ON THE MULTIPLE-SCALE ANALYSIS FOR SOME LINEAR PARTIAL q-DIFFERENCE AND DIFFERENTIAL EQUATIONS WITH HOLOMORPHIC COEFFICIENTS

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Abstract. The analytic and formal solutions of certain family of q-difference-differential equations under the action of a complex perturbation parameter is considered. The previous study [10] provides information in the case when the main equation under study is factorizable, as a product of two equations in the so-called normal form. Each of them gives rise to a single level of q-Gevrey asymptotic expansion. In the present work, the main problem under study does not suffer any factorization, and a different approach is followed. More precisely, we lean on the technique developed in [4], where the first author makes distinction among the different q-Gevrey asymptotic levels by successive applications of two q-Borel-Laplace transforms of different orders both to the same initial problem and which can be described by means of a Newton polygon.

1. Introduction

This work is devoted to the study of a family of linear q-difference-differential problems in the complex domain. It can be arranged into a series of works dedicated to the asymptotic study of holomorphic solutions to different kinds of q-difference-differential problems involving irregular singularities such as [7], [8], [10], [13], and [15]. The study of q-difference and q-difference-differential equations in the complex domain is a promising and fruitful domain of research. In the literature, one may find other interesting approaches to this problems. We refer to [23] as a reference, and contributions in the framework of nonlinear q-analogs of Briot-Bouquet type partial differential equations in [24]. We provide [21, 25] as novel studies in this direction.

The study of q-difference equations has also been under study in different applications in the last years. Some advances in this respect are [17, 18, 19], and the references therein.

The main aim of this work is to study a family of q-difference-differential equations of the form

$$
\begin{aligned}
(1.1)\quad & Q(\partial_z)\sigma_{q,t}u(t,z,\epsilon)\\
&= (\epsilon t)^{d_{D_1}}\sigma_{q,t}^{\frac{d_{D_1}}{k_1}+1}R_{D_1}(\partial_z)u(t,z,\epsilon) + (\epsilon t)^{d_{D_2}}\sigma_{q,t}^{\frac{d_{D_2}}{k_2}+1}R_{D_2}(\partial_z)u(t,z,\epsilon)\\
&\quad + \sum_{\ell=1}^{D-1}\epsilon^{\Delta_\ell}t^{d_\ell}\sigma_{q,t}^{\delta_\ell}(c_\ell(t,z,\epsilon)R_\ell(\partial_z)u(t,z,\epsilon)) + \sigma_{q,t}f(t,z,\epsilon),
\end{aligned}
$$

where D, D_1, D_2 are integer numbers larger than 3, Q, R_{D_1}, R_{D_2} and R_ℓ for $\ell = 1,\dots,D-1$ are polynomials of complex coefficients, and $\Delta_\ell \geq 0$, $\delta_\ell, d_\ell \geq 1$ are nonnegative integers for every

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$1 \leq \ell \leq D-1$. The numbers d_{D_1} and d_{D_2} are positive integers.q stands for a real number with $q > 1$.

We consider the dilation operator $\sigma_{q,t}$ acting on variable t, i.e. $\sigma_{q,t}(t \mapsto f(t)) := f(qt)$, and the generalization of its composition given by

$$\sigma_{q,t}^{\gamma}(t \mapsto f(t)) := f(q^{\gamma}t),$$

for any $\gamma \in \mathbb{R}$. We also fix positive integer numbers k_1 and k_2 with

$$1 \leq k_1 < k_2.$$

As in the former work [16] of the third author, the coefficients $c_\ell(t,z,\epsilon)$ and the forcing term $f(t,z,\epsilon)$ represent bounded holomorphic functions in the vicinity of the origin in $\mathbb{C}^2$ w.r.t (t,ϵ) and on a horizontal strip $H_\beta = \{z \in \mathbb{C}/|\mathrm{Im}(z)| < \beta\}$ of width $\beta > 0$ relatively to the space variable z. However, a new additional constraint is required on the growth of the Taylor expansion of each c_ℓ according to the mixed variable $t\epsilon$, see (4.6). It implies that the functions $c_\ell(t,z,\epsilon)$ can be extended as entire functions in the monomial ϵt in the whole plane $\mathbb{C}$ with so-called q-exponential growth of some order related to k_1 and k_2 (this terminology will be explained later in the paper).

Two singularly perturbed terms on the right hand side of equation (1.1) are distinguished. This makes a crucial difference with respect to the previous work [10] in which only one term appears, whilst the multi-level q-Gevrey asymptotic behavior comes from the forcing term. More precisely, in that previous work we focused on families of q-difference-differential equations that can be factorized as a product of two operators in so-called normal forms each enjoying one single level of q-Gevrey asymptotics. In the present work, the appearance of these two terms would cause a multilevel q-Gevrey phenomenon in the study of the asymptotic solution of (1.1) regarding the perturbation parameter. Our approach is to follow a two-step procedure of summation of the formal solution, which makes the two q-Gevrey asymptotic orders emerge.

Another important difference compared to our previous contribution [10] is that we are now able to handle holomorphic coefficients in time t whilst only polynomial coefficients were considered in [10]. This fact relies on new technical bounds for a q-analog of the convolution of order k displayed in Proposition 2.6.

The point of view we use here is similar to the one performed in the work of the first author, see [4], and is related to direct constraints on the shape of the main equation via a possible description by a Newton polygon. It is important to stress that this approach is specific to the q-difference case. Namely, such a direct procedure for producing two different Gevrey levels in the differential case for the problem stated in the work [9] is impossible due to very strong restrictions related to a formula used in the proof and appearing in [22], see formula (8.7) p. 3630. In that case, only a proposal via factoring the main equation did actually work, as performed in our joint work [11].

Let us briefly review the steps followed in order to achieve our main results in the present work.

Let $0 \leq p \leq \varsigma - 1$. First, we apply q-Borel transformation of order k_1 to equation (1.1) in order to obtain our first auxiliary problem in a Borel plane, problem (4.10). A fixed point result in a complex Banach spaces of functions under an appropriate growth at infinity lead us to an analytic solution, $w_{k_1}^{\mathfrak{d}_p}(\tau,m,\epsilon)$ of (4.10). More precisely, $w_{k_1}^{\mathfrak{d}_p}(\tau,m,\epsilon)$ defines a continuous function defined in $(U_{\mathfrak{d}_p} \cup D(0,\rho)) \times \mathbb{R} \times D(0,\epsilon_0)$, where $U_{\mathfrak{d}_p}$ is an infinite sector of bisecting direction $\mathfrak{d}_p$, and holomorphic with respect to the variables τ and ϵ in $(U_{\mathfrak{d}_p} \cup D(0,\rho))$ and $D(0,\epsilon_0)$, respectively. In addition to that, it holds that this function admits κ q-exponential growth at infinity with respect to τ in $U_{\mathfrak{d}_p}$.This result is described in detail in Proposition 4.2.

A second auxiliary problem in the Borel plane is constructed by applying the formal q-Borel transformation of order k_2 to the main problem (1.1). The second auxiliary equation is stated in (4.25). A second fixed point result in another appropriate Banach spaces of functions allow us to guarantee the existence of an actual solution of the second auxiliary problem, $w_{k_2}^{\mathfrak{d}_p}(\tau, m, \epsilon)$, defined in $S_{\mathfrak{d}_p} \times \mathbb{R} \times D(0, \epsilon_0)$ and holomorphic with respect to τ and ϵ in $S_{\mathfrak{d}_p}$ and $D(0, \epsilon_0)$, respectively. Here, $S_{\mathfrak{d}_p}$ stands for an infinite sector with vertex at the origin and bisecting direction $\mathfrak{d}_p$. Moreover, this function suffers q-exponential growth of order k_2 at infinity w.r.t τ on $S_{\mathfrak{d}_p}$. This statement is proved in Proposition 4.3.

As a matter of fact, the key point in our reasoning is the link between the q-Laplace transform of order κ with respect to τ variable of $w_{k_1}^{\mathfrak{d}_p}$ and $w_{k_2}^{\mathfrak{d}_p}$. In Proposition 4.4, we guarantee that both functions coincide in the intersection of their domain of definition. This would entail that the function $\mathcal{L}^d_{q;1/\kappa}(w^d_{k_1}(\tau, m, \epsilon))$ can be prolongued along direction $\mathfrak{d}_p$, with q-exponential growth of order k_2, see Propoposition 4.4.

The construction of the analytic solution of (1.1), $u^{\mathfrak{d}_p}(t, z, \epsilon)$, is obtained after the application of q-Laplace transformation of order k_2 and inverse Fourier transform, providing a holomorphic function defined in $\mathcal{T} \times H_{\beta'} \times \mathcal{E}_p$, where $\mathcal{T}$ is some well chosen bounded sector centered at 0 and $\{\mathcal{E}_p\}_{0 \le p \le \varsigma - 1}$ represents a good covering in $\mathbb{C}^*$ (see Definition 5.1). This result is described in Theorem 5.3. The following diagram illustrates the procedure to follow. For the attainment of

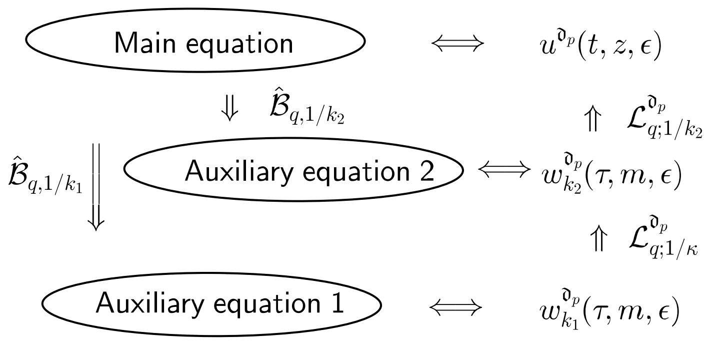

FIGURE 1. Scheme of the different Borel levels attained in the construction of the solution

the asymptotic properties of the analytic solution we make use of a Ramis-Sibuya type theorem in two levels (see Theorem 6.3), and the properties held by the difference of two analytic solutions in the intersection of their domains, whenever it is not empty. The conclusion yields two different q-Gevrey levels of asymptotic behavior of the analytic solution with respect to the formal solution depending on the geometry of the problem. The final main result states the splitting of both, the formal and the analytic solutions to the problem under study, as a sum of three terms. More precisely, if $\mathbb{F}$ denotes the Banach space of holomorphic and bounded functions defined in $\mathcal{T} \times H_{\beta'}$, and $\hat{u}(t, z, \epsilon)$ stands for the formal power series solution of (1.1), then it holds that

$$\hat{u}(t, z, \epsilon) = a(t, z, \epsilon) + \hat{u}_1(t, z, \epsilon) + \hat{u}_2(t, z, \epsilon),$$

where $a(t, z, \epsilon) \in \mathbb{F}\{\epsilon\}$ and $\hat{u}_1(t, z, \epsilon), \hat{u}_2(t, z, \epsilon) \in \mathbb{F}[[\epsilon]]$ and such that for every $0 \le p \le \varsigma - 1$, the function $u^{\mathfrak{d}_p}$ can be written in the form

$$u^{\mathfrak{d}_p}(t, z, \epsilon) = a(t, z, \epsilon) + u_1^{\mathfrak{d}_p}(t, z, \epsilon) + u_2^{\mathfrak{d}_p}(t, z, \epsilon),$$

where $\epsilon \mapsto u_1^{\mathfrak{d}_p}(t, z, \epsilon)$ is a $\mathbb{F}$-valued function that admits $\hat{u}_1(t, z, \epsilon)$ as its q-Gevrey asymptotic expansion of order $1/k_1$ on $\mathcal{E}_p$ and also $\epsilon \mapsto u_2^{\mathfrak{d}_p}(t, z, \epsilon)$ is a $\mathbb{F}$-valued function that admits $\hat{u}_2(t, z, \epsilon)$ as its q-Gevrey asymptotic expansion of order $1/k_2$ on $\mathcal{E}_p$. This corresponds to Theorem 6.4.

The paper is organized as follows.

In Section 2.1, we define some weighted Banach space of continuous functions on the domain $(D(0,\rho)\cup U)\times\mathbb{R}$ with q-exponential growth on the unbounded sector U with respect to the first variable, and exponential decay on $\mathbb{R}$ with respect to the second one. We study the continuity properties of several operators acting of this Banach space. Section 2.2. is concern with the study of a second family of Banach spaces of functions with q-exponential growth on an infinite sector with respect to one variable and exponential decay on $\mathbb{R}$ with respect to the other variable. In Section 3, we recall the definitions and main properties of formal and analytic operators involved in the solution of the main equation. Namely, formal q-Borel transformation and an analytic q-Laplace transform of certain q-Gevrey orders, and inverse Fourier transform. In Section 4.1. and Section 4.2, we study the analytic solutions of two auxiliary problems in two different Borel planes and relate them va q-Laplace transformation (see Theorem 5.3). In Section 5, we describe in detail the main problem under study, and construct its analytic solution and the rate of growth of the difference of two neighboring solutions in their common domain of definition. Finally, Section 6 deal with the existence of a formal solution of the problem, and studies the asymptotic behavior relating the analytic and the formal solutions through a multi-level q-Gevrey asymptotic expansion (Theorem 6.4). This result is attained with the application of a two-level q-version of Ramis-Sibuya theorem (Theorem 6.3).

2. Auxiliary Banach spaces of functions

In this section, we describe auxiliary Banach spaces of functions with certain growth and decay behavior. We also provide important properties of such spaces under certain operators.

Let U_d be an open unbounded sector with vertex at the origin in $\mathbb{C}$, bisecting direction $d\in\mathbb{R}$ and positive opening. We take $\rho>0$ and consider $D(0,\rho)=\{\tau\in\mathbb{C}:|\tau|<\rho\}$.

We fix real numbers $\beta,\mu>0$, $q>1$ and α through the whole section. We assume the distance from $U_d\cup D(0,\rho)$ to the real number $-\delta$ is positive. Let $k>0$. We denote $\overline{D}(0,\rho)$ the closure of $D(0,\rho)$.

The next definition of a Banach space of functions, and subsequent properties have already been studied in previous works. Analogous spaces were treated in [7, 12], inspired by the functional spaces appearing in [20]. We refer the reader to [10, 16] for some of the proofs of the following results, whose enunciates are included for the sake of completeness.

2.1. First family of Banach spaces of functions with q-exponential growth and exponential decay.

Definition 2.1. Let $q>1$. We denote $\mathrm{Exp}^q_{(k,\beta,\mu,\alpha,\rho)}$ the vector space of complex valued continuous functions $(\tau,m)\mapsto h(\tau,m)$ on $(U_d\cup\overline{D}(0,\rho))\times\mathbb{R}$, holomorphic with respect to τ on $U_d\cup D(0,\rho)$ and such that

$$\|h(\tau,m)\|_{(k,\beta,\mu,\alpha,\rho)}=\sup_{\substack{\tau\in U_d\cup\overline{D}(0,\rho),\\ m\in\mathbb{R}}}(1+|m|)^{\mu}e^{\beta|m|}\exp\left(-\frac{k}{2}\frac{\log^2|\tau+\delta|}{\log(q)}-\alpha\log|\tau+\delta|\right)|h(\tau,m)|$$

is finite. The space $(\mathrm{Exp}^q_{(k,\beta,\mu,\alpha,\rho)},\|\cdot\|_{(k,\beta,\mu,\alpha,\rho)})$ is a Banach space.

The proof of the following lemma is straightforward.

Lemma 2.2. *Let $(\tau,m)\mapsto a(\tau,m)$ be a bounded continuous function on $(U_d\cup\overline{D}(0,\rho))\times\mathbb{R}$, holomorphic with respect to τ on $U_d\cup D(0,\rho)$. Then, it holds that*

$$\|a(\tau,m)h(\tau,m)\|_{(k,\beta,\mu,\alpha,\rho)}\le\left(\sup_{\tau\in U_d\cup\overline{D}(0,\rho),m\in\mathbb{R}}|a(\tau,m)|\right)\|h(\tau,m)\|_{(k,\beta,\mu,\alpha,\rho)},$$

for every $h(\tau,m)\in Exp^q_{(k,\beta,\mu,\alpha,\rho)}$.

Proposition 2.3. *Let* $\gamma_1,\gamma_2,\gamma_3 \geq 0$ *such that*

$$\gamma_1 + k\gamma_3 \geq \gamma_2.$$

Let $a_{\gamma_1}(\tau)$ *be a continuous function on* $U_d \cup \overline{D}(0,\rho)$*, holomorphic on* $U_d \cup D(0,\rho)$*, with*

$$|a_{\gamma_1}(\tau)| \leq \frac{1}{(1+|\tau|)^{\gamma_1}},$$

for every $\tau \in (U_d \cup \overline{D}(0,\rho))$*. Then, there exists* $C_1>0$*, depending on* $k,q,\alpha,\gamma_1,\gamma_2,\gamma_3$*, such that*

$$\left\|a_{\gamma_1}(\tau)\tau^{\gamma_2}\sigma_{q,\tau}^{-\gamma_3}f(\tau,m)\right\|_{(k,\beta,\mu,\alpha,\rho)} \leq C_1 \left\|f(\tau,m)\right\|_{(k,\beta,\mu,\alpha,\rho)}$$

for every $f \in Exp^q_{(k,\beta,\mu,\alpha,\rho)}$.

Definition 2.4. We write $E_{(\beta,\mu)}$ for the vector space of continuous functions $h:\mathbb{R}\to\mathbb{C}$ such that

$$\|h(m)\|_{(\beta,\mu)} = \sup_{m\in\mathbb{R}} (1+|m|)^{\mu}\exp(\beta|m|)|h(m)| < \infty.$$

It holds that $(E_{(\beta,\mu)},\|\cdot\|_{(\beta,\mu)})$ is a Banach space.

The Banach space $(E_{(\beta,\mu)},\|\cdot\|_{(\beta,\mu)})$ can be endowed with the structure of a Banach algebra with the following noncomutative product (see Proposition 2 in [16] for further details).

Proposition 2.5. *Let* $Q(X),R(X)\in\mathbb{C}[X]$ *be polynomials such that*

$$\deg(R)\geq\deg(Q), \qquad R(im)\neq 0,$$

for all $m\in\mathbb{R}$*. Let* $m\mapsto b(m)$ *be a continuous function in* $\mathbb{R}$ *such that*

$$|b(m)| \leq 1/|R(im)|, \qquad m\in\mathbb{R}.$$

Assume that $\mu > \deg(Q)+1$*. Then, there exists a constant* $C_2>0$ *(depending on* $Q(X),R(X),\mu$*) such that*

$$\left\|b(m)\int_{-\infty}^{+\infty} f(m-m_1)Q(im_1)g(m_1)dm_1\right\|_{(\beta,\mu)} \leq C_2\left\|f(m)\right\|_{(\beta,\mu)}\left\|g(m)\right\|_{(\beta,\mu)},$$

for every $f(m),g(m)\in E_{(\beta,\mu)}$*. In the sequel, we adopt the notation*

$$f(m)\star^Q g(m) := \int_{-\infty}^{+\infty} f(m-m_1)Q(im_1)g(m_1)dm_1,$$

for every $m\in\mathbb{R}$*, extending the classical convolution product* $\star$ *for* $Q\equiv 1$*. As a result,* $(E_{(\beta,\mu)},\|\cdot\|_{(\beta,\mu)})$ *becomes a Banach algebra for the product* $\star^{b,Q}$ *defined by*

$$f(m)\star^{b,Q} g(m) := b(m)f(m)\star^Q g(m).$$

The next proposition is a slighted modified version of Proposition 3 in [16], adapted to the appearance of two different types of growth of the functions involved, which force holding some positive distance to the origin.

Proposition 2.6. *Let* $b(m),Q(X),R(X)$ *be chosen as in Proposition 2.5. We assume* $1\leq k\leq\kappa$ *is an integer. Let* $c_h(m)\in E_{(\beta,\mu)}$ *for* $h\geq 0$*, such that*

$$\|c_h\|_{(\beta,\mu)} \leq C\left(\frac{1}{T}\right)^h q^{\frac{h^2}{2k}(1-\frac{\kappa}{k})}, \qquad h\geq 0, \tag{2.1}$$

for some $C,T>0$*. Let* $\varphi_k(\tau,m)$ *be the power series*

$$\varphi_k(\tau,m) = \sum_{h\geq 0} c_h(m)\frac{\tau^h}{(q^{1/k})^{h(h-1)/2}} \in E_{(\beta,\mu)}[[\tau]],$$

which defines an entire function with respect to τ with values in $E_{(\beta,\mu)}$, in view of (2.1).

For every $f(\tau,m) \in Exp^q_{(\kappa,\beta,\mu,\alpha,\rho)}$, we define a q-analog of the convolution of order k of $\varphi_k(\tau,m)$ and $f(\tau,m)$ as

$$\varphi_k(\tau,m) \star^Q_{q;1/k} f(\tau,m) := \sum_{h\geq 0} \frac{\tau^h}{(q^{1/k})^{h(h-1)/2}} c_h(m) \star^Q (\sigma_{q,\tau}^{-\frac{h}{k}} f)(\tau,m).$$

Then, the function $b(m)\varphi_k(\tau,m) \star^Q_{q;1/k} f(\tau,m)$ belongs to $Exp^q_{(\kappa,\beta,\mu,\alpha,\rho)}$ and there exists $C_3 > 0$, depending on $\mu, q, \alpha, k, \kappa, Q(X), R(X), \delta$, T, such that

$$\left\| b(m)\varphi_k(\tau,m) \star^Q_{q;1/k} f(\tau,m) \right\|_{(\kappa,\beta,\mu,\alpha,\rho)} \leq C_3 C \left\| f(\tau,m) \right\|_{(\kappa,\beta,\mu,\alpha,\rho)}.$$

Proof. Let $f(\tau,m) \in \mathrm{Exp}^q_{(\kappa,\beta,\mu,\alpha,\rho)}$. From the very definition of the norm $\|\cdot\|_{(\kappa,\beta,\mu,\alpha,\rho)}$, we know that

$$\begin{aligned}
&\left\| b(m)\varphi_k(\tau,m) \star^Q_{q;1/k} f(\tau,m) \right\|_{(\kappa,\beta,\mu,\alpha,\rho)} \\
&= \sup_{\tau \in U_d \cup \overline{D}(0,\rho), m \in \mathbb{R}} (1+|m|)^{\mu} e^{\beta|m|} \exp\left(-\frac{\kappa}{2} \frac{\log^2(|\tau+\delta|)}{\log(q)} - \alpha \log(|\tau+\delta|) \right) |b(m)| \\
&\times \left| \sum_{h\geq 0} \frac{\tau^h}{(q^{1/k})^{\frac{h(h-1)}{2}}} \int_{-\infty}^{+\infty} c_h(m-m_1) Q(im_1) f(\frac{\tau}{q^{h/k}}, m_1) dm_1 \right| = \sup_{\tau \in U_d \cup \overline{D}(0,\rho), m \in \mathbb{R}} L(\tau,m).
\end{aligned}$$

We first give upper estimates for

$$\sup_{\tau \in \overline{D}(0,\rho), m \in \mathbb{R}} L(\tau,m).$$

By construction, there exist two constants $\mathfrak{Q}, \mathfrak{R} > 0$ such that

$$|Q(im_1)| \leq \mathfrak{Q}(1+|m_1|)^{\deg(Q)}, \quad |R(im)| \geq \mathfrak{R}(1+|m|)^{\deg(R)}, \tag{2.2}$$

for all $m \in \mathbb{R}$. Using (2.2) and from Lemma 4 in [14] (see also Lemma 2.2 from [2]), we get a constant $\tilde{C}_2 > 0$ with

$$\begin{aligned}
(1+|m|)^{\mu}|b(m)| &\int_{-\infty}^{+\infty} \frac{|Q(im_1)|}{(1+|m-m_1|)^{\mu}(1+|m_1|)^{\mu}} dm_1 \\
&\leq \sup_{m\in\mathbb{R}} \frac{\mathfrak{Q}}{\mathfrak{R}} (1+|m|)^{\mu-\deg(R)} \times \int_{-\infty}^{+\infty} \frac{1}{(1+|m-m_1|)^{\mu}(1+|m_1|)^{\mu-\deg(Q)}} dm_1 \leq \tilde{C}_2
\end{aligned} \tag{2.3}$$

provided that $\mu > \deg(Q)+1$. From the definition of $\|f\|_{(\kappa,\beta,\mu,\alpha,\rho)}$ and $\|c_h\|_{(\beta,\mu)}$ for all $h \geq 0$, (2.2), and (2.3), we get that for every $\tau \in \overline{D}(0,\rho)$ and $m \in \mathbb{R}$, $L(\tau,m)$ is upper bounded by

$$\begin{aligned}
&(1+|m|)^{\mu} e^{|m|} \exp\left(-\frac{\kappa}{2} \frac{\log^2(|\tau+\delta|)}{\log(q)} - \alpha \log(|\tau+\delta|) \right) |b(m)| \\
&\quad \times \sum_{h\geq 0} \frac{|\tau|^h}{(q^{1/k})^{h(h-1)/2}} \int_{-\infty}^{+\infty} \|c_h\|_{(\beta,\mu)} \frac{1}{(1+|m-m_1|)^{\mu}} e^{-\beta|m-m_1|} |Q(im_1)| \\
&\frac{1}{(1+|m_1|)^{\mu}} e^{-\beta|m_1|} \exp\left(\frac{\kappa}{2} \frac{\log^2(|\tau/q^{h/k}+\delta|)}{\log(q)} + \alpha \log(|\tau/q^{h/k}+\delta|) \right) dm_1 \left\| f(\tau,m) \right\|_{(\kappa,\beta,\mu,\alpha,\rho)}
\end{aligned}$$

$$
\begin{aligned}
&\leq \tilde{C}_2 \exp\left(-\frac{\kappa}{2}\frac{\log^2(|\tau+\delta|)}{\log(q)} - \alpha\log(|\tau+\delta|)\right)\sum_{h\geq 0}\frac{|\tau|^h}{(q^{1/k})^{h(h-1)/2}}\|c_h\|_{(\beta,\mu)}\\
&\qquad\times\exp\left(\frac{\kappa}{2}\frac{\log^2(|\tau/q^{h/k}+\delta|)}{\log(q)} + \alpha\log(|\tau/q^{h/k}+\delta|)\right)\\
&\qquad\qquad\leq \hat{C}_2\sum_{h\geq 0}\frac{\rho^h}{(q^{1/k})^{h(h-1)/2}}\|c_h\|_{(\beta,\mu)}\,\|f(\tau,m)\|_{(\kappa,\beta,\mu,\alpha,\rho)},
\end{aligned}
$$

with $\hat{C}_2 = \tilde{C}_2\exp\left(\frac{\kappa}{2}\frac{\log^2(\rho+\delta)}{\log(q)} + \alpha\log(\rho+\delta)\right)$. The assumption (2.1) on $\|c_h\|_{(\beta,\mu)}$ allows to conclude the result when restricting the domain on the variable τ to the subset $\overline{D}(0,\rho)$.

Let $\widetilde{U}_d$ be the complementary of $\overline{D}(0,\rho)$ in U_d. From what precede we may take the supremum over $\widetilde{U}_d$ instead of $U_d \cup \overline{D}(0,\rho)$.

By inserting terms that correspond to the $||.||_{(\beta,\mu)}$ norm of $c_h(m)$ and to the $||.||_{(\kappa,\beta,\mu,\alpha,\rho)}$ norm of $f(\tau/q^{h/k},m)$, there exists $\tilde{C}_1>0$, such that we can give the bound estimates

$$
\begin{aligned}
&||b(m)\varphi_k(\tau,m)\star^Q_{q;1/k} f(\tau,m)||_{(\kappa,\beta,\mu,\alpha,\rho)}\\
&\quad\leq \tilde{C}_1 \sup_{\tau\in\widetilde{U}_d, m\in\mathbb{R}}(1+|m|)^{\mu}e^{\beta|m|}\exp\left(-\frac{\kappa}{2}\frac{\log^2(|\tau+\delta|)}{\log(q)} - \alpha\log(|\tau+\delta|)\right)|b(m)|\\
&\qquad\times\sum_{h\geq 0}\int_{-\infty}^{+\infty}\left((1+|m-m_1|)^{\mu}e^{\beta|m-m_1|}\frac{|c_h(m-m_1)|}{(q^{1/k})^{h(h-1)/2}}|\tau|^h\right)\\
&\times\left(|f(\frac{\tau}{q^{h/k}},m_1)|(1+|m_1|)^{\mu}e^{\beta|m_1|}\exp\left(-\frac{\kappa}{2}\frac{\log^2(|\tau/q^{h/k}+\delta|)}{\log(q)} - \alpha\log(|\tau/q^{h/k}+\delta|)\right)\right)\\
&\times\left(\frac{e^{-\beta|m-m_1|}}{(1+|m-m_1|)^{\mu}}\frac{|Q(im_1)|}{(1+|m_1|)^{\mu}}e^{-\beta|m_1|}\exp\left(\frac{\kappa}{2}\frac{\log^2(|\tau/q^{h/k}+\delta|)}{\log(q)} + \alpha\log(|\tau/q^{h/k}+\delta|)\right)\right)dm_1.
\end{aligned}
$$

By means of the triangular inequality $|m| \leq |m-m_1| + |m_1|$, we deduce that

$$
||b(m)\varphi_k(\tau,m)\star^Q_{q;1/k} f(\tau,m)||_{(\kappa,\beta,\mu,\alpha,\rho)} \leq \hat{C}||f(\tau,m)||_{(\kappa,\beta,\mu,\alpha,\rho)} \tag{2.4}
$$

where

$$
\begin{aligned}
\hat{C} = \tilde{C}_1 &\sup_{\tau\in\widetilde{U}_d, m\in\mathbb{R}}(1+|m|)^{\mu}\exp\left(-\frac{\kappa}{2}\frac{\log^2(|\tau+\delta|)}{\log(q)} - \alpha\log(|\tau+\delta|)\right)|b(m)|\\
&\times\sum_{h\geq 0}||c_h||_{(\beta,\mu)}\frac{|\tau|^h}{(q^{1/k})^{h(h-1)/2}}\int_{-\infty}^{+\infty}\frac{|Q(im_1)|}{(1+|m-m_1|)^{\mu}(1+|m_1|)^{\mu}}dm_1\\
&\qquad\times\exp\left(\frac{\kappa}{2}\frac{\log^2(|\tau/q^{h/k}+\delta|)}{\log(q)} + \alpha\log(|\tau/q^{h/k}+\delta|)\right).
\end{aligned} \tag{2.5}
$$

Again, we can also apply (2.3) at this point. On the other hand, we can provide upper estimates on the following expression

$$
\begin{aligned}
(2.6)\quad & \exp\left(\frac{\kappa}{2\log(q)}\left(\log^2\left(\left|\frac{\tau}{q^{h/k}}+\delta\right|\right)-\log^2\left(\left|\frac{\tau}{q^{h/k}}\right|\right)\right)\right)\\
&\times\exp\left(\frac{\kappa}{2\log(q)}\left(\log^2\left(\left|\frac{\tau}{q^{h/k}}\right|\right)-\log^2\left(|\tau|\right)\right)\right)\\
&\times\exp\left(\frac{\kappa}{2\log(q)}\left(\log^2\left(|\tau|\right)-\log^2\left(|\tau+\delta|\right)\right)\right)\\
&\times\exp\left(\alpha\log\left(\left|\frac{\tau}{q^{h/k}}\right|\right)-\alpha\log(|\tau|)\right)\\
&\times\exp\left(\alpha\log\left(\left|\frac{\tau}{q^{h/k}}+\delta\right|\right)-\alpha\log\left(|\frac{\tau}{q^{h/k}}|\right)\right)\times\exp\left(\alpha\log(|\tau|)-\alpha\log(|\tau+\delta|)\right)
\end{aligned}
$$

as follows. One can provide the same bounds in the second and fourth lines of (2.6) than in the proof of Proposition 3 [16] which yield to

$$
(2.7)\qquad \exp\left(\frac{\kappa}{2\log(q)}\left(\log^2\left(\left|\frac{\tau}{q^{h/k}}\right|\right)-\log^2\left(|\tau|\right)\right)\right)=q^{\frac{h^2\kappa}{2k^2}}|\tau|^{-\frac{h\kappa}{k}},
$$

and

$$
(2.8)\qquad \exp\left(\alpha\log\left(\left|\frac{\tau}{q^{h/k}}\right|\right)-\alpha\log(|\tau|)\right)=q^{-\frac{\alpha h}{k}},
$$

respectively. It is straight to check that the expression in the fifth line of (2.6) is upper bounded by

$$
(2.9)\qquad C_{31}\left|\frac{q^{h/k}}{\tau}\right|^{\alpha},
$$

for some positive constant C_{31}.

We give upper bounds for the first line in (2.6). In the case that $|\tau/q^{h/k}|\le 1$, this expression is upper bounded by a constant which does not depend on τ nor h. Otherwise, we have

$$
\begin{aligned}
(2.10)\quad & \exp\left(\frac{\kappa}{2\log(q)}\left(\log^2\left(\left|\frac{\tau}{q^{h/k}}+\delta\right|\right)-\log^2\left(\left|\frac{\tau}{q^{h/k}}\right|\right)\right)\right)\\
&\le\exp\left(\frac{\kappa}{2\log(q)}\left(\log^2\left(\left|\frac{\tau}{q^{h/k}}\right|+\delta\right)-\log^2\left(\left|\frac{\tau}{q^{h/k}}\right|\right)\right)\right)\\
&\le\sup_{x>1}\exp\left(\frac{\kappa}{2\log(q)}(\log^2(x+\delta))-\log^2(x)\right)\le C_{32},
\end{aligned}
$$

for some $C_{32}>0$. We finally provide upper bounds on the third line of (2.6). Taking into account that

$$
(2.11)\qquad \log^2|\tau|-\log^2|\tau+\delta|=-\log^2|1+\frac{\delta}{\tau}|-2\log|\tau|\log|1+\frac{\delta}{\tau}|\le C_{33},\quad \tau\in\widetilde{U}_d
$$

for some $C_{33} > 0$. From (2.5), (2.3), (2.6), (2.7), (2.8), (2.9), (2.10), and (2.11) we derive the existence of $\tilde{C}_{31}, \tilde{C}_{32} > 0$ such that

$$\hat{C} \leq \tilde{C}_{31} \sum_{h\geq 0} ||c_h||_{(\beta,\mu)} \left(\sup_{\tau\in\widetilde{U}_d} |\tau|^{h(1-\frac{\kappa}{k})-\alpha} \right) q^{\frac{h}{2k}+\frac{h^2}{2k}(\frac{\kappa}{k}-1)}$$
$$\leq \tilde{C}_{32} C \sum_{h\geq 0} \left(\frac{q^{1/(2k)}}{T} \right)^h \leq C_3 C,$$

which yields the result, when choosing $T > q^{1/(2k)}$. □

Remark: Observe that condition (2.1) on the coefficients c_h is always satisfied in the case that only a finite number of c_h is not identically zero, i.e. $\varphi_k \in E_{(\beta,\mu)}[\tau]$.

2.2. **Second family of Banach spaces of functions with q-exponential growth and exponential decay.** The second family of auxiliary Banach spaces has already been studied in previous works, such as [10, 16]. We refer to these references for the proofs of the related results.

Let S_d be an infinite sector of bisecting direction d and let $\nu \in \mathbb{R}$.

Definition 2.7. We denote $\mathrm{Exp}^q_{(k,\beta,\mu,\nu)}$ the vector space of continuous functions $(\tau, m) \mapsto h(\tau, m)$ on $S_d \times \mathbb{R}$, and holomorphic with respect to τ on S_d, such that

$$\|h(\tau, m)\|_{(k,\beta,\mu,\nu)} = \sup_{\tau\in S_d, m\in\mathbb{R}} (1 + |m|)^{\mu} e^{\beta|m|} \exp\left(-\frac{k \log^2 |\tau|}{2\log(q)} - \nu \log |\tau| \right) |h(\tau, m)|$$

is finite. It holds that $(\mathrm{Exp}^q_{(k,\beta,\mu,\nu)}, \|\cdot\|_{(k,\beta,\mu,\nu)})$ is a Banach space.

Remark 2.8. Let $0 \leq \kappa_1 \leq \kappa_2$. For every $f \in \mathrm{Exp}^q_{(\kappa_1,\beta,\mu,\nu)}$, It holds that $f \in \mathrm{Exp}^q_{(\kappa_2,\beta,\mu,\nu)}$, and

$$\|f(\tau, m)\|_{(\kappa_2,\beta,\mu,\nu)} \leq \|f(\tau, m)\|_{(\kappa_1,\beta,\mu,\nu)}.$$

The proof of the following lemma is a straightforward consequence of the definition.

Lemma 2.9. *Let $a(\tau, m)$ be a bounded continuous function on $S_d \times \mathbb{R}$, holomorphic on S_d with respect to τ. Then,*

$$\|a(\tau, m)f(\tau, m)\|_{(k,\beta,\mu,\nu)} \leq \sup_{\tau\in S_d, m\in\mathbb{R}} |a(\tau, m)| \, \|f(\tau, m)\|_{(k,\beta,\mu,\nu)}$$

for every $f(\tau, m) \in \mathrm{Exp}^q_{(k,\beta,\mu,\nu)}$.

Proposition 2.10. *Let $\gamma_1, \gamma_2 \geq 0$ and $\gamma_3 \in \mathbb{R}$ such that*

$$\gamma_1 + k\gamma_3 \geq \gamma_2, \qquad \gamma_2 \geq k\gamma_3. \tag{2.12}$$

Let $a_{\gamma_1}(\tau)$ be an holomorphic on S_d, with

$$|a_{\gamma_1}(\tau)| \leq \frac{1}{(1 + |\tau|)^{\gamma_1}}, \quad \tau \in S_d.$$

Then, there exists $C_4 > 0$, depending on $k, q, \nu, \gamma_1, \gamma_2, \gamma_3$ such that

$$\left\| a_{\gamma_1}(\tau)\tau^{\gamma_2}\sigma_{q,\tau}^{-\gamma_3} f(\tau, m) \right\|_{(k,\beta,\mu,\nu)} \leq C_4 \|f(\tau, m)\|_{(k,\beta,\mu,\nu)}$$

for every $f \in \mathrm{Exp}^q_{(k,\beta,\mu,\nu)}$.

Proof. For every $f \in \mathrm{Exp}^q_{(k,\beta,\mu,\nu)}$ we have

$$\begin{aligned}
&\left\|a_{\gamma_1}(\tau)\tau^{-\gamma_2}\sigma_{q,\tau}^{\gamma_3} f(\tau,m)\right\|_{(k,\beta,\mu,\nu)} \\
&\quad = \sup_{\tau\in S_d, m\in\mathbb{R}} (1+|m|)^{\mu} e^{\beta|m|} \exp\left(-\frac{k\log^2|\tau|}{2\log(q)} - \nu\log|\tau|\right) \frac{|\tau|^{\gamma_2}}{(1+|\tau|)^{\gamma_1}} |f(\tau/q^{\gamma_3},m)| \\
&\quad\quad \times \exp\left(-\frac{k\log^2|\tau/q^{\gamma_3}|}{2\log(q)} - \nu\log|\tau/q^{\gamma_3}|\right) \exp\left(\frac{k\log^2|\tau/q^{\gamma_3}|}{2\log(q)} + \nu\log|\tau/q^{\gamma_3}|\right) \\
&\quad\quad\quad \le \sup_{\tau\in S_d} q^{\frac{k\gamma_3^2-2\nu\gamma_3}{2}} \frac{|\tau|^{-k\gamma_3+\gamma_2}}{(1+|\tau|)^{\gamma_1}} \|f(\tau,m)\|_{(k,\beta,\mu,\nu)}.
\end{aligned}$$

The result follows from the condition (2.12). □

Following the same lines of arguments as in Proposition 2.6, we deduce the next proposition

Proposition 2.11. *Let $b(m), Q(X), R(X), c_h$ for $h \ge 0$ and $\varphi_k(\tau,m)$ be chosen as in Proposition 2.6. For every $f(\tau,m) \in Exp^q_{(k,\beta,\mu,\nu)}$, it holds that $b(m)\varphi_k(\tau,m) \star^{Q}_{q;1/k} f(\tau,m)$ belongs to $Exp^q_{(k,\beta,\mu,\nu)}$ and there exists $C_4 > 0$, depending on $\mu, q, \nu, k, Q(X), R(X)$, such that*

$$\left\|b(m)\varphi_k(\tau,m) \star^{Q}_{q;1/k} f(\tau,m)\right\|_{(k,\beta,\mu,\nu)} \le CC_4 \|f(\tau,m)\|_{(k,\beta,\mu,\nu)}.$$

3. Formal and analytic operators involved in the study of the problem

The main properties of some formal and analytic transformations are displayed for the sake of completeness. In this section, $\mathbb{E}$ stands for a complex Banach space.

The definition and main properties of the q-analog of Borel and Laplace transformation in several different orders can be found in [4, 20]. The proofs of the following results can be found in [16].

Let $q > 1$ be a real number, and $k \ge 1$ be a rational number.

Definition 3.1. For every $\hat{a}(T) = \sum_{n\ge0} a_n T^n \in \mathbb{E}[[T]]$ we define the formal q-Borel transform of order k of $\hat{a}(T)$ by

$$\hat{\mathcal{B}}_{q;1/k}(\hat{a}(T))(\tau) = \sum_{n\ge0} a_n \frac{\tau^n}{(q^{1/k})^{n(n-1)/2}} \in \mathbb{E}[[\tau]].$$

Proposition 3.2. *Let $\sigma \in \mathbb{N}$ and $j \in \mathbb{Q}$. Then, it holds*

$$\hat{\mathcal{B}}_{q;1/k}(T^{\sigma}\sigma_q^{j}\hat{a}(T))(\tau) = \frac{\tau^{\sigma}}{(q^{1/k})^{\sigma(\sigma-1)/2}} \sigma_q^{j-\frac{\sigma}{k}} \left(\hat{\mathcal{B}}_{q;1/k}(\hat{a}(T))(\tau)\right),$$

for every $\hat{a}(T) \in \mathbb{E}[[T]]$.

The q-analog of Laplace transformation as it is shown was developed in [5]. The associated kernel of such transformation is the Jacobi theta function of order k defined by

$$\Theta_{q^{1/k}}(x) = \sum_{n\in\mathbb{Z}} q^{-\frac{n(n-1)}{2k}} x^n,$$

which turns out to be a holomorphic function in $\mathbb{C}^{\star}$. It turns out to be a solution of the q-difference equation

$$\Theta_{q^{1/k}}(q^{\frac{m}{k}}x) = q^{\frac{m(m+1)}{2k}} x^m \Theta_{q^{1/k}}(x), \tag{3.1}$$

for every $m \in \mathbb{Z}$, valid for all $x \in \mathbb{C}^{\star}$. As a matter of fact, Jacobi theta function of order k is a function of q-Gevrey growth at infinity of order k in the sense that for every $\tilde{\delta} > 0$ there exists $C_{q,k} > 0$, not depending on $\tilde{\delta}$, such that

$$\left|\Theta_{q^{1/k}}(x)\right| \geq C_{q,k}\tilde{\delta} \exp\left(\frac{k}{2}\frac{\log^2|x|}{\log(q)}\right)|x|^{1/2}, \tag{3.2}$$

for $x \in \mathbb{C}^{\star}$ under the condition that $|1 + xq^{\frac{m}{k}}| > \tilde{\delta}$, for every $m \in \mathbb{Z}$.

Definition 3.3. Let $\rho > 0$ and U_d be an unbounded sector with vertex at 0 and bisecting direction $d \in \mathbb{R}$. Let $f : D(0,\rho) \cup U_d \to \mathbb{E}$ be a holomorphic function, continuous on $\overline{D}(0,\rho)$, such that there exist $K > 0$ and $\alpha \in \mathbb{R}$ with

$$\|f(x)\|_{\mathbb{E}} \leq K \exp\left(\frac{k\log^2|x|}{2\log(q)} + \alpha\log|x|\right), \qquad x \in U_d, \quad |x| \geq \rho,$$

and

$$\|f(x)\|_{\mathbb{E}} \leq K, \qquad x \in \overline{D}(0,\rho).$$

Set $\pi_{q^{1/k}} = \frac{\log(q)}{k}\prod_{n\geq 0}(1 - \frac{1}{q^{(n+1)/k}})^{-1}$. We define the q-Laplace transform of order k of f along direction d by

$$\mathcal{L}^d_{q;1/k}(f(x))(T) = \frac{1}{\pi_{q^{1/k}}}\int_{L_d}\frac{f(u)}{\Theta_{q^{1/k}}\left(\frac{u}{T}\right)}\frac{du}{u},$$

where $L_d := \{te^{id} : t \in (0,\infty)\}$.

We refer the reader to Lemma 4 and Proposition 6 in [16] for the proof of the next results. The algebraic property held by q-Laplace transformation would allow to commute some operators with respect to it.

Lemma 3.4. *Let $\tilde{\delta} > 0$. Under the hypotheses of Definition 3.3, we have that $\mathcal{L}^d_{q;1/k}(f(x))(T)$ defines a bounded and holomorphic function on $\mathcal{R}_{d,\tilde{\delta}} \cap D(0,r_1)$ for every $0 < r_1 \leq q^{\left(\frac{1}{2}-\alpha\right)}$, where*

$$\mathcal{R}_{d,\tilde{\delta}} := \left\{T \in \mathbb{C}^{\star} : \left|1 + \frac{re^{id}}{T}\right| > \tilde{\delta},\ \text{for all } r \geq 0\right\}. \tag{3.3}$$

A different choice for d modulo $2\pi\mathbb{Z}$ would provide the same function due to Cauchy formula.

Proposition 3.5. *Let f be a function which satisfies the properties in Definition 3.3, and let $\tilde{\delta} > 0$. Then, for every $\sigma \geq 0$ one has*

$$T^{\sigma}\sigma_q^j(\mathcal{L}^d_{q;1/k}f(x))(T) = \mathcal{L}^d_{q;1/k}\left(\frac{x^{\sigma}}{(q^{1/k})^{\sigma(\sigma-1)/2}}\sigma_q^{j-\frac{\sigma}{k}}\right)(T),$$

for every $T \in \mathcal{R}_{d,\tilde{\delta}} \cap D(0,r_1)$, with $0 < r_1 \leq q^{\left(\frac{1}{2}-\alpha\right)/k}/2$.

Another operator which is used through the work is the inverse Fourier transform.

Proposition 3.6. *Let $f \in E_{(\beta,\mu)}$ with $\beta > 0$, $\mu > 1$. The inverse Fourier transform of f is defined by*

$$\mathcal{F}^{-1}(f)(x) = \frac{1}{(2\pi)^{1/2}}\int_{-\infty}^{\infty} f(m)\exp(ixm)dm$$

for all $x \in \mathbb{R}$. The function $\mathcal{F}^{-1}(f)$ extends to an analytic function on the strip

$$H_{\beta} = \{z \in \mathbb{C}/|\mathrm{Im}(z)| < \beta\}.$$

Let $\phi(m) = imf(m) \in E_{(\beta,\mu-1)}$. Then, we have

$$\partial_z\mathcal{F}^{-1}(f)(z) = \mathcal{F}^{-1}(\phi)(z)$$

for all $z \in H_{\beta}$.

Let $g \in E_{(\beta,\mu)}$ *and let* $\psi(m) = \frac{1}{(2\pi)^{1/2}} f \star g(m)$, *the convolution product of* f *and* g, *for all* $m \in \mathbb{R}$.

From Proposition 2.5, we know that $\psi \in E_{(\beta,\mu)}$. *Moreover, we have*

$$\mathcal{F}^{-1}(f)(z)\mathcal{F}^{-1}(g)(z) = \mathcal{F}^{-1}(\psi)(z)$$

for all $z \in H_{\beta}$.

4. Formal and analytic solutions to some auxiliary convolution initial value problems with complex parameters

Let $1 \leq k_1 < k_2$ and $D, D_1, D_2 \geq 3$ be integers and define $\kappa^{-1} = k_1^{-1} - k_2^{-1}$. Observe that $\kappa > k_1$. Let $q > 1$ be a real number. We also consider the positive integer numbers d_{D_1}, d_{D_2}. For every $1 \leq \ell \leq D-1$ we consider nonnegative integers $d_\ell, \delta_\ell \geq 1$ and $\Delta_\ell \geq 0$. We assume that

$$\delta_1 = 1, \qquad \delta_\ell < \delta_{\ell+1}, \tag{4.1}$$

for $1 \leq \ell \leq D-2$. We also assume that

$$\Delta_\ell \geq d_\ell, \quad \frac{d_{D_1}-1}{\kappa} + \frac{d_\ell}{k_2} + 1 \geq \delta_\ell, \quad \frac{d_\ell}{k_1} + 1 \geq \delta_\ell, \quad \frac{d_{D_2}-1}{k_2} \geq \delta_\ell - 1, \tag{4.2}$$

for every $1 \leq \ell \leq D-1$, and also

$$k_1(d_{D_2}-1) > k_2 d_{D_1}. \tag{4.3}$$

Let $Q(X), R_\ell(X) \in \mathbb{C}[X]$ for $1 \leq \ell \leq D-1$ and $R_{D_1}, R_{D_2} \in \mathbb{C}[X]$, such that

$$\deg(R_{D_2}) = \deg(R_{D_1}), \tag{4.4}$$

and

$$\deg(Q) \geq \deg(R_{D_j}) \geq \deg(R_\ell), \quad \mu - 1 > \deg(R_{D_j}), \quad Q(im) \neq 0, \quad R_{D_j}(im) \neq 0, \tag{4.5}$$

for some $\mu > \deg(R_{D_j}) + 1$ with $j = 1, 2$, for all $m \in \mathbb{R}$, $1 \leq \ell \leq D-1$.

We consider sequences of functions $m \mapsto F_n(m,\epsilon)$ and $m \mapsto C_{\ell,n}(m,\epsilon)$ for $n \geq 0$ belonging to the Banach space $E_{(\beta,\mu)}$ for some $\beta > 0$, depending holomorphically on $\epsilon \in D(0,\epsilon_0)$, for some $\epsilon_0 > 0$. We moreover assume there exist $\tilde{C}_\ell, C_F, T_0 > 0$ such that

$$\|C_{\ell,n}\|_{(\beta,\mu)} \leq \tilde{C}_\ell \left(\frac{1}{T_0}\right)^n q^{-\frac{n^2\kappa}{2k_1k_2}}, \tag{4.6}$$

$$\|F_n\|_{(\beta,\mu)} \leq C_F \left(\frac{1}{T_0}\right)^n, \qquad n \geq 0.$$

We define the formal power series in $E_{(\beta,\mu)}[[T]]$

$$\hat{C}_\ell(T,m,\epsilon) = \sum_{n\geq 0} C_{\ell,n}(m,\epsilon)T^n, \qquad \hat{F}(T,m,\epsilon) = \sum_{n\geq 0} F_n(m,\epsilon)T^n.$$

We consider the following initial value problem

$$
\begin{aligned}
(4.7)\quad & Q(im)\sigma_{q,T}U(T,m,\epsilon) \\
&= T^{d_{D_1}}\sigma_{q,T}^{\frac{d_{D_1}}{k_1}+1}R_{D_1}(im)U(T,m,\epsilon) + T^{d_{D_2}}\sigma_{q,T}^{\frac{d_{D_2}}{k_2}+1}R_{D_2}(im)U(T,m,\epsilon) \\
&+ \sum_{\ell=1}^{D-1}\epsilon^{\Delta_\ell-d_\ell}T^{d_\ell}\sigma_{q,T}^{\delta_\ell}\left(\frac{1}{(2\pi)^{1/2}}\int_{-\infty}^{+\infty}\hat{C}_\ell(T,m-m_1,\epsilon)R_\ell(im_1)U(T,m_1,\epsilon)dm_1\right) \\
&+ \sigma_{q,T}\hat{F}(T,m,\epsilon).
\end{aligned}
$$

Proposition 4.1. *There exists a unique formal power series*

$$
\hat{U}(T,m,\epsilon) = \sum_{n\geq 0} U_n(m,\epsilon)T^n, \tag{4.8}
$$

solution of (4.7), where the coefficients $U_n(m,\epsilon)$ belong to $E_{(\beta,\mu)}$, for $\beta > 0$ and $\mu > \deg(R_{D_j})+1$, $j\in\{1,2\}$, given above and depend holomorphically on $\epsilon \in D(0,\epsilon_0)$.

Proof. We plug the formal power series (4.8) into equation (4.7) to obtain a recursion formula for the coefficients U_n, for $n \geq 0$. We have

$$
\begin{aligned}
&Q(im)U_n(m,\epsilon)q^n = \\
&\quad R_{D_1}(im)U_{n-d_{D_1}}(m,\epsilon)q^{\left(\frac{d_{D_1}}{k_1}+1\right)(n-d_{D_1})} + R_{D_2}(im)U_{n-d_{D_2}}(m,\epsilon)q^{\left(\frac{d_{D_2}}{k_2}+1\right)(n-d_{D_2})} \\
&+ \sum_{\ell=1}^{D-1}\epsilon^{\Delta_\ell-d_\ell}q^{(n-d_\ell)\delta_\ell}\left(\sum_{n_1+n_2=n-d_\ell}\frac{1}{(2\pi)^{1/2}}\int_{-\infty}^{+\infty}C_{\ell,n_1}(m-m_1,\epsilon)R_\ell(im_1)U_{n_2}(m_1,\epsilon)dm_1\right) \\
&+ F_n(m,\epsilon)q^n
\end{aligned}
$$

for every $n \geq \max\{d_{D_1}, d_{D_2}, \max_{1\leq\ell\leq D-1} d_\ell\}$. Due to $C_{\ell,n}, F_n \in E_{(\beta,\mu)}$ for every $n \geq 0$ and $1 \leq \ell \leq D-1$, we get $U_n \in E_{(\beta,\mu)}$ by recursion. □

4.1. **Analytic solutions of a first auxiliary problem in the q-Borel plane.** We proceed to multiply at both sides of equation (4.7) by T^{k_1} and then apply the formal q-Borel transformation of order k_1 with respect to T. Let $\varphi_{k_1,\ell}(\tau,m,\epsilon)$ be the formal q-Borel transform of order k_1 of $\hat{C}_\ell(T,m,\epsilon)$ with respect to T, and $\Psi_{k_1}(\tau,m,\epsilon)$ the formal q-Borel transform of order k_1 of $\hat{F}(T,m,\epsilon)$ with respect to T. More precisely, we have

$$
\varphi_{k_1,\ell}(\tau,m,\epsilon) = \sum_{n\geq 0} C_{\ell,n}(m,\epsilon)\frac{\tau^n}{(q^{1/k_1})^{n(n-1)/2}},
$$

$$
\Psi_{k_1}(\tau,m,\epsilon) = \sum_{n\geq 0} F_n(m,\epsilon)\frac{\tau^n}{(q^{1/k_1})^{n(n-1)/2}}. \tag{4.9}
$$

According to Lemma 5 of [16], the expression $\Psi_{k_1}(\tau,m,\epsilon)$ represents an entire function of q-exponential growth of order k_1 that belongs to the Banach space $\mathrm{Exp}^q_{(\kappa,\beta,\mu,\alpha,\rho)}$ since $\kappa > k_1$, provided that α satisfies $T_0 > q^{\frac{1}{2k_1}}/q^{\alpha/k_1}$, for any unbounded sector U_d and any disc $D(0,\rho)$. More precisely, we have

$$
||\Psi_{k_1}(\tau,m,\epsilon)||_{(\kappa,\beta,\mu,\alpha,\rho)} \leq C_{\Psi_{k_1}}
$$

for some constant $C_{\Psi_{k_1}} > 0$, for all $\epsilon \in D(0,\epsilon_0)$.

In view of the properties of the q-Borel transformation of order k_1, we arrive at the equation

$$
\begin{aligned}
(4.10)\quad & Q(im)\frac{\tau^{k_1}}{(q^{1/k_1})^{\frac{k_1(k_1-1)}{2}}}w_{k_1}(\tau,m,\epsilon) = R_{D_1}(im)\frac{\tau^{d_{D_1}+k_1}}{(q^{1/k_1})^{\frac{(d_{D_1}+k_1)(d_{D_1}+k_1-1)}{2}}}w_{k_1}(\tau,m,\epsilon)\\
&+ R_{D_2}(im)\frac{\tau^{d_{D_2}+k_1}}{(q^{1/k_1})^{\frac{(d_{D_2}+k_1)(d_{D_2}+k_1-1)}{2}}}\sigma_{q,\tau}^{d_{D_2}\left(\frac{1}{k_2}-\frac{1}{k_1}\right)}w_{k_1}(\tau,m,\epsilon) + \frac{\tau^{k_1}}{(q^{1/k_1})^{\frac{k_1(k_1-1)}{2}}}\Psi_{k_1}(\tau,m,\epsilon)\\
&+ \sum_{\ell=1}^{D-1}\epsilon^{\Delta_\ell-d_\ell}\frac{\tau^{d_\ell+k_1}}{(q^{1/k_1})^{\frac{(d_\ell+k_1)(d_\ell+k_1-1)}{2}}}\sigma_{q,\tau}^{\delta_\ell-\frac{d_\ell}{k_1}-1}\left(\frac{1}{(2\pi)^{1/2}}\varphi_{k_1,\ell}(\tau,m,\epsilon)\star^{R_\ell}_{q;1/k_1} w_{k_1}(\tau,m,\epsilon)\right)
\end{aligned}
$$

where $w_{k_1}(\tau,m,\epsilon)$ stands for the formal q-Borel transformation of order k_1 with respect to T. Observe the appearance only of negative powers of the dilation operator in one of the terms in the sum of the right-hand side of the equation.

We assume an unbounded sector of bisecting direction $d_{Q,R_{D_1}} \in \mathbb{R}$ exists,

$$
S_{Q,R_{D_1}} = \left\{z \in \mathbb{C} : |z| \geq r_{Q,R_{D_1}}, |\arg(z) - d_{Q,R_{D_1}}| \leq \nu_{Q,R_{D_1}}\right\},
$$

for some $r_{Q,R_{D_1}}, \nu_{Q,R_{D_1}} > 0$, in such a way that

$$
\frac{Q(im)}{R_{D_1}(im)} \in S_{Q,R_{D_1}},
$$

for every $m \in \mathbb{R}$. We factorize

$$
P_{m,1}(\tau) = \frac{Q(im)}{(q^{1/k_1})^{\frac{k_1(k_1-1)}{2}}} - \frac{R_{D_1}(im)}{(q^{1/k_1})^{\frac{(d_{D_1}+k_1)(d_{D_1}+k_1-1)}{2}}}\tau^{d_{D_1}}
$$

in the form

$$
P_{m,1}(\tau) = -\frac{R_{D_1}(im)}{(q^{1/k_1})^{\frac{(d_{D_1}+k_1)(d_{D_1}+k_1-1)}{2}}}\prod_{\ell=0}^{d_{D_1}-1}(\tau - q_\ell(m)),
$$

with

$$
q_\ell(m) = e^{\frac{2i\pi\ell}{d_{D_1}}}\left(\frac{Q(im)}{R_{D_1}(im)}\right)^{1/d_{D_1}} q^{\frac{d_{D_1}+2k_1-1}{2k_1}},
$$

for every $0 \leq \ell \leq d_{D_1}-1$. Let U_d be an unbounded sector, and $\rho > 0$ such that the following statements hold:

1) There exists $M_1 > 0$ such that $|\tau - q_\ell(m)| \geq M_1(1+|\tau|)$ for every $0 \leq \ell \leq d_{D_1}-1$, $m \in \mathbb{R}$, and $\tau \in U_d \cup \overline{D}(0,\rho)$. An appropriate choice of $r_{Q,R_{D_1}}$ and ρ yields $|q_\ell(m)| > 2\rho$ for every $m \in \mathbb{R}$, and $0 \leq \ell \leq d_{D_1}-1$. In the case that $\nu_{Q,R_{D_1}}$ is small enough, the set $\{q_\ell(m) : m \in \mathbb{R}, 0 \leq \ell \leq d_{D_1}-1\}$ stays at a positive distance to U_d, and it can be chosen with the property that $|q_\ell(m)|/\tau$ has positive distance to $1 \in \mathbb{C}$ for every $\tau \in U_d$, $m \in \mathbb{R}$ and $0 \leq \ell \leq d_{D_1}-1$.
2) There exists $M_2 > 0$ such that $|\tau - q_{\ell_0}(m)| \geq M_2|q_{\ell_0}(m)|$ for some $\ell_0 \in \{0,\dots,d_{D_1}-1\}$, $m \in \mathbb{R}$ and $\tau \in U_d \cup \overline{D}(0,\rho)$. This is a direct consequence of 1), for some small enough $M_2 > 0$.

The previous conditions yield the existence of $C_P > 0$ such that

$$
\begin{aligned}
(4.11)\quad |P_{m,1}(\tau)| & \\
&\geq M_1^{d_{D_1}-1} M_2 \frac{|R_{D_1}(im)|(1+|\tau|)^{d_{D_1}-1}}{(q^{1/k_1})^{\frac{(d_{D_1}+k_1)(d_{D_1}+k_1-1)}{2}}} \left(\frac{|Q(im)|}{|R_{D_1}(im)|}\right)^{1/d_{D_1}} q^{\frac{d_{D_1}+2k_1-1}{2k_1}} \\
&\geq C_P (r_{Q,R_{D_1}})^{1/d_{D_1}} |R_{D_1}(im)|(1+|\tau|)^{d_{D_1}-1},
\end{aligned}
$$

for every $\tau \in U_d \cup \overline{D}(0,\rho)$, and $m \in \mathbb{R}$.

The next result states the existence and uniqueness of a solution of (4.10) in the space $\mathrm{Exp}^q_{(\kappa,\beta,\mu,\alpha,\rho)}$, provided its norm in that space is small enough.

Proposition 4.2. *Under the Assumptions (4.1), (4.2), (4.3), (4.4) and (4.5), there exist $r_{Q,R_{D_1}} > 0$, a constant $\varpi > 0$ and constants $\varsigma_\varphi, \varsigma_\Psi > 0$ such that if*

$$
\tilde{C}_\ell \leq \varsigma_\varphi \qquad C_{\Psi_1} \leq \varsigma_\Psi, \tag{4.12}
$$

for all $1 \leq \ell \leq D-1$ (see 4.6), then the equation (4.10) admits a unique solution $w^d_{k_1}(\tau,m,\epsilon) \in Exp^q_{(\kappa,\beta,\mu,\alpha,\rho)}$ with $\left\|w^d_{k_1}(\tau,m,\epsilon)\right\|_{(\kappa,\beta,\mu,\alpha,\rho)} \leq \varpi$, for every $\epsilon \in D(0,\epsilon_0)$.

Proof. Let $\epsilon \in D(0,\epsilon_0)$ and consider the operator $\mathcal{H}_\epsilon$ defined by

$$
\begin{aligned}
\mathcal{H}^1_\epsilon(w(\tau,m)) &:= \frac{R_{D_2}(im)}{P_{m,1}(\tau)} \frac{\tau^{d_{D_2}}}{(q^{1/k_1})^{\frac{(d_{D_2}+k_1)(d_{D_2}+k_1-1)}{2}}} \sigma_{q,\tau}^{d_{D_2}\left(\frac{1}{k_2}-\frac{1}{k_1}\right)} w(\tau,m) \\
&+ \sum_{\ell=1}^{D-1} \epsilon^{\Delta_\ell - d_\ell} \frac{\tau^{d_\ell}}{P_{m,1}(\tau)(q^{1/k_1})^{\frac{(d_\ell+k_1)(d_\ell+k_1-1)}{2}}} \sigma_{q,\tau}^{\delta_\ell - \frac{d_\ell}{k_1}-1} \left(\frac{1}{(2\pi)^{1/2}} \varphi_{k_1,\ell}(\tau,m,\epsilon) \star^{R_\ell}_{q;1/k_1} w(\tau,m)\right) \\
&+ \frac{1}{P_{m,1}(\tau)(q^{1/k_1})^{\frac{k_1(k_1-1)}{2}}} \Psi_{k_1}(\tau,m,\epsilon).
\end{aligned}
$$

Note that a fixed point of $\mathcal{H}^1_\epsilon(w(\tau,m))$ will lead to a convenient solution of (4.10). To apply the fixed point theorem, we are going ot prove successively two facts.

(1) One may choose small enough $\varsigma_\varphi, \varsigma_\Psi, \varpi > 0$, and large enough $r_{Q,R_{D_1}} > 0$ such that

$$
\mathcal{H}^1_\epsilon(\overline{B}(0,\varpi)) \subseteq \overline{B}(0,\varpi), \tag{4.13}
$$

where $\overline{B}(0,\varpi)$ stands for the closed disc centered at 0, with radius ϖ in the Banach space $\mathrm{Exp}^q_{(\kappa,\beta,\mu,\alpha,\rho)}$.

(2) It holds

$$
\left\|\mathcal{H}^1_\epsilon(w_1(\tau,m)) - \mathcal{H}^1_\epsilon(w_2(\tau,m))\right\|_{(\kappa,\beta,\mu,\alpha,\rho)} \leq \frac{1}{2} \left\|w_1(\tau,m) - w_2(\tau,m)\right\|_{(\kappa,\beta,\mu,\alpha,\rho)}, \tag{4.14}
$$

for every $w_1(\tau,m), w_2(\tau,m) \in \overline{B}(0,\varpi)$.

Proof of (4.13).

We first check (4.13). Let $w(\tau,m) \in \mathrm{Exp}^q_{(\kappa,\beta,\mu,\alpha,\rho)}$.

With (4.2) and the definition of κ, we find that $d_{D_1} - 1 + \kappa(d_\ell/k_1 + 1 - \delta_\ell) \geq d_\ell$ and $d_\ell/k_1 + 1 - \delta_\ell \geq 0$. Thus, taking into account assumptions (4.1), (4.5), regarding (4.11) together with Proposition 2.3 and Proposition 2.6 we get

(4.15)
$$\left\| \epsilon^{\Delta_\ell - d_\ell} \frac{\tau^{d_\ell}}{P_{m,1}(\tau)(q^{1/k_1})^{\frac{(d_\ell+k_1)(d_\ell+k_1-1)}{2}}} \sigma_{q,\tau}^{\delta_\ell - \frac{d_\ell}{k_1} - 1} \left(\frac{1}{(2\pi)^{1/2}} \varphi_{k_1,\ell}(\tau,m,\epsilon) \star^{R_\ell}_{q;1/k_1} w(\tau,m) \right) \right\|_{(\kappa,\beta,\mu,\alpha,\rho)}$$
$$\leq \epsilon_0^{\Delta_\ell - d_\ell} \frac{C_1 C_3 \varsigma_\varphi}{(q^{1/k_1})^{\frac{(d_\ell+k_1)(d_\ell+k_1-1)}{2}} C_P(r_{Q,R_{D_1}})^{1/d_{D_1}} (2\pi)^{1/2}} \left\| w(\tau,m) \right\|_{(\kappa,\beta,\mu,\alpha,\rho)}.$$

Gathering Lemma 2.2, we get

$$\left\| \frac{1}{P_{m,1}(\tau)(q^{1/k_1})^{k_1(k_1-1)/2}} \Psi_{k_1}(\tau,m,\epsilon) \right\|_{(\kappa,\beta,\mu,\alpha,\rho)}$$
$$\leq \frac{1}{(q^{1/k_1})^{k_1(k_1-1)/2} C_P(r_{Q,R_{D_1}})^{1/d_{D_1}}} \sup_{m\in\mathbb{R}} \frac{1}{|R_{D_1}(im)|} \varsigma_\Psi. \tag{4.16}$$

Condition (4.4) and the application of Proposition 2.3 and Lemma 2.2 yields

$$\left\| \frac{R_{D_2}(im)}{P_{m,1}(\tau)} \frac{\tau^{d_{D_2}}}{(q^{1/k_1})^{\frac{(d_{D_2}+k_1)(d_{D_2}+k_1-1)}{2}}} \sigma_{q,\tau}^{d_{D_2}\left(\frac{1}{k_2}-\frac{1}{k_1}\right)} w(\tau,m) \right\|_{(\kappa,\beta,\mu,\alpha,\rho)}$$
$$\leq \sup_{m\in\mathbb{R}} \frac{|R_{D_2}(im)|}{|R_{D_1}(im)|} \frac{C_1}{(q^{1/k_1})^{\frac{(d_{D_2}+k_1)(d_{D_2}+k_1-1)}{2}} C_P(r_{Q,R_{D_1}})^{1/d_{D_1}}} \varpi. \tag{4.17}$$

An appropriate choice of $r_{Q,R_{D_1}} > 0$, $\varpi, \varsigma_\Psi, \varsigma_\varphi > 0$ gives

$$\sum_{\ell=1}^{D-1} \epsilon_0^{\Delta_\ell - d_\ell} \frac{C_3 \varsigma_\varphi C_1}{(q^{1/k_1})^{\frac{(d_\ell+k_1)(d_\ell+k_1-1)}{2}} C_P(r_{Q,R_{D_1}})^{1/d_{D_1}} (2\pi)^{1/2}} \varpi$$
$$+ \frac{1}{(q^{1/k_1})^{k_1(k_1-1)/2} C_P(r_{Q,R_{D_1}})^{1/d_{D_1}}} \sup_{m\in\mathbb{R}} \frac{1}{|R_{D_1}(im)|} \varsigma_\Psi$$
$$+ \sup_{m\in\mathbb{R}} \frac{|R_{D_2}(im)|}{|R_{D_1}(im)|} \frac{C_1 \varpi}{(q^{1/k_1})^{\frac{(d_{D_2}+k_1)(d_{D_2}+k_1-1)}{2}} C_P(r_{Q,R_{D_1}})^{1/d_{D_1}}} \leq \varpi. \tag{4.18}$$

Regarding (4.15), (4.16), (4.17) and (4.18) one concludes (4.13).

Proof of (4.14).

We proceed to prove (4.14). Let $w_1, w_2 \in \mathrm{Exp}^q_{(\kappa,\beta,\mu,\alpha,\rho)}$. We assume $\|w_\ell(\tau,m)\|_{(\kappa,\beta,\mu,\alpha,\rho)} \leq \varpi$, $\ell = 1,2$, for some $\varpi > 0$. Let $E(\tau,m) := w_1(\tau,m) - w_2(\tau,m)$. On one hand, from (4.15) one has

(4.19)
$$\left\| \frac{\epsilon^{\Delta_\ell - d_\ell} \tau^{d_\ell}}{P_{m,1}(\tau)(q^{1/k_1})^{\frac{(d_\ell+k_1)(d_\ell+k_1-1)}{2}}} \sigma_{q,\tau}^{\delta_\ell - \frac{d_\ell}{k_1} - 1} \left(\frac{1}{(2\pi)^{1/2}} \varphi_{k_1,\ell}(\tau,m,\epsilon) \star^{R_\ell}_{q;1/k_1} E(\tau,m) \right) \right\|_{(\kappa,\beta,\mu,\alpha,\rho)}$$
$$\leq \epsilon_0^{\Delta_\ell - d_\ell} \frac{C_3 \varsigma_\varphi C_1}{(q^{1/k_1})^{\frac{(d_\ell+k_1)(d_\ell+k_1-1)}{2}} C_P(r_{Q,R_{D_1}})^{1/d_{D_1}} (2\pi)^{1/2}} \left\| E(\tau,m) \right\|_{(\kappa,\beta,\mu,\alpha,\rho)}.$$

On the other hand, (4.17) yields

$$
\begin{aligned}
&\left\| \frac{R_{D_2}(im)}{P_{m,1}(\tau)} \frac{\tau^{d_{D_2}}}{(q^{1/k_1})^{\frac{(d_{D_2}+k_1)(d_{D_2}+k_1-1)}{2}}} \sigma_{q,\tau}^{d_{D_2}\left(\frac{1}{k_2}-\frac{1}{k_1}\right)} E(\tau,m) \right\|_{(\kappa,\beta,\mu,\alpha,\rho)} \\
&\leq \sup_{m\in\mathbb{R}} \frac{|R_{D_2}(im)|}{|R_{D_1}(im)|} \frac{C_1}{(q^{1/k_1})^{\frac{(d_{D_2}+k_1)(d_{D_2}+k_1-1)}{2}} C_P(r_{Q,R_{D_1}})^{1/d_{D_1}}} \|E(\tau,m)\|_{(\kappa,\beta,\mu,\alpha,\rho)} . \qquad (4.20)
\end{aligned}
$$

We choose $r_{Q,R_{D_1}} > 0$, $\varsigma_\varphi > 0$ such that

$$
\begin{aligned}
&\sum_{\ell=1}^{D-1} \epsilon_0^{\Delta_\ell-d_\ell} \frac{C_3\varsigma_\varphi C_1}{(q^{1/k_1})^{\frac{(d_\ell+k_1)(d_\ell+k_1-1)}{2}} C_P(r_{Q,R_{D_1}})^{1/d_{D_1}}(2\pi)^{1/2}} \\
&+ \sup_{m\in\mathbb{R}} \frac{|R_{D_2}(im)|}{|R_{D_1}(im)|} \frac{C_1}{(q^{1/k_1})^{\frac{(d_{D_2}+k_1)(d_{D_2}+k_1-1)}{2}} C_P(r_{Q,R_{D_1}})^{1/d_{D_1}}} \leq \frac{1}{2}. \qquad (4.21)
\end{aligned}
$$

The statement (4.14) is a direct consequence of condition (4.21) applied to (4.19) and (4.20).

Let us finish the proof of the proposition. At this point, in view of (4.13) and (4.14), one can choose $\varpi > 0$ such that $\overline{B}(0,\varpi) \subseteq \mathrm{Exp}^q_{(\kappa,\beta,\mu,\alpha,\rho)}$, which defines a complete metric space for the norm $\|\cdot\|_{(\kappa,\beta,\mu,\alpha,\rho)}$. The map $\mathcal{H}^1_\epsilon$ is contractive from $\overline{B}(0,\varpi)$ into itself. The fixed point theorem states that $\mathcal{H}^1_\epsilon$ admits a unique fixed point $w^d_{k_1}(\tau,m,\epsilon) \in \overline{B}(0,\varpi) \subseteq \mathrm{Exp}^q_{(\kappa,\beta,\mu,\alpha,\rho)}$, for every $\epsilon \in D(0,\epsilon_0)$. The construction of $w^d_{k_1}(\tau,m,\epsilon)$ allows us to conclude that it turns out to be a solution of (4.10). □

The next step consists on studying the solutions of a second auxiliary problem. This problem lies in a second q-Borel plane and its solution would guarantee the extension, with appropriate growth, of the acceleration of the solution to our first auxiliary problem, described in (4.10).

We set

$$
\Psi_{k_2}(\tau,m,\epsilon) = \sum_{n\geq 0} F_n(m,\epsilon) \frac{\tau^n}{(q^{1/k_2})^{n(n-1)/2}} \qquad (4.22)
$$

the q-Borel transform of order k_2 of $F(T,m,\epsilon)$. According to the second condition of (4.6), the expression $\Psi_{k_2}(\tau,m,\epsilon)$ stands for an entire function of q-exponential growth of order k_2 wich belongs to the Banach space $\mathrm{Exp}^q_{(k_2,\beta,\mu,\alpha,\nu)}$ provided that $\nu \in \mathbb{R}$ satisfies $T_0 > q^{\frac{1}{2k_2}}/q^{\nu/k_2}$ for any unbounded sector S_d. More precisely, we have

$$
\|\Psi_{k_2}(\tau,m,\epsilon)\|_{(k_2,\beta,\mu,\nu)} \leq C_{\Psi_{k_2}} \qquad (4.23)
$$

for some constant $C_{\Psi_{k_2}} > 0$, for all $\epsilon \in D(0,\epsilon_0)$.

4.2. **Analytic solutions of a second auxiliary problem in the q-Borel plane.** We multiply both sides of equation (4.7) by T^{k_2} and apply formal q-Borel transformation of order k_2 with respect to T. In view of the properties of q-Borel transformation, the resulting problem is determined by

$$\begin{aligned}(4.24)\quad Q(im)\frac{\tau^{k_2}}{(q^{1/k_2})^{\frac{k_2(k_2-1)}{2}}}\hat{w}_{k_2}(\tau,m,\epsilon)\\ = R_{D_1}(im)\frac{\tau^{d_{D_1}+k_2}}{(q^{1/k_2})^{\frac{(d_{D_1}+k_2)(d_{D_1}+k_2-1)}{2}}}\sigma_{q,\tau}^{d_{D_1}\left(\frac{1}{k_1}-\frac{1}{k_2}\right)}\hat{w}_{k_2}(\tau,m,\epsilon)\\ + R_{D_2}(im)\frac{\tau^{d_{D_2}+k_2}}{(q^{1/k_2})^{\frac{(d_{D_2}+k_2)(d_{D_2}+k_2-1)}{2}}}\hat{w}_{k_2}(\tau,m,\epsilon)+\frac{\tau^{k_2}}{(q^{1/k_2})^{\frac{k_2(k_2-1)}{2}}}\Psi_{k_2}(\tau,m,\epsilon)\\ +\sum_{\ell=1}^{D-1}\epsilon^{\Delta_\ell-d_\ell}\frac{\tau^{d_\ell+k_2}}{(q^{1/k_2})^{\frac{(d_\ell+k_2)(d_\ell+k_2-1)}{2}}}\sigma_{q,\tau}^{\delta_\ell-\frac{d_\ell}{k_2}-1}\left(\frac{1}{(2\pi)^{1/2}}\varphi_{k_2,\ell}(\tau,m,\epsilon)\star_{q;1/k_2}^{R_\ell}\hat{w}_{k_2}(\tau,m,\epsilon)\right).\end{aligned}$$

Here $\hat{w}_{k_2}(\tau,m,\epsilon)$, $\Psi_{k_2}(\tau,m,\epsilon)$ and $\varphi_{k_2,l}(\tau,m,\epsilon)$ stand for the formal q-Borel transform of order k_2 of $\hat{U}(T,m,\epsilon)$, $F(T,m,\epsilon)$ and $\hat{C}_l(T,m,\epsilon)$.

We consider our second auxiliary problem, namely

$$\begin{aligned}(4.25)\quad Q(im)\frac{\tau^{k_2}}{(q^{1/k_2})^{\frac{k_2(k_2-1)}{2}}}w_{k_2}(\tau,m,\epsilon)\\ = R_{D_1}(im)\frac{\tau^{d_{D_1}+k_2}}{(q^{1/k_2})^{\frac{(d_{D_1}+k_2)(d_{D_1}+k_2-1)}{2}}}\sigma_{q,\tau}^{d_{D_1}\left(\frac{1}{k_1}-\frac{1}{k_2}\right)}w_{k_2}(\tau,m,\epsilon)\\ + R_{D_2}(im)\frac{\tau^{d_{D_2}+k_2}}{(q^{1/k_2})^{\frac{(d_{D_2}+k_2)(d_{D_2}+k_2-1)}{2}}}w_{k_2}(\tau,m,\epsilon)+\frac{\tau^{k_2}}{(q^{1/k_2})^{\frac{k_2(k_2-1)}{2}}}\Psi_{k_2}(\tau,m,\epsilon)\\ +\sum_{\ell=1}^{D-1}\epsilon^{\Delta_\ell-d_\ell}\frac{\tau^{d_\ell+k_2}}{(q^{1/k_2})^{\frac{(d_\ell+k_2)(d_\ell+k_2-1)}{2}}}\sigma_{q,\tau}^{\delta_\ell-\frac{d_\ell}{k_2}-1}\left(\frac{1}{(2\pi)^{1/2}}\varphi_{k_2,\ell}(\tau,m,\epsilon)\star_{q;1/k_2}^{R_\ell}w_{k_2}(\tau,m,\epsilon)\right).\end{aligned}$$

We assume an unbounded sector of bisecting direction $d_{Q,R_{D_2}}\in\mathbb{R}$ exists,

$$S_{Q,R_{D_2}}=\left\{z\in\mathbb{C}:|z|\geq r_{Q,R_{D_2}},|\arg(z)-d_{Q,R_{D_2}}|\leq\nu_{Q,R_{D_2}}\right\},$$

for some $\nu_{Q,R_{D_2}}>0$, in such a way that

$$\frac{Q(im)}{R_{D_2}(im)}\in S_{Q,R_{D_2}},$$

for every $m\in\mathbb{R}$. We factorize

$$P_{m,2}(\tau)=\frac{Q(im)}{(q^{1/k_2})^{\frac{k_2(k_2-1)}{2}}}-\frac{R_{D_2}(im)}{(q^{1/k_2})^{\frac{(d_{D_2}+k_2)(d_{D_2}+k_2-1)}{2}}}\tau^{d_{D_2}}$$

in the form

$$P_{m,2}(\tau)=-\frac{R_{D_2}(im)}{(q^{1/k_2})^{\frac{(d_{D_2}+k_2)(d_{D_2}+k_2-1)}{2}}}\prod_{\ell=0}^{d_{D_2}-1}(\tau-q_{\ell,2}(m)).$$

Let S_d be an unbounded sector with small enough aperture in such a way that:

1) There exists $M_{12}>0$ such that $|\tau-q_{\ell,2}(m)|\geq M_{12}(1+|\tau|)$ for every $0\leq\ell\leq d_{D_2}-1$, $m\in\mathbb{R}$, and $\tau\in S_d$.
2) There exists $M_{22}>0$ such that we have $|\tau-q_{\ell_0,2}(m)|\geq M_{22}|q_{\ell_0}(m)|$ for some $\ell_0\in\{0,\dots,d_{D_2}-1\}$, $m\in\mathbb{R}$ and $\tau\in S_d$.

The previous conditions yield the existence of $C_{P,2} > 0$ such that

$$|P_{m,2}(\tau)| \geq C_{P,2}(r_{Q,R_{D_2}})^{1/d_{D_2}}|R_{D_2}(im)|(1+|\tau|)^{d_{D_2}-1}, \tag{4.26}$$

for every $\tau \in S_d$, and $m \in \mathbb{R}$.

Proposition 4.3. *Let $\varpi > 0$. Under the hypotheses (4.1), (4.2), (4.3), (4.4), (4.5) and those in the geometry of the problem described in this subsection concerning the construction of the elements appearing in (4.25). If (4.12) holds, then, for every $\epsilon \in D(0,\epsilon_0)$, the equation (4.25) admits a unique solution $w^d_{k_2}(\tau,m,\epsilon)$ in the space $Exp^q_{(k_2,\beta,\mu,\nu)}$ for $\nu \in \mathbb{R}$ and depends holomorphically with respect to $\epsilon \in D(0,\epsilon_0)$. Moreover, $\left\|w^d_{k_2}(\tau,m,\epsilon)\right\|_{(k_2,\beta,\mu,\nu)} \leq \varpi$.*

Proof. Let $\epsilon \in D(0,\epsilon_0)$. We consider the map $\mathcal{H}^2_\epsilon$ defined by

$$\begin{aligned}\mathcal{H}^2_\epsilon(w(\tau,m)) := &\frac{R_{D_1}(im)}{P_{m,2}(\tau)}\frac{\tau^{d_{D_1}}}{(q^{1/k_2})^{\frac{(d_{D_1}+k_2)(d_{D_1}+k_2-1)}{2}}}\sigma_{q,\tau}^{d_{D_1}\left(\frac{1}{k_1}-\frac{1}{k_2}\right)}w(\tau,m)\\ &+\sum_{\ell=1}^{D-1}\epsilon^{\Delta_\ell-d_\ell}\frac{\tau^{d_\ell}}{P_{m,2}(\tau)(q^{1/k_2})^{\frac{(d_\ell+k_2)(d_\ell+k_2-1)}{2}}}\sigma_{q,\tau}^{\delta_\ell-\frac{d_\ell}{k_2}-1}\left(\frac{1}{(2\pi)^{1/2}}\varphi_{k_2,\ell}(\tau,m,\epsilon)\star^{R_\ell}_{q;1/k_2}w(\tau,m)\right)\\ &+\frac{1}{P_{m,2}(\tau)(q^{1/k_2})^{\frac{k_2(k_2-1)}{2}}}\Psi_{k_2}(\tau,m,\epsilon).\end{aligned} \tag{4.27}$$

Note that a fixed point of $\mathcal{H}^2_\epsilon(w(\tau,m))$ will lead to a convenient solution of (4.25). To apply the fixed point theorem, we are going ot prove successively two facts.

(1) One may choose small enough $\varsigma_\varphi,\varsigma_\Psi,\varpi > 0$, and large enough $r_{Q,R_{D_2}} > 0$ such that

$$\mathcal{H}^2_\epsilon(\overline{B}(0,\varpi)) \subseteq \overline{B}(0,\varpi), \tag{4.28}$$

where $\overline{B}(0,\varpi)$ stands for the closed disc centered at 0, with radius ϖ in the Banach space $\mathrm{Exp}^q_{(k_2,\beta,\mu,\nu)}$.

(2) It holds

$$\left\|\mathcal{H}^2_\epsilon(w_1(\tau,m))-\mathcal{H}^2_\epsilon(w_2(\tau,m))\right\|_{(k_2,\beta,\mu,\nu)} \leq \frac{1}{2}\left\|w_1(\tau,m)-w_2(\tau,m)\right\|_{(k_2,\beta,\mu,\nu)}, \tag{4.29}$$

for every $w_1(\tau,m), w_2(\tau,m) \in \overline{B}(0,\varpi)$.

Proof of (4.28).

We first check (4.28). Let $w(\tau,m) \in \mathrm{Exp}^q_{(k_2,\beta,\mu,\nu)}$.

With (4.2), we find that $d_{D_2}-1+k_2(d_\ell/k_2+1-\delta_\ell) \geq d_\ell$. Taking into account assumptions (4.1), (4.4), (4.5), regarding (4.26) together with Lemma 2.9, Proposition 2.10 and Proposition 2.11 we get

$$\begin{aligned}&\left\|\epsilon^{\Delta_\ell-d_\ell}\frac{\tau^{d_\ell}}{P_{m,2}(\tau)(q^{1/k_2})^{\frac{(d_\ell+k_2)(d_\ell+k_2-1)}{2}}}\sigma_{q,\tau}^{\delta_\ell-\frac{d_\ell}{k_2}-1}\left(\frac{1}{(2\pi)^{1/2}}\varphi_{k_2,\ell}(\tau,m,\epsilon)\star^{R_\ell}_{q;1/k_2}w(\tau,m)\right)\right\|_{(k_2,\beta,\mu,\nu)}\\ &\leq \epsilon_0^{\Delta_\ell-d_\ell}\frac{C_4C_3\varsigma_\varphi}{(q^{1/k_2})^{\frac{(d_\ell+k_2)(d_\ell+k_2-1)}{2}}C_{P,2}(r_{Q,R_{D_2}})^{1/d_{D_2}}(2\pi)^{1/2}}\left\|w(\tau,m)\right\|_{(k_2,\beta,\mu,\nu)}.\end{aligned} \tag{4.30}$$

Gathering Lemma 2.9 we get

$$
\begin{aligned}
&\left\| \frac{1}{P_{m,2}(\tau)(q^{1/k_2})^{k_2(k_2-1)/2}} \Psi_{k_2}(\tau,m,\epsilon) \right\|_{(k_2,\beta,\mu,\nu)} \\
&\leq \frac{1}{(q^{1/k_2})^{k_2(k_2-1)/2} C_{P,2}(r_{Q,R_{D_2}})^{1/d_{D_2}}} \sup_{m\in\mathbb{R}} \frac{1}{|R_{D_2}(im)|} \varsigma_{\Psi_2}, \tag{4.31}
\end{aligned}
$$

for some ς_{Ψ_2}. Observe that ς_{Ψ_2} tends to 0 when ς_{Ψ} does.

Condition (4.4), and the application of Proposition 2.10 and Lemma 2.9 yields

$$
\begin{aligned}
&\left\| \frac{R_{D_1}(im)}{P_{m,2}(\tau)} \frac{\tau^{d_{D_1}}}{(q^{1/k_2})^{\frac{(d_{D_1}+k_2)(d_{D_1}+k_2-1)}{2}}} \sigma_{q,\tau}^{d_{D_1}\left(\frac{1}{k_1}-\frac{1}{k_2}\right)} w(\tau,m) \right\|_{(k_2,\beta,\mu,\nu)} \\
&\leq \sup_{m\in\mathbb{R}} \frac{|R_{D_1}(im)|}{|R_{D_2}(im)|} \frac{C_4}{(q^{1/k_2})^{\frac{(d_{D_1}+k_2)(d_{D_1}+k_2-1)}{2}} C_{P,2}(r_{Q,R_{D_2}})^{1/d_{D_2}}} \varpi. \tag{4.32}
\end{aligned}
$$

An appropriate choice of $r_{Q,R_{D_2}} > 0$, $\varpi, \varsigma_{\varphi}, \varsigma_{\Psi} > 0$ gives

$$
\begin{aligned}
&\sum_{\ell=1}^{D-1} \epsilon_0^{\Delta_\ell - d_\ell} \frac{C_3 \varsigma_{\varphi} C_4}{(q^{1/k_2})^{\frac{(d_\ell+k_2)(d_\ell+k_2-1)}{2}} C_{P,2}(r_{Q,R_{D_2}})^{1/d_{D_2}} (2\pi)^{1/2}} \varpi \\
&+ \frac{1}{(q^{1/k_2})^{k_2(k_2-1)/2} C_{P,2}(r_{Q,R_{D_2}})^{1/d_{D_2}}} \sup_{m\in\mathbb{R}} \frac{1}{|R_{D_2}(im)|} \varsigma_{\Psi} \\
&+ \sup_{m\in\mathbb{R}} \frac{|R_{D_1}(im)|}{|R_{D_2}(im)|} \frac{C_4 \varpi}{(q^{1/k_2})^{\frac{(d_{D_1}+k_2)(d_{D_1}+k_2-1)}{2}} C_{P,2}(r_{Q,R_{D_2}})^{1/d_{D_2}}} \leq \varpi. \tag{4.33}
\end{aligned}
$$

Regarding (4.30), (4.31), (4.32) and (4.33) one concludes (4.28).

Proof of (4.29).

We proceed to prove (4.29). Let $w_1, w_2 \in \mathrm{Exp}^q_{(k_2,\beta,\mu,\nu)}$. We assume $\|w_\ell(\tau,m)\|_{(k_2,\beta,\mu,\nu)} \leq \varpi$, $\ell = 1,2$, for some $\varpi > 0$. Let $E(\tau,m) = w_1(\tau,m) - w_2(\tau,m)$. On one hand, from (4.30) one has

$$
\begin{aligned}
&\left\| \epsilon^{\Delta_\ell - d_\ell} \frac{\tau^{d_\ell}}{P_{m,2}(\tau)(q^{1/k_2})^{\frac{(d_\ell+k_2)(d_\ell+k_2-1)}{2}}} \sigma_{q,\tau}^{\delta_\ell - \frac{d_\ell}{k_2} - 1} \left(\frac{1}{(2\pi)^{1/2}} \varphi_{k_2,\ell}(\tau,m,\epsilon) \star^{R_\ell}_{q;1/k_2} E(\tau,m) \right) \right\|_{(k_2,\beta,\mu,\nu)} \\
&\leq \epsilon_0^{\Delta_\ell - d_\ell} \frac{C_3 \varsigma_{\varphi} C_4}{(q^{1/k_2})^{\frac{(d_\ell+k_2)(d_\ell+k_2-1)}{2}} C_{P,2}(r_{Q,R_{D_2}})^{1/d_{D_2}} (2\pi)^{1/2}} \|E(\tau,m)\|_{(k_2,\beta,\mu,\nu)}.
\end{aligned}
$$

On the other hand, (4.32) yields

$$
\begin{aligned}
&\left\| \frac{R_{D_1}(im)}{P_{m,2}(\tau)} \frac{\tau^{d_{D_1}}}{(q^{1/k_2})^{\frac{(d_{D_1}+k_2)(d_{D_1}+k_2-1)}{2}}} \sigma_{q,\tau}^{d_{D_1}\left(\frac{1}{k_1}-\frac{1}{k_2}\right)} E(\tau,m) \right\|_{(k_2,\beta,\mu,\nu)} \\
&\leq \sup_{m\in\mathbb{R}} \frac{|R_{D_1}(im)|}{|R_{D_2}(im)|} \frac{C_4}{(q^{1/k_2})^{\frac{(d_{D_1}+k_2)(d_{D_1}+k_2-1)}{2}} C_{P,2}(r_{Q,R_{D_2}})^{1/d_{D_2}}} \|E(\tau,m)\|_{(k_2,\beta,\mu,\nu)}.
\end{aligned}
$$

We choose $r_{Q,R_{D_2}} > 0$, $\varsigma_{\varphi} > 0$ such that

$$\sum_{\ell=1}^{D-1} \epsilon_0^{\Delta_\ell - d_\ell} \frac{C_3 \varsigma_\varphi C_4}{(q^{1/k_2})^{\frac{(d_\ell+k_2)(d_\ell+k_2-1)}{2}} C_{P,2}(r_{Q,R_{D_2}})^{1/d_{D_2}} (2\pi)^{1/2}} + \sup_{m\in\mathbb{R}} \frac{|R_{D_1}(im)|}{|R_{D_2}(im)|} \frac{C_4}{(q^{1/k_2})^{\frac{(d_{D_1}+k_2)(d_{D_1}+k_2-1)}{2}} C_{P,2}(r_{Q,R_{D_2}})^{1/d_{D_2}}} \leq \frac{1}{2}.$$

We conclude (4.29). Let us finish the proof of the proposition. At this point, in view of (4.28) and (4.29), one can choose $\varpi > 0$ such that $\overline{B}(0,\varpi) \subseteq \mathrm{Exp}^q_{(k_2,\beta,\mu,\nu)}$, which defines a complete metric space for the norm $\|\cdot\|_{(k_2,\beta,\mu,\nu)}$. The map $\mathcal{H}^2_\epsilon$ is contractive from $\overline{B}(0,\varpi)$ into itself. The fixed point theorem states that $\mathcal{H}^2_\epsilon$ admits a unique fixed point $w^d_{k_2}(\tau,m,\epsilon) \in \overline{B}(0,\varpi) \subseteq \mathrm{Exp}^q_{(k_2,\beta,\mu,\nu)}$, for every $\epsilon \in D(0,\epsilon_0)$. The construction of $w^d_{k_2}(\tau,m,\epsilon)$ allow us to conclude it turns out to be a solution of (4.25). □

The existing link between the acceleration of w_{k_1} and w_{k_2} is now provided. Both functions coincide in the intersection of their domain of definition. This fact assures the extension of the acceleration of w_{k_1} along direction d, with appropriate q-exponential growth in order to apply q-Laplace transformation of that order to recover the analytic solution of the main problem under study.

Proposition 4.4. *We consider $w^d_1(\tau,m,\epsilon)$ constructed in Proposition 4.2. The function*

$$\tau \mapsto \mathcal{L}^d_{q;1/\kappa}(w^d_{k_1}(\tau,m,\epsilon)) := \mathcal{L}^d_{q;1/\kappa}(h \mapsto w^d_{k_1}(h,m,\epsilon))(\tau)$$

defines a bounded holomorphic function in $\mathcal{R}_{d,\tilde{\delta}} \cap D(0,r_1)$, for $0 < r_1 \leq q^{\left(\frac{1}{2}-\alpha\right)/\kappa}/2$. Moreover, it holds that

$$\mathcal{L}^d_{q;1/\kappa}(w^d_{k_1}(\tau,m,\epsilon)) = w^d_{k_2}(\tau,m,\epsilon), \qquad (\tau,m,\epsilon) \in S^b_d \times \mathbb{R} \times D(0,\epsilon_0), \tag{4.34}$$

where S^b_d is a finite sector of bisecting direction d.

Proof. We recall from Proposition 4.2 that $w^d_{k_1} \in \mathrm{Exp}^q_{(\kappa,\beta,\mu,\alpha,\rho)}$. This guarantees appropriate bounds on $\tau \in U_d$ in order to apply q-Laplace transformation of order κ along direction d. This yields that for every $\tilde{\delta} > 0$, the function $\mathcal{L}^d_{q;1/\kappa}(w^d_{k_1}(\tau,m,\epsilon))$ defines a bounded and holomorphic function in $\mathcal{R}_{d,\tilde{\delta}} \cap D(0,r_1)$ for $0 < r_1 \leq q^{\left(\frac{1}{2}-\alpha\right)/\kappa}/2$.

In order to prove that (4.34) holds, it is sufficient to prove that $\mathcal{L}^d_{q;1/\kappa}(w^d_{k_1}(\tau,m,\epsilon))$ and $w^d_{k_2}$ are both solutions of some problem, with unique solution in certain Banach space, so they must coincide. For that purpose, we multiply both sides of equation (4.10) by τ^{-k_1} and take q-Laplace transformation of order κ along direction d.

The properties of q-Laplace transformation yield

$$\mathcal{L}^d_{q;1/\kappa}\left(\tau^{d_{D_1}} w^d_{k_1}(\tau,m,\epsilon)\right) = (q^{1/\kappa})^{d_{D_1}(d_{D_1}-1)/2} \tau^{d_{D_1}} \sigma_{q,\tau}^{\frac{d_{D_1}}{\kappa}} \mathcal{L}^d_{q;1/\kappa}(w^d_{k_1})(\tau,m,\epsilon), \tag{4.35}$$

$$\mathcal{L}^d_{q;1/\kappa}\left(\tau^{d_{D_2}} \sigma_{q,\tau}^{d_{D_2}\left(\frac{1}{k_2}-\frac{1}{k_1}\right)} w^d_{k_1}(\tau,m,\epsilon)\right) = (q^{1/\kappa})^{d_{D_2}(d_{D_2}-1)/2} \tau^{d_{D_2}} \mathcal{L}^d_{q;1/\kappa}(w^d_{k_1})(\tau,m,\epsilon), \tag{4.36}$$

and

$$\mathcal{L}^{d}_{q;1/\kappa}\left(\tau^{d_\ell}\sigma_{q,\tau}^{\delta_\ell+\frac{d_\ell}{k_1}-1}\left(\frac{1}{(2\pi)^{1/2}}\varphi_{k_1,\ell}(\tau,m,\epsilon)\star^{R_\ell}_{q;1/k_1} w^d_{k_1}(\tau,m,\epsilon)\right)\right) = (q^{1/\kappa})^{d_\ell(d_\ell-1)/2}\tau^{d_\ell}\sigma_{q,\tau}^{\delta_\ell-\frac{d_\ell}{k_2}-1}\mathcal{L}^{d}_{q;1/\kappa}\left(\frac{1}{(2\pi)^{1/2}}\varphi_{k_1,\ell}(\tau,m,\epsilon)\star^{R_\ell}_{q;1/k_1} w^d_{k_1}(\tau,m,\epsilon)\right). \tag{4.37}$$

We claim that we have

$$\mathcal{L}^{d}_{q;1/\kappa}\left(\varphi_{k_1,\ell}(\tau,m,\epsilon)\star^{R_\ell}_{q;1/k_1} w^d_{k_1}(\tau,m,\epsilon)\right) = \varphi_{k_2,\ell}(\tau,m,\epsilon)\star^{R_\ell}_{q;1/k_2}\mathcal{L}^{d}_{q;1/\kappa}(w^d_{k_1}(\tau,m,\epsilon)). \tag{4.38}$$

This is a consequence of the change in the order of integration in the operators involved in (4.38). This situation is different from that of (60) in the proof of Proposition 12 in [10]. Assume the variable of integration with respect to Laplace operator is r. After the change of variable $\tilde{r} = r/q^{h/k_1}$, we reduce the study to that of Ξ in the proof of Proposition 12 in [10], with r replaced by r^{1-h}. This last argument guarantees the availability of the change of order in the integration operators involved in (4.38). We now give proof of (4.38) under this consideration.

We have

$$\mathcal{L}^{d}_{q;1/\kappa}\left(\tfrac{1}{(2\pi)^{1/2}}\varphi_{k_1,\ell}(\tau,m,\epsilon)\star^{R_\ell}_{q;1/k_1} w^d_{k_1}(\tau,m,\epsilon)\right)$$

$$= \frac{1}{\pi_{q^{1/\kappa}}}\int_0^\infty \left(\varphi_{k_1,\ell}(re^{id},m,\epsilon)\star^{R_\ell}_{q;1/k_1} w^d_{k_1}(re^{id},m,\epsilon)\right)\frac{1}{\Theta_{q^{1/\kappa}}\left(\frac{re^{id}}{\tau}\right)}\frac{dr}{r}$$

$$= \frac{1}{\pi_{q^{1/\kappa}}}\int_0^\infty \left(\sum_{n\geq 0}\frac{(re^{id})^n}{(q^{1/k_1})^{n(n-1)/2}}C_{\ell,n}(m,\epsilon)\star^{R_\ell}(\sigma_{q,\tau}^{-\frac{n}{k_1}}w_{k_1})(re^{id},m,\epsilon)\right)\frac{1}{\Theta_{q^{1/\kappa}}\left(\frac{re^{id}}{\tau}\right)}\frac{dr}{r}$$

$$= \frac{1}{\pi_{q^{1/\kappa}}}\int_0^\infty \left(\sum_{n\geq 0}\frac{(re^{id})^n}{(q^{1/k_1})^{n(n-1)/2}}\int_{-\infty}^{\infty}C_{\ell,n}(m-m_1,\epsilon)R_\ell(im_1)w_{k_1}(re^{id}q^{-\frac{n}{k_1}},m_1,\epsilon)dm_1\right) \times \frac{1}{\Theta_{q^{1/\kappa}}\left(\frac{re^{id}}{\tau}\right)}\frac{dr}{r}.$$

We make the change of variable $\tilde{r} = r/q^{n/k_1}$ to get that the previous expression equals

$$\frac{1}{\pi_{q^{1/\kappa}}}\int_0^\infty \left(\int_{-\infty}^{\infty}\frac{(\tilde{r}e^{id})^n q^{n(n-1)/k_1}}{(q^{1/k_1})^{n(n-1)/2}}\sum_{n\geq 0}C_{\ell,n}(m-m_1,\epsilon)R_\ell(im_1)w_{k_1}(\tilde{r}e^{id},m_1,\epsilon)dm_1\right) \times \frac{1}{\Theta_{q^{1/\kappa}}\left(\frac{\tilde{r}e^{id}q^{n/k_1}}{\tau}\right)}\frac{d\tilde{r}}{\tilde{r}}.$$

In view of (3.1), $k_1^{-1} = \kappa^{-1} + k_2^{-1}$, the change of order of the integrals and the dominated convergence theorem, the previous equation equals

$$
\begin{aligned}
&= \frac{1}{\pi_{q^{1/\kappa}}} \int_0^\infty \left(\int_{-\infty}^{\infty} \frac{(\tilde{r}e^{id})^n q^{n(n-1)/k_1}}{(q^{1/k_1})^{n(n-1)/2}} \sum_{n\geq 0} C_{\ell,n}(m-m_1,\epsilon) R_\ell(im_1) w_{k_1}(\tilde{r}e^{id}, m_1,\epsilon) dm_1 \right) \\
&\quad \times \frac{1}{\Theta_{q^{1/\kappa}}\left(\frac{\tilde{r}e^{id}q^{n/k_2}}{\tau}\right) q^{\frac{n(n+1)}{2\kappa}} \left(\frac{\tilde{r}e^{id}q^{n/k_2}}{\tau}\right)^n} \frac{d\tilde{r}}{\tilde{r}} \\
&= \frac{1}{\pi_{q^{1/\kappa}}} \int_0^\infty \left(\int_{-\infty}^{\infty} \frac{\tau^n q^{n(n-1)/\kappa}}{(q^{1/k_1})^{n(n-1)/2}} \sum_{n\geq 0} C_{\ell,n}(m-m_1,\epsilon) R_\ell(im_1) w_{k_1}(\tilde{r}e^{id}, m_1,\epsilon) dm_1 \right) \\
&\quad \times \frac{1}{\Theta_{q^{1/\kappa}}\left(\frac{\tilde{r}e^{id}q^{n/k_2}}{\tau}\right)} \frac{d\tilde{r}}{\tilde{r}} \\
&= \int_{-\infty}^{\infty} \left(\sum_{n\geq 0} \frac{\tau^n}{(q^{1/k_2})^{n(n-1)/2}} C_{\ell,n}(m-m_1,\epsilon) \right) R_\ell(im_1) \left[\frac{1}{\pi_{q^{1/\kappa}}} \int_0^\infty \frac{w_{k_1}(\tilde{r}e^{id}, m_1,\epsilon)}{\Theta_{q^{1/\kappa}}\left(\frac{\tilde{r}e^{id}q^{n/k_2}}{\tau}\right)} \frac{d\tilde{r}}{\tilde{r}} \right] dm_1 \\
&\quad = \varphi_{k_2,\ell}(\tau,m,\epsilon) \star^{R_\ell}_{q;1/k_2} \mathcal{L}^d_{q;1/\kappa}(w^d_{k_1}(\tau,m,\epsilon)),
\end{aligned}
$$

from where we conclude (4.38).

On the other hand, we observe by direct computation that

$$
\mathcal{L}^d_{q;1/\kappa}(\Psi_{k_1}(\tau,m,\epsilon)) = \Psi_{k_2}(\tau,m,\epsilon) \tag{4.39}
$$

for every $(\tau,m,\epsilon) \in (\mathcal{R}_{d,\tilde{\delta}} \cap D(0,r_1)) \times \mathbb{R} \times D(0,\epsilon_0)$.

In view of (4.35), (4.36), (4.37), (4.38), and the last formula above (4.39), we derive that

$$
\begin{aligned}
&\frac{Q(im)}{(q^{1/k_1})^{\frac{k_1(k_1-1)}{2}}} \mathcal{L}^d_{q;1/\kappa}(w^d_{k_1})(\tau,m,\epsilon) \\
&\qquad = R_{D_1}(im) \frac{(q^{1/\kappa})^{d_{D_1}(d_{D_1}-1)/2}}{(q^{1/k_1})^{\frac{(d_{D_1}+k_1)(d_{D_1}+k_1-1)}{2}}} \tau^{d_{D_1}} \sigma_{q,\tau}^{\frac{d_{D_1}}{\kappa}} \mathcal{L}^d_{q;1/\kappa}(w^d_{k_1})(\tau,m,\epsilon) \\
&\quad + R_{D_2}(im) \frac{(q^{1/\kappa})^{\frac{d_{D_2}(d_{D_2}-1)}{2}}}{(q^{1/k_1})^{\frac{(d_{D_2}+k_1)(d_{D_2}+k_1-1)}{2}}} \tau^{d_{D_2}} \mathcal{L}^d_{q;1/\kappa}(w^d_{k_1})(\tau,m,\epsilon) + \frac{1}{(q^{1/k_1})^{\frac{k_1(k_1-1)}{2}}} \Psi_{k_2}(\tau,m,\epsilon) \\
&+ \sum_{\ell=1}^{D-1} \epsilon^{\Delta_\ell - d_\ell} \tau^{d_\ell} \frac{(q^{1/\kappa})^{\frac{d_\ell(d_\ell-1)}{2}}}{(q^{1/k_1})^{\frac{(d_\ell+k_1)(d_\ell+k_1-1)}{2}}} \sigma_{q,\tau}^{\delta_\ell - \frac{d_\ell}{k_2} - 1} \left(\frac{1}{(2\pi)^{1/2}} \varphi_{k_2,\ell}(\tau,m,\epsilon) \star^{R_\ell}_{q;1/k_2} \mathcal{L}^d_{q;1/\kappa}(w^d_{k_1})(\tau,m,\epsilon) \right),
\end{aligned}
$$

for every $(\tau,m,\epsilon) \in (\mathcal{R}_{d,\tilde{\delta}} \cap D(0,r_1)) \times \mathbb{R} \times D(0,\epsilon_0)$. We multiply at both sides of the previous equation by $(q^{1/k_1})^{k_1(k_1-1)/2}/(q^{1/k_2})^{k_2(k_2-1)/2}$. The fact that

$$
\frac{(q^{1/\kappa})^{\frac{d_\ell(d_\ell-1)}{2}} (q^{1/k_1})^{\frac{k_1(k_1-1)}{2}}}{(q^{1/k_1})^{\frac{(d_\ell+k_1)(d_\ell+k_1-1)}{2}} (q^{1/k_2})^{\frac{k_2(k_2-1)}{2}}} = \frac{1}{(q^{1/k_2})^{\frac{(\mathcal{D}+k_2)(\mathcal{D}+k_2-1)}{2}}},
$$

with $\mathcal{D} \in \{d_{D_1}, d_{D_2}, d_\ell\}$ entails that $\mathcal{L}^d_{q;1/\kappa}(w^d_{k_1}(\tau,m,\epsilon))$ is a solution of (4.25) in its domain of definition.

Let S^b_d be a bounded sector of bisecting direction d such that $S^b_d \subseteq (\mathcal{R}_{d,\tilde{\delta}} \cap D(0,r_1)) \cap S_d$, which is a nonempty set due to the assumptions on the construction of these sets. The functions

$\mathcal{L}^d_{q;1/\kappa}(w^d_{k_1}(\tau,m,\epsilon))$ and $w^d_{k_2}(\tau,m,\epsilon)$ are continuous complex functions defined on $S^b_d\times\mathbb{R}\times D(0,\epsilon_0)$ and holomorphic with respect to τ (resp. ϵ) on S^b_d (resp. $D(0,\epsilon_0)$).

Let $\epsilon\in D(0,\epsilon_0)$ and put $\Omega=\min\{\alpha,\nu\}$. It is straight to check that both functions belong to the complex Banach space $H_{(k_2,\beta,\mu,\Omega)}$, of all continuous functions $(\tau,m)\mapsto h(\tau,m)$, defined on $S^b_d\times\mathbb{R}$, holomorphic with respect to τ in S^b_d such that

$$\|h(\tau,m)\|_{H_{(k_2,\beta,\mu,\Omega)}}=\sup_{\tau\in S^b_d,m\in\mathbb{R}}(1+|m|)^{\mu}e^{\beta|m|}\exp\left(-\frac{k_2}{2}\frac{\log^2|\tau|}{\log(q)}-\Omega\log|\tau|\right)|h(\tau,m)|$$

is finite. It holds that $\mathcal{L}^d_{q;1/\kappa}(w^d_{k_1}(\tau,m,\epsilon))$, $w^d_{k_2}(\tau,m,\epsilon)$ and $\Psi_{k_2}(\tau,m,\epsilon)$ belong to $H_{(k_2,\beta,\mu,\Omega)}$ due to Proposition 4.2, Proposition 4.3. As we can see in the proof of Proposition 4.3, the operator $\mathcal{H}^2_\epsilon$ defined in (4.27) has a unique fixed point in $H_{(k_2,\beta,\mu,\Omega)}$ provided small enough constants $\varsigma_\Psi,\varsigma_\varphi>0$, for $1\le\ell\le D-1$. Indeed, this fixed point is a solution of the auxiliary problem (4.25) in the disc $D(0,\varsigma)$ of $H_{(k_2,\beta,\mu,\Omega)}$, whilst $\mathcal{L}^d_{q;1/\kappa}(w^d_{k_1}(\tau,m,\epsilon))$, $w^d_{k_2}(\tau,m,\epsilon)$ are both solutions of the same problem, in the disc $D(0,\varsigma)$ of $H_{(k_2,\beta,\mu,\Omega)}$, so they do coincide in the domain $S^b_d\times\mathbb{R}\times D(0,\epsilon_0)$. Identity (4.34) follows from here. □

5. Analytic solutions to a q-difference-differential equation

This section is devoted to determine in detail the main problem under study, and provide an analytic solution to it. It is worth mentioning that, although the techniques developed in previous sections are essentially novel, once the tools have been implemented, the procedure of construction of the solution coincides with that explained in Section 5 of [10]. For the sake of completeness and a self contained work, we describe every step of the construction in detail, whilst we have decided to pass over the proofs which can be found in [10].

Let $1\le k_1<k_2$. We define $1/\kappa=1/k_1-1/k_2$ and take integers D,D_1,D_2 larger than 3. Let $q>1$ be a real number. We also consider positive integers d_{D_1},d_{D_2}, and for every $1\le\ell\le D-1$ we choose non negative integers $d_\ell,\delta_\ell\ge1$ and $\Delta_\ell\ge0$. We make the following assumptions on the previous constants:

Assumption (A): $\delta_1=1$ and $\delta_\ell<\delta_{\ell+1}$ for every $1\le\ell\le D-2$.

Assumption (B): We have

$$\Delta_\ell\ge d_\ell,\quad \frac{d_{D_1}-1}{\kappa}+\frac{d_\ell}{k_2}+1\ge\delta_\ell,\quad \frac{d_\ell}{k_1}+1\ge\delta_\ell,\quad \frac{d_{D_2}-1}{k_2}\ge\delta_\ell-1,$$

for every $1\le\ell\le D-1$, and

$$k_1(d_{D_2}-1)>k_2d_{D_1}.$$

Let Q,R_{D_1},R_{D_2}, and R_ℓ for $1\le\ell\le D-1$ be polynomials with complex coefficients such that

Assumption (C): $\deg(R_{D_2})=\deg(R_{D_1})$, $\deg(Q)\ge\deg(R_{D_1})\ge\deg(R_\ell)$. Moreover, we assume $Q(im)\neq0$ and $R_{D_j}(im)\neq0$ for all $m\in\mathbb{R}$, $1\le\ell\le D-1$.

Let $S_{Q,R_{D_1}}$ and $S_{Q,R_{D_2}}$ be unbounded sectors of bisecting directions $d_{Q,R_{D_1}}\in\mathbb{R}$ and $d_{Q,R_{D_2}}\in\mathbb{R}$ respectively, with

$$S_{Q,R_{D_j}}=\left\{z\in\mathbb{C}:|z|\ge r_{Q,R_{D_j}},|\arg(z)-d_{Q,R_{D_j}}|\le\nu_{Q,R_{D_j}}\right\},$$

for some $\nu_{Q,R_{D_j}}>0$, and such that

$$\frac{Q(im)}{R_{D_j}(im)}\in S_{Q,R_{D_j}},$$

for every $m\in\mathbb{R}$.

Definition 5.1. Let $\varsigma \geq 2$ be an integer. A family $(\mathcal{E}_p)_{0\leq p\leq \varsigma-1}$ is said to be a good covering in $\mathbb{C}^\star$ (in the ϵ plane) if the next hypotheses hold:

- $\mathcal{E}_p$ is an open sector of finite radius $\epsilon_0>0$, and vertex at the origin for every $0\leq p\leq \varsigma-1$.
- $\mathcal{E}_j\cap\mathcal{E}_k\neq\emptyset$ for $0\leq j,k\leq\varsigma-1$ if and only if $|j-k|\leq 1$ (we put $\mathcal{E}_\varsigma:=\mathcal{E}_0$).
- $\cup_{p=0}^{\varsigma-1}\mathcal{E}_p=\mathcal{U}\setminus\{0\}$ for some neighborhood of the origin $\mathcal{U}$.

Definition 5.2. Let $(\mathcal{E}_p)_{0\leq p\leq\varsigma-1}$ be a good covering. Let $\mathcal{T}$ be an open bounded sector with vertex at the origin and radius $r_{\mathcal{T}}>0$. Given $\alpha\in\mathbb{R}$ and $\nu\in\mathbb{R}$ we assume that

$$0<\epsilon_0,r_{\mathcal{T}}<1,\quad \nu+\frac{k_2}{\log(q)}\log(r_{\mathcal{T}})<0,\quad \alpha+\frac{\kappa}{\log(q)}\log(\epsilon_0 r_{\mathcal{T}})<0,\quad \epsilon_0 r_{\mathcal{T}}\leq q^{\left(\frac{1}{2}-\nu\right)/k_2}/2.$$

We consider a family of unbounded sectors $U_{\mathfrak{d}_p}$, $0\leq p\leq\varsigma-1$, with bisecting direction $\mathfrak{d}_p\in\mathbb{R}$, and a family of open domains $\mathcal{R}^b_{\mathfrak{d}_p}:=\mathcal{R}_{\mathfrak{d}_p,\tilde{\delta}}\cap D(0,\epsilon_0 r_{\mathcal{T}})$, with

$$\mathcal{R}_{\mathfrak{d}_p,\tilde{\delta}}:=\left\{T\in\mathbb{C}^\star:\left|1+\frac{re^{i\mathfrak{d}_p}}{T}\right|>\tilde{\delta},\text{ for every } r\geq 0\right\}.$$

We assume $\mathfrak{d}_p$, $0\leq p\leq\varsigma-1$ is chosen to satisfy the following conditions: there exist $S_{\mathfrak{d}_p}\cup\overline{D}(0,\rho)$ and $\rho>0$ such that

- Conditions 1), 2), Page 14 in Section 4.1 hold. Observe that, under this assumption, Conditions 1), 2), Page 18 in Section 4.2 hold for $S_{\mathfrak{d}_p}$.
- For every $0\leq p\leq\varsigma-1$ we have $\mathcal{R}^b_{\mathfrak{d}_p}\cap\mathcal{R}^b_{\mathfrak{d}_{p+1}}\neq\emptyset$, and for every $t\in\mathcal{T}$ and $\epsilon\in\mathcal{E}_p$ we have $\epsilon t\in\mathcal{R}^b_{\mathfrak{d}_p}$ (where $\mathcal{R}_{\mathfrak{d}_\varsigma}:=\mathcal{R}_{\mathfrak{d}_0}$).

The family $\{(\mathcal{R}_{\mathfrak{d}_p,\tilde{\delta}})_{0\leq p\leq\varsigma-1},D(0,\rho),\mathcal{T}\}$ is said to be associated to the good covering $(\mathcal{E}_p)_{0\leq p\leq\varsigma-1}$.

Let $(\mathcal{E}_p)_{0\leq p\leq\varsigma-1}$ be a good covering, and a family $\{(\mathcal{R}_{\mathfrak{d}_p,\tilde{\delta}})_{0\leq p\leq\varsigma-1},D(0,\rho),\mathcal{T}\}$ associated to it. For every $0\leq p\leq\varsigma-1$ we study the following equation

$$(5.1)\quad Q(\partial_z)\sigma_{q,t}u^{\mathfrak{d}_p}(t,z,\epsilon)\\ =(\epsilon t)^{d_{D_1}}\sigma_{q,t}^{\frac{d_{D_1}}{k_1}+1}R_{D_1}(\partial_z)u^{\mathfrak{d}_p}(t,z,\epsilon)+(\epsilon t)^{d_{D_2}}\sigma_{q,t}^{\frac{d_{D_2}}{k_2}+1}R_{D_2}(\partial_z)u^{\mathfrak{d}_p}(t,z,\epsilon)\\ +\sum_{\ell=1}^{D-1}\epsilon^{\Delta_\ell}t^{d_\ell}\sigma_{q,t}^{\delta_\ell}(c_\ell(t,z,\epsilon)R_\ell(\partial_z)u^{\mathfrak{d}_p}(t,z,\epsilon))+\sigma_{q,t}f(t,z,\epsilon).$$

The terms $c_\ell(t,z,\epsilon)$ are determined as follows, for every $1\leq\ell\leq D-1$. Let $C_\ell(T,m,\epsilon)$ be the entire function in T, with coefficients in $E_{(\beta,\mu)}$ for some $\beta>0$ and $\mu\in\mathbb{R}$, given by

$$C_\ell(T,m,\epsilon)=\sum_{n\geq 0}C_{\ell,n}(m,\epsilon)T^n,$$

such that $\mu-1\geq\deg(R_{D_j})$, for $j\in\{1,2\}$. Assume this function depends holomorphically on $\epsilon\in D(0,\epsilon_0)$and also the existence of $\tilde{C}_\ell,T_0>0$ such that the left-hand side of (4.6) holds for all $n\geq 0$ and $\epsilon\in D(0,\epsilon_0)$. We put

$$c_\ell(t,z,\epsilon):=\mathcal{F}^{-1}\left(m\mapsto C_\ell(\epsilon t,m,\epsilon)\right)(z),$$

which is a holomorphic and bounded function on $\mathcal{T}\times H_{\beta'}\times D(0,\epsilon_0)$. Indeed, one can substitute $\mathcal{T}$ by any bounded set in $\mathbb{C}$ in the previous product domain.

The function $f(t,z,\epsilon)$ is constructed as follows. Let $m\mapsto F_n(m,\epsilon)$ be a function in $E_{(\beta,\mu)}$ for every $n\geq 0$, depending holomorphically on $\epsilon\in D(0,\epsilon_0)$. We also assume there exist C_F,T_0 such that (4.6) holds and define $\hat{F}(T,m,\epsilon)=\sum_{n\geq 0}F_nT^n$.

By construction, $\hat{F}(T,m,\epsilon)$ represents a holomorphic function in T on the disc $D(0,T_0/2)$ with values in the Banach space $E_{(\beta,\mu)}$, for all $\epsilon \in D(0,\epsilon_0)$. We define

$$f(t,z,\epsilon) = \mathcal{F}^{-1}(m \mapsto \hat{F}(\epsilon t,m,\epsilon))(z)$$

which stands for a holomorphic and bounded function on $D(0,\epsilon_0 T_0/2) \times H_{\beta'} \times D(0,\epsilon_0)$, for all $0 < \beta' < \beta$.

Theorem 5.3. *Under the construction made at the beginning of this section of the elements involved in the problem (5.1), assume that the above conditions hold. Let $(\mathcal{E}_p)_{0\leq p\leq \varsigma-1}$ be a good covering in $\mathbb{C}^\star$, for which a family $\{(\mathcal{R}_{\mathfrak{d}_p,\tilde{\delta}})_{0\leq p\leq\varsigma-1}, D(0,\rho), \mathcal{T}\}$ associated to this covering is considered.*

Then, there exist large enough $r_{Q,R_{D_1}}, r_{Q,R_{D_2}} > 0$ and constants $\varsigma_\Psi > 0$ and $\varsigma_\varphi > 0$ such that if

$$\tilde{C}_\ell \leq \varsigma_\varphi, \qquad C_{\Psi_1} \leq \varsigma_\psi$$

for all $1 \leq \ell \leq D-1$, then for every $0 \leq p \leq \varsigma - 1$, one can construct a solution $u^{\mathfrak{d}_p}(t,z,\epsilon)$ of (5.1), which defines a holomorphic function on $\mathcal{T} \times H_{\beta'} \times \mathcal{E}_p$, for every $0 < \beta' < \beta$.

Proof. Let $0 \leq p \leq \varsigma - 1$ and consider the equation

$$Q(im)\sigma_{q,T}U^{\mathfrak{d}_p}(T,m,\epsilon) = T^{d_{D_1}}\sigma_{q,T}^{\frac{d_{D_1}}{k_1}+1}R_{D_1}(im)U^{\mathfrak{d}_p}(T,m,\epsilon) + T^{d_{D_2}}\sigma_{q,T}^{\frac{d_{D_2}}{k_2}+1}R_{D_2}(im)U^{\mathfrak{d}_p}(T,m,\epsilon)$$
$$+\sum_{\ell=1}^{D-1}\epsilon^{\Delta_\ell-d_\ell}T^{d_\ell}\sigma_{q,T}^{\delta_\ell}\left(\frac{1}{(2\pi)^{1/2}}\int_{-\infty}^{+\infty}C_\ell(T,m-m_1,\epsilon)R_\ell(im_1)U^{\mathfrak{d}_p}(T,m_1,\epsilon)dm_1\right)$$
$$+\sigma_{q,T}\hat{F}(T,m,\epsilon). \tag{5.2}$$

Under an appropriate choice of the constants ς_Ψ and ς_φ one can follow the construction in Section 4.1 and apply Proposition 4.2 to obtain a solution $U^{\mathfrak{d}_p}(T,m,\epsilon)$ of (5.2).

Regarding the properties of q-Laplace transformation, and from the results obtained in Section 4.2, $U^{\mathfrak{d}_p}(T,m,\epsilon)$ is the q-Laplace transformation of order k_2 of a function $w_{k_2}^{\mathfrak{d}_p}$ along direction $\mathfrak{d}_p$, which depends on T. Indeed,

$$U^{\mathfrak{d}_p}(T,m,\epsilon) = \frac{1}{\pi_{q^{1/k_2}}}\int_{L_{\mathfrak{d}_p}}\frac{w_{k_2}^{\mathfrak{d}_p}(u,m,\epsilon)}{\Theta_{q^{1/k_2}}\left(\frac{u}{T}\right)}\frac{du}{u}, \tag{5.3}$$

for some $L_{\mathfrak{d}_p} \subseteq S_{\mathfrak{d}_p} \cup \{0\}$, and $w_{k_2}^{\mathfrak{d}_p}(\tau,m,\epsilon)$ defines a continuous function on $S_{\mathfrak{d}_p} \times \mathbb{R} \times D(0,\epsilon_0)$, and holomorphic with respect to (τ,ϵ) in $S_{\mathfrak{d}_p} \times D(0,\epsilon_0)$. In addition to this, there exists $C_{w_{k_2}^{\mathfrak{d}_p}} > 0$ such that

$$|w_{k_2}^{\mathfrak{d}_p}(\tau,m,\epsilon)| \leq C_{w_{k_2}^{\mathfrak{d}_p}}\frac{1}{(1+|m|)^\mu}e^{-\beta|m|}\exp\left(\frac{k_2}{2\log(q)}\log^2|\tau| + \nu\log|\tau|\right), \tag{5.4}$$

for some $\nu \in \mathbb{R}$. This holds for $\tau \in S_{\mathfrak{d}_p}$, $m \in \mathbb{R}$, and $\epsilon \in D(0,\epsilon_0)$. Moreover, in view of Proposition 4.4, the function $w_{k_2}^{\mathfrak{d}_p}(\tau,m,\epsilon)$ and the q-Laplace transformation of order κ of the function $w_{k_1}^{\mathfrak{d}_p}(\tau,m,\epsilon)$ along direction $\mathfrak{d}_p^1$, where $e^{i\mathfrak{d}_p^1}\mathbb{R}_+ \subseteq S_{\mathfrak{d}_p} \cup \{0\}$, depending on τ, coincide in $(S_{\mathfrak{d}_p} \cap D(0,r_1)) \times \mathbb{R} \times D(0,\epsilon_0)$, for $0 < r_1 \leq q^{\left(\frac{1}{2}-\alpha\right)/\kappa}/2$, for some $\alpha \in \mathbb{R}$. The function $w_{k_1}^{\mathfrak{d}_p}(\tau,m,\epsilon)$ is such that

$$|w_{k_1}^{\mathfrak{d}_p}(\tau,m,\epsilon)| \leq C_{w_{k_1}^{\mathfrak{d}_p}}\frac{1}{(1+|m|)^\mu}e^{-\beta|m|}\exp\left(\frac{\kappa}{2\log(q)}\log^2|\tau+\delta| + \alpha\log|\tau+\delta|\right), \tag{5.5}$$

for some $C_{w_{k_1}^{\mathfrak{d}_p}},\delta>0$, valid for $\tau\in(D(0,\rho)\cup U_{\mathfrak{d}_p})$, $m\in\mathbb{R}$ and $\epsilon\in D(0,\epsilon_0)$. This function is the extension of a function $w_{k_1}(\tau,m,\epsilon)$, common for every $0\le p\le\varsigma-1$, continuous on $\overline{D}(0,\rho)\times\mathbb{R}\times D(0,\epsilon_0)$ and holomorphic with respect to (τ,ϵ) in $D(0,\rho)\times D(0,\epsilon_0)$.

The bounds in (5.4) with respect to m variable are transmitted to $U^{\mathfrak{d}_p}(T,m,\epsilon)$ as defined in (5.3). This allows to define the function

$$u^{\mathfrak{d}_p}(t,z,\epsilon):=\mathcal{F}^{-1}(m\mapsto U^{\mathfrak{d}_p}(\epsilon t,m,\epsilon))(z)$$
$$=\frac{1}{(2\pi)^{1/2}}\frac{1}{\pi_q^{1/k_2}}\int_{-\infty}^{\infty}\int_{L_{\mathfrak{d}_p}}\frac{w_{k_2}^{\mathfrak{d}_p}(u,m,\epsilon)}{\Theta_{q^{1/k_2}}\left(\frac{u}{\epsilon t}\right)}\frac{du}{u}\exp(izm)dm,$$

which turns out to be holomorphic on $\mathcal{T}\times H_{\beta'}\times\mathcal{E}_p$. The properties of inverse Fourier transform allow us to conclude that $u^{\mathfrak{d}_p}(t,z,\epsilon)$ is a solution of equation (5.1) defined on $\mathcal{T}\times H_{\beta'}\times\mathcal{E}_p$. □

Proposition 5.4. *Let $0\le p\le\varsigma-1$. Under the hypotheses of Theorem 5.3, assume that the unbounded sectors $U_{\mathfrak{d}_p}$ and $U_{\mathfrak{d}_{p+1}}$ are wide enough so that $U_{\mathfrak{d}_p}\cap U_{\mathfrak{d}_{p+1}}$ contains the sector $U_{\mathfrak{d}_p,\mathfrak{d}_{p+1}}=\{\tau\in\mathbb{C}^\star:\arg(\tau)\in[\mathfrak{d}_p,\mathfrak{d}_{p+1}]\}$. Then, there exist $K_1>0$ and $K_2\in\mathbb{R}$ such that*

$$|u^{\mathfrak{d}_{p+1}}(t,z,\epsilon)-u^{\mathfrak{d}_p}(t,z,\epsilon)|\le K_1\exp\left(-\frac{k_2}{2\log(q)}\log^2|\epsilon|\right)|\epsilon|^{K_2}, \tag{5.6}$$

for every $t\in\mathcal{T}$, $z\in H_{\beta'}$, and $\epsilon\in\mathcal{E}_p\cap\mathcal{E}_{p+1}$.

Proof. Let $0\le p\le\varsigma-1$. Taking into account that $U_{\mathfrak{d}_p,\mathfrak{d}_{p+1}}\subseteq U_{\mathfrak{d}_p}\cap U_{\mathfrak{d}_{p+1}}$, we observe from the construction of the functions $U^{\mathfrak{d}_p}$ and $U^{\mathfrak{d}_{p+1}}$ that $\mathcal{L}_{q;1/\kappa}^{\mathfrak{d}_p}(w_{k_1}^{\mathfrak{d}_p})(\tau,m,\epsilon)$ and $\mathcal{L}_{q;1/\kappa}^{\mathfrak{d}_{p+1}}(w_{k_1}^{\mathfrak{d}_{p+1}})(\tau,m,\epsilon)$ coincide in the domain $(\mathcal{R}_{\mathfrak{d}_p}^b\cap\mathcal{R}_{\mathfrak{d}_{p+1}}^b)\times\mathbb{R}\times D(0,\epsilon_0)$. This entails the existence of $w_{k_2}^{\mathfrak{d}_p,\mathfrak{d}_{p+1}}(\tau,m,\epsilon)$, holomorphic with respect to τ on $\mathcal{R}_{\mathfrak{d}_p}^b\cup\mathcal{R}_{\mathfrak{d}_{p+1}}^b$, continuous with respect to $m\in\mathbb{R}$ and holomorphic with respect to ϵ in $D(0,\epsilon_0)$ which coincides with $\mathcal{L}_{q;1/\kappa}^{\mathfrak{d}_p}(w_{k_1}^{\mathfrak{d}_p})(\tau,m,\epsilon)$ on $\mathcal{R}_{\mathfrak{d}_p}^b\times\mathbb{R}\times D(0,\epsilon_0)$ and also with $\mathcal{L}_{q;1/\kappa}^{\mathfrak{d}_{p+1}}(w_{k_1}^{\mathfrak{d}_{p+1}})(\tau,m,\epsilon)$ on $\mathcal{R}_{\mathfrak{d}_{p+1}}^b\times\mathbb{R}\times D(0,\epsilon_0)$.

Let $\widetilde{\rho}>0$ be such that $\widetilde{\rho}e^{i\mathfrak{d}_p}\subseteq\mathcal{R}_{\mathfrak{d}_p}^b$ and $\widetilde{\rho}e^{i\mathfrak{d}_{p+1}}\subseteq\mathcal{R}_{\mathfrak{d}_{p+1}}^b$. The function

$$u\mapsto\frac{w_{k_2}^{\mathfrak{d}_p,\mathfrak{d}_{p+1}}(u,m,\epsilon)}{\Theta_{q^{1/k_2}}\left(\frac{u}{\epsilon t}\right)}$$

is holomorphic on $\mathcal{R}_{\mathfrak{d}_p}^b\cup\mathcal{R}_{\mathfrak{d}_{p+1}}^b$ for all $(m,\epsilon)\in\mathbb{R}\times(\mathcal{E}_p\cap\mathcal{E}_{p+1})$ and its integral along the closed path constructed by concatenation of the segment starting at the origin and with ending point fixed at $\widetilde{\rho}e^{i\mathfrak{d}_p}$, the arc of circle with radius $\widetilde{\rho}$ connecting $\widetilde{\rho}e^{i\mathfrak{d}_p}$ with $\widetilde{\rho}e^{i\mathfrak{d}_{p+1}}\subseteq\mathcal{R}_{\mathfrak{d}_{p+1}}^b$, and the segment from $\widetilde{\rho}e^{i\mathfrak{d}_{p+1}}$ to 0, vanishes. The difference $u^{\mathfrak{d}_{p+1}}-u^{\mathfrak{d}_p}$ can be written in the form

$$\begin{aligned}
&u^{\mathfrak{d}_{p+1}}(t,z,\epsilon)-u^{\mathfrak{d}_p}(t,z,\epsilon)\\
&=\frac{1}{(2\pi)^{1/2}}\frac{1}{\pi_{q^{1/k_2}}}\int_{-\infty}^{\infty}\int_{L_{\mathfrak{d}_{p+1},\widetilde{\rho}}}\frac{w_{k_2}^{\mathfrak{d}_{p+1}}(u,m,\epsilon)}{\Theta_{q^{1/k_2}}\left(\frac{u}{\epsilon t}\right)}\exp(izm)\frac{du}{u}dm,\\
&\quad-\frac{1}{(2\pi)^{1/2}}\frac{1}{\pi_{q^{1/k_2}}}\int_{-\infty}^{\infty}\int_{L_{\mathfrak{d}_p,\widetilde{\rho}}}\frac{w_{k_2}^{\mathfrak{d}_p}(u,m,\epsilon)}{\Theta_{q^{1/k_2}}\left(\frac{u}{\epsilon t}\right)}\exp(izm)\frac{du}{u}dm\\
&\quad+\frac{1}{(2\pi)^{1/2}}\frac{1}{\pi_{q^{1/k_2}}}\int_{-\infty}^{\infty}\int_{\mathcal{C}_{\widetilde{\rho},\mathfrak{d}_p,\mathfrak{d}_{p+1}}}\frac{w_{k_2}^{\mathfrak{d}_p,\mathfrak{d}_{p+1}}(u,m,\epsilon)}{\Theta_{q^{1/k_2}}\left(\frac{u}{\epsilon t}\right)}\exp(izm)\frac{du}{u}dm,
\end{aligned}\tag{5.7}$$

where $L_{\mathfrak{d}_j,\widetilde{\rho}}=[\widetilde{\rho},+\infty)e^{i\mathfrak{d}_j}$ for $j\in\{p,p+1\}$ and $\mathcal{C}_{\widetilde{\rho},\mathfrak{d}_p,\mathfrak{d}_{p+1}}$ is the arc of circle connecting $\widetilde{\rho}e^{i\mathfrak{d}_p}$ with $\widetilde{\rho}e^{i\mathfrak{d}_{p+1}}$ (see Figure 2).

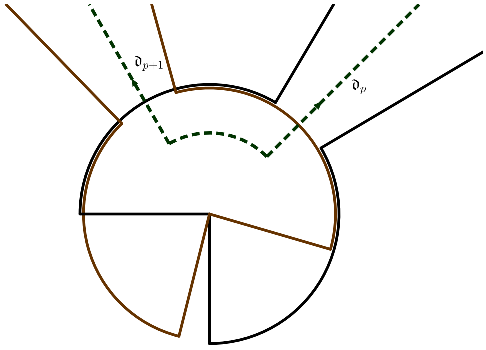

FIGURE 2. Deformation of the path of integration, first case.

Let us put

$$I_1 := \left| \frac{1}{(2\pi)^{1/2}} \frac{1}{\pi_{q^{1/k_2}}} \int_{-\infty}^{\infty} \int_{L_{\mathfrak{d}_{p+1},\tilde{\rho}}} \frac{w_{k_2}^{\mathfrak{d}_{p+1}}(u,m,\epsilon)}{\Theta_{q^{1/k_2}}\left(\frac{u}{\epsilon t}\right)} \exp(izm) \frac{du}{u} dm \right|.$$

In view of (5.4) and (3.2), one has

$$I_1 \leq \frac{\tilde{C}_{w_{k_2}^{\mathfrak{d}_{p+1}}}}{C_{q,k_2}\tilde{\delta}(2\pi)^{1/2}} \frac{|\epsilon t|^{1/2}}{\pi_{q^{1/k_2}}} \int_{-\infty}^{\infty} e^{-\beta|m|-m\Im(z)} \frac{dm}{(1+|m|)^{\mu}}$$
$$\times \int_{\tilde{\rho}}^{\infty} \exp\left(\frac{k_2 \log^2|u|}{2\log(q)} + \nu \log|u|\right) |u|^{-3/2} \exp\left(-\frac{k_2 \log^2\left(\frac{|u|}{|\epsilon t|}\right)}{2\log(q)}\right) d|u|.$$

We recall that we have restricted the domain on the variable z such that $|\Im(z)| \leq \beta' < \beta$. Then, the first integral in the previous expression in convergent, and one derives

$$I_1 \leq \frac{\tilde{C}_{w_{k_2}^{\mathfrak{d}_{p+1}}}}{(2\pi)^{1/2}} \frac{(\epsilon_0 r_{\mathcal{T}})^{1/2}}{\pi_{q^{1/k_2}}} \int_{\tilde{\rho}}^{\infty} \exp\left(\frac{k_2\log^2|u|}{2\log(q)}\right) \exp\left(-\frac{k_2\log^2\left(\frac{|u|}{|\epsilon t|}\right)}{2\log(q)}\right) |u|^{\nu-3/2} d|u|,$$

for some $\tilde{C}_{w_{k_2}^{\mathfrak{d}_{p+1}}} > 0$. We derive

$$\exp\left(\frac{k_2\log^2|u|}{2\log(q)}\right) \exp\left(-\frac{k_2\log^2\left(\frac{|u|}{|\epsilon t|}\right)}{2\log(q)}\right) = \exp\left(\frac{k_2}{2\log(q)}(-\log^2|\epsilon| - 2\log|\epsilon|\log|t| - \log^2|t|)\right)$$
$$\times \exp\left(\frac{k_2}{\log(q)}(\log|u|\log|\epsilon| + \log|u|\log|t|)\right).$$

From the assumption that $0 < \epsilon_0 < 1$ and $0 < r_{\mathcal{T}} < 1$, we get

$$\exp\left(-\frac{k_2}{\log(q)}\log|\epsilon|\log|t|\right) \leq |\epsilon|^{-\frac{k_2}{\log(q)}\log(r_{\mathcal{T}})}, \tag{5.8}$$
$$\exp\left(\frac{k_2}{\log(q)}\log|u|\log|\epsilon|\right) \leq |\epsilon|^{\frac{k_2}{\log(q)}\log(\tilde{\rho})},$$

for $t \in \mathcal{T}$, $\epsilon \in \mathcal{E}_p \cap \mathcal{E}_{p+1}$, $|u| \geq \widetilde{\rho}$, and also

$$\exp\left(\frac{k_2}{\log(q)} \log|u| \log|t|\right) \leq |t|^{\frac{k_2}{\log(q)}\log(\widetilde{\rho})}, \text{ if } \widetilde{\rho} \leq |u| \leq 1$$
$$\exp\left(\frac{k_2}{\log(q)} \log|u| \log|t|\right) \leq |u|^{\frac{k_2}{\log(q)}\log(r_{\mathcal{T}})}, \text{ if } |u| \geq 1, \tag{5.9}$$

for $t \in \mathcal{T}$. In addition to that, there exists $K_{k_2,\widetilde{\rho},q} > 0$ such that

$$\sup_{x>0} x^{\frac{k_2}{\log(q)}\log(\widetilde{\rho})} \exp\left(-\frac{k_2}{2\log(q)}\log^2(x)\right) \leq K_{k_2,\widetilde{\rho},q}. \tag{5.10}$$

In view of (5.8), (5.9), (5.10), and bearing in mind that the inequalities of Definition 5.2 hold, we deduce there exist $\tilde{K}^1 \in \mathbb{R}$, $\tilde{K}^2 > 0$ such that

$$\exp\left(\frac{k_2 \log^2|u|}{2\log(q)}\right)\exp\left(-\frac{k_2\log^2\left(\frac{|u|}{|\epsilon t|}\right)}{2\log(q)}\right)|u|^{\nu} \leq \tilde{K}^2 \exp\left(-\frac{k_2}{2\log(q)}\log^2|\epsilon|\right)|\epsilon|^{\tilde{K}^1},$$

for $t \in \mathcal{T}$, $r \geq \widetilde{\rho}$, and $\epsilon \in \mathcal{E}_p \cap \mathcal{E}_{p+1}$. Provided this last inequality, we arrive at

$$I_1 \leq \frac{\tilde{K}^2 C_{w_{k_2}^{\mathfrak{d}_{p+1}}}}{C_{q,k_2}\tilde{\delta}(2\pi)^{1/2}} \frac{(\epsilon_0 r_{\mathcal{T}})^{1/2}}{\pi_{q^{1/k_2}}} \int_{\widetilde{\rho}}^{\infty} \frac{d|u|}{|u|^{3/2}} \exp\left(-\frac{k_2}{2\log(q)}\log^2|\epsilon|\right)|\epsilon|^{\tilde{K}^1}$$
$$= \tilde{K}^3 \exp\left(-\frac{k_2}{2\log(q)}\log^2|\epsilon|\right)|\epsilon|^{\tilde{K}^1}, \tag{5.11}$$

for some $\tilde{K}^3 > 0$, for all $t \in \mathcal{T}$, $z \in H_{\beta'}$, and $\epsilon \in \mathcal{E}_p \cap \mathcal{E}_{p+1}$.

We can estimate in the same manner the expression

$$I_2 := \left| \frac{1}{(2\pi)^{1/2}} \frac{1}{\pi_{q^{1/k_2}}} \int_{-\infty}^{\infty} \int_{L_{\mathfrak{d}_p,\widetilde{\rho}}} \frac{w_{k_2}^{\mathfrak{d}_p}(u,m,\epsilon)}{\Theta_{q^{1/k_2}}\left(\frac{u}{\epsilon t}\right)} \exp(izm)\frac{du}{u}dm \right|.$$

to arrive at the existence of $\tilde{K}^4 > 0$ such that

$$I_2 \leq \tilde{K}^4 \exp\left(-\frac{k_2}{2\log(q)}\log^2|\epsilon|\right)|\epsilon|^{\tilde{K}^1}, \tag{5.12}$$

for all $t \in \mathcal{T}$, $z \in H_{\beta'}$, and $\epsilon \in \mathcal{E}_p \cap \mathcal{E}_{p+1}$. We now provide upper bounds for the quantity

$$I_3 := \left| \frac{1}{(2\pi)^{1/2}} \frac{1}{\pi_{q^{1/k_2}}} \int_{-\infty}^{\infty} \int_{\mathcal{C}_{\widetilde{\rho},\mathfrak{d}_p,\mathfrak{d}_{p+1}}} \frac{w_{k_2}^{\mathfrak{d}_p,\mathfrak{d}_{p+1}}(u,m,\epsilon)}{\Theta_{q^{1/k_2}}\left(\frac{u}{\epsilon t}\right)} \exp(izm)\frac{du}{u}dm \right|.$$

From the construction of $w_{k_2}^{\mathfrak{d}_p,\mathfrak{d}_{p+1}}(\tau,m,\epsilon)$, we have

$$|w_{k_2}^{\mathfrak{d}_p,\mathfrak{d}_{p+1}}(u,m,\epsilon)| \leq \tilde{C}_{w_{k_1}^{\mathfrak{d}_p}} \frac{1}{(1+|m|)^{\mu}} e^{-\beta|m|},$$

for some $\tilde{C}_{w_{k_1}^{\mathfrak{d}_p}} > 0$, valid for $u \in \mathcal{C}_{\widetilde{\rho},\mathfrak{d}_p,\mathfrak{d}_{p+1}}$, $m \in \mathbb{R}$ and $\epsilon \in D(0,\epsilon_0)$.

The estimates with (3.2) allow us to obtain the existence of $\tilde{C}_{w_{k_2}}^{\mathfrak{d}_p,\mathfrak{d}_{p+1}} > 0$ such that

$$I_3 \leq \tilde{C}_{w_{k_2}}^{\mathfrak{d}_p,\mathfrak{d}_{p+1}} \int_{-\infty}^{\infty} \frac{e^{-\beta|m| - m\Im(z)}}{(1+|m|)^{\mu}} dm |\mathfrak{d}_{p+1} - \mathfrak{d}_p| |t|^{1/2} \exp\left(-\frac{k_2\log^2\left(\frac{\widetilde{\rho}}{|\epsilon t|}\right)}{2\log(q)}\right),$$

for all $t \in \mathcal{T}$, $z \in H_{\beta'}$, and $\epsilon \in \mathcal{E}_p \cap \mathcal{E}_{p+1}$. We can follow analogous arguments as in the previous steps to provide upper estimates of the expression

$$|t|^{1/2} \exp\left(-\frac{k_2 \log^2\left(\frac{\tilde{\rho}}{|\epsilon t|}\right)}{2\log(q)}\right).$$

Indeed,

$$|t|^{1/2} \exp\left(-\frac{k_2 \log^2\left(\frac{\tilde{\rho}}{|\epsilon t|}\right)}{2\log(q)}\right) = \exp\left(-\frac{k_2 \log^2(\tilde{\rho})}{2\log(q)}\right) |\epsilon|^{\frac{k_2 \log(\tilde{\rho})}{\log(q)}} |t|^{\frac{k_2 \log(\tilde{\rho})}{\log(q)}}$$
$$\times \exp\left(\frac{k_2}{2\log(q)}(-\log^2|\epsilon| - 2\log|\epsilon|\log|t| - \log^2|t|)\right)|t|^{1/2}.$$

From the assumption $0 \leq \epsilon_0 < 1$ we check that

$$\exp\left(-\frac{k_2}{\log(q)}\log|\epsilon|\log|t|\right) \leq |\epsilon|^{-\frac{k_2}{\log(q)}\log(r_{\mathcal{T}})},$$

for $t \in \mathcal{T}$, $\epsilon \in \mathcal{E}_p \cap \mathcal{E}_{p+1}$. Gathering (5.10), we get the existence of $\tilde{K}^5 \in \mathbb{R}$, $\tilde{K}^6 > 0$ such that

$$|t|^{1/2} \exp\left(-\frac{k_2 \log^2\left(\frac{\tilde{\rho}}{|\epsilon t|}\right)}{2\log(q)}\right) \leq \tilde{K}^6 \exp\left(-\frac{k_2}{2\log(q)}\log^2|\epsilon|\right)|\epsilon|^{\tilde{K}^5},$$

to conclude that

$$I_3 \leq \tilde{K}^7 \exp\left(-\frac{k_2}{2\log(q)}\log^2|\epsilon|\right)|\epsilon|^{\tilde{K}^5}, \tag{5.13}$$

for some $\tilde{K}^7 > 0$, all $t \in \mathcal{T}$, $z \in H_{\beta'}$, and $\epsilon \in \mathcal{E}_p \cap \mathcal{E}_{p+1}$. We conclude the proof of this result in view of (5.11), (5.12), (5.13) and the decomposition (5.7).

□

Lemma 5.5. *Let $0 \leq p \leq \varsigma - 1$. Under the hypotheses of Theorem 5.3, assume that $U_{\mathfrak{d}_p} \cap U_{\mathfrak{d}_{p+1}} = \emptyset$. Then, there exist $K_p^{\mathcal{L}} > 0$, $M_p^{\mathcal{L}} \in \mathbb{R}$ such that*

$$\left|\mathcal{L}^{\mathfrak{d}_{p+1}}_{q;1/\kappa}(w^{\mathfrak{d}_{p+1}}_{k_1})(\tau,m,\epsilon) - \mathcal{L}^{\mathfrak{d}_p}_{q;1/\kappa}(w^{\mathfrak{d}_p}_{k_1})(\tau,m,\epsilon)\right|$$
$$\leq K_p^{\mathcal{L}} e^{-\beta|m|}(1+|m|)^{-\mu}\exp\left(-\frac{\kappa}{2\log(q)}\log^2|\tau|\right)|\tau|^{M_p^{\mathcal{L}}},$$

for every $\epsilon \in (\mathcal{E}_p \cap \mathcal{E}_{p+1})$, $\tau \in (\mathcal{R}^b_{\mathfrak{d}_p} \cap \mathcal{R}^b_{\mathfrak{d}_{p+1}})$ and $m \in \mathbb{R}$.

Proof. We first recall that, without loss of generality, the intersection $\mathcal{R}^d_{\mathfrak{d}_p} \cap \mathcal{R}^d_{\mathfrak{d}_{p+1}}$ can be assumed to be a nonempty set because one can vary $\tilde{\delta}$ in advance to be as close to 0 as desired.

Analogous arguments as in the beginning of the proof of Proposition 5.4 allow us to write

$$\begin{aligned}\mathcal{L}_{q;1/\kappa}^{\mathfrak{d}_{p+1}}(w_{k_1}^{\mathfrak{d}_{p+1}})(\tau,m,\epsilon)-\mathcal{L}_{q;1/\kappa}^{\mathfrak{d}_{p}}(w_{k_1}^{\mathfrak{d}_{p}})(\tau,m,\epsilon)\\ =\frac{1}{\pi_{q^{1/\kappa}}}\int_{L_{\mathfrak{d}_{p+1},\tilde{\rho}}}\frac{w_{k_1}^{\mathfrak{d}_{p+1}}(u,m,\epsilon)}{\Theta_{q^{1/\kappa}}\left(\frac{u}{\tau}\right)}\frac{du}{u},\\ -\frac{1}{\pi_{q^{1/\kappa}}}\int_{L_{\mathfrak{d}_{p},\tilde{\rho}}}\frac{w_{k_1}^{\mathfrak{d}_{p}}(u,m,\epsilon)}{\Theta_{q^{1/\kappa}}\left(\frac{u}{\tau}\right)}\frac{du}{u}\\ +\frac{1}{\pi_{q^{1/\kappa}}}\int_{\mathcal{C}_{\tilde{\rho},\mathfrak{d}_p,\mathfrak{d}_{p+1}}}\frac{w_{k_1}(u,m,\epsilon)}{\Theta_{q^{1/\kappa}}\left(\frac{u}{\tau}\right)}\frac{du}{u},\end{aligned}\tag{5.14}$$

where $\tilde{\rho}$, $L_{\mathfrak{d}_p,\tilde{\rho}}$, $L_{\mathfrak{d}_{p+1},\tilde{\rho}}$ and $\mathcal{C}_{\tilde{\rho},\mathfrak{d}_p,\mathfrak{d}_{p+1}}$ are constructed in Proposition 5.4.

In view of (5.5) and (3.2), one has

$$\begin{aligned}I_1^{\mathcal{L}}:=\left|\frac{1}{\pi_{q^{1/\kappa}}}\int_{L_{\mathfrak{d}_p,\tilde{\rho}}}\frac{w_{k_1}^{\mathfrak{d}_p}(u,m,\epsilon)}{\Theta_{q^{1/\kappa}}\left(\frac{u}{\tau}\right)}\frac{du}{u}\right|\\ \le\frac{C_{w_{k_1}^{\mathfrak{d}_p}}}{C_{q,\kappa}\tilde{\delta}}\frac{|\tau|^{1/2}}{(1+|m|)^{\mu}}e^{-\beta|m|}\int_{\tilde{\rho}}^{\infty}\frac{\exp\left(\frac{\kappa\log^2|re^{i\mathfrak{d}_p}+\delta|}{2\log(q)}+\alpha\log|re^{i\mathfrak{d}_p}+\delta|\right)}{\exp\left(\frac{\kappa}{2}\frac{\log^2\left(\frac{r}{|\tau|}\right)}{\log(q)}\right)}\frac{dr}{r^{3/2}}\\ \le K_{p,1}^{\mathcal{L}}|\tau|^{1/2}(1+|m|)^{-\mu}e^{-\beta|m|}\int_{\tilde{\rho}}^{\infty}\frac{\exp\left(\frac{\kappa\log^2 r}{2\log(q)}+\alpha\log r\right)}{\exp\left(\frac{\kappa}{2}\frac{\log^2\left(\frac{r}{|\tau|}\right)}{\log(q)}\right)}\frac{dr}{r^{3/2}}\end{aligned}$$

for some $K_{p,1}^{\mathcal{L}}>0$. Usual calculations, and taking into account the choice of α in Definition 5.2, one derives the previous expression equals

$$K_{p,1}^{\mathcal{L}}|\tau|^{1/2}(1+|m|)^{-\mu}e^{-\beta|m|}\exp\left(-\frac{\kappa}{2\log(q)}\log^2|\tau|\right)\int_{\tilde{\rho}}^{\infty}r^{\frac{\kappa\log|\tau|}{\log(q)}+\alpha-3/2}dr,$$

which yields

$$I_1^{\mathcal{L}}\le K_{p,2}^{\mathcal{L}}(1+|m|)^{-\mu}e^{-\beta|m|}\exp\left(-\frac{\kappa}{2\log(q)}\log^2|\tau|\right),\tag{5.15}$$

for some $K_{p,2}^{\mathcal{L}}>0$. Analogous arguments allow us to obtain the existence of $K_{p,3}^{\mathcal{L}}>0$ such that

$$\begin{aligned}I_2^{\mathcal{L}}:=\left|\frac{1}{\pi_{q^{1/\kappa}}}\int_{L_{\mathfrak{d}_{p+1},\tilde{\rho}}}\frac{w_{k_1}^{\mathfrak{d}_{p+1}}(u,m,\epsilon)}{\Theta_{q^{1/\kappa}}\left(\frac{u}{\tau}\right)}\frac{du}{u}\right|\\ \le K_{p,3}^{\mathcal{L}}(1+|m|)^{-\mu}e^{-\beta|m|}\exp\left(-\frac{\kappa}{2\log(q)}\log^2|\tau|\right).\end{aligned}\tag{5.16}$$

We write

$$I_3^{\mathcal{L}}:=\left|\frac{1}{\pi_{q^{1/\kappa}}}\int_{\mathcal{C}_{\tilde{\rho},\mathfrak{d}_p,\mathfrak{d}_{p+1}}}\frac{w_{k_1}(u,m,\epsilon)}{\Theta_{q^{1/\kappa}}\left(\frac{u}{\tau}\right)}\frac{du}{u}\right|.$$

Regarding (5.5) and (3.2), one derives that

$$I_3^{\mathcal{L}} \leq \frac{C_{w_{k_1}^{\mathfrak{d}_p}}}{\pi_{q^{1/\kappa}}} \frac{e^{-\beta|m|}}{(1+|m|)^{\mu}} \frac{|\tau|^{1/2}}{\tilde{\rho}^{1/2} C_{q,\kappa} \tilde{\delta}} \int_{\mathfrak{d}_p}^{\mathfrak{d}_{p+1}} \frac{\exp\left(\frac{\kappa \log^2|\tilde{\rho}e^{i\theta}+\delta|}{2\log(q)} + \alpha\log|\tilde{\rho}e^{i\theta}+\delta|\right)}{\exp\left(\frac{\kappa}{2}\frac{\log^2\left(\frac{\tilde{\rho}}{|\tau|}\right)}{\log(q)}\right)} d\theta$$

$$\leq K_{p,4}^{\mathcal{L}} |\tau|^{1/2} \frac{e^{-\beta|m|}}{(1+|m|)^{\mu}} \exp\left(-\frac{\kappa}{2}\frac{\log^2\left(\frac{\tilde{\rho}}{|\tau|}\right)}{\log(q)}\right)$$

with

$$K_{p,4}^{\mathcal{L}} = |\mathfrak{d}_{p+1} - \mathfrak{d}_p| \frac{C_{w_{k_1}^{\mathfrak{d}_p}}}{\pi_{q^{1/\kappa}}} \frac{1}{\tilde{\rho}^{1/2} C_{q,\kappa} \tilde{\delta}} \exp\left(\frac{\kappa \log^2(\tilde{\rho}+\delta)}{2\log(q)} + \alpha\log(\tilde{\rho}+\delta)\right).$$

Let $K_{p,5}^{\mathcal{L}} = K_{p,4}^{\mathcal{L}} \exp(-\frac{\kappa}{2\log(q)}\log^2(\tilde{\rho}))$. It is straightforward to check that

$$I_3^{\mathcal{L}} \leq K_{p,5}^{\mathcal{L}} |\tau|^{1/2+\frac{\kappa\log(\tilde{\rho})}{\log(q)}} \frac{e^{-\beta|m|}}{(1+|m|)^{\mu}} \exp\left(-\frac{\kappa}{2}\frac{\log^2|\tau|}{\log(q)}\right). \tag{5.17}$$

From (5.15), (5.16) and (5.17), put into (5.14), we conclude the result. □

Proposition 5.6. *Let $0 \leq p \leq \varsigma - 1$. Under the hypotheses of Theorem 5.3, assume that $U_{\mathfrak{d}_p} \cap U_{\mathfrak{d}_{p+1}} = \emptyset$. Then, there exist $K_3 > 0$ and $K_4 \in \mathbb{R}$ such that*

$$|u^{\mathfrak{d}_{p+1}}(t,z,\epsilon) - u^{\mathfrak{d}_p}(t,z,\epsilon)| \leq K_3 \exp\left(-\frac{k_1}{2\log(q)}\log^2|\epsilon|\right)|\epsilon|^{K_4},$$

for every $t \in \mathcal{T}$, $z \in H_{\beta'}$, and $\epsilon \in \mathcal{E}_p \cap \mathcal{E}_{p+1}$.

Proof. Let $0 \leq p \leq \varsigma - 1$. Under the assumptions of the enunciate, we observe that one can not proceed as in the proof of Proposition 5.4 for there does not exist a common function for both indices p and $p+1$, defined in $\mathcal{R}_{\mathfrak{d}_p}^b \cup \mathcal{R}_{\mathfrak{d}_{p+1}}^b$ in the variable of integration, when applying q-Laplace transform. However, one can use the analytic continuation property and write the difference $u^{\mathfrak{d}_{p+1}} - u^{\mathfrak{d}_p}$ as follows. Let $\tilde{\rho} > 0$ be such that $\tilde{\rho}e^{i\mathfrak{d}_p} \in \mathcal{R}_{\mathfrak{d}_p}^b$ and $\tilde{\rho}e^{i\mathfrak{d}_{p+1}} \in \mathcal{R}_{\mathfrak{d}_{p+1}}^b$, and let $\theta_{p,p+1} \in \mathbb{R}$ be such that $\tilde{\rho}e^{\theta_{p,p+1}}$ lies in both $\mathcal{R}_{\mathfrak{d}_p}^b$ and $\mathcal{R}_{\mathfrak{d}_{p+1}}^b$. We write $u^{\mathfrak{d}_{p+1}}(t,z,\epsilon) - u^{\mathfrak{d}_p}(t,z,\epsilon)$ as follows

$$\begin{aligned} u^{\mathfrak{d}_{p+1}}&(t,z,\epsilon) - u^{\mathfrak{d}_p}(t,z,\epsilon) \\ &= \frac{1}{(2\pi)^{1/2}}\frac{1}{\pi_{q^{1/k_2}}}\int_{-\infty}^{\infty}\int_{L_{\mathfrak{d}_{p+1},\tilde{\rho}}} \frac{w_{k_2}^{\mathfrak{d}_{p+1}}(u,m,\epsilon)}{\Theta_{q^{1/k_2}}\left(\frac{u}{\epsilon t}\right)}\exp(izm)\frac{du}{u}dm \\ &\quad - \frac{1}{(2\pi)^{1/2}}\frac{1}{\pi_{q^{1/k_2}}}\int_{-\infty}^{\infty}\int_{L_{\mathfrak{d}_p,\tilde{\rho}}} \frac{w_{k_2}^{\mathfrak{d}_p}(u,m,\epsilon)}{\Theta_{q^{1/k_2}}\left(\frac{u}{\epsilon t}\right)}\exp(izm)\frac{du}{u}dm \\ &\quad - \frac{1}{(2\pi)^{1/2}}\frac{1}{\pi_{q^{1/k_2}}}\int_{-\infty}^{\infty}\int_{\mathcal{C}_{\tilde{\rho},\theta_{p,p+1},\mathfrak{d}_{p+1}}} \frac{w_{k_2}^{\mathfrak{d}_p,\mathfrak{d}_{p+1}}(u,m,\epsilon)}{\Theta_{q^{1/k_2}}\left(\frac{u}{\epsilon t}\right)}\exp(izm)\frac{du}{u}dm \\ &\quad + \frac{1}{(2\pi)^{1/2}}\frac{1}{\pi_{q^{1/k_2}}}\int_{-\infty}^{\infty}\int_{\mathcal{C}_{\tilde{\rho},\theta_{p,p+1},\mathfrak{d}_p}} \frac{w_{k_2}^{\mathfrak{d}_p,\mathfrak{d}_{p+1}}(u,m,\epsilon)}{\Theta_{q^{1/k_2}}\left(\frac{u}{\epsilon t}\right)}\exp(izm)\frac{du}{u}dm \\ &+ \frac{1}{(2\pi)^{1/2}}\frac{1}{\pi_{q^{1/k_2}}}\int_{-\infty}^{\infty}\int_{L_{0,\tilde{\rho},\theta_{p,p+1}}} \frac{\mathcal{L}_{q;1/\kappa}^{\mathfrak{d}_{p+1}}(w_{k_1}^{\mathfrak{d}_{p+1}})(\tau,m,\epsilon) - \mathcal{L}_{q;1/\kappa}^{\mathfrak{d}_p}(w_{k_1}^{\mathfrak{d}_p})(\tau,m,\epsilon)}{\Theta_{q^{1/k_2}}\left(\frac{u}{\epsilon t}\right)}\exp(izm)\frac{du}{u}dm. \end{aligned} \tag{5.18}$$

Here, we have denoted $L_{\mathfrak{d}_j,\tilde{\rho}} = [\tilde{\rho},+\infty)e^{i\mathfrak{d}_j}$ for $j \in \{p,p+1\}$, $\mathcal{C}_{\tilde{\rho},\theta_{p,p+1},\mathfrak{d}_{p+1}}$ is the arc of circle connecting $\tilde{\rho}e^{i\mathfrak{d}_{p+1}}$ with $\tilde{\rho}e^{i\theta_{p,p+1}}$, $\mathcal{C}_{\tilde{\rho},\theta_{p,p+1},\mathfrak{d}_p}$ is the arc of circle connecting $\tilde{\rho}e^{i\mathfrak{d}_p}$ with $\tilde{\rho}e^{i\theta_{p,p+1}}$, $L_{0,\tilde{\rho},\theta_{p,p+1}} = [0,\tilde{\rho}]e^{i\theta_{p,p+1}}$, as it is shown in the following figure.

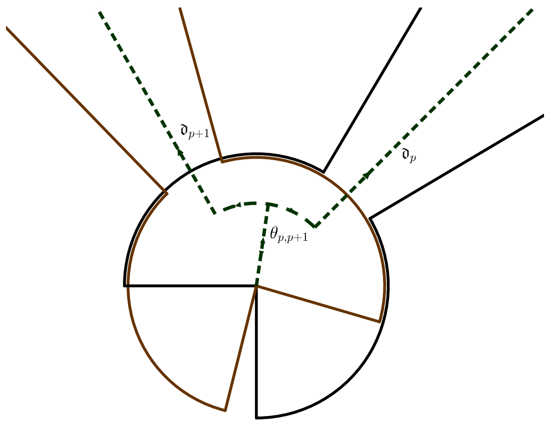

FIGURE 3. Deformation of the path of integration, second case.

Following the same line of arguments as those in the proof of Proposition 5.4, we can guarantee the existence of $\hat{K}^j > 0$ and $\hat{K}^k \in \mathbb{R}$ for $1 \leq j \leq 4$ and $5 \leq k \leq 8$ such that

$$J_1 := \left| \frac{1}{(2\pi)^{1/2}} \frac{1}{\pi_{q^{1/k_2}}} \int_{-\infty}^{\infty} \int_{L_{\mathfrak{d}_{p+1},\tilde{\rho}}} \frac{w_{k_2}^{\mathfrak{d}_{p+1}}(u,m,\epsilon)}{\Theta_{q^{1/k_2}}\left(\frac{u}{\epsilon t}\right)} \exp(izm) \frac{du}{u} dm \right| \\ \leq \hat{K}_1 \exp\left(-\frac{k_2}{2\log(q)} \log^2|\epsilon|\right) |\epsilon|^{\hat{K}^5},$$

$$J_2 := \left| \frac{1}{(2\pi)^{1/2}} \frac{1}{\pi_{q^{1/k_2}}} \int_{-\infty}^{\infty} \int_{L_{\mathfrak{d}_p,\tilde{\rho}}} \frac{w_{k_2}^{\mathfrak{d}_p}(u,m,\epsilon)}{\Theta_{q^{1/k_2}}\left(\frac{u}{\epsilon t}\right)} \exp(izm) \frac{du}{u} dm \right| \\ \leq \hat{K}_2 \exp\left(-\frac{k_2}{2\log(q)} \log^2|\epsilon|\right) |\epsilon|^{\hat{K}^6},$$

$$J_3 := \left| \frac{1}{(2\pi)^{1/2}} \frac{1}{\pi_{q^{1/k_2}}} \int_{-\infty}^{\infty} \int_{\mathcal{C}_{\tilde{\rho},\theta_{p,p+1},\mathfrak{d}_{p+1}}} \frac{w_{k_2}^{\mathfrak{d}_p,\mathfrak{d}_{p+1}}(u,m,\epsilon)}{\Theta_{q^{1/k_2}}\left(\frac{u}{\epsilon t}\right)} \exp(izm) \frac{du}{u} dm \right| \\ \leq \hat{K}_3 \exp\left(-\frac{k_2}{2\log(q)} \log^2|\epsilon|\right) |\epsilon|^{\hat{K}^7},$$

$$J_4 := \left| \frac{1}{(2\pi)^{1/2}} \frac{1}{\pi_{q^{1/k_2}}} \int_{-\infty}^{\infty} \int_{\mathcal{C}_{\tilde{\rho},\theta_{p,p+1},\mathfrak{d}_p}} \frac{w_{k_2}^{\mathfrak{d}_p,\mathfrak{d}_{p+1}}(u,m,\epsilon)}{\Theta_{q^{1/k_2}}\left(\frac{u}{\epsilon t}\right)} \exp(izm) \frac{du}{u} dm \right| \\ \leq \hat{K}_4 \exp\left(-\frac{k_2}{2\log(q)} \log^2|\epsilon|\right) |\epsilon|^{\hat{K}^8}$$

We now give estimates for

$$J_5 := \frac{1}{(2\pi)^{1/2}} \frac{1}{\pi_{q^{1/k_2}}} \times \left| \int_{-\infty}^{\infty} \int_{L_{0,\tilde{\rho},\theta_{p,p+1}}} \frac{\mathcal{L}^{\mathfrak{d}_{p+1}}_{q;1/\kappa}(w^{\mathfrak{d}_{p+1}}_{k_1})(u,m,\epsilon) - \mathcal{L}^{\mathfrak{d}_p}_{q;1/\kappa}(w^{\mathfrak{d}_p}_{k_1})(u,m,\epsilon)}{\Theta_{q^{1/k_2}}\left(\frac{u}{\epsilon t}\right)} \exp(izm) \frac{du}{u} dm \right|.$$

In view of Lemma 5.5 and (3.2), one has

$$J_5 \leq \frac{K^{\mathcal{L}}_p}{(2\pi)^{1/2}} \frac{1}{\pi_{q^{1/k_2}}} \int_{-\infty}^{\infty} e^{-\beta|m| - \Im(z)m} \frac{dm}{(1+|m|)^{\mu}} \int_0^{\tilde{\rho}} \frac{\exp\left(-\frac{\kappa}{2\log(q)} \log^2|u|\right) |u|^{M^{\mathcal{L}}_p}}{C_{q,k_2}\tilde{\delta} \exp\left(\frac{k_2}{2} \frac{\log^2\left|\frac{u}{\epsilon t}\right|}{\log(q)}\right) \left|\frac{u}{\epsilon t}\right|^{1/2}} \frac{d|u|}{|u|}.$$

We recall that $z \in H_{\beta'}$ for some $\beta' < \beta$. Then, there exists $K_{31} > 0$ such that

$$J_5 \leq \frac{K^{\mathcal{L}}_p K_{31}}{(2\pi)^{1/2}} \frac{|\epsilon|^{1/2} r_{\mathcal{T}}^{1/2}}{\pi_{q^{1/k_2}} C_{q,k_2}\tilde{\delta}} \int_0^{\tilde{\rho}} \frac{\exp\left(-\frac{\kappa}{2\log(q)} \log^2|u|\right) |u|^{M^{\mathcal{L}}_p}}{\exp\left(\frac{k_2}{2} \frac{\log^2\left|\frac{u}{\epsilon t}\right|}{\log(q)}\right)} \frac{d|u|}{|u|^{3/2}}.$$

We now proceed to prove the expression

$$\int_0^{\tilde{\rho}} \frac{\exp\left(-\frac{\kappa}{2\log(q)} \log^2|u|\right)}{\exp\left(\frac{k_2}{2} \frac{\log^2\left|\frac{u}{\epsilon t}\right|}{\log(q)}\right)} \exp\left(\frac{k_1}{2\log(q)} \log^2|\epsilon|\right) \frac{d|u|}{|u|^{3/2 - M^{\mathcal{L}}_p}}$$

is upper bounded by a positive constant times a certain power of $|\epsilon|$ for every $\epsilon \in (\mathcal{E}_p \cap \mathcal{E}_{p+1})$ and $t \in \mathcal{T}$. This concludes the existence of $K_{32} > 0$ such that

$$J_5 \leq K_{32}|\epsilon|^{1/2} \exp\left(-\frac{k_1}{2\log(q)} \log^2|\epsilon|\right), \tag{5.19}$$

for every $\epsilon \in (\mathcal{E}_p \cap \mathcal{E}_{p+1})$, $t \in \mathcal{T}$ and $z \in H_{\beta'}$.

Indeed, we have

$$\int_0^{\tilde{\rho}} \frac{\exp\left(-\frac{\kappa}{2\log(q)} \log^2|u|\right)}{\exp\left(\frac{k_2}{2} \frac{\log^2\left(\frac{|u|}{|\epsilon t|}\right)}{\log(q)}\right)} \exp\left(\frac{k_1}{2\log(q)} \log^2|\epsilon|\right) \frac{d|u|}{|u|^{3/2 - M^{\mathcal{L}}_p}}$$

equals

$$\exp\left(\frac{k_1}{2\log(q)} \log^2|\epsilon| - \frac{k_2}{2\log(q)} \log^2|\epsilon t|\right) \times \int_0^{\tilde{\rho}} \exp\left(-\frac{(\kappa + k_2)}{2\log(q)} \log^2|u|\right) |u|^{\frac{k_2 \log|\epsilon t|}{\log(q)} - \frac{3}{2} + M^{\mathcal{L}}_p} d|u|. \tag{5.20}$$

Given $m_1 \in \mathbb{R}$ and $m_2 > 0$, the function $[0,\infty) \ni x \mapsto H(x) = x^{m_1}\exp(-m_2\log^2(x))$ attains its maximum value at $x_0 = \exp(\frac{m_1}{2m_2})$ with $H(x_0) = \exp(\frac{m_1^2}{4m_2})$. This yields and upper bound for the integrand in (5.20); the expression in (5.20) is estimated from above by

$$\begin{aligned}(5.21)\quad \tilde{\rho}\exp\left(\frac{(M_p^{\mathcal{L}}-3/2)^2\log(q)}{2(\kappa+k_2)}\right)\exp\left(\frac{1}{2\log(q)}(\frac{k_2^2}{\kappa+k_2}-k_2+k_1)\log^2|\epsilon|\right)\\ \times\exp\left(\frac{1}{2\log(q)}(\frac{k_2^2}{\kappa+k_2}-k_2)\log^2|t|\right)|t|^{\frac{k_2(M_p^{\mathcal{L}}-3/2)}{\kappa+k_2}}\\ \times\exp\left(\frac{1}{\log(q)}(\frac{k_2^2}{\kappa+k_2}-k_2)\log|\epsilon|\log|t|\right)|\epsilon|^{\frac{k_2(M_p^{\mathcal{L}}-3/2)}{\kappa+k_2}}.\end{aligned}$$

The second line in (5.21) is upper bounded for every t because $\frac{k_2^2}{\kappa+k_2}<k_2$ and also, one has an upper bound for $\exp\left(\frac{1}{\log(q)}(\frac{k_2^2}{\kappa+k_2}-k_2)\log|\epsilon|\log|t|\right)$ is 1. Regarding Definition 5.2, and taking into account that

$$\frac{k_2^2}{\kappa+k_2}-k_2=-k_1,$$

the expression (5.21) is upper bounded by

$$K_{33}|\epsilon|^{\frac{k_2(M_p^{\mathcal{L}}-3/2)}{\kappa+k_2}}$$

for some $K_{33}>0$. The conclusion is achieved. The result follows from (5.18), the estimates J_1 to J_4, and (5.19). □

6. Existence of formal series solutions in the complex parameter and asymptotic expansion in two levels

In the first part of this section, we remind two q-analogs of Ramis-Sibuya theorem from [10, 16]. This result provides the tool to guarantee the existence of a formal power series in the perturbation parameter which formally solves the main problem and such that it asymptotically represents the analytic solution of that equation.

This asymptotic representation is held in the sense of q-asymptotic expansions of certain positive order.

Definition 6.1. Let V be a bounded open sector with vertex at 0 in $\mathbb{C}$. Let $(\mathbb{F},\|\cdot\|_{\mathbb{F}})$ be a complex Banach space. Let $q\in\mathbb{R}$ with $q>1$ and let k be a positive integer. We say that a holomorphic function $f:V\to\mathbb{F}$ admits the formal power series $\hat{f}(\epsilon)=\sum_{n\geq 0}f_n\epsilon^n\in\mathbb{F}[[\epsilon]]$ as its q-Gevrey asymptotic expansion of order $1/k$ if for every open subsector U with $(\overline{U}\setminus\{0\})\subseteq V$, there exist $A,C>0$ such that

$$\left\|f(\epsilon)-\sum_{n=0}^{N}f_n\epsilon^n\right\|_{\mathbb{F}}\leq CA^{N+1}q^{\frac{N(N+1)}{2k}}|\epsilon|^{N+1},$$

for every $\epsilon\in U$, and $N\geq 0$.

The set of functions which admit null q-Gevrey asymptotic expansion of certain positive order are characterized as follows. The proof of this result, already stated in [16], provides the q-analog of Theorem XI-3-2 in [6].

Lemma 6.2. *A holomorphic function $f:V\to\mathbb{F}$ admits the null formal power series $\hat{0}\in\mathbb{F}[[\epsilon]]$ as its q-Gevrey asymptotic expansion of order $1/k$ if and only if for every open subsector U with $(\overline{U}\setminus\{0\})\subseteq V$ there exist constants $K_1\in\mathbb{R}$ and $K_2>0$ with*

$$\|f(\epsilon)\|_{\mathbb{F}}\leq K_2\exp\left(-\frac{k}{2\log(q)}\log^2|\epsilon|\right)|\epsilon|^{K_1},$$

for all $\epsilon\in U$.

The next result leans on the one level version of the q-analog of Ramis Sibuya theorem, stated in [16], provides a two level result in this framework. See [10] for a proof.

Theorem 6.3. *Let $(\mathbb{F}, \|\cdot\|_{\mathbb{F}})$ be a Banach space and $(\mathcal{E}_p)_{0\le p\le\varsigma-1}$ be a good covering in $\mathbb{C}^\star$. Let $0<k_1<k_2$, consider a holomorphic function $G_p:\mathcal{E}_i\to\mathbb{F}$ for every $0\le p\le\varsigma-1$ and put $\Delta_p(\epsilon)=G_{p+1}(\epsilon)-G_p(\epsilon)$ for every $\epsilon\in Z_p:=\mathcal{E}_p\cap\mathcal{E}_{p+1}$. Moreover, we assume:*

1) *The functions $G_p(\epsilon)$ are bounded as ϵ tends to 0 on $\mathcal{E}_p$ for every $0\le p\le\varsigma-1$.*

2) *There exist nonempty sets $I_1, I_2\subseteq\{0,1,\ldots,\varsigma-1\}$ such that $I_1\cup I_2=\{0,1,\ldots,\varsigma-1\}$ and $I_1\cap I_2=\emptyset$. Also,*

- *for every $p\in I_1$ there exist constants $K_1>0$, $M_1\in\mathbb{R}$ such that*

$$\|\Delta_p(\epsilon)\|_{\mathbb{F}}\le K_1|\epsilon|^{M_1}\exp\left(-\frac{k_1}{2\log(q)}\log^2|\epsilon|\right),\quad \epsilon\in Z_p,$$

- *and, for every $p\in I_2$ there exist constants $K_2>0$, $M_2\in\mathbb{R}$ such that*

$$\|\Delta_p(\epsilon)\|_{\mathbb{F}}\le K_2|\epsilon|^{M_2}\exp\left(-\frac{k_2}{2\log(q)}\log^2|\epsilon|\right),\quad \epsilon\in Z_p.$$

Then, there exists a convergent power series $a(\epsilon)\in\mathbb{F}\{\epsilon\}$ defined on some neighborhood of the origin and $\hat{G}^1(\epsilon),\hat{G}^2(\epsilon)\in\mathbb{F}[[\epsilon]]$ such that G_p can be written in the form

$$G_p(\epsilon)=a(\epsilon)+G_p^1(\epsilon)+G_p^2(\epsilon).$$

$G_p^1(\epsilon)$ is holomorphic on $\mathcal{E}_p$ and admits $\hat{G}^1(\epsilon)$ as its q-Gevrey asymptotic expansion of order $1/k_1$ on $\mathcal{E}_p$, for every $p\in I_1$; whilst $G_p^2(\epsilon)$ is holomorphic on $\mathcal{E}_p$ and admits $\hat{G}^2(\epsilon)$ as its q-Gevrey asymptotic expansion of order $1/k_2$ on $\mathcal{E}_p$, for every $p\in I_2$.

We conclude this section with the main result in the work in which we guarantee the existence of a formal solution of the main problem (5.1), written as a formal power series in the perturbation parameter, with coefficients in an appropriate Banach space, say $\hat{u}(t,z,\epsilon)$. Moreover, it represents, in some sense to be precised, each solution $u^{\mathfrak{d}_p}(t,z,\epsilon)$ of the problem (5.1).

From now on, $\mathbb{F}$ stands for the Banach space of bounded holomorphic functions defined on $\mathcal{T}\times H_{\beta'}$, with the supremum norm, where $\beta'<\beta$, as above.

Theorem 6.4. *Under the hypotheses of Theorem 5.3, there exists a formal power series*

$$\hat{u}(t,z,\epsilon)=\sum_{m\ge0}h_m(t,z)\frac{\epsilon^m}{m!}\in\mathbb{F}[[\epsilon]], \tag{6.1}$$

formal solution of the equation

$$\begin{aligned}Q(\partial_z)\sigma_{q,t}\hat{u}(t,z,\epsilon) &= (\epsilon t)^{d_{D_1}}\sigma_{q,t}^{\frac{d_{D_1}}{k_1}+1}R_{D_1}(\partial_z)\hat{u}(t,z,\epsilon)+(\epsilon t)^{d_{D_2}}\sigma_{q,t}^{\frac{d_{D_2}}{k_2}+1}R_{D_2}(\partial_z)\hat{u}(t,z,\epsilon)\\ &\quad+\sum_{\ell=1}^{D-1}\epsilon^{\Delta_\ell}t^{d_\ell}\sigma_{q,t}^{\delta_\ell}(c_\ell(t,z,\epsilon)R_\ell(\partial_z)\hat{u}(t,z,\epsilon))+\sigma_{q,t}f(t,z,\epsilon).\end{aligned} \tag{6.2}$$

Moreover, $\hat{u}(t,z,\epsilon)$ turns out to be the common q-Gevrey asymptotic expansion of order $1/k_1$ on $\mathcal{E}_p$ of the function $u^{\mathfrak{d}_p}$, seen as holomorphic function from $\mathcal{E}_p$ into $\mathbb{F}$, for $0\le p\le\varsigma-1$. In addition to that, $\hat{u}$ is of the form

$$\hat{u}(t,z,\epsilon)=a(t,z,\epsilon)+\hat{u}_1(t,z,\epsilon)+\hat{u}_2(t,z,\epsilon),$$

where $a(t,z,\epsilon) \in \mathbb{F}\{\epsilon\}$ *and* $\hat{u}_1(t,z,\epsilon), \hat{u}_2(t,z,\epsilon) \in \mathbb{F}[[\epsilon]]$ *and such that for every* $0 \le p \le \varsigma - 1$, *the function* $u^{\mathfrak{d}_p}$ *can be written in the form*

$$u^{\mathfrak{d}_p}(t,z,\epsilon) = a(t,z,\epsilon) + u_1^{\mathfrak{d}_p}(t,z,\epsilon) + u_2^{\mathfrak{d}_p}(t,z,\epsilon),$$

where $\epsilon \mapsto u_1^{\mathfrak{d}_p}(t,z,\epsilon)$ *is a* $\mathbb{F}$*-valued function that admits* $\hat{u}_1(t,z,\epsilon)$ *as its q-Gevrey asymptotic expansion of order* $1/k_1$ *on* $\mathcal{E}_p$ *and also* $\epsilon \mapsto u_2^{\mathfrak{d}_p}(t,z,\epsilon)$ *is a* $\mathbb{F}$*-valued function that admits* $\hat{u}_2(t,z,\epsilon)$ *as its q-Gevrey asymptotic expansion of order* $1/k_2$ *on* $\mathcal{E}_p$*.*

Proof. For every $0 \le p \le \varsigma - 1$, one can consider the function $u^{\mathfrak{d}_p}(t,z,\epsilon)$ constructed in Theorem 5.3. We define $G_p(\epsilon) := (t,z) \mapsto u^{\mathfrak{d}_p}(t,z,\epsilon)$, which is a holomorphic and bounded function from $\mathcal{E}_p$ into $\mathbb{F}$. In view of Proposition 5.4 and Proposition 5.6, one can split the set $\{0,1,\dots,\varsigma-1\}$ in two nonempty subsets of indices, I_1 and I_2 with $\{0,1,\dots,\varsigma-1\} = I_1 \cup I_2$ and such that I_1 (resp. I_2) consists of all the elements in $\{0,1,\dots,\varsigma-1\}$ such that $U_{\mathfrak{d}_p} \cap U_{\mathfrak{d}_{p+1}}$ contains the sector $U_{\mathfrak{d}_p,\mathfrak{d}_{p+1}}$, as defined in Proposition 5.4 (resp. $U_{\mathfrak{d}_p} \cap U_{\mathfrak{d}_{p+1}} = \emptyset$). From (5.6) and (5.18) one can apply Theorem 6.3 and deduce the existence of formal power series $\hat{G}^1(\epsilon), \hat{G}^2(\epsilon) \in \mathbb{F}[[\epsilon]]$, a convergent power series $a(\epsilon) \in \mathbb{F}\{\epsilon\}$ and holomorphic functions $G_p^1(\epsilon), G_p^2(\epsilon)$ defined on $\mathcal{E}_p$ and with values in $\mathbb{F}$ such that

$$G_p(\epsilon) = a(\epsilon) + G_p^1(\epsilon) + G_p^2(\epsilon),$$

and for $j = 1,2$, one has $G_p^j(\epsilon)$ admits $\hat{G}^j(\epsilon)$ as its q-Gevrey asymptotic expansion or order $1/k_j$ on $\mathcal{E}_p$. We put

$$\hat{u}(t,z,\epsilon) = \sum_{m \ge 0} h_m(t,z)\frac{\epsilon^m}{m!} := a(\epsilon) + \hat{G}_p^1(\epsilon) + \hat{G}_p^2(\epsilon).$$

It only rests to prove that $\hat{u}(t,z,\epsilon)$ is the solution of (6.2). Indeed, since $u^{\mathfrak{d}_p}$ admits $\hat{u}(t,z,\epsilon)$ as its q-Gevrey asymptotic expansion of order $1/k_1$ on $\mathcal{E}_p$, we have that

$$\lim_{\epsilon \to 0, \epsilon \in \mathcal{E}_p} \sup_{t \in \mathcal{T}, z \in H_{\beta'}} |\partial_\epsilon^m u^{\mathfrak{d}_p}(t,z,\epsilon) - h_m(t,z)| = 0,$$

for every $0 \le p \le \varsigma - 1$ and $m \ge 0$. Let $p \in \{0,1,\dots,\varsigma-1\}$. By construction, the function $u^{\mathfrak{d}_p}(t,z,\epsilon)$ solves equation (6.2). We take derivatives of order $m \ge 0$ with respect to ϵ at both sides of equation (5.1) and deduce that

$$(6.3)\quad Q(\partial_z)\sigma_{q,t}(\partial_\epsilon^m u^{\mathfrak{d}_p})(t,z,\epsilon)$$
$$= \sum_{m_1+m_2=m} \frac{m!}{m_1!m_2!}\partial_\epsilon^{m_1}(\epsilon^{d_{D_1}})t^{d_{D_1}}\sigma_{q,t}^{\frac{d_{D_1}}{k_1}+1} R_{D_1}(\partial_z)(\partial_\epsilon^{m_2}u^{\mathfrak{d}_p})$$
$$+ \sum_{m_1+m_2=m} \frac{m!}{m_1!m_2!}\partial_\epsilon^{m_1}(\epsilon^{d_{D_2}})t^{d_{D_2}}\sigma_{q,t}^{\frac{d_{D_2}}{k_2}+1} R_{D_2}(\partial_z)(\partial_\epsilon^{m_2}u^{\mathfrak{d}_p})$$
$$+ \sum_{\ell=1}^{D-1} \sum_{m_1+m_2+m_3=m} \frac{m!}{m_1!m_2!m_3!}(\partial_\epsilon^{m_1}\epsilon^{\Delta_\ell})t^{d_\ell}\sigma_{q,t}^{\delta_\ell}(\partial_\epsilon^{m_2}c_\ell(t,z,\epsilon)R_\ell(\partial_z)\partial_\epsilon^{m_3}u^{\mathfrak{d}_p}(t,z,\epsilon))$$
$$+ \sigma_{q,t}(\partial_\epsilon^m f)(t,z,0),$$

for every $(t, z, \epsilon) \in \mathcal{T} \times H_{\beta'} \times \mathcal{E}_p$. We let $\epsilon \to 0$ in (6.3) and obtain the recursion formula

$$Q(\partial_z)\sigma_{q,t}h_m(t,z) = \frac{m!}{(m-d_{D_1})!}t^{d_{D_1}}\sigma_{q,t}^{\frac{d_{D_1}}{k_1}+1}R_{D_1}(\partial_z)(h_{m-d_{D_1}}(t,z)) + \frac{m!}{(m-d_{D_2})!}t^{d_{D_2}}\sigma_{q,t}^{\frac{d_{D_2}}{k_2}+1}R_{D_2}(\partial_z)(h_{m-d_{D_2}}(t,z)) + \sum_{\ell=1}^{D-1}\sum_{m_2+m_3=m-\Delta_\ell}\frac{m!}{m_2!m_3!}t^{d_\ell}\sigma_{q,t}^{\delta_\ell}(\partial_\epsilon^{m_2}c_\ell(t,z,0)R_\ell(\partial_z)h_{m_3}(t,z)) + \sigma_{q,t}(\partial_\epsilon^m f)(t,z,0), \tag{6.4}$$

for every $m \geq \max\{d_{D_1}, d_{D_2}, \max_{1\leq \ell \leq D-1}\Delta_\ell\}$, and all $(t,z) \in \mathcal{T} \times H_{\beta'}$.

Bearing in mind that both c_l and f are holomorphic w.r.t ϵ in a neighborhood of the origin, in such neighborhood one has

$$c_\ell(t,z,\epsilon) = \sum_{m\geq 0}\frac{(\partial_\epsilon^m c_m)(t,z,0)}{m!}\epsilon^m \quad , \quad f(t,z,\epsilon) = \sum_{m\geq 0}\frac{(\partial_\epsilon^m f)(t,z,0)}{m!}\epsilon^m \tag{6.5}$$

for every $1 \leq \ell \leq D-1$.

By plugging (6.1) into (6.2) and bearing in mind (6.4) and (6.5) one concludes that the formal power series $\hat{u}(t,z,\epsilon) = \sum_{m\geq 0} h_m(t,z)\epsilon^m/m!$ is a solution of equation (6.2).

□

References

[1] W. Balser, *Formal power series and linear systems of meromorphic ordinary differential equations*, Universitext, Springer-Verlag, New York, 2000.

[2] O. Costin and S. Tanveer, *Short time existence and Borel summability in the Navier-Stokes equation in* $\mathbb{R}^3$, Comm. Partial Differential Equations **34** (2009), no. 7-9, 785–817.

[3] T. Dreyfus and Anton Eloy, *q-Borel-Laplace summation for q-difference equations with two slopes*, J. Difference Equ. Appl. **22** (2016), no. 10, 1501–1511.

[4] T. Dreyfus, *Building meromorphic solutions of q-difference equations using a Borel-Laplace summation*, Int. Math. Res. Not. IMRN (2015), no. 15, 6562–6587.

[5] L. Di Vizio and Changgui Zhang, *On q-summation and confluence*, Ann. Inst. Fourier (Grenoble) **59** (2009), no. 1, 347–392.

[6] P.-F. Hsieh and Y. Sibuya, *Basic theory of ordinary differential equations*, Universitext, Springer-Verlag, New York, 1999.

[7] A. Lastra and S. Malek, *On q-Gevrey asymptotics for singularly perturbed q-difference-differential problems with an irregular singularity*, Abstr. Appl. Anal. (2012), Art. ID 860716, 35.

[8] ______, *On parametric Gevrey asymptotics for singularly perturbed partial differential equations with delays*, Abstr. Appl. Anal. (2013), Art. ID 723040, 18. MR 3129346

[9] ______, *On parametric Gevrey asymptotics for singularly perturbed partial differential equations with delays*, Abstr. Appl. Anal. (2013), Art. ID 723040, 18.

[10] ______, *On parametric multilevel q-Gevrey asymptotics for some linear q-difference-differential equations*, Adv. Difference Equ. (2015), 2015:344, 52.

[11] ______, *On parametric multisummable formal solutions to some nonlinear initial value Cauchy problems*, Adv. Difference Equ. (2015), 2015:200, 78.

[12] ______, *On multiscale gevrey and q-gevrey asymptotics for some linear q-difference differential initial value Cauchy problems*, J. Difference Equ. Appl. 23 (2017), no. 8, 1397–1457.

[13] A. Lastra, S. Malek, and J. Sanz, *On q-asymptotics for linear q-difference-differential equations with Fuchsian and irregular singularities*, J. Differential Equations **252** (2012), no. 10, 5185–5216.

[14] S. Malek, *On Gevrey asymptotics for some nonlinear integro-differential equations*, J. Dyn. Control Syst. **16** (2010), no. 3, 377–406.

[15] ______, *On singularly perturbed q-difference-differential equations with irregular singularity*, J. Dyn. Control Syst. **17** (2011), no. 2, 243–271.

[16] S. Malek, *Parametric gevrey asymptotics for a q-analog of some linear initial value problem*, Funkcial. Ekvac. 60 (2017), no. 1, 21–63.

[17] D. Pravica, N. Randriampiry, M. Spurr, *On q-advanced spherical Bessel functions of the first kind and perturbations of the Haar wavelet.* Appl. Comput. Harmon. Anal. 44 (2018), no. 2, 350–413.

[18] D. Pravica, N. Randriampiry, M. Spurr, *q-Advanced models for tsunamis and rogue waves.*Abstr. Appl. Anal., 2012 (2012), Article 414060.

[19] D. Pravica, N. Randriampiry, M. Spurr, *Solutions of a class of multiplicatively advanced differential equations.* C. R. Math. Acad. Sci. Paris 356 (2018), no. 7, 776–817.

[20] J.-P. Ramis, *About the growth of entire functions solutions of linear algebraic q-difference equations*, Ann. Fac. Sci. Toulouse Math. (6) **1** (1992), no. 1, 53–94.

[21] H. Tahara, *q-analogues of Laplace and Borel transforms by means of q-exponentials*, Ann. Inst. Fourier (Grenoble) 67 (2017), no. 5, 1865–1903.

[22] H. Tahara and H. Yamazawa, *Multisummability of formal solutions to the Cauchy problem for some linear partial differential equations*, J. Differential Equations **255** (2013), no. 10, 3592–3637.

[23] ———, *q-analogue of summability of formal solutions of some linear q-difference-differential equations*, Opuscula Math. **35** (2015), no. 5, 713–738.

[24] H. Yamazawa, *Holomorphic and singular solutions of q-difference-differential equations of Briot-Bouquet type*, Funkcial. Ekvac. **59** (2016), no. 2, 185–197.

[25] ———, *Gevrey and q-Gevrey asymptotics for some linear q-difference-differential equations*, communication in “Formal and analytic solutions of functional equations on the complex domain”, Kyoto, 2018.

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