A brief account of the Cauchy problem in General Relativity

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Abstract

The Cauchy problem in PDE theory has a lot of importance in General Relativity. Some of the prominent aspects are the inclusion of a metric and a given hypersurface to determine the evolution of a given manifold. Some of the aspects we shall discuss in this review are the features of an initial value set in Cosmology, and the significance of some of the results of the Cauchy problem. Our discussion will focus towards stability, singularities, gravitational radiation, and the geometric features of singularities in General Relativity.

General Relativity gained a significant amount of interest from the mathematical community particularly from the time the question of the geometry of Space-time was asked. This question composed of a particularly interesting question, which has many beautiful features in the physical arena of General Relativity. This question originated from a primary backstory of the time physicists were discussing the implications of the Field equations,

$$G_{\mu\nu} \equiv R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = kT_{\mu\nu}, \qquad (k = 8\pi G)$$

The equation above is a system of PDEs describing the $3 + 1$ manifold, Space-time. The Field equations show the relation between the nature of the geometry of a manifold (that is the curvature and the nature of the manifold in Differential Geometry) and the physical aspects (that is the matter content of the manifold). The Field equations were solved by understanding the nature of a model constructed from a metric, and conditions on the manifold. An example is that of the Schwarzschild model, where the metric is static and describes a spherically symmetric model with the conditions on the matter content being that the solution describes a vacuum solution, that is to say that $T_{\mu\nu} = 0$.

Einstein wanted to describe the nature of a particular Cosmology using the Field equations, which is called the *Einstein Static Universe*, which has the feature of being static, i.e. the model has zero acceleration. But in order to describe this, he had to introduce an additional term, the "Cosmological constant" Λ in the EFEs (Einstein Field Equations), $G_{\mu\nu} + g_{\mu\nu}\Lambda = kT_{\mu\nu}$. This term showed one problem. The model was such that the Cosmological constant was fine-tuned to the nature of the model, that is being static. If the value of this was perturbed even a little, the model

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would have a non-zero acceleration, and would therefore *not* be static. From here, the problem of identifying a set of terms that described the nature of a manifold and it's evolution felt the need to be addressed [1]. Particularly, this was for three points:

- Nature of Stability: As seen above, the Einstein static Universe is an unstable model, since a minor perturbation to the Cosmological constant would render the nature of the model to be disturbed. In a similar way, there are many models that are otherwise stable – the order of symmetry of the model is not disturbed even if one of the terms that describe the model are perturbed [1].
- Singularities: In highly symmetric models, there is often the point of geodesic incompleteness that is observed. The question then is to find out how to determine the nature of singularities, and in particular the problem of understanding how they appear in apparently geodesically complete models. Also, the nature of how singularities behave, in that they are covered by an event horizon, and that they can never be timelike [14].
- Propagation of gravitational waves: After the idea of gravitational waves came up, the question of how fast they propagate came up. The linearised form of a perturbed Minkowskian metric allows us to consider the nature of the waves, and by writing out a wave equation for this perturbation, Einstein showed that the propagation of gravitational waves was at the speed of light. However, the aspect of coordinate choice stands out to be considerably significant.[1]

The Cauchy problem lies fundamental in understanding many geometric aspects of General Relativity. Although the origins of the Cauchy problem lie in pure PDE theory, there are many geometric aspects of this problem that are quite important in understanding various physical properties of the results of General Relativity. Consider a metric of the form $g = -dt^2 + R\widetilde{g}$, where R is a certain spatial factor. A simple example of a model such as this is that of the Friedmann-Lemaitre-Robertson-Walker (FLRW) model M^4 , which has a metric of the following form, (where $d\Sigma^2$ is the metric of the 3-Riemann manifold V^3 of the spatial components r, θ, ϕ)

$$ds^2 = -dt^2 + d\Sigma^2 = -dt^2 + a^2\left(t\right)\left[\frac{dr^2}{1 - Fr^2} + r^2 d\Omega^2\right]$$

Where F is the sectional curvature, that is for $V^3 = R^3$, we have $F = 0$. Likewise, for the manifold $V^3 = S^3$, we have $F = +1$, and for $V^3 = H^3$, we have $F = -1$. In general, we have the parameter F as a function of curvature, which describes the behaviour of nearby geodesics. For a flat Universe, we have the equations to be of the form $ds^2 = -dt^2 + dr^2 + r^2 d\Omega^2$. An example of an initial data problem here would be to identify a set H for the metric above such that the manifold M^4 satisfies the Cosmological principle and evolves under the description of the set H . Our discussion will be oriented towards such examples of situations where the Cauchy problem has physical significance in General Relativity.

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The Cauchy problem in General Relativity is to solve for a set that describes the evolution of a manifold on which the unknown terms are laid [11]. The inception of this problem in General Relativity was when the description of the Einstein static Universe) was written, but with a *Cosmological* constant that caused the model to be unstable in that the constant, when perturbed would no longer yield a static Universe with zero acceleration. This meant that the description of a model had a set of terms that described the evolution of the manifold. Our attention is towards the Cauchy problem in GR, and some results that have been given from this.

Consider a manifold (M, g) that has a set $H\left(h, \widetilde{g}_{\mu\nu}\right)$ such that h is a given Riemannian manifold, and $\widetilde{g}_{\mu\nu}$ a Riemann metric such that H is a collection of such $(h, \widetilde{g})$ that exist on M and describe the evolution of M . That is to say that there is a set H which shows the evolution of M . Our goal is to understand the properties of such H that describe the nature of a Cosmology [6]. An example is the usual topology of a manifold described by a metric of the form $ds^2 = -dt^2 + a^2\left(t\right)d\Sigma^2$, where $d\Sigma^2 = dr^2 + r^2 d\Omega^2$ as discussed above previously. We can denote this as the Riemann metric $\widetilde{g}$. We shall assume a constant sectional curvature to be that of a Euclidean form, that is to say that we assume the curvature represents that of R^3 , and is therefore spatially flat. We are also assuming that the manifold M satisfies the Cosmological principle, that is to say that all observers in M would observe a homogeneous and isotropic spatial distribution. Solving the Einstein Field Equations (EFEs)

$$G_{\mu\nu} = kT_{\mu\nu}$$

for a given form of matter content, we get the solutions to be of four forms: one for the 00 , $0i$, ii and $ij\left(i \neq j\right)$ components. The problem is to solve for the terms such as (a, ρ) such that we can understand the evolution of M [1]. There are two sets of equations that result from this – one are the constraint equations, and the other are the evolution equations. The field equations, can be re-written as $R_{\mu\nu} = k\left(T_{\mu\nu} - \frac{1}{2}Tg_{\mu\nu}\right)$. If the matter content is such that $T_{\mu\nu} = 0$, then we have the *vacuum constraint* equation, which reads $R_{\mu\nu} = 0$. This is a form of a constraint equation, where the given initial value set has to satisfy a given constraint in order to exist and describe M . We also have the evolution equations, which describe the behaviour of the initial value set as describing the evolution of the manifold. An initial value set then can be defined as a collection of terms $H\left(h, \widetilde{g}_{\mu\nu},\ A_{\mu\nu}\right)$ that satisfies the constraint and evolution equations. $A_{\mu\nu}$ is a symmetric tensor of rank $[0, 2]$, that is to say that it is a covariant tensor that satisfies

$$A_{\mu\nu} = \frac{1}{2}\left(A_{\mu\nu} + A_{\nu\mu}\right)$$

Yvonne Choquet Bruhat, in 1952 under John Leray showed that there is an initial data set $H\left(h, \widetilde{g}_{\mu\nu},\ A_{\mu\nu}\right)$ that satisfies the vacuum constraint $R_{\mu\nu} = 0$, and is such that the Riemann manifold h is a space-like hypersurface $h \subset M$ with an induced Riemann metric $\widetilde{g}$ [3].

There are three primary results from the Cauchy problem. In our short introduction above, we notice that for a symmetric M , it is reasonable to ask if there could be instabilities in the evolution of the manifold merely due to minor perturbations in the initial data set. Consider the example of the Einstein static model, where we have the description of a static (that is to say that it has zero acceleration) model of Space-time. Einstein had to introduce the Cosmological constant Λ in his field equations to describe the static Universe, but soon it was apparent that a minor change in the value of Λ would render the model to be non-static. This means that there is the problem of stability in the Cauchy problem. Another example of a Stability problem is that of showing the nature of the Minkowski space-time. There are many aspects of this, and one of them is the idea of using "isothermal coordinates" which were idealised by de Donder and continued in Yvonne Choquet Bruhat's works [3]. The condition of the isothermal coordinates is by laying out the condition $\Box_g g_{\mu\nu} = F(g, \partial g)$, where F is a function of only g and the first-order derivatives of g . There are many ways of proving the stability of the Minkowski space, such as the Christodoulou-Klainerman approach and the Rodnianski method (the latter has some problems with the null condition) [3][7].

Another feature of General Relativity is that of singularities arising in certain models. In the Schwarzschild solution, it is apparent that there are certain singularities in the model, at $R = 2GM$ and at $R = 0$. It is now apparent to ask whether there are certain features of a model that show the possibility of singularities in the model. We now know the sufficient conditions to understand how singularities form in a given manifold based on the matter content and the focusing effect of nearby geodesics. Energy conditions that described the nature of the matter content are used to determine the convergence of (time-like) geodesics. The most basic energy condition is the Strong energy condition, which says that from the field equations where we have $R_{\mu\nu} = k\left(T_{\mu\nu} - \frac{1}{2}Tg_{\mu\nu}\right)$, we can ensure that the curvature is such that nearby geodesics focus, and that is described as a positive value of the $\left(T_{\mu\nu} - \frac{1}{2}Tg_{\mu\nu}\right)$ term. From the relation as mentioned, we can state the Strong energy condition to be that $R_{\mu\nu}u^\mu u^\nu \geq 0$. Alongside this condition, there are other conditions that are linked directly to the Cauchy problem. A suitable set of initial conditions show the condition describing the "trueness" of a possible singularity. This is because there exist models that are globally hyperbolic (that is to say that there exists a Cauchy hypersurface $F \subset M$) and satisfy the energy conditions, but are not geodesically incomplete. In general, singularity theorems emphasise on understanding the nature of the manifold (under matter content and curvature) to understand the focusing of geodesics. A brief synopsis of the points are [10]:

- Energy conditions, as discussed above,
- Conditions on the convergence of geodesics,
- Existence of a Cauchy hypersurface (although it is not an important point in the Hawking-Penrose theorem),
- Trapped surface and change of the light cone sign (Penrose and Hawking theorems).

Singularity theorems are also significant in understanding the nature of the Universe. If there is sufficient amount of matter in a Universe, we can argue that the light cone of a point changes sign somewhere.

One of the most prominent results of the Cauchy problem is that of gravitational radiation, and in particular gravitational radiation emitted from a merging event of two dense bodies, to which the rest of this essay is devoted to.

Gravitational radiation has many origins, particularly from the following sources:

- Pulsations from a Supernova,
- Inspiral of two dense objects (BNS and BBH systems),
- Coalescence of two dense objects (BNS and BBH systems),
- Stochastic gravitational wave background.

In particular, we have the pulsations from a Supernova, which takes place during the collapse of the core of a star while forming a Supernova. The stochastic gravitational wave background is formed by the chaotic interference of events emitting gravitational waves throughout the history of the Universe. (The upper limit to the energy-density is cast upon the gravitational wave background by the critical density parameter.)

Gravitational waves can also be emitted as a result of the accelerated motion of two dense objects merging. An example is the inspiral of two black holes, which emits gravitational waves. In order to write down a theory of gravitational waves, we can linearize our theory by writing a perturbed Minkowski metric of the form

$$g_{\mu\nu} = \eta_{\mu\nu} + \epsilon h_{\mu\nu}, \quad 0 < \epsilon \ll 1$$

The metric is then of the form $ds^2 = g_{\mu\nu} dx^\mu dx^\nu = (\eta_{\mu\nu} + h_{\mu\nu}) dx^\mu dx^\nu$. We have chosen the coefficient ϵ to be small such that it linearizes out theory by reducing all higher order terms to be negligible. Here, we will imply the Lorenz gauge condition, which states that $\partial_\mu H^{\mu\nu} = 0$. In fact, this form is quite analogous to what we would do to write down the vacuum constraint to write down the wave-like equation, $\Box_g g_{\mu\nu} = F$, where F is a function as previously seen on the metric and *only* the first-order derivatives of g .

By computing the value of the Riemann tensor, we can evaluate the nature of the perturbation tensor, where we would have defined a TR and TT conditions, which respectively refer to Trace-Reversed and Transverse-Traceless conditions of the perturbation tensor. The TR form of $h_{\mu\nu}$ is of the form $H_{\mu\nu} = h_{\mu\nu} - \frac{1}{2} h \eta_{\mu\nu}$. The TT condition reduces to the following type:

$$\Box h_{\mu\nu} \equiv \left(-\partial_0^2 + \nabla^2\right) h_{\mu\nu} = -16\pi T_{\mu\nu}$$

Gravitational waves have a polarisation that is orthogonal, in that the effect on two perpendicular "arms" of an object would experience orthogonal effects. The two components of the polarisation are those that are left out after the respective conditions (TT and TR respectively, a result of the Gauge conditions we used on $h_{\mu\nu}$), and the fact that we have $H_{\mu\nu}^{TT} = h_{\mu\nu}$. The only two components left are those of the form h_{cross} and h_{plus} polarised. This is used in detecting gravitational waves in laser-based instruments, which rely on this orthogonal polarisation of a propagating gravitational wave. (Einstein, Infeld, Hoffman *et al* focused on the description of the theory of gravitational wave when the perturbed metric was allowed to range into higher order terms. This was developed as the *Post-Newtonian approximation*, which held for models where the motion of the composite masses of a merger was less than the speed of light.)

The Cauchy problem has the aspect of Global existence, which relates into the nature of the radiation. The Christodoulou-Klainerman observation of the nature of geodesic (in)completeness in 1993 showed the nature of the nature of the Minkowski Space-time. The problem had to do with identifying the geometric properties of the Minkowski Space-time under the global nonlinearity problem. This also had implications into the nature of radiation from a certain model that is developed under the Christodoulou-Klainerman theorem. Gravitational waves are structured to be planar, travelling along (future) null infinity with orthogonal effects on a ring of particles. (Null infinity is the collection of null geodesics at r approaches infinity, and are future-directed, i.e. have endpoints a in C^- and b in C^+ , where C^- and C^+ are the past and future light-cones respectively.)

The Initial data problem also has another significant result – that of the Cosmic Censorship hypothesis [14]. This has to do with the conjecture drawn by Roger Penrose which said that *all singularities are necessarily covered by an event horizon.* This however, is only the "weak" form of the problem. This was considered for (vacuum condition is implied) spherical symmetry by Christodoulou. The "strong" version of the Censorship hypothesis has to do with showing that timelike singularities are impossible. This version of the Censorship hypothesis has to do with showing that GR is a deterministic theory, and is deeply connected to the mathematical initial data. The Cauchy problem has a lot of implications, particularly from the aspect of the nature of Cauchy development. The conjecture states that a set $H\left(h,\widetilde{g},A\right)$ that is a vacuum initial data set, i.e. an initial data set satisfying the vacuum constraint, then if the Cauchy development is inextendible, then h is either a closed manifold or is asymptotically flat. This is shown as a physical property by noting that for such a set as above, there is a form $R^{abcd}R_{abcd}$ (often referred to as the *Kretschmann scalar*) that is such that the existence of timelike singularities is prevented. The nature of this conjecture is held in two ways. Firstly, the positivity of the previous geometric definition of the strong censorship conjecture is such that the examples considered so far are based on symmetric, cosmological states or isolated backgrounds (symmetric such as spherical symmetry and a cosmological setting as in satisfying the cosmological principle).

The matter content is also set in a certain way. We can consider our model to be of a vacuum, that is to say that $T_{\mu\nu}=0$. Noting that we wish the condition of asymptotic flatness to hold, the only vacuum model is the Schwarzschild solution, described by the metric

$$ds^2=-\left(1-\frac{2GM}{c^2R}\right)dt^2+\left(1-\frac{2GM}{c^2R}\right)^{-1}dr^2+r^2d\Omega^2$$

Apart from the vacuum setting, there is also the massless Klein-Gordon scalar field that can be chosen. In fact, this was the setting for Christodoulou's paper on the weak censorship conjecture. One of the many "relatives" of the weak censorship conjecture is the Penrose's inequality. This was drawn up as a part of realising the importance of the weak censorship conjecture. Roger Penrose wrote an inequality of the form $M_t \geq \sqrt{A/16\pi}$, where M_t is the total mass and A is the area of a black hole on the Space-time pair (M,g) . If the inequality is false, then this has a deep implication on the nature of the weak cosmic censorship conjecture. Also, the nature of the geometry of Space-times is also conjectured based on this inequality.

The Cauchy problem in General Relativity has many implications – some of them leading to questions that explain many intrinsic factors of the nature of Space-time. Our discussion holds here, but the geometric features of the initial data problem are many, from Stability of initial data sets to the nature of Space-times containing singularities, the problem has a vivid range of aspects, some of the questions arising from which still need to be addressed [13].

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