p)}{|p| - a - ibp}\right\|_{L^\infty(\mathbb{R})} \leq \frac{C}{\sqrt{2\pi}}\left\|\frac{dG_m(x)}{dx} - \frac{dG(x)}{dx}\right\|_{L^1(\mathbb{R})} \to 0$$

as $m \to \infty$ due to the one of our assumptions, such that (5.13) holds.

In order to establish (5.12) in the situation when $a > 0$, we need to consider

$$\frac{|\widehat{G}_m(p) - \widehat{G}(p)|}{\sqrt{(|p| - a)^2 + b^2p^2}}. \tag{5.18}$$

Apparently, the denominator in fraction (5.18) can be estimated from below by a positive constant and the numerator in (5.18) can be bounded from above by virtue of (5.17). Thus

$$\left\|\frac{\widehat{G}_m(p)}{|p| - a - ibp} - \frac{\widehat{G}(p)}{|p| - a - ibp}\right\|_{L^\infty(\mathbb{R})} \leq C\|G_m(x) - G(x)\|_{L^1(\mathbb{R})} \to 0$$

as $m \to \infty$ according to the one of the assumptions, such that (5.12) holds in the case a) of our lemma.

Let us proceed to establishing (5.12) in the situation when a vanishes. In this case orthogonality relations (5.10) hold as assumed. We easily demonstrate that the analogous statement will be valid in the limit. Clearly,

$$|(G(x), 1)_{L^2(\mathbb{R})}| = |(G(x) - G_m(x), 1)_{L^2(\mathbb{R})}| \leq \|G_m(x) - G(x)\|_{L^1(\mathbb{R})} \to 0$$

as $m \to \infty$ via the one of our assumptions, so that

$$(G(x), 1)_{L^2(\mathbb{R})} = 0 \tag{5.19}$$

holds. Evidently, we have

$$\widehat{G}(p) = \widehat{G}(0) + \int_0^p \frac{d\widehat{G}(s)}{ds}ds, \quad \widehat{G}_m(p) = \widehat{G}_m(0) + \int_0^p \frac{d\widehat{G}_m(s)}{ds}ds, \quad m \in \mathbb{N}.$$

Formulas (5.19) and (5.10) gives us

$$\widehat{G}(0) = 0, \quad \widehat{G}_m(0) = 0, \quad m \in \mathbb{N}.$$

Therefore,

$$\left| \frac{\widehat{G}_m(p)}{|p| - ibp} - \frac{\widehat{G}(p)}{|p| - ibp} \right| = \left| \frac{\int_0^p \left[\frac{d\widehat{G}_m(s)}{ds} - \frac{d\widehat{G}(s)}{ds} \right] ds}{|p| - ibp} \right|. \tag{5.20}$$

The definition of the standard Fourier transform (5.1) yields

$$\left| \frac{d\widehat{G}_m(p)}{dp} - \frac{d\widehat{G}(p)}{dp} \right| \leq \frac{1}{\sqrt{2\pi}} \|xG_m(x) - xG(x)\|_{L^1(\mathbb{R})}.$$

This enables us to obtain the upper bound on the right side of (5.20) given by

$$\frac{\|xG_m(x) - xG(x)\|_{L^1(\mathbb{R})}}{\sqrt{2\pi(1 + b^2)}},$$

so that

$$\left\| \frac{\widehat{G}_m(p)}{|p| - ibp} - \frac{\widehat{G}(p)}{|p| - ibp} \right\|_{L^\infty(\mathbb{R})} \leq \frac{\|xG_m(x) - xG(x)\|_{L^1(\mathbb{R})}}{\sqrt{2\pi(1 + b^2)}} \to 0$$

as $m \to \infty$ due to the one of the given conditions. Thus (5.12) holds when $a = 0$. Apparently, under the assumptions of our lemma

$$N_{a,\ b} < \infty, \quad N_{a,\ b,\ m} < \infty, \quad m \in \mathbb{N}, \quad a \geq 0, \quad b \in \mathbb{R}, \quad b \neq 0$$

by virtue of the result of Lemma A1 above. We have upper bounds (5.9) when $a > 0$ and (5.11) if a vanishes. An easy limiting argument using (5.14) and (5.15) yields (5.16). ■

Consider the function $G(x) : I \to \mathbb{R}$, such that $G(0) = G(2\pi)$. Its Fourier transform on our finite interval is defined as

$$G_n := \int_0^{2\pi} G(x) \frac{e^{-inx}}{\sqrt{2\pi}} dx, \quad n \in \mathbb{Z}, \tag{5.21}$$

so that $G(x) = \sum_{n=-\infty}^{\infty} G_n \frac{e^{inx}}{\sqrt{2\pi}}$. Evidently, the inequality

$$\|G_n\|_{l^\infty} \leq \frac{1}{\sqrt{2\pi}} \|G(x)\|_{L^1(I)} \tag{5.22}$$

holds. Clearly, if our function is continuous on the interval I, we have the upper bound

$$\|G(x)\|_{L^1(I)} \le 2\pi \|G(x)\|_{C(I)}. \tag{5.23}$$

The estimate from above

$$\|nG_n\|_{l^\infty} \le \frac{1}{\sqrt{2\pi}} \left\| \frac{dG(x)}{dx} \right\|_{L^1(I)} \tag{5.24}$$

easily follows from (5.22). Similarly to the whole real line case, we introduce

$$\mathcal{N}_{a,\ b} := max\left\{ \left\| \frac{G_n}{|n| - a - ibn} \right\|_{l^\infty}, \quad \left\| \frac{n^2 G_n}{|n| - a - ibn} \right\|_{l^\infty} \right\}, \tag{5.25}$$

where $a \ge 0,\ b \in \mathbb{R},\ b \ne 0$ are the constants.

We have the following technical proposition.

Lemma A3. *Let* $G(x) : I \to \mathbb{R},\ G(x) \in C(I),\ \dfrac{dG(x)}{dx} \in L^1(I)$ *and* $G(0) = G(2\pi)$.

a) If $a > 0,\ b \in \mathbb{R},\ b \ne 0$ *then* $\mathcal{N}_{a,\ b} < \infty$.

b) If $a = 0,\ b \in \mathbb{R},\ b \ne 0$ *then* $\mathcal{N}_{0,\ b} < \infty$ *if and only if the orthogonality condition*

$$(G(x), 1)_{L^2(I)} = 0. \tag{5.26}$$

is valid.

Proof. It can be easily verified that in both cases a) and b) of the lemma under the given assumptions we have

$$\frac{n^2 G_n}{|n| - a - ibn} \in l^\infty. \tag{5.27}$$

Indeed, $\dfrac{n}{|n| - a - ibn} \in l^\infty$ and $nG_n \in l^\infty$ due to inequality (5.24) along with the one of our conditions, such that (5.27) holds.

To establish the statement of the part a) of our lemma, we need to consider the expression

$$\frac{|G_n|}{\sqrt{(|n| - a)^2 + b^2 n^2}}. \tag{5.28}$$

Apparently, the denominator in (5.28) can be trivially estimated below by a positive constant and the numerator in (5.28) can be easily bounded above by virtue of (5.22) along with (5.23). Thus, $\mathcal{N}_{a,\ b} < \infty$ in the situation when $a > 0$. To demonstrate the validity of the result of our lemma in the case when a vanishes, we note that

$$\left| \frac{G_n}{|n| - ibn} \right| \tag{5.29}$$

is bounded if and only if $G_0 = 0$. This is equivalent to orthogonality relation (5.26). In this case we can estimate expression (5.29) from above by

$$\frac{1}{\sqrt{2\pi}|n|}\frac{\|G(x)\|_{L^1(I)}}{\sqrt{1+b^2}} \leq \sqrt{2\pi}\frac{\|G(x)\|_{C(I)}}{\sqrt{1+b^2}} < \infty$$

by means of (5.22) and (5.23) under the given conditions. ∎

For the purpose of the studies of problems (2.9), we define

$$\mathcal{N}_{a,\ b,\ m} := max\left\{\left\|\frac{G_{m,n}}{|n|-a-ibn}\right\|_{l^\infty}, \quad \left\|\frac{n^2 G_{m,n}}{|n|-a-ibn}\right\|_{l^\infty}\right\} \tag{5.30}$$

with the constants $a \geq 0,\ b \in \mathbb{R},\ b \neq 0$ and $m \in \mathbb{N}$. Let us conclude the article with the following technical statement.

Lemma A4. *Let us assume that for* $m \in \mathbb{N}$

$$G_m(x) : I \to \mathbb{R}, \quad G_m(x) \in C(I), \quad \frac{dG_m(x)}{dx} \in L^1(I), \quad G_m(0) = G_m(2\pi),$$

and

$$G_m(x) \to G(x) \quad in \quad C(I), \quad \frac{dG_m(x)}{dx} \to \frac{dG(x)}{dx} \quad in \quad L^1(I)$$

as $m \to \infty$.

a) If $a > 0,\ b \in \mathbb{R},\ b \neq 0,$ *let*

$$2\sqrt{\pi}\mathcal{N}_{a,\ b,\ m} l \leq 1 - \varepsilon \tag{5.31}$$

for all $m \in \mathbb{N}$ *with some fixed* $0 < \varepsilon < 1$.

b) If $a = 0,\ b \in \mathbb{R},\ b \neq 0,$ *let the orthogonality relation*

$$(G_m(x), 1)_{L^2(I)} = 0, \quad m \in \mathbb{N} \tag{5.32}$$

hold. Let in addition

$$2\sqrt{\pi}\mathcal{N}_{0,\ b,\ m} l \leq 1 - \varepsilon \tag{5.33}$$

for all $m \in \mathbb{N}$ *with a certain fixed* $0 < \varepsilon < 1$.

Then

$$\frac{G_{m,n}}{|n|-a-ibn} \to \frac{G_n}{|n|-a-ibn}, \quad m \to \infty, \tag{5.34}$$

$$\frac{n^2 G_{m,n}}{|n|-a-ibn} \to \frac{n^2 G_n}{|n|-a-ibn}, \quad m \to \infty \tag{5.35}$$

in l^∞, *so that*

$$\left\|\frac{G_{m,n}}{|n|-a-ibn}\right\|_{l^\infty} \to \left\|\frac{G_n}{|n|-a-ibn}\right\|_{l^\infty}, \quad m \to \infty, \tag{5.36}$$

$$\left\| \frac{n^2 G_{m,n}}{|n| - a - ibn} \right\|_{l^\infty} \to \left\| \frac{n^2 G_n}{|n| - a - ibn} \right\|_{l^\infty}, \quad m \to \infty. \tag{5.37}$$

Moreover, the estimate

$$2\sqrt{\pi}\mathcal{N}_{a,\ b} l \le 1 - \varepsilon \tag{5.38}$$

is valid.

Proof. Evidently, under the given conditions, the limiting kernel function $G(x)$ will be periodic as well. Indeed, we easily derive

$$|G(0) - G(2\pi)| \le |G(0) - G_m(0)| + |G_m(2\pi) - G(2\pi)| \le 2\|G_m(x) - G(x)\|_{C(I)} \to 0, \quad m \to \infty$$

due to the one of our assumptions. Hence, $G(0) = G(2\pi)$. By means of (5.22) along with (5.23) we arrive at

$$\|G_{m,n} - G_n\|_{l^\infty} \le \frac{1}{\sqrt{2\pi}} \|G_m - G\|_{L^1(I)} \le \sqrt{2\pi} \|G_m - G\|_{C(I)} \to 0, \quad m \to \infty \tag{5.39}$$

via the one of the given conditions. It can be easily verified that the statements of (5.34) and (5.35) will yield (5.36) and (5.37) respectively by virtue of the triangle inequality. By means of (5.24), we arrive at the upper bound

$$\left\| \frac{n^2 G_{m,n}}{|n| - a - ibn} - \frac{n^2 G_n}{|n| - a - ibn} \right\|_{l^\infty} \le \frac{1}{\sqrt{2\pi}} \left\| \frac{n}{|n| - a - ibn} \right\|_{l^\infty} \left\| \frac{dG_m(x)}{dx} - \frac{dG(x)}{dx} \right\|_{L^1(I)} \to 0$$

as $m \to \infty$ as assumed, so that (5.35) holds. In order to demonstrate the validity of (5.34) in the situation when $a > 0$, we need to consider

$$\frac{|G_{m,n} - G_n|}{\sqrt{(|n| - a)^2 + b^2 n^2}}. \tag{5.40}$$

Clearly, the denominator of (5.40) can be estimated from below by a positive constant and the numerator bounded from above by virtue of (5.39). This implies (5.34) for $a > 0$.

Let us conclude the proof of the lemma by establishing (5.34) in the case when $a = 0$. According to the one of the given assumptions, we have orthogonality relation (5.32). It can be easily verified that the analogous condition is valid in the limit. Indeed,

$$|(G(x), 1)_{L^2(I)}| = |(G(x) - G_m(x), 1)_{L^2(I)}| \le 2\pi \|G_m(x) - G(x)\|_{C(I)} \to 0, \quad m \to \infty$$

as assumed. Hence,

$$(G(x), 1)_{L^2(I)} = 0,$$

which is equivalent to $G_0 = 0$. Clearly, $G_{m,0} = 0$, $m \in \mathbb{N}$ due to orthogonality relation (5.32). By virtue of (5.39), we derive

$$\left| \frac{G_{m,n} - G_n}{|n| - ibn} \right| \le \frac{\sqrt{2\pi} \|G_m(x) - G(x)\|_{C(I)}}{\sqrt{1 + b^2}}.$$

Because the norm in the right side of this upper bound converges to zero as $m \to \infty$, (5.34) is valid in the situation when $a = 0$ as well. Apparently, under the given conditions we have

$$\mathcal{N}_{a,\ b} < \infty, \quad \mathcal{N}_{a,\ b,\ m} < \infty, \quad m \in \mathbb{N}, \quad a \geq 0, \quad b \in \mathbb{R}, \quad b \neq 0$$

by means of the result of our Lemma A3. We have inequalities (5.31) for $a > 0$ and (5.33) when $a = 0$. A trivial limiting argument relying on (5.36) and (5.37) yields (5.38). ■

References

[1] M.S. Agranovich, *Elliptic boundary problems*, Encyclopaedia Math. Sci., **79**, Partial Differential Equations, IX, Springer, Berlin (1997), 1–144

[2] N. Apreutesei, N. Bessonov, V. Volpert, V. Vougalter, *Spatial structures and generalized travelling waves for an integro- differential equation*, Discrete Contin. Dyn. Syst. Ser. B, **13** (2010), no. 3, 537–557

[3] H. Berestycki, G. Nadin, B. Perthame, L. Ryzhik, *The non-local Fisher-KPP equation: travelling waves and steady states*, Nonlinearity, **22** (2009), no. 12, 2813–2844

[4] H. Berestycki, F. Hamel, N. Nadirashvili, *The speed of propagation for KPP type problems. I. Periodic framework*, J. Eur. Math. Soc. (JEMS), **7** (2005), no. 2, 173–213

[5] H. Brezis, L. Oswald, *Remarks on sublinear elliptic equations*, Nonlinear Anal., **10** (1986), no. 1, 55–64

[6] A. Ducrot, M. Marion, V. Volpert, *Reaction-diffusion problems with non-Fredholm operators*, Adv. Differential Equations, **13** (2008), no. 11-12, 1151–1192

[7] M.A. Efendiev, *Fredholm structures, topological invariants and applications.* AIMS Series on Differential Equations & Dynamical Systems, **3**. American Institute of Mathematical Sciences (AIMS), Springfield, MO, 2009, 205 pp.

[8] M.A. Efendiev, *Finite and infinite dimensional attractors for evolution equations of mathematical physics.* Gakuto International Series. Mathematical Sciences and Applications, **33**. *Gakkōtosho Co., Ltd., Tokyo*, (2010). 239 pp.

[9] M.A. Efendiev, *Symmetrization and stabilization of solutions of nonlinear elliptic equations,* Fields Institute Monographs, **36**. Fields Institute for Research in Mathematical Sciences, Toronto, ON; Springer, Cham (2018), 258 pp.

[10] M.A. Efendiev, L.A. Peletier, *On the large time behavior of solutions of fourth order parabolic equations and ε-entropy of their attractors,* C. R. Math. Acad. Sci. Paris, **344** (2007), no. 2, 93–96

[11] M.A. Efendiev, V. Vougalter, *Solvability of some integro-differential equations with drift,* Osaka J. Math., **57** (2020), no. 2, 247–265

[12] M.A. Efendiev, V. Vougalter, *Solvability in the sense of sequences for some fourth order non-Fredholm operators,* J. Differential Equations, **271** (2021), 280–300

[13] M.A. Efendiev, V. Vougalter, *Verification of Biomedical Processes with Anomalous Diffusion, Transport and Interaction of Species,* DOI: 10.1007/978-981-15-9034-4_5, Book: Nonlinear Dynamics, Chaos, and Complexity, Springer, (2021)

[14] M.A. Efendiev, V. Vougalter, *Existence of solutions for some non-Fredholm integro-differential equations with mixed diffusion,* J. Differential Equations, **284** (2021), 83–101

[15] M.A. Efendiev, S.V. Zelik, *The attractor for a nonlinear reaction-diffusion system in an unbounded domain,* Comm. Pure Appl. Math., **54** (2001), no. 6, 625-688

[16] H.G. Gebran, C.A. Stuart, *Fredholm and properness properties of quasilinear elliptic systems of second order,* Proc. Edinb. Math. Soc. (2), **48** (2005), no. 1, 91-124

[17] H.G. Gebran, C.A. Stuart, *Exponential decay and Fredholm properties in second-order quasilinear elliptic systems,* J. Differential Equations, **249** (2010), no. 1, 94–117

[18] P.D. Hislop, I.M. Sigal, *Introduction to spectral theory. With applications to Schrödinger operators.* Applied Mathematical Sciences, **113**. Springer-Verlag, New York, 1996. 337 pp.

[19] M.A. Krasnosel'skii, *Topological methods in the theory of nonlinear integral equations.* Pergamon Press The Macmillan Co., New York (1964). 395 pp.

[20] J.-L. Lions, E. Magenes, *Problèmes aux limites non homogènes et applications.* Vol. 1. (French) Travaux et Recherches Mathematiques, No. 17 Dunod, Paris, 1968. 372 pp.

[21] V.G. Maz'ja, *Weak solutions of the Dirichlet and Neumann problems,* (Russian) Trudy Moskov. Mat. Obsc., **20** (1969), 137–172

[22] V.G. Maz'ja, M. Otelbaev, *Imbedding theorems and the spectrum of a certain pseudodifferential operator,* (Russian) Sibirsk. Mat. Z., **18** (1977), no. 5, 1073–1087, 1206

[23] R. Metzler, J. Klafter, *The random walk's guide to anomalous diffusion: a fractional dynamics approach,* Phys. Rep., **339** (2000), 1–77

[24] P.J. Rabier, C.A. Stuart, *Fredholm and properness properties of quasilinear elliptic operators on* $\mathbb{R}^N$, Math. Nachr., **231** (2001), 129–168

[25] L.R. Volevich, *Solubility of boundary value problems for general elliptic systems,* (Russian) Mat. Sb. (N.S.), **68 (110)** (1965), 373–416

[26] V. Volpert, *Elliptic partial differential equations.* Volume 1: Fredholm theory of elliptic problems in unbounded domains. Monographs in Mathematics, **101**. Birkhäuser/Springer Basel AG, Basel, (2011). 639 pp.

[27] V. Volpert, B. Kazmierczak, M. Massot, Z. Peradzynski, *Solvability conditions for elliptic problems with non-Fredholm operators,* Appl. Math. (Warsaw), **29** (2002), no. 2, 219–238

[28] V. Volpert, V. Vougalter, *Emergence and propagation of patterns in nonlocal reaction-diffusion equations arising in the theory of speciation.* Dispersal, individual movement and spatial ecology, Lecture Notes in Math., **2071**, Springer, Heidelberg (2013), 331–353

[29] V. Volpert, V. Vougalter, *Solvability in the sense of sequences to some non-Fredholm operators,* Electron. J. Differential Equations, **2013**, No. 160, 16 pp. (2013)

[30] V. Vougalter, V. Volpert, *Solvability conditions for some non-Fredholm operators,* Proc. Edinb. Math. Soc. (2), **54** (2011), no. 1, 249–271

[31] V. Vougalter, V. Volpert, *On the existence of stationary solutions for some non-Fredholm integro-differential equations,* Doc. Math., **16** (2011), 561–580

[32] V. Vougalter, V. Volpert, *On the solvability conditions for the diffusion equation with convection terms,* Commun. Pure Appl. Anal., **11** (2012), no. 1, 365–373

[33] V. Vougalter, V. Volpert, *Solvability conditions for some linear and nonlinear non-Fredholm elliptic problems,* Anal. Math. Phys., **2** (2012), no. 4, 473–496

[34] V. Vougalter, V. Volpert, *Existence of stationary solutions for some integro-differential equations with superdiffusion,* Rend. Semin. Mat. Univ. Padova, **137** (2017), 185–201

[35] V. Vougalter, V. Volpert, *Existence in the sense of sequences of stationary solutions for some non-Fredholm integro- differential equations,* J. Math. Sci. (N.Y.), **228** (2018), no. 6, Problems in mathematical analysis. No. 90 (Russian), 601–632

[36] V. Vougalter, V. Volpert, *Existence of stationary solutions for some non-Fredholm integro-differential equations with superdiffusion,* J. Pseudo-Differ. Oper. Appl., **9** (2018), no. 1, 1–24

[37] V. Vougalter, V. Volpert, *Solvability of some integro- differential equations with anomalous diffusion and transport,* Anal. Math. Phys., **11** (2021), no. 3, Paper No. 135, 26 pp.