EIGENVALUE BOUNDS FOR SCHRÖDINGER OPERATORS WITH RANDOM COMPLEX POTENTIALS

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Abstract. We consider the Schrödinger operator perturbed by a random complex-valued potential. For this operator, we consider its eigenvalues situated in the unit disk. We obtain an estimate on the rate of accumulation of these eigenvalues to the positive half-line.

1. Introduction. Main results

In this paper, we study the behavior of eigenvalues of the operator $H = -\Delta + V$ acting on a Hilbert space $L^2(\mathbb{R}^d)$. The potential V is assumed to be a complex-valued function of the form

$$V(x) = \sum_{n\in\mathbb{Z}^d} \omega_n v_n \chi(x-n), \qquad v_n \in \mathbb{C}, \quad x \in \mathbb{R}^d,$$

where ω_n are independent random variables taking values in the interval $[-1,1]$ and χ is the characteristic function of the unit cube $[0,1)^d$.

The probability space in our theorems is the set Σ of all infinite sequences $\omega = \{\omega_n\}_{n\in\mathbb{Z}^d}$. The probability measure is defined on Σ as the infinite product of corresponding measures on intervals $[-1,1]$. Since ω_n can be viewed as a function on Σ whose value is equal to the n-th coordinate of ω, its expectation $\mathbb{E}[\omega_n]$ can be viewed as an integral over Σ. We impose the condition

$$\mathbb{E}[\omega_n] = 0$$

on ω_n guaranteeing oscillations of V. The coefficients v_n do not have to be real.

To formulate the main result, we set

$$\tilde{V}(x) = \sum_{n\in\mathbb{Z}^d} |v_n|\chi(x-n).$$

Note that $\tilde{V}$ is a non-negative function such that $|V| \leqslant \tilde{V}$.

Theorem 1.1. *Let $d \geqslant 3$, let $0 < R_0 \leqslant 1$, and let $1 < \nu < q < 2$. Then the eigenvalues λ_j of the operator $-\Delta + V$ satisfy*

$$\mathbb{E}\Big[\sum_{|\lambda_j|<R_0^2} \operatorname{Im}\sqrt{\lambda_j}\,|\lambda_j|^{(q-1)/2}\Big] \leqslant C|R_0|^{q-\nu}\Big(\int_{\mathbb{R}^d} |\tilde{V}(x)|^p dx\Big)^2, \tag{1.1}$$

with

$$p = \frac{d}{2} + \frac{d-q}{2(d-2)}. \tag{1.2}$$

It is assumed that $\operatorname{Im}\sqrt{\lambda_j} \geqslant 0$. The constant C in (1.1) depends only on d, ν and q.

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It is known that, if $v_n \in \mathbb{R}$, then the eigenvalues λ_j obey the Lieb-Thirring estimate (see [11], [16], and [17])

$$\sum_j |\lambda_j|^\gamma \leqslant C \int_{\mathbb{R}^d} |V(x)|^{d/2+\gamma}\, dx, \qquad V = \bar{V},\ d \geqslant 3, \gamma \geqslant 0. \tag{1.3}$$

Theorem 1.1 alows one to consider real potentials V for which the right hand side of (1.3) is infinite, while the left hand side is finite almost surely. Indeed, let $1 < 2\gamma = q < d/(d-1)$. Then the parameter p in (1.2) satisfies the inequality

$$p > d/2 + \gamma. \tag{1.4}$$

Similar results for real random potentils $V = \bar{V}$ were obtained earlier in [18]. However, there is a big difference between Theorem 1.1 and the results of [18], since the only point of accumulation of eigenvalues of the operator H in the case considered by the authors of [18] is the point $\lambda = 0$. When one studies complex-valued potentials, the fact that the eigenvalues λ_j might accumulate to points other than $\lambda = 0$ should not be excluded. Examples of decaying complex potentials V such that eigenvalues of $H = -\Delta + V$ accumulate to points of the positive real line $\mathbb{R}_+$ are constructed in [1]. Because of the difference between the cases of real and complex potentials, it would be more appropriate to ask what new information Theorem 1.1 provides compared to [6] and [8], rather than realize that this theorem does not follow from the Lieb-Thirring estimate even in the selfadjoint case.

The related result of [8] says that there is a constant C that depends on d, p and γ such that

$$\sum_j \operatorname{dist}(\lambda_j, \mathbb{R}_+)|\lambda_j|^{\gamma-1} \leqslant C\Big(\int_{\mathbb{R}^d} |V|^p dx\Big)^{2\gamma/(2p-d)},$$

under conditions on γ and p implying that $p < \gamma + d/2$. One can now refer to (1.4) to conclude that our results do give new information about the distribution of eigenvalues in the complex plane.

The same conclusion could be made by an analysis of the results of [6], where the eigenvalues in the disk $\mathbb{D}_V = \{z \in \mathbb{C} : \quad |z|^{p-d/2} \leqslant C_{p,d} \int |V|^p dx\}$ are considered separately from the rest of the eigenvalues (here $p > d/2$). The author of [6] proves that under some restrictions on p,

$$\Big(\sum_{\lambda_j \in \mathbb{D}_V} \operatorname{dist}(\lambda_j, \mathbb{R}_+)^\gamma\Big)^\sigma \leqslant C \int_{\mathbb{R}^d} |V|^p dx, \tag{1.5}$$

for γ equal either to p or $2p - d + \varepsilon$. The constants $C > 0$ and $\sigma > 0$, depending only on d and p in the first case, also depend on $\varepsilon > 0$ in the second. In its turn, $\varepsilon > 0$ belongs to the interval whose size depends on p. The observation we make is that $p < \gamma + d/2$ in (1.5). On the other hand, in deterministic results, p simply can not be larger than $\gamma + d/2$.

The next statement is an improvement of Theorem 1.1 for $3 \leqslant d \leqslant 5$.

Theorem 1.2. *Let $3 \leqslant d \leqslant 5$ and let $0 < R_0 \leqslant 1$. Assume that τ_1 satisfies*

$$0 \leqslant \Big(\Big(\frac{d}{2} + \frac{(\eta-1)(d+1)}{7d} + \frac{d-\eta}{2(d-2)}\Big) - 2\Big)\tau_1 \leqslant \frac{(\nu-1)(d+1)}{7d}$$

with η and ν such that $1 < \nu < \eta < 2$. If $d = 3$, then we assume additionally that $8\nu+9\eta < 26$. Let p, q and r be the numbers defined by

$$p = \frac{d}{7\tau_1}, \quad \frac{1}{q} = \frac{1-\theta}{p} + \frac{\theta}{2}, \qquad \text{and} \qquad \frac{1}{r} = \frac{1-\theta}{2p} + \frac{\theta}{2},$$

where θ is the solution of the equation

$$\tau_1(1-\theta) + \frac{\theta}{2}\Big(\frac{d}{2} + \frac{d-\eta}{2(d-2)}\Big) = 1.$$

Then the eigenvalues λ_j of the operator $-\Delta + V$ satisfy

$$\mathbb{E}\Big[\sum_{|\lambda_j|\leqslant R_0^2} \operatorname{Im}\sqrt{\lambda_j}|\lambda_j|^{(\sigma-1)/2}\Big] \leqslant C_{\tau_1,\sigma}|R_0|^{\sigma-\theta q\nu/2}\Big(\int_{\mathbb{R}^d} |\tilde{V}(x)|^r dx\Big)^{2q/r}, \qquad \sigma > \theta q\nu/2.$$

Besides its dependence on d, the constant $C_{\tau_1,\sigma}$ in this inequality depends on a choice of the parameters τ_1 and σ.

Theorem 1.2 gives new information about eigenvaues of H. Even in the case $V = \bar{V}$, this theorem does not follow from the Lieb-Thirring estimates. It turns into Theorem 1.1 for dimensions $3 \leqslant d \leqslant 5$ once we set $\tau_1 = 0$. On the other hand, since it allows to consider ratios σ/r smaller than ratios q/p allowed by Theorem 1.1, Theorem 1.2 is an improvement of Theorem 1.1 for dimensions $3 \leqslant d \leqslant 5$.

One of the difficulties we encountered in this paper is that our statements can not be derived by taking expectations in the inequalities obtained by Borichev, Golinskii, and Kupin [3]. The reason is that operators of the Birman-Schwinger type we are dealing with might have different properties for different ω. This difficulty was overcome through an application of the Joukowsky transform to a half-plane with a removed semi-disk and consecutive integration with respect to the radius.

2. Preliminaries

Everywhere below, $\mathfrak{S}_p$ denotes the class of compact operators K obeying

$$\|K\|_p^p = \operatorname{Tr}(K^*K)^{p/2} < \infty, \qquad p > 1.$$

Note that if $K \in \mathfrak{S}_p$ for some $p > 1$, then $K \in \mathfrak{S}_q$ for $q > p$ and $\|K\|_q \leqslant \|K\|_p$.

Let z_j be the eigenvalues of a compact operator $K \in \mathfrak{S}_n$ where $n \in \mathbb{N} \setminus \{0\}$. We define the n-th determinant of $I + K$ by

$$\det{}_n(I+K) = \prod_j (1+z_j)\exp\Big(\sum_{m=1}^{n-1} \frac{(-1)^m z_j^m}{m}\Big), \qquad n \geqslant 2;$$

$$\det(I+K) = \prod_j (1+z_j), \qquad n = 1.$$

There exists a constant $C_n > 0$ depending only on n such that

$$\big|\det{}_n(I+X)\big| \leqslant e^{C_n\|X\|_n^n}, \qquad \forall X \in \mathfrak{S}_n.$$

Moreover, the following statement holds (see Proposition 2.1 of [14]):

Proposition 2.1. *Let $n \geqslant 2$. Then for any $n-1 \leqslant p \leqslant n$, there exists a constant $C_{p,n} > 0$ depending only on p and n such that*

$$\big|\det_n(I+X)\big| \leqslant e^{C_{p,n}\|X\|^p_{\mathfrak{S}_p}}, \qquad \forall X \in \mathfrak{S}_p. \tag{2.1}$$

The way the eigenvalue bounds are obtained in [14] goes through applications of the following abstract result.

Theorem 2.2. *Let H_0 be a selfadjoint operator on a Hilbert space $\mathfrak{H}$. Let W_1 and W_2 be two bounded operators on $\mathfrak{H}$ and let $V = W_2W_1$. Assume that the function*

$$\mathbb{C}_+ \quad \ni \quad z \mapsto W_1(H_0-z)^{-1}W_2 \quad \in \quad \mathfrak{S}_p, \quad 1 \leqslant p < \infty,$$

is analytic in the upper half plane $\mathbb{C}_+ = \{z \in \mathbb{C}: \ \operatorname{Im} z > 0\}$ and continuous up to the real line $\mathbb{R}$. Assume also that

$$\|W_1(H_0-z)^{-1}W_2\|^p_{\mathfrak{S}_p} = o\Big(\frac{1}{|z|}\Big), \qquad \text{as} \quad |z| \to \infty. \tag{2.2}$$

Then the eigenvalues λ_j of H_0+V in $\mathbb{C}_+$ satisfy

$$\sum_j \operatorname{Im}\lambda_j \leqslant C_p \int_{-\infty}^{\infty} \|W_1(H_0-\lambda-i0)^{-1}W_2\|^p_{\mathfrak{S}_p} d\lambda. \tag{2.3}$$

where C_p depends only on the parameter p.

Proof. The proof of this statement relies on Jensen's inequality for zeros of an analytic function which is (also) justified in Proposition 3.11 of [14].

Proposition 2.3. *Let $a(z)$ be an analytic function on $\mathbb{C}_+$ satisfying the condition*

$$a(z) = 1 + o\Big(\frac{1}{|z|}\Big), \qquad \text{as} \quad |z| \to \infty.$$

Assume that for some $\gamma > 0$,

$$\ln|a(\lambda+i\gamma)| \leqslant f(\lambda), \qquad \forall \lambda \in \mathbb{R}.$$

Then zeros of $a(z)$ situated above the line $\operatorname{Im} z = \gamma$ *satisfy the inequality*

$$\sum_j (\operatorname{Im}\lambda_j - \gamma)_+ \leqslant \frac{1}{2\pi}\int_{-\infty}^{\infty} f(\lambda)\, d\lambda. \tag{2.4}$$

The statement also holds for $\gamma = 0$, if $a(z)$ is continuous up to the real line $\mathbb{R}$.

The bound (2.3) follows from (2.1) and the estimate (2.4) with $\gamma = 0$ once we set

$$a(z) = \det_n(I - W_1(H_0-z)^{-1}W_2), \quad \text{and} \quad f(\lambda) = C_{p,n}\|W_1(H_0-\lambda-i0)^{-1}W_2\|^p_{\mathfrak{S}_p}$$

This completes the proof of Theorem 2.2. □

One of the tools used in the present paper is an interpolation. Interpolation has been also used to prove Theorem 1.2 of [14], which could be generalized and formulated as follows:

Theorem 2.4. *Let (Ω, μ) be a space with a σ-finite measure μ such that $L^2(\Omega, \mu)$ is separable. Let H_0 be a selfadjoint operator on the Hilbert space $L^2(\Omega, \mu)$. Assume that the integral kernel of the operator e^{-itH_0} satisfies the estimate*

$$\left|e^{-itH_0}(x,y)\right| \leqslant \frac{C}{t^{\varkappa}}, \qquad \forall t > 0, \qquad \forall x, y \in \Omega,$$

for some $\varkappa > 0$. Let $V \in L^p(\Omega, \mu) \cap L^\infty(\Omega, \mu)$ for $p > \varkappa$ such that $p \geqslant 1$. Assume also that (2.2) holds for all W_1 and W_2 that belong to a class of functions dense in $L^{2p}(\Omega, \mu)$. Then eigenvalues of the operator $H = H_0 + V$ satisfy

$$\sum_j |\mathrm{Im}\lambda_j|^r \leqslant C_{p,r} \Big(\int_\Omega |V(x)|^p d\mu \Big)^{\frac{r}{p-\varkappa}}$$

for any $r > \max\{2(p - \varkappa), 1\}$.

The proof this result is literally a counterpart of the proof of Theorem 1.2 from [14] with the only difference that the value of the parameter $\varkappa$ in Theorem 1.2 of [14] is $3/2$ and $\Omega = \mathbb{R}^3$. However, one can consider different $\varkappa$ and spaces Ω which are different from $\mathbb{R}^d$. Especially interesting are spaces of fractional dimensions for which $2\varkappa$ is not integer.

Another object that we will work with is the operator

$$X(k) = |V|^{1/2}(-\Delta - z)^{-1}V(-\Delta - z)^{-1}V|V|^{-1/2}, \qquad z = k^2, \quad k \in \mathbb{C}_+.$$

If V is a bounded compactly supported function, then $X(k)$ is a trace class operator for $d \leqslant 3$ and $X(k) \in \mathfrak{S}_p$ for $p > d/4$ and $d \geqslant 4$. In this case, we set

$$D_n(k) = \det{}_n(I - X(k)), \qquad n > d/4, \; n \in \mathbb{N}.$$

Proposition 2.5. *Let V be a compactly supported function on $\mathbb{R}^d$. If a point $\lambda \in \mathbb{C} \setminus \mathbb{R}_+$ is an eigenvalue of $H = -\Delta + V$, then $D_n(k) = 0$ for $k = \sqrt{\lambda}$. The algebraic multiplicity of the eigenvalue λ does not exceed the multiplicity of the root of the function $D_n(\cdot)$.*

Proof. According to the Birman-Schwinger principle, a point λ is an eigenvalue of H if and only if -1 is an eigenvalue of $|V|^{1/2}(-\Delta - \lambda)^{-1}V|V|^{-1/2}$. Therefore, 1 is an eigenvalue of $X(k_0)$ with $k_0^2 = \lambda$. On the other hand, if 1 is an eigenvalue of $X(k_0)$, then $D_n(k_0) = 0$.

The statement about the multiplicity follows from the fact that an isolated eigenvalue of H whose multiplicity m is larger than 1 can be turned into m simple eigenvalues by an arbitrarily small perturbation of finite rank (which does not have to be a function). For any $\varepsilon > 0$ there is a finite rank operator K_ε such that $\|K_\varepsilon\| < \varepsilon$ and that all eigenvalues of $-\Delta + K_\varepsilon + V$ near λ are simple. Define now the function

$$d_\varepsilon(k) = \det{}_n(I - |V|^{1/2}(-\Delta + K_\varepsilon - z)^{-1}V(-\Delta + K_\varepsilon - z)^{-1}V|V|^{-1/2})$$

analytic in the neighborhood of $k_0 = \sqrt{\lambda}$ for sufficiently small $\varepsilon > 0$. In this neighbourhood of the point k_0, we have $d_\varepsilon(k) \to D_n(k)$ uniformly, as $\varepsilon \to 0$. Since the function $d_\varepsilon(k)$ has at least m zeros near k_0, the multiplicity of the zero of function $D_n(k)$ at $k = k_0$ can not be smaller than m by the argument principle. □

3. Large values of $\operatorname{Re}\zeta$ without projections

The following proposition gives an important estimate for the integral kernel of $(-\Delta - z)^{-\zeta}$.

Proposition 3.1. *Let $d \geqslant 2$ and let $(d-1)/2 \leqslant \operatorname{Re}\zeta \leqslant (d+1)/2$. The integral kernel of the operator $(-\Delta - z)^{-\zeta}$ satisfies the estimate*

$$\big|(-\Delta - z)^{-\zeta}(x,y)\big| \leqslant \beta e^{\alpha(\operatorname{Im}\zeta)^2}|k|^{(d-1)/2-\operatorname{Re}\zeta}|x-y|^{\operatorname{Re}\zeta-(d+1)/2} \tag{3.1}$$

for $z \notin \mathbb{R}_+$. The positive constants β and α in this inequality depend only on d and $\operatorname{Re}\zeta$.

The proof of this proposition, as well as related references, can be found in [8].

Corollary 3.2. *Let $(d-1)/2 \leqslant \operatorname{Re}\zeta < (d+1)/2$, where $d \geqslant 2$. Let $2 \leqslant r < \frac{2d}{2\operatorname{Re}\zeta-1}$. Suppose that W is a function of the form*

$$W(x) = \sum_{n\in\mathbb{Z}^d} w_n \chi(x-n), \qquad w_n \in \mathbb{C}, \quad x \in \mathbb{R}^d.$$

Then

$$\big\|W(-\Delta - z)^{-\zeta}\chi_l\big\|_{\mathfrak{S}_2} \leqslant \beta e^{\alpha(\operatorname{Im}\zeta)^2}|k|^{(d-1)/2-\operatorname{Re}\zeta}||W||_r, \tag{3.2}$$

for $z \notin \mathbb{R}_+$. The positive constants β and α in this inequality depend only on d and $\operatorname{Re}\zeta$. If $\operatorname{Re}\zeta = (d+1)/2$ and $d \geqslant 2$, then (3.2) holds with $r = 2$.

Proof. It follows from (3.1) that

$$\big\|W(-\Delta - z)^{-\zeta}\chi_l\big\|^2_{\mathfrak{S}_2} \leqslant C e^{2\alpha(\operatorname{Im}\zeta)^2}|k|^{(d-1)-2\operatorname{Re}\zeta}\sum_{n\in\mathbb{Z}^d}\big(|n-l|+1\big)^{2\operatorname{Re}\zeta-(d+1)}|w_n|^2.$$

A simple application of Hölder's inequality leads to (3.2). □

We need to turn (3.2) into a similar estimate for the $\mathfrak{S}_4$-norm of the operator corresponding to smaller values of $\operatorname{Re}\zeta$. For that purpose, we employ the following inequality:

$$\big\|W(-\Delta - z)^{-\zeta}\chi_l\big\| \leqslant \beta e^{\alpha(\operatorname{Im}\zeta)^2}||W||_\infty, \tag{3.3}$$

for $\operatorname{Re}\zeta = 0$.

By interpolation we obtain from (3.2) and (3.3) that

Proposition 3.3. *Let $(d-1)/2 \leqslant \varkappa < (d+1)/2$, where $d \geqslant 2$. Let $2 \leqslant r < \frac{2d}{2\varkappa-1}$. Suppose that W is a function of the form*

$$W(x) = \sum_{n\in\mathbb{Z}^d} w_n \chi(x-n), \qquad w_n \in \mathbb{C}, \quad x \in \mathbb{R}^d.$$

Then, for any $\operatorname{Re}\zeta = \tau \in (0,\varkappa]$ and $z \notin \mathbb{R}_+$,

$$\big\|W(-\Delta - z)^{-\zeta}\chi_l\big\|_{\mathfrak{S}_{2\varkappa/\tau}} \leqslant \beta e^{\alpha(\operatorname{Im}\zeta)^2}|k|^{((d-1)/(2\varkappa)-1)\tau}||W||_{r\varkappa/\tau}. \tag{3.4}$$

The positive constants β and α in this inequality depend only on d and τ. If $\varkappa = (d+1)/2$ and $d \geqslant 2$, then (3.5) holds with $r = 2$.

Proof. Indeed, let $\operatorname{Re}\zeta_0 = \tau$ and let

$$A = \Omega\,|A|$$

be the polar decomposition of the operator

$$A = |W|^{\zeta_0/\tau}(-\Delta - z)^{-\zeta_0}\chi_l.$$

Consider the function

$$f(\zeta) = e^{\alpha\zeta^2}\operatorname{Tr}\Big(|W|^{\zeta/\tau}(-\Delta - z)^{-\zeta}\chi_l|A|^{(2\varkappa-\zeta+i\operatorname{Im}\zeta_0)/\tau}\Omega^*\Big).$$

If $\operatorname{Re}\zeta = 0$, then

$$|f(\zeta)| \leqslant C_1\|A\|^{2\varkappa/\tau}_{\mathfrak{S}_{2\varkappa/\tau}}.$$

If $\operatorname{Re}\zeta = \varkappa$, then

$$|f(\zeta)| \leqslant C_2|k|^{(d-1)/2-\varkappa}\|A\|^{\varkappa/\tau}_{\mathfrak{S}_{2\varkappa/\tau}}\|W\|^{\varkappa/\tau}_{r\varkappa/\tau}.$$

Consequently, by the three lines lemma,

$$|f(\zeta_0)| \leqslant C|k|^{\theta((d-1)/2-\varkappa)}\|W\|^{\theta\varkappa/\tau}_{r\varkappa/\tau}\|A\|^{(2-\theta)\varkappa/\tau}_{\mathfrak{S}_{2\varkappa/\tau}}, \qquad \theta = \tau/\varkappa.$$

Put differently,

$$|e^{\alpha\zeta_0^2}|\|A\|^{2\varkappa/\tau}_{\mathfrak{S}_{2\varkappa/\tau}} \leqslant C|k|^{\theta((d-1)/2-\varkappa)}\|W\|^{\theta\varkappa/\tau}_{r\varkappa/\tau}\|A\|^{(2-\theta)\varkappa/\tau}_{\mathfrak{S}_{2\varkappa/\tau}}, \qquad \theta = \tau/\varkappa.$$

The latter inequality implies (3.5). The proof is completed. □

In particular, once we set $r\varkappa/\tau = 4$, we obtain

Corollary 3.4. *Let $(d-1)/2 \leqslant \varkappa < (d+1)/2$, where $d \geqslant 2$. Suppose that W is a function of the form*

$$W(x) = \sum_{n\in\mathbb{Z}^d} w_n\chi(x-n), \qquad w_n \in \mathbb{C}, \quad x \in \mathbb{R}^d.$$

Then

$$\big\|W(-\Delta - z)^{-\zeta}\chi_l\big\|_{\mathfrak{S}_4} \leqslant \beta e^{\alpha(\operatorname{Im}\zeta)^2}|k|^{((d-1)/(2\varkappa)-1)\operatorname{Re}\zeta}\|W\|_4, \tag{3.5}$$

for any $\varkappa/2 \leqslant \operatorname{Re}\zeta < \min\{\varkappa, d\varkappa/(4\varkappa-2)\}$ and $z \notin \mathbb{R}_+$. The positive constants β and α in this inequality depend only on d and $\operatorname{Re}\zeta$. If $\varkappa = (d+1)/2$ and $d \geqslant 2$, then (3.5) holds with $\operatorname{Re}\zeta = \varkappa/2$.

Let us now consider the operator

$$\mathfrak{X}(\zeta) = e^{\alpha_0\zeta^2}W(-\Delta - z)^{-\zeta}V(-\Delta - z)^{-\zeta}W,$$

where W is a **fixed** function independent of ω. The proof of the following proposition is based on the fact that $\mathbb{E}[\omega_n] = 0$.

Proposition 3.5. *Let $(d-1)/2 \leqslant \varkappa < (d+1)/2$, where $d \geqslant 2$. Let $\varkappa/2 \leqslant \operatorname{Re}\zeta < \min\{\varkappa, d\varkappa/(4\varkappa-2)\}$. Assume that $\tilde V \in L^2(\mathbb{R}^d)$, $W \in L^4(\mathbb{R}^d)$ and $\alpha_0 > 2\alpha$. Then*

$$\Big(\mathbb{E}(\|\mathfrak{X}(\zeta)\|^2_{\mathfrak{S}_2})\Big)^{1/2} \leqslant C_{\operatorname{Re}\zeta}e^{(2\alpha-\alpha_0)(\operatorname{Im}\zeta)^2}|k|^{((d-1)/\varkappa-2)\operatorname{Re}\zeta}\|\tilde V\|_2\|W\|_4^2. \tag{3.6}$$

If $\varkappa = (d+1)/2$ and $d \geqslant 2$, then (3.6) holds with $\operatorname{Re}\zeta = \varkappa/2$.

Proof. Obviously,

$$\mathbb{E}\Big(\|\mathfrak{X}(\zeta)\|^2_{\mathfrak{S}_2}\Big)=\mathbb{E}\Big(\operatorname{Tr}\mathfrak{X}(\zeta)^*\mathfrak{X}(\zeta)\Big)\leqslant$$
$$e^{2\alpha_0\operatorname{Re}\zeta^2}\sum_{l\in\mathbb{Z}^d}|v_l|^2\|W(-\Delta-z)^{-\zeta}\chi_l\|^2_{\mathfrak{S}_4}\,\|\chi_l(-\Delta-z)^{-\zeta}W\|^2_{\mathfrak{S}_4}.$$

Together with Corollary 3.4, this implies (3.6). □

Corollary 3.6. *Let $(d-1)/2\leqslant\varkappa<(d+1)/2$, where $d\geqslant 2$. Let $\varkappa/2\leqslant\operatorname{Re}\zeta<\min\{\varkappa, d\varkappa/(4\varkappa-2)\}$. Assume that $\tilde V\in L^2(\mathbb{R}^d)$, $W=\tilde V^{1/2}$, and $\alpha_0>2\alpha$. Then*

$$\Big(\mathbb{E}(\|\mathfrak{X}(\zeta)\|^2_{\mathfrak{S}_2})\Big)^{1/2}\leqslant C_{\operatorname{Re}\zeta}e^{(2\alpha-\alpha_0)(\operatorname{Im}\zeta)^2}|k|^{((d-1)/\varkappa-2)\operatorname{Re}\zeta}\|\tilde V\|^2_2. \tag{3.7}$$

If $\varkappa=(d+1)/2$ and $d\geqslant 2$, then (3.7) holds with $\operatorname{Re}\zeta=\varkappa/2$.

4. An estimate for the square of the Birman-Schwinger operator

According to our observations that we made, if $W=\sqrt{\tilde V}$, then $\mathfrak{X}(\zeta)$ is a function that obeys (3.7) for some rather large values of $\operatorname{Re}\zeta$ and it also obeys

$$\|\mathfrak{X}(\zeta)\|\leqslant C\|\tilde V\|^2_\infty$$

for $\operatorname{Re}\zeta=0$. To obtain our first result about eigenvalues, we can interpolate between these two cases. Let

$$\tilde X(k)=W(-\Delta-z)^{-1}V(-\Delta-z)^{-1}W,\qquad z=k^2,\quad k\in\mathbb{C}_+,$$

where, W is a fixed function independent of ω. Here is the result of the interpolation (which does not work for $d=2$):

Proposition 4.1. *Let $(d-1)/2\leqslant\varkappa<(d+1)/2$ where $d\geqslant 3$. Let*

$$\max\{2,\varkappa\}\leqslant p<\min\{2\varkappa, d\varkappa/(2\varkappa-1)\}. \tag{4.1}$$

Let $W=\tilde V^{1/2}$. Assume that $\tilde V\in L^p(\mathbb{R}^d)$. Then

$$\Big(\mathbb{E}(\|\tilde X(k)\|^p_{\mathfrak{S}_p})\Big)^{1/p}\leqslant C|k|^{(d-1)/\varkappa-2}\|\tilde V\|^2_p \tag{4.2}$$

If $\varkappa=(d+1)/2$ and $d\geqslant 3$, then (4.2) holds with $p=\varkappa$.

Proof. Note that $X(k)=\mathfrak{X}(1)$. The logic of interpolation says that (4.2) holds for p defined as

$$p=2/\theta,\quad \text{for}\quad \theta\quad \text{such that}\quad 1=\theta\tau,$$

where $\varkappa/2\leqslant\tau<\min\{\varkappa, d\varkappa/(4\varkappa-2)\}$. Of course, this interpolation works only if $\tau>1$, which is impossible for $d=2$. Observe that, in these notations, $p=2\tau$.

Let

$$X(k)=\Omega\,|X(k)|$$

be the polar decomposition of the operator $X(k)$. Consider the function

$$f(\zeta)=e^{\alpha_0\zeta^2}\mathbb{E}\Big(\operatorname{Tr}\Big(|W|^\zeta(-\Delta-z)^{-\zeta}V_\zeta(-\Delta-z)^{-\zeta}|W|^\zeta|X(k)|^{2\tau-\zeta}\Omega^*\Big)\Big),$$

where

$$V_\zeta(x):=\sum_n\omega_n|v_n|^\zeta e^{i\arg v_n}\chi(x-n).$$

If $\operatorname{Re}\zeta = 0$, then

$$|f(\zeta)| \leqslant C_1 \mathbb{E}\Big(\|X(k)\|_{\mathfrak{S}_{2\tau}}^{2\tau}\Big).$$

If $\operatorname{Re}\zeta = \tau$, then

$$|f(\zeta)| \leqslant C_2 |k|^{((d-1)/\varkappa-2)\tau}\Big(\mathbb{E}\Big(\|X(k)\|_{\mathfrak{S}_{2\tau}}^{2\tau}\Big)\Big)^{1/2}\|\tilde V\|_{2\tau}^{2\tau}.$$

Consequently, by the three lines lemma,

$$|f(1)| \leqslant C|k|^{(d-1)/\varkappa-2}\|\tilde V\|_{2\tau}^{2}\Big(\mathbb{E}\Big(\|X(k)\|_{\mathfrak{S}_{2\tau}}^{2\tau}\Big)\Big)^{1-1/(2\tau)}.$$

Put differently,

$$\mathbb{E}\Big(\|X(k)\|_{\mathfrak{S}_{2\tau}}^{2\tau}\Big) \leqslant C|k|^{(d-1)/\varkappa-2}\|\tilde V\|_{2\tau}^{2}\Big(\mathbb{E}\Big(\|X(k)\|_{\mathfrak{S}_{2\tau}}^{2\tau}\Big)\Big)^{1-1/(2\tau)}.$$

The latter inequality implies (4.2) because $2\tau = p$. The proof is completed. □

Now, we can formulate and prove the following result.

Theorem 4.2. *Let $d \geqslant 3$ and let $1 < \nu < q < 2$. Assume that $W = |V|^{1/2}$. Then*

$$\mathbb{E}(\|X(k)\|_{\mathfrak{S}_p}^p) \leqslant C|k|^{-\nu}\|\tilde V\|_p^{2p} \tag{4.3}$$

for p defined by

$$p = \frac{d(d-1)-q}{2(d-2)} = \frac{d}{2} + \frac{d-q}{2(d-2)}. \tag{4.4}$$

Proof. Observe that the assumption $\nu < q < 2$ leads to the inequalities

$$\frac{d+1}{2} < p < \frac{d(d-1)-\nu}{2(d-2)}. \tag{4.5}$$

We will show that conditions of Proposition 4.1 are fulfilled for the parameter $\varkappa$ defined by

$$\varkappa = \frac{(d-1)p}{2p-\nu},$$

The latter relation simply means that

$$\nu = \Big(2 - \frac{(d-1)}{\varkappa}\Big)p. \tag{4.6}$$

Consequently (4.3) follows from (4.2). The second inequality in (4.5) implies

$$\varkappa > \frac{d(d-1)-\nu}{2(d-\nu)} > \frac{d-1}{2}, \tag{4.7}$$

while the first inequality in (4.5) combined with the condition $\nu < 2$ implies that

$$\varkappa < \frac{d+1}{2}.$$

One can also see that the first inequality in (4.7) is equivalent to the estimate

$$p = \frac{\varkappa\nu}{2\varkappa-(d-1)} < \frac{d\varkappa}{2\varkappa-1},$$

Finally, note that in $d \geqslant 3$, the condition $p < 2\varkappa$ follows from the fact that $\nu + q > 2$. □

5. Proof of Theorem 1.1

We will work with the function

$$d(z) = \det_n(I - X(k)), \qquad n = [p] + 1,$$

where z is related to k via the Joukowski mapping

$$z = \frac{R}{k} + \frac{k}{R}, \qquad R > 0,$$

which maps the set $\{k \in \mathbb{C} : \ \operatorname{Im} k > 0, \quad |k| > R\}$ onto the upper half-plane $\{z \in \mathbb{C} : \ \operatorname{Im} z > 0\}$. Rather standard arguments lead to the estimate

$$\sum_j \operatorname{Im} z_j \leqslant C \int_{-\infty}^{\infty} \ln |d(z)| dz, \tag{5.1}$$

where z_j are zeros of the function $d(z)$ situated in the upper half-plane $\mathbb{C}_+$. In fact, (5.1) could be established in the same way as Jensen's inequality for zeros of an analytic function on a unit disk. In (5.1) we assume that V is compactly supported. The relation (5.1) leads to the estimate

$$\sum_j \Big(\frac{|k_j|^2 - R^2}{|k_j|^2 R}\Big)_+ \operatorname{Im} k_j \leqslant C\Big(\int_{-\infty}^{\infty} \|X(k)\|_p^p \Big(\frac{1}{R} - \frac{R}{k^2}\Big)_+ dk + \int_0^{\pi} \|X(R\cdot e^{i\theta})\|_p^p \sin\theta d\theta\Big),$$

Taking the expectation we obtain

$$\sum_j \mathbb{E}\Big[\frac{\operatorname{Im} k_j (|k_j|^2 - R^2)_+}{|k_j|^2 R}\Big] \leqslant C\Big(\int_{-\infty}^{\infty} \mathbb{E}[\|X(k)\|_p^p] \Big(\frac{1}{R} - \frac{R}{k^2}\Big)_+ dk + \int_0^{\pi} \mathbb{E}[\|X(R\cdot e^{i\theta})\|_p^p] \sin\theta d\theta\Big).$$

Due to Theorem 4.2, the latter inequality leads to

$$\sum_j \mathbb{E}\Big[\frac{\operatorname{Im} k_j (|k_j|^2 - R^2)_+}{|k_j|^2 R}\Big] \leqslant C|R|^{-\nu} \|\tilde V\|_p^{2p}. \tag{5.2}$$

Now, suppose that we consider only the eigenvalues $\lambda_j = k_j^2$ that satisfy the inequality

$$|k_j| \leqslant R_0.$$

Multiplying (5.2) by R^{q-1} and integrating with respect to R from 0 to R_0, we obtain

$$\sum_{|k_j|\leqslant R_0} \mathbb{E}[\operatorname{Im} k_j |k_j|^{q-1}] \leqslant C|R_0|^{q-\nu} \|\tilde V\|_p^{2p}, \qquad q > \nu. \tag{5.3}$$

This implies Theorem 1.1

6. Operators of the Birman-Schwinger type

Let a, b and V be functions on $\mathbb{R}^d$. Define

$$A_\zeta = |a|^\zeta F V_\zeta F^* |b|^\zeta,$$

where F is the unitary Fourier transform operator. For any complex number z, we understand V_z as the sum

$$V_z(x) := \sum_n \omega_n |v_n|^z e^{i \arg v_n} \chi(x - n).$$

Note that the operator A_ζ can be viewed as a sum over the lattice $\mathbb{Z}^d$

$$A_\zeta = \sum_{n\in\mathbb{Z}^d} A_{\zeta,n} \tag{6.1}$$

where

$$A_{\zeta,n} = \omega_n |a|^\zeta F |v_n|^\zeta e^{i\arg v_n} \chi(\cdot - n) F^* |b|^\zeta.$$

We will show that while A_ζ might be not bounded at some points ω, it is still a compact operator almost surely if a, b and $\tilde V$ are in L^2. We remind the reader that $\tilde V$ was defined as the function

$$\tilde V(x) = \sum_n |v_n| \chi(x-n).$$

Remark. Operators of the form $aFWF^*b$ do not have to be bounded for all a, b and W from L^2. Indeed, let

$$W(x) = (|x|+1)^{-s}, \qquad \text{with} \quad d/2 < s < 2d/3,$$

and let

$$a(\xi) = b(\xi) = \begin{cases} |\xi|^{-3s/4}, & \text{if} \quad |\xi| \leqslant 1, \\ 0, & \text{if} \quad |\xi| > 1. \end{cases}$$

If $aFWF^*b$ was bounded, the operator $T = aF\sqrt{W}$ would be bounded as well. The latter is not true, simply because $T\psi \notin L^2$ for $\psi = W$ (the singularity of $T\psi$ at zero is $|\xi|^{3s/4-d}$).

Proposition 6.1. *Let $a \in L^2$, $b \in L^2$ and $\tilde V \in L^2$. Let also $p \geqslant 2$. Then the sum* (6.1) *with* $\operatorname{Re}\zeta = 2/p$ *converges almost surely in $\mathfrak{S}_p$. Moreover,*

$$\Big(\mathbb{E}\big[\|A_\zeta\|_{\mathfrak{S}_p}^p\big]\Big)^{1/p} \leqslant (2\pi)^{-2d/p} \|a\|_2^{2/p} \|b\|_2^{2/p} \|\tilde V\|_2^{2/p}, \qquad \operatorname{Re}\zeta = 2/p. \tag{6.2}$$

Proof. We are going to prove (6.2) for one point ζ_0 such that $\operatorname{Re}\zeta_0 = 2/p$. For that purpose, we define the operator $K(\omega) = |A_{\zeta_0}|^{p/2}$. Then, obviously,

$$\beta := \mathbb{E}\Big(\|K\|_{\mathfrak{S}_2}^2\Big) = \mathbb{E}\big[\|A_{\zeta_0}\|_{\mathfrak{S}_p}^p\big].$$

Let $\Omega = \Omega(\omega)$ be the partially isometric operator appearing in the polar decomposition of A_{ζ_0}

$$A_{\zeta_0} = \Omega(\omega)|A_{\zeta_0}|.$$

We introduce the analytic function

$$f(\zeta) = \mathbb{E}[\operatorname{Tr} A_\zeta |K|^{2-\zeta} |K|^{i\operatorname{Im}\zeta_0} \Omega^*],$$

which will be treated by the three lines lemma. Since $||A_\zeta|| \leqslant 1$ for $\operatorname{Re}\zeta = 0$, and $\||K|^{i\operatorname{Im}\zeta_0}\Omega^*\| \leqslant 1$, we obtain that

$$|f(\zeta)| \leqslant \beta, \qquad \text{for} \quad \operatorname{Re}\zeta = 0. \tag{6.3}$$

On the other hand,

$$|f(\zeta)| \leqslant (2\pi)^{-d} \beta^{1/2} \|\tilde V\|_2 \, \|a\|_2 \, \|b\|_2, \qquad \text{for} \quad \operatorname{Re}\zeta = 1, \tag{6.4}$$

by an analogue of Hölder's inequality valid for Shatten classes. Indeed, for $\operatorname{Re}\zeta = 1$,

$$|f(\zeta)|^2 \leqslant \mathbb{E}[\|A_\zeta\|_{\mathfrak{S}_2}^2] \cdot \mathbb{E}[\|K\|_{\mathfrak{S}_2}^2],$$

and

$$\mathbb{E}[\|A_\zeta\|^2_{\mathfrak{S}_2}] = \mathbb{E}[\operatorname{Tr} A^*_\zeta A_\zeta] = \sum_{n\in\mathbb{Z}^d} \mathbb{E}[\operatorname{Tr} A^*_{\zeta,n} A_{\zeta,n}] \leqslant (2\pi)^{-2d}\|\tilde V\|_2^2\,\|a\|_2^2\,\|b\|_2^2.$$

Using the three lines lemma, we obtain from (6.3) and (6.4) that

$$|f(\zeta)| \leqslant (2\pi)^{-d\operatorname{Re}\zeta}\beta^{1-\operatorname{Re}\zeta/2}\|\tilde V\|_2^{\operatorname{Re}\zeta}\|a\|_2^{\operatorname{Re}\zeta}\|b\|_2^{\operatorname{Re}\zeta}$$

Note now that $f(\zeta_0) = \beta$. Consequently,

$$\beta^{1/p} \leqslant (2\pi)^{-2d/p}\|\tilde V\|_2^{2/p}\|a\|_2^{2/p}\|b\|_2^{2/p}.$$

□

Corollary 6.2. *Let T be a random operator of the form*

$$T = |a|FVF^*|b|,$$

with

$$V(x) := \sum_n \omega_n v_n \chi(x-n).$$

Let $a \in L^p$, $b \in L^p$, $v_n \in \ell^p$ and $p \geqslant 2$. Then

$$\Big(\mathbb{E}\big[\|T\|^p_{\mathfrak{S}_p}\big]\Big)^{1/p} \leqslant (2\pi)^{-2d/p}\|a\|_p\|b\|_p\|\tilde V\|_p.$$

Proof. Observe that the functions $|a|^{p/2}$, $|b|^{p/2}$ and $\tilde V^{p/2}$ belong to L^2. Therefore, according to the proposition, the $\mathfrak{S}_p$-norm of the operator

$$\tilde K = |a|^{p\zeta/2}FV_{p\zeta/2}F^*|b|^{p\zeta/2}$$

obeys the inequality

$$\Big(\mathbb{E}\big[\|\tilde K\|^p_{\mathfrak{S}_p}\big]\Big)^{1/p} \leqslant (2\pi)^{-2d/p}\||a|^{p/2}\|_2^{2/p}\||b|^{p/2}\|_2^{2/p}\|\tilde V^{p/2}\|_2^{2/p}, \qquad \operatorname{Re}\zeta = 2/p.$$

□

The following result is a very well known bound obtained by E. Seiler and B. Simon [19]. Moreover, the reader can easily prove it using standard interpolation.

Proposition 6.3. *Let a and W be two functions from $L^p(\mathbb{R}^d)$ with $p \geqslant 2$. Let T be the operator*

$$T = aFW,$$

where F is the operator of Fourier transform. Then

$$\|T\|_{\mathfrak{S}_p} \leqslant (2\pi)^{-d/p}\|a\|_p\|W\|_p, \qquad p \geqslant 2.$$

Corollary 6.4. *Let $q \geqslant p \geqslant 2$. Let T be a random operator of the form*

$$T = |a|FVF^*|b|,$$

with

$$V(x) := \sum_n \omega_n v_n \chi(x-n).$$

Let $a \in L^p$, $b \in L^q$ and $v_n \in \ell^p$. Then

$$\Big(\mathbb{E}\big[\|T\|^q_{\mathfrak{S}_p}\big]\Big)^{1/q} \leqslant (2\pi)^{-d/p-d/q}\|a\|_p\|b\|_q\|\tilde V\|_p.$$

Proof. According to Proposition 6.3,

$$\|T\|_{\mathfrak{S}_p} \leqslant (2\pi)^{-d/p}\|a\|_p\|b\|_\infty\|\tilde{V}\|_p, \qquad p \geqslant 2.$$

On the other hand, according to Corollary 6.2,

$$\Big(\mathbb{E}\big[\|T\|_{\mathfrak{S}_p}^p\big]\Big)^{1/p} \leqslant (2\pi)^{-2d/p}\|a\|_p\|b\|_p\|\tilde{V}\|_p.$$

It remains to interpolate between the two cases.

For that purpose, we introduce the function

$$f(\zeta) = \mathbb{E}\Big[\Big(\operatorname{Tr} K^p\Big)^{(1+q-p)(1-\zeta)/p+\zeta(p-1)(q-p)/p^2} \operatorname{Tr}|a|FVF^*|b|^{q\zeta/p}K^{p-1}\Omega^*\Big],$$

where $K = \big||a|FVF^*|b|\big|$ and Ω is the partially isometric operator appearing in the polar decomposition

$$|a|FVF^*|b| = \Omega K.$$

For our convenience, we denote

$$\beta := \mathbb{E}\big[(\operatorname{Tr} K^p)^{q/p}\big]$$

If $\operatorname{Re}\zeta = 0$, then by Hölder's inequality,

$$|f(\zeta)| \leqslant (2\pi)^{-d/p}\beta\|a\|_p\|\tilde{V}\|_p.$$

If $\operatorname{Re}\zeta = 1$, then

$$|f(\zeta)| \leqslant \mathbb{E}\Big[\Big(\operatorname{Tr} K^p\Big)^{(p-1)(q-p)/p^2} \||a|FVF^*|b|^{q/p}\|_{\mathfrak{S}_p}\big(\operatorname{Tr} K^p\big)^{(p-1)/p}\Big],$$

which leads to

$$|f(\zeta)| \leqslant \beta^{1-1/p}(2\pi)^{-2d/p}\|a\|_p\|b\|_q^{q/p}\|\tilde{V}\|_p.$$

Observe also that

$$f(p/q) = \beta.$$

Thus, by the three lines lemma, we obtain that

$$\beta \leqslant \beta^{1-1/q}(2\pi)^{-d/p-d/q}\|a\|_p\|b\|_q\|\tilde{V}\|_p.$$

The proof is completed. □

7. Large values of $\operatorname{Re}\zeta$

Let $0 < R \leqslant 1$. Let $\chi_{0,k}$ be the characteristic function of the ball

$$\mathfrak{B} = \Big\{\xi \in \mathbb{R}^d: \ |\xi| \leqslant \frac{2|k|}{R}\Big\},$$

and let $\chi_{1,k} = 1 - \chi_{0,k}$ be the characteristic function of its complement

$$\mathbb{R}^d \setminus \mathfrak{B} = \Big\{\xi \in \mathbb{R}^d: \ |\xi| > \frac{2|k|}{R}\Big\}.$$

We introduce the operators

$$P_{n,k} = F\chi_{n,k}F^*,$$

which are the spectral projections of $-\Delta$ corresponding to the intervals $[0, 4|k|^2/R^2]$ and $(4|k|^2/R^2, \infty)$.

Besides depending on the properties of $(-\Delta - z)^{-\zeta}$, the arguments of this paper also rely on the properties of the operators $P_{n,k}(-\Delta - z)^{-\zeta}$ for different values of ζ. In this section, we discuss relatively large values of $\operatorname{Re}\zeta$. The following proposition gives an important estimate for the integral kernel of $P_{n,k}(-\Delta - z)^{-\zeta}$.

Proposition 7.1. *Let $R \leqslant 1$. Let $d \geqslant 2$ and let $(d-1)/2 < \operatorname{Re}\zeta \leqslant (d+1)/2$. The integral kernel of the operator $P_{j,k}(\Delta - z)^{-\zeta}$ satisfies the estimate*

$$\left|P_{j,k}(-\Delta - z)^{-\zeta}(x,y)\right| \leqslant \beta e^{\alpha(\operatorname{Im}\zeta)^2}|k|^{(d-1)/2-\operatorname{Re}\zeta}|x-y|^{\operatorname{Re}\zeta-(d+1)/2} \tag{7.1}$$

for $z \notin \mathbb{R}_+$ and $j = 0, 1$. The positive constants β and α in this inequality depend only on d and $\operatorname{Re}\zeta$.

Proof. Due to Proposition 3.1, it is sufficient to prove only one of the inequalities (7.1). Let us first estimate the integrals

$$I_n = \int_{2^n|k|<R|\xi|<2^{n+1}|k|} \frac{e^{i\xi(x-y)}\,d\xi}{(|\xi|^2-k^2)^\zeta} = -|x-y|^{-2}\int_{2^n|k|<R|\xi|<2^{n+1}|k|} \frac{\Delta_\xi e^{i\xi(x-y)}\,d\xi}{(|\xi|^2-k^2)^\zeta} =$$

$$|x-y|^{-2}\int_{\mathbb{S}_{2^{n+1}|k|/R}\cup\mathbb{S}_{2^n|k|/R}} \frac{\pm i(x-y)\xi e^{i\xi(x-y)}\,dS_\xi}{|\xi|(|\xi|^2-k^2)^\zeta} - \zeta|x-y|^{-2}\int_{2^n|k|<R|\xi|<2^{n+1}|k|} \frac{2i\xi(x-y)e^{i\xi(x-y)}\,d\xi}{(|\xi|^2-k^2)^{\zeta+1}}, \tag{7.2}$$

for $n \geqslant 1$. We will show that

$$\left|I_n\right| \leqslant \beta e^{\alpha(\operatorname{Im}\zeta)^2}\left(2^n|k|/R\right)^{(d-1)/2-\operatorname{Re}\zeta}|x-y|^{\operatorname{Re}\zeta-(d+1)/2} \tag{7.3}$$

for some $\beta > 0$ and $\alpha > 0$. A priori,

$$|I_n| \leqslant C_d e^{2\pi|\operatorname{Im}\zeta|}(2^n|k|/R)^{d-2\operatorname{Re}\zeta}, \tag{7.4}$$

but the representation (7.2) leads to

$$|I_n| \leqslant C_d e^{2\pi|\operatorname{Im}\zeta|}(2^n|k|/R)^{d-2\operatorname{Re}\zeta-1}|x-y|^{-1}. \tag{7.5}$$

The first estimate (7.4) implies (7.3) for $2^n|k||x-y| < R$, because in this case,

$$|I_n| \leqslant C_d e^{2\pi|\operatorname{Im}\zeta|}(2^n|k|/R)^{d-2\operatorname{Re}\zeta}\left(2^n|k||x-y|/R\right)^{\operatorname{Re}\zeta-(d+1)/2}.$$

The second inequality (7.5) implies (7.3) for $2^n|k||x-y| \geqslant R$, because $(d+1)/2 - \operatorname{Re}\zeta \leqslant 1$ and, therefore,

$$(2^n|k|/R)^{d-2\operatorname{Re}\zeta-1}|x-y|^{-1} \leqslant (2^n|k|/R)^{d-2\operatorname{Re}\zeta+\operatorname{Re}\zeta-(d+1)/2}|x-y|^{\operatorname{Re}\zeta-(d+1)/2}.$$

The estimates (7.3) imply (7.1) for $j = 1$, because

$$P_{1,k}(-\Delta - z)^{-\zeta}(x,y) = (2\pi)^{-d}\sum_{n=1}^{\infty} I_n.$$

□

Corollary 7.2. *Let $(d-1)/2 < \operatorname{Re}\zeta < (d+1)/2$, where $d \geqslant 2$. Let $2 \leqslant r < \frac{2d}{2\operatorname{Re}\zeta - 1}$. Suppose that W is a function of the form*

$$W(x) = \sum_{n\in\mathbb{Z}^d} w_n\chi(x-n), \qquad w_n \in \mathbb{C}, \quad x \in \mathbb{R}^d.$$

Then

$$\left\|WP_{j,k}(-\Delta - z)^{-\zeta}\chi_l\right\|_{\mathfrak{S}_2} \leqslant \beta e^{\alpha(\operatorname{Im}\zeta)^2}|k|^{(d-1)/2-\operatorname{Re}\zeta}||W||_r, \tag{7.6}$$

for $z \notin \mathbb{R}_+$ and $j = 0, 1$. The positive constants β and α in this inequality depend only on d and $\operatorname{Re}\zeta$. If $\operatorname{Re}\zeta = (d+1)/2$ and $d \geqslant 2$, then (7.6) holds with $r = 2$.

Proof. It follows from (7.1) that

$$\big\|WP_{j,k}(-\Delta - z)^{-\zeta}\chi_l\big\|_{\mathfrak{S}_2}^2 \leqslant Ce^{2\alpha(\operatorname{Im}\zeta)^2}|k|^{(d-1)-2\operatorname{Re}\zeta}\sum_{n\in\mathbb{Z}^d}\big(|n-l|+1\big)^{2\operatorname{Re}\zeta-(d+1)}|w_n|^2.$$

A simple application of Hölder's inequality leads to (7.6). □

On the other hand, we have the following inequality:

$$\big\|WP_{j,k}(-\Delta - z)^{-\zeta}\chi_l\big\| \leqslant \beta e^{\alpha(\operatorname{Im}\zeta)^2}||W||_\infty, \tag{7.7}$$

for $\operatorname{Re}\zeta = 0$.

By interpolation we obtain from (7.6) and (7.7) that

Proposition 7.3. *Let $(d-1)/2 < \varkappa < (d+1)/2$, where $d \geqslant 2$. Let $2 \leqslant r < \frac{2d}{2\varkappa-1}$. Suppose that W is a function of the form*

$$W(x) = \sum_{n\in\mathbb{Z}^d} w_n\chi(x-n), \qquad w_n \in \mathbb{C}, \quad x \in \mathbb{R}^d.$$

Then, for any $\operatorname{Re}\zeta = \tau \in (0, \varkappa)$, $z \notin \mathbb{R}_+$ and $j = 0, 1$,

$$\big\|WP_{j,k}(-\Delta - z)^{-\zeta}\chi_l\big\|_{\mathfrak{S}_{2\varkappa/\tau}} \leqslant \beta e^{\alpha(\operatorname{Im}\zeta)^2}|k|^{((d-1)/(2\varkappa)-1)\tau}||W||_{r\varkappa/\tau}. \tag{7.8}$$

The positive constants β and α in this inequality depend only on d and τ. If $\varkappa = (d+1)/2$ and $d \geqslant 2$, then (7.9) holds with $r = 2$.

Proof. Indeed, let $\operatorname{Re}\zeta_0 = \tau$ and let

$$A = \Omega\,|A|$$

be the polar decomposition of the operator

$$A = |W|^{\zeta_0/\tau}P_{j,k}(-\Delta - z)^{-\zeta_0}\chi_l.$$

Consider the function

$$f(\zeta) = e^{\alpha\zeta^2}\operatorname{Tr}\Big(|W|^{\zeta/\tau}P_{j,k}(-\Delta - z)^{-\zeta}\chi_l|A|^{(2\varkappa-\zeta+i\operatorname{Im}\zeta_0)/\tau}\Omega^*\Big).$$

If $\operatorname{Re}\zeta = 0$, then

$$|f(\zeta)| \leqslant C_1\|A\|_{\mathfrak{S}_{2\varkappa/\tau}}^{2\varkappa/\tau}.$$

If $\operatorname{Re}\zeta = \varkappa$, then

$$|f(\zeta)| \leqslant C_2|k|^{(d-1)/2-\varkappa}\|A\|_{\mathfrak{S}_{2\varkappa/\tau}}^{\varkappa/\tau}\|W\|_{r\varkappa/\tau}^{\varkappa/\tau}.$$

Consequently, by the three lines lemma,

$$|f(\zeta_0)| \leqslant C|k|^{\theta((d-1)/2-\varkappa)}\|W\|_{r\varkappa/\tau}^{\theta\varkappa/\tau}\|A\|_{\mathfrak{S}_{2\varkappa/\tau}}^{(2-\theta)\varkappa/\tau}, \qquad \theta = \tau/\varkappa.$$

Put differently,

$$|e^{\alpha\zeta_0^2}|\|A\|_{\mathfrak{S}_{2\varkappa/\tau}}^{2\varkappa/\tau} \leqslant C|k|^{\theta((d-1)/2-\varkappa)}\|W\|_{r\varkappa/\tau}^{\theta\varkappa/\tau}\|A\|_{\mathfrak{S}_{2\varkappa/\tau}}^{(2-\theta)\varkappa/\tau}, \qquad \theta = \tau/\varkappa.$$

The latter inequality implies (7.9). The proof is completed. □

In particular, once we set $r\varkappa/\tau = 4$, we obtain

Corollary 7.4. *Let $(d-1)/2 < \varkappa < (d+1)/2$, where $d \geqslant 2$. Suppose that W is a function of the form*

$$W(x) = \sum_{n\in\mathbb{Z}^d} w_n \chi(x-n), \qquad w_n \in \mathbb{C}, \quad x \in \mathbb{R}^d.$$

Then

$$\big\| W P_{j,k}(-\Delta - z)^{-\zeta}\chi_l \big\|_{\mathfrak{S}_4} \leqslant \beta e^{\alpha(\operatorname{Im}\zeta)^2} |k|^{((d-1)/(2\varkappa)-1)\operatorname{Re}\zeta} ||W||_4, \tag{7.9}$$

for any $\varkappa/2 \leqslant \operatorname{Re}\zeta < \min\{\varkappa, d\varkappa/(4\varkappa-2)\}$, $z \notin \mathbb{R}_+$ and $j = 0, 1$. The positive constants β and α in this inequality depend only on d and $\operatorname{Re}\zeta$. If $\varkappa = (d+1)/2$ and $d \geqslant 2$, then (7.9) holds with $\operatorname{Re}\zeta = \varkappa/2$.

We will now discuss the properties of the random operators

$$X_{n,m}(\zeta) = e^{\alpha_0\zeta^2}\Big(W P_{n,k}(-\Delta - z)^{-\zeta} V (-\Delta - z)^{-\zeta} P_{m,k} W \Big).$$

Here W is a **fixed** function which does not depend on ω.

Proposition 7.5. *Let $(d-1)/2 < \varkappa < (d+1)/2$, where $d \geqslant 2$. Let $\varkappa/2 \leqslant \operatorname{Re}\zeta < \min\{\varkappa, d\varkappa/(4\varkappa-2)\}$. Assume that $\tilde V \in L^2(\mathbb{R}^d)$, $W \in L^4(\mathbb{R}^d)$ and $\alpha_0 > 2\alpha$. Then*

$$\Big(\mathbb{E}(\|X_{n,m}(\zeta)\|^2_{\mathfrak{S}_2}) \Big)^{1/2} \leqslant C_{\operatorname{Re}\zeta} e^{(2\alpha-\alpha_0)(\operatorname{Im}\zeta)^2} |k|^{((d-1)/\varkappa-2)\operatorname{Re}\zeta} \|\tilde V\|_2 \|W\|_4^2. \tag{7.10}$$

If $\varkappa = (d+1)/2$ and $d \geqslant 2$, then (7.10) holds with $\operatorname{Re}\zeta = \varkappa/2$.

Proof. Obviously,

$$\mathbb{E}\Big(\|X_{n,m}(\zeta)\|^2_{\mathfrak{S}_2} \Big) = \mathbb{E}\Big(\operatorname{Tr} X_{n,m}(\zeta)^* X_{n,m}(\zeta) \Big) \leqslant$$

$$e^{2\alpha_0 \operatorname{Re}\zeta^2} \sum_{l\in\mathbb{Z}^d} |v_l|^2 \|W P_{n,k}(-\Delta - z)^{-\zeta}\chi_l\|^2_{\mathfrak{S}_4} \, \|\chi_l(-\Delta - z)^{-\zeta} P_{m,k} W\|^2_{\mathfrak{S}_4}.$$

Together with Corollary 7.4, this implies (7.10). □

We will also study the spectral properties of the operator

$$Y(\zeta) = X_{0,0}(\zeta) + X_{0,1}(\zeta) + X_{1,0}(\zeta).$$

Corollary 7.6. *Let $(d-1)/2 < \varkappa < (d+1)/2$, where $d \geqslant 2$. Let $\varkappa/2 \leqslant \operatorname{Re}\zeta < \min\{\varkappa, d\varkappa/(4\varkappa-2)\}$. Assume that $\tilde V \in L^2(\mathbb{R}^d)$, $W = \tilde V^{1/2}$, and $\alpha_0 > 2\alpha$. Then*

$$\Big(\mathbb{E}(\|Y(\zeta)\|^2_{\mathfrak{S}_2}) \Big)^{1/2} \leqslant C_{\operatorname{Re}\zeta} e^{(2\alpha-\alpha_0)(\operatorname{Im}\zeta)^2} |k|^{((d-1)/\varkappa-2)\operatorname{Re}\zeta} \|\tilde V\|_2^2. \tag{7.11}$$

If $\varkappa = (d+1)/2$ and $d \geqslant 2$, then (7.11) holds with $\operatorname{Re}\zeta = \varkappa/2$.

8. Small values of $\operatorname{Re}\zeta$

The notations we use in this section are the same as in the previous one. In particular, the projections $P_{n,k}$ are the same as before. As it was mentioned, the arguments of this paper rely on the properties of the operators $P_{n,k}(-\Delta - z)^{-\zeta}$ for different values of ζ. In this section, we discuss the case $0 \leqslant \operatorname{Re}\zeta < 1$.

In the next two propositions, we discuss the properties of the random operators

$$X_{n,m}(\zeta) = e^{\alpha_0\zeta^2}\Big(WP_{n,k}(-\Delta - z)^{-\zeta}V(-\Delta - z)^{-\zeta}P_{m,k}W\Big)$$

for $\operatorname{Re}\zeta = \gamma/2$ and $0 < \gamma < 3/2$. Here W is a **fixed** function which does not depend on ω. The value of the parameter α_0 should be sufficiently large as in Corollary 7.6.

Later, we will also study the spectral properties of the operator

$$Y(\zeta) = X_{0,0}(\zeta) + X_{0,1}(\zeta) + X_{1,0}(\zeta).$$

However, the terms in this representation will be studied separately. A this point, we do not discuss $X_{1,1}(\zeta)$ at all.

Proposition 8.1. *Let $d \geqslant 2$. Let $z \in \mathbb{C} \setminus \mathbb{R}_+$ and let $2 \leqslant 2p < 3/\gamma$. Assume that $0 < R \leqslant 1$. If $\operatorname{Re}\zeta = \gamma/2$, $W \in L^{4p}$, and $\tilde V \in L^{2p}$, then $X_{0,0}(\zeta) \in \mathfrak{S}_p$ almost surely. Moreover,*

$$\mathbb{E}\big(||X_{0,0}(\zeta)||^p_{\mathfrak{S}_p}\big)^{1/p} \leqslant C_{p,\gamma}e^{-\alpha_0|\operatorname{Im}\zeta|^2/2}\Big(\frac{|k|}{R}\Big)^{3d/2p-2\gamma}\|\tilde V\|_{2p}\|W\|^2_{4p}. \tag{8.1}$$

Proof. This statement follows from Corollary 6.2 and Proposition 6.3. If $r = q/2 = 2p$, then $1/r + 2/q = 1/p$. Moreover, since

$$X_{0,0}(\zeta) = e^{\alpha_0\zeta^2}\Big(W(-\Delta - z)^{-\zeta/3}P_{0,k}(-\Delta - z)^{-2\zeta/3}V(-\Delta - z)^{-2\zeta/3}P_{0,k}(-\Delta - z)^{-\zeta/3}W\Big),$$

we obtain the estimate

$$\|\tilde X_{0,0}(\zeta)\|_p \leqslant |e^{\alpha_0\zeta^2}| \cdot \|W(-\Delta - z)^{-\zeta/3}P_{0,k}\|_q$$
$$\|\tilde P_{0,k}(-\Delta - z)^{-2\zeta/3}V(-\Delta - z)^{-2\zeta/3}P_{0,k}\|_r\|P_{0,k}(-\Delta - z)^{-\zeta/3}W\|_q.$$

It remains to realize that

$$\Big(\int_{\mathbb{R}^d}\frac{\chi_{0,k}d\xi}{|(|\xi|^2 - z)^{2\zeta/3}|^r}\Big)^{2/r} \leqslant \Big(\int_{|\xi|<2|k|}\frac{d\xi}{|(|\xi|^2 - z)^{2\zeta/3}|^r}\Big)^{2/r} + c_{p,\gamma}e^{c|\operatorname{Im}\zeta|}\Big(\int_{|\xi|<2|k|/R}\frac{d\xi}{|\xi|^{2\gamma r/3}}\Big)^{2/r} \leqslant$$
$$C_{p,\gamma}e^{c|\operatorname{Im}\zeta|}\Big(\frac{|k|}{R}\Big)^{2(d-2r\gamma/3)/r} = C_{p,\gamma}e^{c|\operatorname{Im}\zeta|}\Big(\frac{|k|}{R}\Big)^{d/p-4\gamma/3}, \quad \gamma r < 3,$$

while a similar argument shows that

$$\Big(\int_{\mathbb{R}^d}\frac{\chi_{0,k}d\xi}{|(|\xi|^2 - z)^{\zeta/3}|^q}\Big)^{2/q} \leqslant \tilde C_{p,\gamma}e^{c|\operatorname{Im}\zeta|}\Big(\frac{|k|}{R}\Big)^{2(d-q\gamma/3)/q} = \tilde C_{p,\gamma}e^{c|\operatorname{Im}\zeta|}\Big(\frac{|k|}{R}\Big)^{d/2p-2\gamma/3}.$$

□

Proposition 8.2. *Let $2 \leqslant d \leqslant 5$. Let $z \in \mathbb{C} \setminus \mathbb{R}_+$ and let $2 \leqslant 2p < 3/\gamma$. Assume that $4p\gamma > d$ and $0 < R \leqslant 1$. If $\operatorname{Re}\zeta = \gamma/2$, $W \in L^{4p}$, and $\tilde V \in L^{2p}$, then $X_{0,1}(\zeta) \in \mathfrak{S}_p$ for all ω. Moreover,*

$$||X_{0,1}(\zeta)||_{\mathfrak{S}_p} \leqslant C_{p,\gamma}e^{-\alpha_0|\operatorname{Im}\zeta|^2/2}\Big(\frac{|k|}{R}\Big)^{d/p-2\gamma}\|\tilde V\|_{2p}\|W\|^2_{4p}. \tag{8.2}$$

Proof. Since

$$X_{0,1}(\zeta) = e^{\alpha_0\zeta^2}\Big(W(-\Delta - z)^{-\zeta/3}P_{0,k}(-\Delta - z)^{-2\zeta/3}VP_{1,k}(-\Delta - z)^{-\zeta}W\Big),$$

we obtain the estimate

$$\|X_{0,1}(\zeta)\|_p \leqslant |e^{\alpha_0\zeta^2}| \cdot \|W(-\Delta - z)^{-\zeta/3}P_{0,k}\|_{4p}$$
$$\|P_{0,k}(-\Delta - z)^{-2\zeta/3}V\|_{2p}\|P_{1,k}(-\Delta - z)^{-\zeta}W\|_{4p}.$$

It remains to realize that

$$\Big(\int_{\mathbb{R}^d} \frac{\chi_{0,k}d\xi}{|(|\xi|^2 - z)^{2\zeta/3}|^{2p}}\Big)^{1/(2p)} \leqslant \tilde{C}_{p,\gamma}e^{c|\operatorname{Im}\zeta|}\Big(\frac{|k|}{R}\Big)^{d/(2p)-2\gamma/3},$$

while

$$\Big(\int_{\mathbb{R}^d} \frac{\chi_{0,k}d\xi}{|(|\xi|^2 - z)^{\zeta/3}|^{4p}}\Big)^{1/(4p)} \leqslant \tilde{C}_{p,\gamma}e^{c|\operatorname{Im}\zeta|}\Big(\frac{|k|}{R}\Big)^{d/(4p)-\gamma/3}.$$

Finally,

$$\Big(\int_{\mathbb{R}^d} \frac{\chi_{1,k}d\xi}{|(|\xi|^2 - z)^{\zeta}|^{4p}}\Big)^{1/(4p)} \leqslant 2e^{c|\operatorname{Im}\zeta|}\Big(\int_{|\xi|>2|k|/R} \frac{d\xi}{(\frac{3}{4}|\xi|^2)^{2\gamma p}}\Big)^{1/(4p)} \leqslant \tilde{C}_{p,\gamma}e^{c|\operatorname{Im}\zeta|}\Big(\frac{|k|}{R}\Big)^{d/(4p)-\gamma}.$$

□

Let us now talk about the operator $Y(\zeta)$. The study of this operator has to be harder compared to the study of $X_{1,1}(\zeta)$ simply because $P_{1,k}(-\Delta - z)^{-\zeta}$ is bounded uniformly in z, while it is not true about $P_{0,k}(-\Delta - z)^{-\zeta}$.

Corollary 8.3. *Let $2 \leqslant d \leqslant 5$. Let $|k| \geqslant R$ where $0 < R \leqslant 1$. Let also $W = \sqrt{\tilde{V}}$. Assume that $2 \leqslant 2p < 3/\gamma$ and $4p\gamma > d$. If $\operatorname{Re}\zeta = \gamma/2$ and $\tilde{V} \in L^{2p}$, then*

$$\mathbb{E}\big(||Y(\zeta)||^p_{\mathfrak{S}_p}\big)^{1/p} \leqslant C_{p,\gamma}e^{-\alpha_0|\operatorname{Im}\zeta|^2/2}\Big(\frac{|k|}{R}\Big)^{3d/2p-2\gamma}\|\tilde{V}\|^2_{2p}.$$

In particular, we can set $p = 1$ and prove the following statement.

Proposition 8.4. *Let $2 \leqslant d \leqslant 5$. Let $|k| \geqslant R$ where $0 < R \leqslant 1$. Let also $W = \sqrt{\tilde{V}}$. Assume that*

$$\frac{d}{8} < \frac{\gamma}{2} = \operatorname{Re}\zeta < \frac{3}{4}.$$

Then

$$\mathbb{E}\big(||Y(\zeta)||_{\mathfrak{S}_1}\big) \leqslant C_{\operatorname{Re}\zeta}\, e^{-\alpha_0|\operatorname{Im}\zeta|^2/2}\Big(\frac{|k|}{R}\Big)^{3d/2-4\operatorname{Re}\zeta}\|\tilde{V}\|^2_2.$$

9. Another interpolation between small and large values of $\operatorname{Re}\zeta$

Let us recall two theorems that hold for the operator

$$Y(\zeta) = X_{0,0}(\zeta) + X_{0,1}(\zeta) + X_{1,0}(\zeta),$$

with $W = \tilde{V}^{1/2}$. By small values of $\operatorname{Re}\zeta$ we mean the values that are considered in Corollary 8.3, which states that, for any $p \geqslant 1$ and $d/(8p) < \operatorname{Re}\zeta < 3/(4p)$,

$$\mathbb{E}\big(||Y(\zeta)||^p_{\mathfrak{S}_p}\big)^{1/p} \leqslant C_{\operatorname{Re}\zeta,\,p}e^{-\alpha_0|\operatorname{Im}\zeta|^2/2}\Big(\frac{|k|}{R}\Big)^{3d/2p-4\operatorname{Re}\zeta}\|\tilde{V}\|^2_{2p}. \tag{9.1}$$

In this corollary, we had to assume that $2 \leqslant d \leqslant 5$ and $|k| \geqslant R$ where $0 < R \leqslant 1$. One should not forget also that our assumptions about $\gamma = 2\operatorname{Re}\zeta$ imply that $\operatorname{Re}\zeta < 3/4$.

In the next result, we only replace $4\operatorname{Re}\zeta$ by $d/(2p)$ in the right hand side of (9.1).

Theorem 9.1. *Let* $2 \leqslant d \leqslant 5$. *Let* $W = \tilde{V}^{1/2}$. *Let*

$$0 < \operatorname{Re}\zeta < 3/4.$$

Assume that

$$\frac{d}{8\operatorname{Re}\zeta} < p < \frac{3}{4\operatorname{Re}\zeta}, \qquad p \geqslant 1,$$

and $0 < R \leqslant 1$. *Then*

$$\mathbb{E}\big(||Y(\zeta)||^p_{\mathfrak{S}_p}\big)^{1/p} \leqslant C_{\operatorname{Re}\zeta,p} e^{-\alpha_0|\operatorname{Im}\zeta|^2/2}\Big(\frac{|k|}{R}\Big)^{d/p}\|\tilde{V}\|^2_{2p}$$

for $|k| \geqslant R$.

For the sake of simplicity, we choose

$$p = \frac{d}{7\operatorname{Re}\zeta}$$

In this case, because of the assumption $p \geqslant 1$ that we made, we have to assume that

$$0 < \operatorname{Re}\zeta \leqslant \frac{d}{7}.$$

Note that $d/7 < 3/4$. Thus, we can formulate the following assertion:

Corollary 9.2. *Let* $2 \leqslant d \leqslant 5$. *Let* $0 < \operatorname{Re}\zeta \leqslant d/7$ *and let* $p = \frac{d}{7\operatorname{Re}\zeta}$. *Assume that* $0 < R \leqslant 1$. *Then*

$$\mathbb{E}\big(||Y(\zeta)||^p_{\mathfrak{S}_p}\big)^{1/p} \leqslant C_{\operatorname{Re}\zeta,p} e^{-\alpha_0|\operatorname{Im}\zeta|^2/2}\Big(\frac{|k|}{R}\Big)^{d/p}\|\tilde{V}\|^2_{2p}$$

for $|k| \geqslant R$.

By the large values of $\operatorname{Re}\zeta$ we mean the values appearing in Corollary 7.6. We will use only a simpler version of this result.

Theorem 9.3. *Let* $d \geqslant 3$. *Let* $1 < \nu < \eta < 2$. *Let*

$$2\operatorname{Re}\zeta = \frac{d}{2} + \frac{d-\eta}{2(d-2)} \tag{9.2}$$

Assume that $V \in L^2(\mathbb{R}^d)$ *and* $\alpha_0 > 2\alpha$. *Then*

$$\Big(\mathbb{E}(\|Y(\zeta)\|^2_{\mathfrak{S}_2})\Big)^{1/2} \leqslant C_{\operatorname{Re}\zeta} e^{(2\alpha-\alpha_0)(\operatorname{Im}\zeta)^2}|k|^{-\nu/2}\|\tilde{V}\|^2_2. \tag{9.3}$$

Proof. For $\operatorname{Re}\zeta$ defined in (9.2), the assumption $\nu < \eta < 2$ leads to the inequalities

$$\frac{d+1}{2} < 2\operatorname{Re}\zeta < \frac{d(d-1)-\nu}{2(d-2)}. \tag{9.4}$$

Let us now introduce the parameter $\varkappa$ setting

$$\varkappa = \frac{2(d-1)\operatorname{Re}\zeta}{4\operatorname{Re}\zeta - \nu}.$$

The latter relation simply means that

$$\nu = \Big(2 - \frac{(d-1)}{\varkappa}\Big) 2\operatorname{Re}\zeta. \tag{9.5}$$

Thus, (9.3) coincides with (7.11). Let us check that all conditions of Corollary 7.6 are fulfilled. The second inequality in (9.4) implies

$$\varkappa > \frac{d(d-1)-\nu}{2(d-\nu)} > \frac{d-1}{2}, \tag{9.6}$$

while the first inequality in (9.4) combined with the condition $\nu < 2$ implies that

$$\varkappa < \frac{d+1}{2}.$$

One can also see that the first inequality in (9.6) is equivalent to the estimate

$$2\operatorname{Re}\zeta = \frac{\varkappa\nu}{2\varkappa-(d-1)} < \frac{d\varkappa}{2\varkappa-1},$$

Finally, note that in $d \geqslant 3$, the condition $\operatorname{Re}\zeta < \varkappa$ follows from the fact that $\nu + \eta > 2$. Consequently, Corollary 7.6 implies Theorem 9.3. □

We interpolate between Corollary 9.2 and Theorem 9.3.

Theorem 9.4. *Let $3 \leqslant d \leqslant 5$. Assume that τ_1 satisfies*

$$0 \leqslant \Big(\Big(\frac{d}{2} + \frac{(\eta-1)(d+1)}{7d} + \frac{d-\eta}{2(d-2)}\Big) - 2\Big)\tau_1 \leqslant \frac{(\nu-1)(d+1)}{7d}. \tag{9.7}$$

with η and ν such that $1 < \nu < \eta < 2$. If $d = 3$, then we assume additionally that $8\nu + 9\eta < 26$. Let p, q and r be the numbers defined by

$$p = \frac{d}{7\tau_1}, \quad \frac{1}{q} = \frac{1-\theta}{p} + \frac{\theta}{2}, \qquad \text{and} \qquad \frac{1}{r} = \frac{1-\theta}{2p} + \frac{\theta}{2}, \tag{9.8}$$

where θ is the solution of the equation

$$\tau_1(1-\theta) + \frac{\theta}{2}\Big(\frac{d}{2} + \frac{d-\eta}{2(d-2)}\Big) = 1. \tag{9.9}$$

Then

$$\Big(\mathbb{E}\big(||Y(1)||^q_{\mathfrak{S}_q}\big)\Big)^{1/q} \leqslant C_q\Big(\frac{|k|}{R}\Big)^{d(1-\theta)/p}|k|^{-\theta\nu/2}\|\tilde V\|^2_r, \tag{9.10}$$

for $|k| \geqslant R$ and $0 < R \leqslant 1$.

Proof. Observe that

$$\tau_1 < \begin{cases} \frac{2(\nu-1)(d+1)}{7(d-3)d} \leqslant \frac{d}{7}, & \text{if} \quad d > 3, \\ \frac{8(\nu-1)}{21(2-\eta)} \leqslant \frac{d}{7}, & \text{if} \quad 8\nu + 9\eta < 26, \text{ and } d = 3, \end{cases}$$

In both cases, τ_1 obeys

$$0 < \tau_1 \leqslant d/7.$$

Consider $Y(\zeta)$ for ζ running over the strip

$$\tau_1 \leqslant \operatorname{Re}\zeta \leqslant \frac{d}{4} + \frac{d-\eta}{4(d-2)}.$$

Since we have some information about the values of this function on the boundary of the strip, we obtain (9.10) by interpolation between Corollary 9.2 and Theorem 9.3. □

Remark. We need to explain why the parameters were selected the way described in Theorem 9.4. The work with perturbation determinants requires convergence of integrals of the form

$$\int_\varepsilon^\infty \mathbb{E}\big(||Y(1)||^q_{\mathfrak{S}_q}\big)dk, \quad \varepsilon > 0,$$

so we need the parameters to satisfy the condition

$$qd(1-\theta)/p - q\theta\nu/2 < -1,$$

which is equivalent to the inequality

$$\tau_1(1-\theta) < \frac{\theta\nu}{14} - \frac{1}{7q} = \frac{\theta(\nu-1)}{14} - \frac{(1-\theta)\tau_1}{d},$$

implying that

$$\tau_1(1-\theta) < \frac{\theta(\nu-1)(d+1)}{14d}.$$

The latter can be written differently as follows

$$1 - \frac{\theta}{2}\Big(\frac{d}{2} + \frac{d-\eta}{2(d-2)}\Big) < \frac{\theta(\nu-1)(d+1)}{14d}.$$

In other words,

$$2 < \theta\Big(\frac{d}{2} + \frac{(\nu-1)(d+1)}{7d} + \frac{d-\eta}{2(d-2)}\Big). \tag{9.11}$$

The condition that θ is large can be converted into an inequality showing that τ_1 is small. The relation (9.11) is satisfied, if

$$\Big(\Big(\frac{d}{2} + \frac{(\nu-1)(d+1)}{7d} + \frac{d-\eta}{2(d-2)}\Big) - 2\Big)\tau_1 < \frac{(\nu-1)(d+1)}{7d}$$

Since $\eta > \nu$, that condition is obviously fulfilled, if (9.7) holds.

In the next statement, we estimate the remainder $X_{1,1}(\zeta)$ for $\zeta = 1$.

Theorem 9.5. *Let $p > 3d/4 \geqslant 2$ and let $\zeta = 1$. Then*

$$\mathbb{E}[||X_{1,1}(\zeta)||^p_{\mathfrak{S}_{p/2}}]^{1/p} \leqslant C\Big(\frac{|k|}{R}\Big)^{-4}\|\tilde V\|_p^2.$$

Proof. In this theorem, we deal with the operator

$$W(-\Delta - z)^{-1}P_{1,k}V(-\Delta - z)^{-1}P_{1,k}W$$

On the one hand, we see that

$$\mathbb{E}[||(-\Delta - z)^{-2/3}P_{1,k}V(-\Delta - z)^{-2/3}P_{1,k}||^p_{\mathfrak{S}_p}]^{1/p} \leqslant C\Big(\int_{|\xi|>2|k|/R}\Big||\xi|^2 - z\Big|^{-2p/3}d\xi\Big)^{2/p}\|\tilde V\|_p,$$

which implies the inequality

$$\mathbb{E}[||(-\Delta - z)^{-2/3}P_{1,k}V(-\Delta - z)^{-2/3}P_{1,k}||^p_{\mathfrak{S}_p}]^{1/p} \leqslant C\Big(\frac{|k|}{R}\Big)^{-8/3}\|\tilde V\|_p, \qquad p > 3d/4.$$

On the other hand,

$$\|W(-\Delta-z)^{-1/3}P_{1,k}\|^2_{\mathfrak{S}_{2p}} \leqslant C\Big(\frac{|k|}{R}\Big)^{-4/3}\|\tilde V\|_p, \qquad p>3d/4.$$

Consequently,

$$\mathbb{E}[||W(-\Delta-z)^{-1}P_{1,k}V(-\Delta-z)^{-1}P_{1,k}W||^p_{\mathfrak{S}_{p/2}}]^{1/p} \leqslant C\Big(\frac{|k|}{R}\Big)^{-4}\|\tilde V\|_p^2, \qquad p>3d/4.$$

□

The next statement follows by Hölder's inequality.

Corollary 9.6. *Let $q>3d/8\geqslant 1$ and let $\zeta=1$. Then*

$$\mathbb{E}[||X_{1,1}(\zeta)||^q_{\mathfrak{S}_q}]^{1/q} \leqslant C\Big(\frac{|k|}{R}\Big)^{-4}\|\tilde V\|^2_{2q}.$$

Surprisingly, q in (9.8) satisfies the inequality $q>3d/8\geqslant 1$. Thus, we obtain the following result.

Theorem 9.7. *Let $3\leqslant d\leqslant 5$. Assume that τ_1 satisfies (9.7) with η and ν such that $1<\nu<\eta<2$. If $d=3$, then we assume additionally that $8\nu+9\eta<26$. Let p, q and r be the numbers defined by*

$$p=\frac{d}{7\tau_1}, \quad \frac1q=\frac{1-\theta}{p}+\frac{\theta}{2}, \qquad \text{and} \qquad \frac1r=\frac{1-\theta}{2p}+\frac{\theta}{2}, \tag{9.12}$$

where θ is the solution of the equation

$$\tau_1(1-\theta)+\frac{\theta}{2}\Big(\frac d2+\frac{d-\eta}{2(d-2)}\Big)=1. \tag{9.13}$$

Then

$$\Big(\mathbb{E}\big(||X(k)||^q_{\mathfrak{S}_q}\big)\Big)^{1/q} \leqslant C_q\Big[\Big(\frac{|k|}{R}\Big)^{d(1-\theta)/p}|k|^{-\theta\nu/2}+\Big(\frac{|k|}{R}\Big)^{-4}\Big]\|\tilde V\|_r^2,$$

for $|k|\geqslant R$ and $0<R\leqslant 1$.

10. Proof of Theorem 1.2

We will work with the function

$$d(z)=\det_n(I-X(k)), \qquad n=[q]+1,$$

where z is related to k via the Joukowski mapping

$$z=\frac Rk+\frac kR, \qquad R>0,$$

which maps the set $\{k\in\mathbb{C}:\ \operatorname{Im}k>0,\quad |k|>R\}$ onto the upper half-plane $\{z\in\mathbb{C}:\ \operatorname{Im}z>0\}$. Rather standard arguments lead to the estimate

$$\sum_j \operatorname{Im}z_j \leqslant C\int_{-\infty}^{\infty}\ln|d(z)|dz, \tag{10.1}$$

where z_j are zeros of the function $d(z)$ situated in the upper half-plane $\mathbb{C}_+$. In fact, (10.1) could be established in the same way as Jensen's inequality for zeros of an analytic function on a unit disk. In (10.1) we assume that V is compactly supported. The relation (10.1) leads to the estimate

$$\sum_j \Big(\frac{|k_j|^2 - R^2}{|k_j|^2 R}\Big)_+ \operatorname{Im} k_j \leqslant C\Big(\int_{-\infty}^{\infty} \|X(k)\|_q^q \Big(\frac{1}{R} - \frac{R}{k^2}\Big)_+ dk + \int_0^\pi \|X(R\cdot e^{i\theta})\|_q^q \sin\theta d\theta\Big),$$

Taking the expectation we obtain

$$\sum_j \mathbb{E}\Big[\frac{\operatorname{Im} k_j (|k_j|^2 - R^2)_+}{|k_j|^2 R}\Big] \leqslant C\Big(\int_{-\infty}^{\infty} \mathbb{E}[\|X(k)\|_q^q] \Big(\frac{1}{R} - \frac{R}{k^2}\Big)_+ dk + \int_0^\pi \mathbb{E}[\|X(R\cdot e^{i\theta})\|_q^q] \sin\theta d\theta\Big).$$

Due to Theorem 9.7, the latter inequality leads to

$$\sum_j \mathbb{E}\Big[\frac{\operatorname{Im} k_j (|k_j|^2 - R^2)_+}{|k_j|^2 R}\Big] \leqslant C|R|^{-\theta q\nu/2} \|\tilde V\|_r^{2q}. \tag{10.2}$$

Now, suppose that we consider only the eigenvalues $\lambda_j = k_j^2$ that satisfy the inequality

$$|k_j| \leqslant R_0.$$

Multiplying (10.2) by $R^{\sigma-1}$ and integrating with respect to R from 0 to R_0, we obtain

$$\sum_{|k_j|\leqslant R_0} \mathbb{E}[\operatorname{Im} k_j |k_j|^{\sigma-1}] \leqslant C|R_0|^{\sigma-\theta q\nu/2} \|\tilde V\|_r^{2q}, \qquad \sigma > \theta q\nu/2. \tag{10.3}$$

That completes the proof. □

While we do not intend to describe all results related to the theory of operators with complex valued potentials, we would like to mention articles [1] -[10], [12]-[15] which could be viewed as valueable contributions in this area. Some of these papers were already mentioned in Introduction.

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