SOLVABILITY IN THE SENSE OF SEQUENCES FOR SOME NON-FREDHOLM OPERATORS IN HIGHER DIMENSIONS

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Abstract: We study solvability of certain linear nonhomogeneous elliptic problems and establish that under reasonable technical assumptions the convergence in $L^2(\mathbb{R}^d)$ of their right sides implies the existence and the convergence in $H^1(\mathbb{R}^d)$ of the solutions. The equations involve the square roots of the sums of second order non-Fredholm differential operators and we rely on the methods of the spectral and scattering theory for Schrödinger type operators similarly to our earlier work [26].
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1. Introduction

Consider the problem

$$\sqrt{-\Delta + V(x)}u - au = f, \tag{1.1}$$

where $u \in E = H^1(\mathbb{R}^d)$ and $f \in F = L^2(\mathbb{R}^d)$, $d \in \mathbb{N}$, a is a constant and the scalar potential function $V(x)$ tends to 0 at infinity. For $a \geq 0$, the essential spectrum of the operator $A : E \to F$ which corresponds to the left side of problem (1.1) contains the origin. Consequently, this operator fails to satisfy the Fredholm property. Its image is not closed, for $d > 1$ the dimensions of its kernel and the codimension of its image are not finite. The present work deals with the studies of certain properties of the operators of this kind. Let us recall that elliptic equations containing non Fredholm operators were treated extensively in recent years (see [15], [16], [17], [19], [20], [21], [22], [23], [24], [25], also [6]) along with their potential applications to the theory of reaction-diffusion problems (see [8], [9]). Non-Fredholm operators are also important when studying wave systems with an infinite number of localized traveling waves (see [1]). In the particular case when $a = 0$ the operator A^2 satisfies the Fredholm property in some properly chosen weighted spaces (see [2], [3], [4], [5], [6]). However, the case of $a \neq 0$ is significantly different and the approach developed in these works cannot be applied.

One of the important issues about problems with non-Fredholm operators concerns their solvability. We address it in the following setting. Let f_n be a sequence of functions in the image of the operator A, such that $f_n \to f$ in $L^2(\mathbb{R}^d)$ as $n \to \infty$. Denote by u_n a sequence of functions from $H^1(\mathbb{R}^d)$ such that

$$Au_n = f_n, \ n \in \mathbb{N}.$$

Since the operator A does not satisfy the Fredholm property, the sequence u_n may not be convergent. Let us call a sequence u_n such that $Au_n \to f$ a solution in the sense of sequences of problem $Au = f$ (see [15]). If such sequence converges to a function u_0 in the norm of the space E, then u_0 is a solution of this equation. Solution in the sense of sequences is equivalent in this case to the usual solution. However, in the case of the non-Fredholm operators, this convergence may not hold or it can occur in some weaker sense. In such case, solution in the sense of sequences may not imply the existence of the usual solution. In the present work we will find sufficient conditions of equivalence of solutions in the sense of sequences and the usual solutions. In the other words, the conditions on sequences f_n under which the corresponding sequences u_n are strongly convergent. Solvability in the sense of sequences for the sums of Schrödinger type operators without Fredholm property was treated in [26].

In the first part of the article we study the problem

$$\sqrt{-\Delta_x + V(x) - \Delta_y + U(y)}u - au = f(x, y), \quad x, y \in \mathbb{R}^3. \tag{1.2}$$

The operator

$$H_{U,\,V} := \sqrt{-\Delta_x + V(x) - \Delta_y + U(y)} \tag{1.3}$$

here is defined via the spectral calculus. Here and further down the Laplacian operators Δ_x and Δ_y are with respect to the x and y variables respectively, such that cumulatively $\Delta := \Delta_x + \Delta_y$. Similarly, for the gradients

$$\nabla := \nabla_x + \nabla_y,$$

where ∇_x and ∇_y act on x and y variables respectively. The square roots of second order differential operators are actively used, for instance in the studies of the superdiffusion problems (see e.g. [27] and the references therein), in relativistic Quantum Mechanics (see e.g. [18]). The scalar potential functions involved in (1.3) are assumed to be shallow and short-range, satisfying the assumptions analogous to the ones of [19] and [21].

Assumption 1. *The potential functions $V(x), U(y) : \mathbb{R}^3 \to \mathbb{R}$ satisfy the bounds*

$$|V(x)| \leq \frac{C}{1 + |x|^{3.5+\varepsilon}}, \quad |U(y)| \leq \frac{C}{1 + |y|^{3.5+\varepsilon}}$$

with some $\varepsilon > 0$ and $x, y \in \mathbb{R}^3$ a.e. such that

$$4^{\frac{1}{9}}\frac{9}{8}(4\pi)^{-\frac{2}{3}}\|V\|_{L^\infty(\mathbb{R}^3)}^{\frac{1}{9}}\|V\|_{L^{\frac{4}{3}}(\mathbb{R}^3)}^{\frac{8}{9}} < 1, \tag{1.4}$$

$$4^{\frac{1}{9}}\frac{9}{8}(4\pi)^{-\frac{2}{3}}\|U\|_{L^\infty(\mathbb{R}^3)}^{\frac{1}{9}}\|U\|_{L^{\frac{4}{3}}(\mathbb{R}^3)}^{\frac{8}{9}} < 1 \tag{1.5}$$

and

$$\sqrt{c_{HLS}}\|V\|_{L^{\frac{3}{2}}(\mathbb{R}^3)} < 4\pi, \quad \sqrt{c_{HLS}}\|U\|_{L^{\frac{3}{2}}(\mathbb{R}^3)} < 4\pi.$$

Here and below C denotes a finite positive constant and c_{HLS} given on p.98 of [12] is the constant in the Hardy-Littlewood-Sobolev inequality

$$\left|\int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\frac{f_1(x)f_1(y)}{|x-y|^2}dxdy\right| \leq c_{HLS}\|f_1\|_{L^{\frac{3}{2}}(\mathbb{R}^3)}^2, \quad f_1 \in L^{\frac{3}{2}}(\mathbb{R}^3).$$

The norm of a function $f_1 \in L^p(\mathbb{R}^d),\ 1 \leq p \leq \infty,\ d \in \mathbb{N}$ is denoted as $\|f_1\|_{L^p(\mathbb{R}^d)}$. We designate the inner product of two functions as

$$(f(x), g(x))_{L^2(\mathbb{R}^d)} := \int_{\mathbb{R}^d} f(x)\bar{g}(x)dx, \tag{1.6}$$

with a slight abuse of notations when such functions are not square integrable. Indeed, if $f(x) \in L^1(\mathbb{R}^d)$ and $g(x)$ is bounded like, for example the functions of the continuos spectrum of the Schrödinger operators discussed below, then the integral in the right side of (1.6) is well defined. By means of Lemma 2.3 of [21], under Assumption 1 above on the scalar potentials, operator (1.3) considered as acting in $L^2(\mathbb{R}^6)$ with domain $H^1(\mathbb{R}^6)$ is self-adjoint and is unitarily equivalent to $\sqrt{-\Delta_x - \Delta_y}$ on $L^2(\mathbb{R}^6)$ via the product of the wave operators (see [11], [14])

$$\Omega_V^\pm := s - \lim_{t\to\mp\infty} e^{it(-\Delta_x+V(x))}e^{it\Delta_x}, \quad \Omega_U^\pm := s - \lim_{t\to\mp\infty} e^{it(-\Delta_y+U(y))}e^{it\Delta_y},$$

with the limits here understood in the strong L^2 sense (see e.g. [13] p.34, [7] p.90). Hence, operator (1.3) has no nontrivial $L^2(\mathbb{R}^6)$ eigenfunctions. Its essential spectrum fills the nonnegative semi-axis $[0, +\infty)$. Therefore, operator (1.3) does not satisfy the Fredholm property. The functions of the continuos spectrum of the first operator involved in (1.3) are the solutions the Schrödinger equation

$$[-\Delta_x + V(x)]\varphi_k(x) = k^2\varphi_k(x), \quad k \in \mathbb{R}^3,$$

in the integral form the Lippmann-Schwinger equation

$$\varphi_k(x) = \frac{e^{ikx}}{(2\pi)^{\frac{3}{2}}} - \frac{1}{4\pi}\int_{\mathbb{R}^3}\frac{e^{i|k||x-y|}}{|x-y|}(V\varphi_k)(y)dy \tag{1.7}$$

and the orthogonality conditions $(\varphi_k(x), \varphi_{k_1}(x))_{L^2(\mathbb{R}^3)} = \delta(k - k_1), \ k, k_1 \in \mathbb{R}^3$. The integral operator involved in (1.7)

$$(Q\varphi)(x) := -\frac{1}{4\pi}\int_{\mathbb{R}^3} \frac{e^{i|k||x-y|}}{|x-y|}(V\varphi)(y)dy, \quad \varphi(x) \in L^\infty(\mathbb{R}^3).$$

We consider $Q : L^\infty(\mathbb{R}^3) \to L^\infty(\mathbb{R}^3)$ and its norm $\|Q\|_\infty < 1$ under Assumption 1 via Lemma 2.1 of [21]. In fact, this norm is bounded above by the k-independent quantity $I(V)$, which is the left side of inequality (1.4). Analogously, for the second operator involved in (1.3) the functions of its continuous spectrum solve

$$[-\Delta_y + U(y)]\eta_q(y) = q^2\eta_q(y), \quad q \in \mathbb{R}^3,$$

in the integral formulation

$$\eta_q(y) = \frac{e^{iqy}}{(2\pi)^{\frac{3}{2}}} - \frac{1}{4\pi}\int_{\mathbb{R}^3} \frac{e^{i|q||y-z|}}{|y-z|}(U\eta_q)(z)dz, \tag{1.8}$$

such that the the orthogonality relations $(\eta_q(y), \eta_{q_1}(y))_{L^2(\mathbb{R}^3)} = \delta(q-q_1), \ q, q_1 \in \mathbb{R}^3$ hold. The integral operator involved in (1.8) is

$$(P\eta)(y) := -\frac{1}{4\pi}\int_{\mathbb{R}^3} \frac{e^{i|q||y-z|}}{|y-z|}(U\eta)(z)dz, \quad \eta(y) \in L^\infty(\mathbb{R}^3).$$

For $P : L^\infty(\mathbb{R}^3) \to L^\infty(\mathbb{R}^3)$ its norm $\|P\|_\infty < 1$ under Assumption 1 by virtue of Lemma 2.1 of [21]. As before, this norm can be estimated above by the q-independent quantity $I(U)$, which is the left side of inequality (1.5). Let us denote by the double tilde sign the generalized Fourier transform with the product of these functions of the continuous spectrum

$$\tilde{\tilde{f}}(k, q) := (f(x, y), \varphi_k(x)\eta_q(y))_{L^2(\mathbb{R}^6)}, \quad k, q \in \mathbb{R}^3. \tag{1.9}$$

(1.9) is a unitary transform on $L^2(\mathbb{R}^6)$. We will be using the Sobolev space

$$H^1(\mathbb{R}^d) = \{u(x) : \mathbb{R}^d \to \mathbb{C} \mid u(x) \in L^2(\mathbb{R}^d), \ \nabla u \in L^2(\mathbb{R}^d)\}$$

equipped with the norm

$$\|u\|^2_{H^1(\mathbb{R}^d)} = \|u\|^2_{L^2(\mathbb{R}^d)} + \|\nabla u\|^2_{L^2(\mathbb{R}^d)}, \quad d \in \mathbb{N}.$$

Our first main proposition is as follows.

Theorem 2. *Let Assumption 1 hold and* $f(x, y) \in L^2(\mathbb{R}^6)$.

a) When $a = 0$*, let in addition* $f(x, y) \in L^1(\mathbb{R}^6)$*. Then equation (1.2) admits a unique solution* $u(x, y) \in H^1(\mathbb{R}^6)$.

b) When $a > 0$, let in addition $xf(x,y),\ yf(x,y) \in L^1(\mathbb{R}^6)$. Then problem (1.2) possesses a unique solution $u(x,y) \in H^1(\mathbb{R}^6)$ if and only if

$$(f(x,y), \varphi_k(x)\eta_q(y))_{L^2(\mathbb{R}^6)} = 0, \quad (k,q) \in S_a^6. \tag{1.10}$$

Here and below S_a^d stands for the sphere in $\mathbb{R}^d$ of radius a centered at the origin. Such unit sphere will be denoted as S^d and its Lebesgue measure as $|S^d|$. Note that in the case of $a = 0$ in the theorem above no orthogonality conditions are needed to solve equation (1.2) in $H^1(\mathbb{R}^6)$.

Then we turn our attention to the issue of the solvability in the sense of sequences for our problem. The corresponding sequence of equations with $n \in \mathbb{N}$ is given by

$$\sqrt{-\Delta_x + V(x) - \Delta_y + U(y)}u_n - au_n = f_n(x,y), \quad x,y \in \mathbb{R}^3 \tag{1.11}$$

with the right sides convergent to the right side of (1.2) in $L^2(\mathbb{R}^6)$ as $n \to \infty$.

Theorem 3. *Let Assumption 1 hold, $n \in \mathbb{N}$ and $f_n(x,y) \in L^2(\mathbb{R}^6)$, such that $f_n(x,y) \to f(x,y)$ in $L^2(\mathbb{R}^6)$ as $n \to \infty$.*

a) When $a = 0$, let in addition $f_n(x,y) \in L^1(\mathbb{R}^6),\ n \in \mathbb{N}$, such that $f_n(x,y) \to f(x,y)$ in $L^1(\mathbb{R}^6)$ as $n \to \infty$. Then equations (1.2) and (1.11) have unique solutions $u(x,y) \in H^1(\mathbb{R}^6)$ and $u_n(x,y) \in H^1(\mathbb{R}^6)$ respectively, such that $u_n(x,y) \to u(x,y)$ in $H^1(\mathbb{R}^6)$ as $n \to \infty$.

b) When $a > 0$, let in addition $xf_n(x,y),\ yf_n(x,y) \in L^1(\mathbb{R}^6),\ n \in \mathbb{N}$, such that $xf_n(x,y) \to xf(x,y),\ yf_n(x,y) \to yf(x,y)$ in $L^1(\mathbb{R}^6)$ as $n \to \infty$ and the orthogonality conditions

$$(f_n(x,y), \varphi_k(x)\eta_q(y))_{L^2(\mathbb{R}^6)} = 0, \quad (k,q) \in S_a^6. \tag{1.12}$$

hold for all $n \in \mathbb{N}$. Then problems (1.2) and (1.11) admit unique solutions $u(x,y) \in H^1(\mathbb{R}^6)$ and $u_n(x,y) \in H^1(\mathbb{R}^6)$ respectively, such that $u_n(x,y) \to u(x,y)$ in $H^1(\mathbb{R}^6)$ as $n \to \infty$.

In the second part of the article we consider the problem

$$\sqrt{-\Delta_x - \Delta_y + U(y)}u - au = \phi(x,y), \quad x \in \mathbb{R}^d, \quad y \in \mathbb{R}^3 \tag{1.13}$$

with $d \in \mathbb{N}$ and the scalar potential function involved in (1.13) is shallow and short-range under Assumption 1 as before. The operator

$$L_U := \sqrt{-\Delta_x - \Delta_y + U(y)} \tag{1.14}$$

here is defined by means of the spectral calculus. Similarly to (1.3), under our assumptions operator (1.14) considered as acting in $L^2(\mathbb{R}^{d+3})$ with domain $H^1(\mathbb{R}^{d+3})$ is self-adjoint and is unitarily equivalent to $\sqrt{-\Delta_x - \Delta_y}$. Therefore, operator (1.14) has no nontrivial $L^2(\mathbb{R}^{d+3})$ eigenfunctions. Its essential spectrum fills the nonnegative semi-axis $[0, +\infty)$ and such that operator (1.14) fails to satisfy the Fredholm property. Let us consider another generalized Fourier transform with the standard Fourier harmonics and the perturbed plane waves

$$\tilde{\hat{\phi}}(k,q) := \left(\phi(x,y), \frac{e^{ikx}}{(2\pi)^{\frac{d}{2}}}\eta_q(y)\right)_{L^2(\mathbb{R}^{d+3})}, \quad k \in \mathbb{R}^d, \quad q \in \mathbb{R}^3. \tag{1.15}$$

(1.15) is a unitary transform on $L^2(\mathbb{R}^{d+3})$. We have the following statement.

Theorem 4. *Let the potential function $U(y)$ satisfy Assumption 1 and $\phi(x,y) \in L^2(\mathbb{R}^{d+3}),\ d \in \mathbb{N}$.*

a) When $a = 0$, let in addition $\phi(x,y) \in L^1(\mathbb{R}^{d+3})$. Then equation (1.13) admits a unique solution $u(x,y) \in H^1(\mathbb{R}^{d+3})$.

b) When $a > 0$, let in addition $x\phi(x,y),\ y\phi(x,y) \in L^1(\mathbb{R}^{d+3})$. Then problem (1.13) has a unique solution $u(x,y) \in H^1(\mathbb{R}^{d+3})$ if and only if

$$\left(\phi(x,y), \frac{e^{ikx}}{(2\pi)^{\frac{d}{2}}}\eta_q(y)\right)_{L^2(\mathbb{R}^{d+3})} = 0, \quad (k,q) \in S_a^{d+3}. \tag{1.16}$$

Note that in the case of $a = 0$ of this theorem no orthogonality relations are needed to solve problem (1.13) in $H^1(\mathbb{R}^{d+3})$.

Our final main proposition deals with the issue of the solvability in the sense of sequences for our problem. The corresponding sequence of equations with $n \in \mathbb{N}$ is given by

$$\sqrt{-\Delta_x - \Delta_y + U(y)}u_n - au_n = \phi_n(x,y), \quad x \in \mathbb{R}^d, \quad d \in \mathbb{N}, \quad y \in \mathbb{R}^3 \tag{1.17}$$

with the right sides convergent to the right side of (1.13) in $L^2(\mathbb{R}^{d+3})$ as $n \to \infty$.

Theorem 5. *Let the potential function $U(y)$ satisfy Assumption 1, $n \in \mathbb{N}$ and $\phi_n(x,y) \in L^2(\mathbb{R}^{d+3}),\ d \in \mathbb{N}$, such that $\phi_n(x,y) \to \phi(x,y)$ in $L^2(\mathbb{R}^{d+3})$ as $n \to \infty$.*

a) When $a = 0$, let in addition $\phi_n(x,y) \in L^1(\mathbb{R}^{d+3}),\ n \in \mathbb{N}$, such that $\phi_n(x,y) \to \phi(x,y)$ in $L^1(\mathbb{R}^{d+3})$ as $n \to \infty$. Then equations (1.13) and (1.17) possess unique solutions $u(x,y) \in H^1(\mathbb{R}^{d+3})$ and $u_n(x,y) \in H^1(\mathbb{R}^{d+3})$ respectively, such that $u_n(x,y) \to u(x,y)$ in $H^1(\mathbb{R}^{d+3})$ as $n \to \infty$.

b) When $a > 0$, let in addition $x\phi_n(x,y),\ y\phi_n(x,y) \in L^1(\mathbb{R}^{d+3})$, such that $x\phi_n(x,y) \to x\phi(x,y),\ y\phi_n(x,y) \to y\phi(x,y)$ in $L^1(\mathbb{R}^{d+3})$ as $n \to \infty$ and the orthogonality relations

$$\left(\phi_n(x,y), \frac{e^{ikx}}{(2\pi)^{\frac{d}{2}}}\eta_q(y)\right)_{L^2(\mathbb{R}^{d+3})} = 0, \quad (k,q) \in S_a^{d+3}. \tag{1.18}$$

hold for all $n \in \mathbb{N}$. Then problems (1.13) and (1.17) admit unique solutions $u(x,y) \in H^1(\mathbb{R}^{d+3})$ and $u_n(x,y) \in H^1(\mathbb{R}^{d+3})$ respectively, such that $u_n(x,y) \to u(x,y)$ in $H^1(\mathbb{R}^{d+3})$ as $n \to \infty$.

Let us note that (1.10), (1.12), (1.16), (1.18) are the orthogonality conditions involving the functions of the continuous spectrum of our Schrödinger operators, as distinct from the Limiting Absorption Principle in which one orthogonalizes to the standard Fourier harmonics (see e.g. Lemma 2.3 and Proposition 2.4 of [10]). We proceed to the proof of our statements.

2. Solvability in the sense of sequences with two potentials

Proof of Theorem 2. Let us note that it is sufficient to solve equation (1.2) in $L^2(\mathbb{R}^6)$, because its square integrable solution will belong to $H^1(\mathbb{R}^6)$ as well. Indeed, using definition (1.3) it can be trivially verified that $\|H_{U,\,V}u\|^2_{L^2(\mathbb{R}^6)}$ equals to

$$\|\nabla u\|^2_{L^2(\mathbb{R}^6)} + \int_{\mathbb{R}^6} V(x)|u(x,y)|^2 dxdy + \int_{\mathbb{R}^6} U(y)|u(x,y)|^2 dxdy, \tag{2.19}$$

where $u(x,y)$ is a square integrable solution of (1.2), the scalar potentials $V(x)$ and $U(y)$ are bounded by means of Assumption 1 and $f(x,y) \in L^2(\mathbb{R}^6)$ by virtue of the one of our assumptions. Then (2.19) yields $\nabla u(x,y) \in L^2(\mathbb{R}^6)$, such that $u(x,y) \in H^1(\mathbb{R}^6)$.

To prove the uniqueness of solutions for our problem, we suppose that equation (1.2) has two square integrable solutions $u_1(x,y)$ and $u_2(x,y)$. Then their difference $w(x,y) := u_1(x,y) - u_2(x,y) \in L^2(\mathbb{R}^6)$ satisfies the equation

$$H_{U,\,V}w = aw.$$

Since operator (1.3) has no nontrivial square integrable eigenfunctions in the whole space as discussed above, we have $w(x,y) = 0$ a.e. in $\mathbb{R}^6$.

First of all, we consider the case of our theorem when $a = 0$. Let us apply the generalized Fourier transform (1.9) to both sides of problem (1.2). This yields

$$\tilde{\tilde{u}}(k,q) = \frac{\tilde{\tilde{f}}(k,q)}{\sqrt{k^2+q^2}}\chi_{\left\{\sqrt{k^2+q^2}\le 1\right\}} + \frac{\tilde{\tilde{f}}(k,q)}{\sqrt{k^2+q^2}}\chi_{\left\{\sqrt{k^2+q^2}>1\right\}} \tag{2.20}$$

with $k, q \in \mathbb{R}^3$. Here and throughout the paper χ_A will denote the characteristic function of a set $A \subseteq \mathbb{R}^d$. Obviously, the second term in the right side of (2.20) can be estimated from above in the absolute value by $\tilde{\tilde{f}}(k,q) \in L^2(\mathbb{R}^6)$ due to the one of our assumptions. The first term in the right side of (2.20) can be easily estimated from above in the absolute value by virtue of Corollary 2.2 of [21] by

$$\frac{1}{(2\pi)^3}\frac{1}{1-I(V)}\frac{1}{1-I(U)}\|f\|_{L^1(\mathbb{R}^6)}\frac{\chi_{\left\{\sqrt{k^2+q^2}\leq 1\right\}}}{\sqrt{k^2+q^2}}.$$

Therefore,

$$\left\|\frac{\tilde{\tilde{f}}(k,q)}{\sqrt{k^2+q^2}}\chi_{\left\{\sqrt{k^2+q^2}\leq 1\right\}}\right\|_{L^2(\mathbb{R}^6)} \leq \frac{1}{(2\pi)^3}\frac{1}{1-I(V)}\frac{1}{1-I(U)}\|f\|_{L^1(\mathbb{R}^6)}\frac{\sqrt{|S^6|}}{2},$$

which is finite as assumed in the theorem. Hence the unique solution $u(x,y) \in L^2(\mathbb{R}^6)$.

We conclude the proof with treating the case b) of the theorem. We apply the generalized Fourier transform (1.9) to both sides of equation (1.2) and arrive at

$$\tilde{\tilde{u}}(k,q) = \frac{\tilde{\tilde{f}}(k,q)}{\sqrt{k^2+q^2}-a}.$$

Let us introduce the set

$$A_\delta := \{(k,q) \in \mathbb{R}^6 \mid a-\delta \leq \sqrt{k^2+q^2} \leq a+\delta\}, \quad 0<\delta<a, \tag{2.21}$$

such that

$$\tilde{\tilde{u}}(k,q) = \frac{\tilde{\tilde{f}}(k,q)}{\sqrt{k^2+q^2}-a}\chi_{A_\delta} + \frac{\tilde{\tilde{f}}(k,q)}{\sqrt{k^2+q^2}-a}\chi_{A_\delta^c}. \tag{2.22}$$

Note that for a set $A \subseteq \mathbb{R}^d$ we denote its complement as A^c. Evidently, the second term in the right side of (2.22) can be bounded from above in the absolute value by $\frac{|\tilde{\tilde{f}}(k,q)|}{\delta} \in L^2(\mathbb{R}^6)$ due to the one of our assumptions. Clearly, we have the representation

$$\tilde{\tilde{f}}(k,q) = \tilde{\tilde{f}}(a,\sigma) + \int_a^{\sqrt{k^2+q^2}} \frac{\partial \tilde{\tilde{f}}(s,\sigma)}{\partial s} ds.$$

Here and below σ will denote the angle variables on the sphere. This enables us to express the first term in the right side of (2.22) as

$$\frac{\tilde{\tilde{f}}(a,\sigma)}{\sqrt{k^2+q^2}-a}\chi_{A_\delta} + \frac{\int_a^{\sqrt{k^2+q^2}} \frac{\partial \tilde{\tilde{f}}(s,\sigma)}{\partial s} ds}{\sqrt{k^2+q^2}-a}\chi_{A_\delta}. \tag{2.23}$$

Evidently, we can estimate the second term in (2.23) from above in the absolute value by

$$\|(\nabla_k + \nabla_q)\tilde{\tilde{f}}(k,q)\|_{L^\infty(\mathbb{R}^6)}\chi_{A_\delta} \in L^2(\mathbb{R}^6),$$

where the gradients ∇_k and ∇_q act on variables k and q respectively. Note that under our assumptions $(\nabla_k + \nabla_q)\tilde{\tilde{f}}(k,q) \in L^\infty(\mathbb{R}^6)$ by means of Lemma 11 of [19]. Apparently, the first term in (2.23) is square integrable if and only if $\tilde{\tilde{f}}(a,\sigma)$ vanishes, which is equivalent to orthogonality condition (1.10). ∎

Let us turn our attention to establishing the solvability in the sense of sequences for our equation in the case of two scalar potentials.

Proof of Theorem 3. Suppose $u(x,y)$ and $u_n(x,y),\ n \in \mathbb{N}$ are the unique solutions of equations (1.2) and (1.11) in $H^1(\mathbb{R}^6)$ with $a \geq 0$ respectively and it is known that $u_n(x,y) \to u(x,y)$ in $L^2(\mathbb{R}^6)$ as $n \to \infty$. Then, it will follow that $u_n(x,y) \to u(x,y)$ in $H^1(\mathbb{R}^6)$ as $n \to \infty$ as well. Indeed, from (1.2) and (1.11) we easily derive that

$$H_{U,\,V}(u_n(x,y) - u(x,y)) = a(u_n(x,y) - u(x,y)) + [f_n(x,y) - f(x,y)],$$

which clearly implies

$$\|H_{U,\,V}(u_n(x,y) - u(x,y))\|_{L^2(\mathbb{R}^6)} \leq a\|u_n(x,y) - u(x,y)\|_{L^2(\mathbb{R}^6)} +$$

$$+\|f_n(x,y) - f(x,y)\|_{L^2(\mathbb{R}^6)} \to 0, \quad n \to \infty$$

by means of our assumptions. We express

$$\|H_{U,\,V}(u_n(x,y) - u(x,y))\|^2_{L^2(\mathbb{R}^6)} = \|\nabla(u_n(x,y) - u(x,y))\|^2_{L^2(\mathbb{R}^6)} +$$

$$+\int_{\mathbb{R}^6} V(x)|u_n(x,y) - u(x,y)|^2 dxdy + \int_{\mathbb{R}^6} U(y)|u_n(x,y) - u(x,y)|^2 dxdy$$

with the bounded scalar potentials $V(x)$ and $U(y)$ due to Assumption 1. Thus, in the identity above the left side along with the second and the last term in the right side tend to zero as $n \to \infty$. This yields that $\nabla u_n(x,y) \to \nabla u(x,y)$ in $L^2(\mathbb{R}^6)$ as $n \to \infty$, such that $u_n(x,y) \to u(x,y)$ in $H^1(\mathbb{R}^6)$ as $n \to \infty$ as well.

In the case a) problems (1.2) and (1.11) have unique solutions $u(x,y), u_n(x,y)$ belonging to $H^1(\mathbb{R}^6)$ respectively with $n \in \mathbb{N}$ by virtue of the part a) of Theorem 2 above. Let us apply the generalized Fourier transform (1.9) to both sides of equations (1.2) and (1.11). This yields

$$\tilde{\tilde{u}}(k,q) = \frac{\tilde{\tilde{f}}(k,q)}{\sqrt{k^2+q^2}}, \quad \tilde{\tilde{u}}_n(k,q) = \frac{\tilde{\tilde{f}}_n(k,q)}{\sqrt{k^2+q^2}}, \quad n \in \mathbb{N}.$$

Thus $\tilde{\tilde{u}}_n(k,q) - \tilde{\tilde{u}}(k,q)$ can be written as

$$\frac{\tilde{\tilde{f}}_n(k,q) - \tilde{\tilde{f}}(k,q)}{\sqrt{k^2+q^2}} \chi_{\left\{\sqrt{k^2+q^2} \le 1\right\}} + \frac{\tilde{\tilde{f}}_n(k,q) - \tilde{\tilde{f}}(k,q)}{\sqrt{k^2+q^2}} \chi_{\left\{\sqrt{k^2+q^2} > 1\right\}}. \tag{2.24}$$

Evidently, the second term in (2.24) can be easily bounded from above in the absolute value by $|\tilde{\tilde{f}}_n(k,q) - \tilde{\tilde{f}}(k,q)|$, such that

$$\left\| \frac{\tilde{\tilde{f}}_n(k,q) - \tilde{\tilde{f}}(k,q)}{\sqrt{k^2+q^2}} \chi_{\left\{\sqrt{k^2+q^2} > 1\right\}} \right\|_{L^2(\mathbb{R}^6)} \le \|f_n(x,y) - f(x,y)\|_{L^2(\mathbb{R}^6)} \to 0$$

as $n \to \infty$ due to the one of our assumptions. We estimate the first term in (2.24) from above in the absolute value by means of the Corollary 2.2 of [21] by

$$\frac{1}{(2\pi)^3} \frac{1}{1 - I(V)} \frac{1}{1 - I(U)} \|f_n(x,y) - f(x,y)\|_{L^1(\mathbb{R}^6)} \frac{\chi_{\left\{\sqrt{k^2+q^2} \le 1\right\}}}{\sqrt{k^2+q^2}},$$

such that

$$\left\| \frac{\tilde{\tilde{f}}_n(k,q) - \tilde{\tilde{f}}(k,q)}{\sqrt{k^2+q^2}} \chi_{\left\{\sqrt{k^2+q^2} \le 1\right\}} \right\|_{L^2(\mathbb{R}^6)} \le$$

$$\le \frac{1}{(2\pi)^3} \frac{1}{1 - I(V)} \frac{1}{1 - I(U)} \|f_n(x,y) - f(x,y)\|_{L^1(\mathbb{R}^6)} \frac{\sqrt{|S^6|}}{2} \to 0, \quad n \to \infty$$

according to the one of our assumptions. Therefore, $u_n(x,y) \to u(x,y)$ in $L^2(\mathbb{R}^6)$ as $n \to \infty$ in the case when the parameter $a = 0$.

Then we proceed to the proof of the part b) of the theorem. For each $n \in \mathbb{N}$ equation (1.11) admits a unique solution $u_n(x,y) \in H^1(\mathbb{R}^6)$ by means of the result of the part b) of Theorem 2 above. By virtue of (1.12) along with Corollary 2.2 of [21], we estimate for $(k,q) \in S_a^6$

$$|(f(x,y), \varphi_k(x)\eta_q(y))_{L^2(\mathbb{R}^6)}| = |(f(x,y) - f_n(x,y), \varphi_k(x)\eta_q(y))_{L^2(\mathbb{R}^6)}| \le$$

$$\le \frac{1}{(2\pi)^3} \frac{1}{1 - I(V)} \frac{1}{1 - I(U)} \|f_n(x,y) - f(x,y)\|_{L^1(\mathbb{R}^6)} \to 0, \quad n \to \infty.$$

Note that under our assumptions $f_n(x,y) \to f(x,y)$ in $L^1(\mathbb{R}^6)$ via the simple argument on p.114 of [26]. Hence, we obtain

$$(f(x,y), \varphi_k(x)\eta_q(y))_{L^2(\mathbb{R}^6)} = 0, \quad (k,q) \in S_a^6. \tag{2.25}$$

Therefore, equation (1.2) admits a unique solution $u(x,y) \in H^1(\mathbb{R}^6)$ due to the result of the part b) of Theorem 2 above. We apply the generalized Fourier transform (1.9) to both sides of problems (1.2) and (1.11). This gives us

$$\tilde{\tilde{u}}_n(k,q) - \tilde{\tilde{u}}(k,q) = \frac{\tilde{\tilde{f}}_n(k,q) - \tilde{\tilde{f}}(k,q)}{\sqrt{k^2+q^2} - a} \chi_{A_\delta} + \frac{\tilde{\tilde{f}}_n(k,q) - \tilde{\tilde{f}}(k,q)}{\sqrt{k^2+q^2} - a} \chi_{A_\delta^c} \tag{2.26}$$

with the set A_δ defined in (2.21). Clearly, the second term in the right side of (2.26) can be bounded from above in the absolute value by $\dfrac{|\tilde{\tilde{f}}_n(k,q)-\tilde{\tilde{f}}(k,q)|}{\delta}$, such that

$$\left\|\frac{\tilde{\tilde{f}}_n(k,q)-\tilde{\tilde{f}}(k,q)}{\sqrt{k^2+q^2}-a}\chi_{A_\delta^c}\right\|_{L^2(\mathbb{R}^6)} \leq \frac{\|f_n(x,y)-f(x,y)\|_{L^2(\mathbb{R}^6)}}{\delta}\to 0, \quad n\to\infty$$

due to the one of our assumptions. Orthogonality conditions (1.12) and (2.25) yield

$$\tilde{\tilde{f}}(a,\sigma)=0, \quad \tilde{\tilde{f}}_n(a,\sigma)=0, \quad n\in\mathbb{N},$$

such that

$$\tilde{\tilde{f}}(k,q)=\int_a^{\sqrt{k^2+q^2}}\frac{\partial\tilde{\tilde{f}}(s,\sigma)}{\partial s}ds, \quad \tilde{\tilde{f}}_n(k,q)=\int_a^{\sqrt{k^2+q^2}}\frac{\partial\tilde{\tilde{f}}_n(s,\sigma)}{\partial s}ds, \quad n\in\mathbb{N}.$$

This enables us to write the first term in the right side of (2.26) as

$$\frac{\int_a^{\sqrt{k^2+q^2}}\left[\frac{\partial\tilde{\tilde{f}}_n(s,\sigma)}{\partial s}-\frac{\partial\tilde{\tilde{f}}(s,\sigma)}{\partial s}\right]ds}{\sqrt{k^2+q^2}-a}\chi_{A_\delta}. \tag{2.27}$$

Obviously, (2.27) can be bounded from above in the absolute value by

$$\|(\nabla_k+\nabla_q)(\tilde{\tilde{f}}_n(k,q)-\tilde{\tilde{f}}(k,q))\|_{L^\infty(\mathbb{R}^6)}\chi_{A_\delta}.$$

This allows us to estimate the $L^2(\mathbb{R}^6)$ norm of (2.27) from above by

$$C\|(\nabla_k+\nabla_q)(\tilde{\tilde{f}}_n(k,q)-\tilde{\tilde{f}}(k,q))\|_{L^\infty(\mathbb{R}^6)}\to 0, \quad n\to\infty$$

by means of the part a) of Lemma 5 of [26] under our assumptions. Therefore, $u_n(x,y)\to u(x,y)$ in $L^2(\mathbb{R}^6)$ as $n\to\infty$. ∎

In the last section of the article we treat the situation when a free Laplace operator is added to the three dimensional Schrödinger operator.

3. Solvability in the sense of sequences with Laplacian and a single potential

Proof of Theorem 4. Evidently, it is sufficient to solve problem (1.13) in $L^2(\mathbb{R}^{d+3})$, since its square integrable solution will belong to $H^1(\mathbb{R}^{d+3})$ as well. Indeed, by means of definition (1.14) it can be easily verified that $\|L_U u\|^2_{L^2(\mathbb{R}^{d+3})}$ is equal to

$$\|\nabla u\|^2_{L^2(\mathbb{R}^{d+3})}+\int_{\mathbb{R}^{d+3}}U(y)|u(x,y)|^2dxdy, \quad d\in\mathbb{N} \tag{3.28}$$

where $u(x,y)$ is a square integrable solution of (1.13), the scalar potential function $U(y)$ is bounded due to Assumption 1 and $\phi(x,y) \in L^2(\mathbb{R}^{d+3})$ by means of the one of our assumptions. Then (3.28) implies that $\nabla u(x,y) \in L^2(\mathbb{R}^{d+3})$, such that $u(x,y) \in H^1(\mathbb{R}^{d+3})$.

To establish the uniqueness of solutions for our equation, let us suppose that (1.13) admits two square integrable solutions $u_1(x,y)$ and $u_2(x,y)$. Then their difference $w(x,y) := u_1(x,y) - u_2(x,y) \in L^2(\mathbb{R}^{d+3})$ is a solution of the equation

$$L_U w = aw.$$

Since operator (1.14) does not have nontrivial square integrable eigenfunctions in the whole space as mentioned above, we have $w(x,y) = 0$ a.e. in $\mathbb{R}^{d+3}$.

Let us first treat the case of our theorem when $a = 0$. We apply the generalized Fourier transform (1.15) to both sides of equation (1.13). This yields

$$\tilde{\hat{u}}(k,q) = \frac{\tilde{\hat{\phi}}(k,q)}{\sqrt{k^2+q^2}}\chi_{\left\{\sqrt{k^2+q^2}\le 1\right\}} + \frac{\tilde{\hat{\phi}}(k,q)}{\sqrt{k^2+q^2}}\chi_{\left\{\sqrt{k^2+q^2}>1\right\}} \tag{3.29}$$

with $k \in \mathbb{R}^d,\ q \in \mathbb{R}^3$. Clearly, the second term in (3.29) can be bounded from above in the abosolute value by $|\tilde{\hat{\phi}}(k,q)| \in L^2(\mathbb{R}^{d+3})$ due to the one of our assumptions. Corollary 2.2 of [21] yields

$$|\tilde{\hat{\phi}}(k,q)| \le \frac{1}{(2\pi)^{\frac{d+3}{2}}}\frac{1}{1-I(u)}\|\phi(x,y)\|_{L^1(\mathbb{R}^{d+3})},$$

such that the first term in (3.29) can be estimated from above in the abosolute value by

$$\frac{1}{(2\pi)^{\frac{d+3}{2}}}\frac{1}{1-I(u)}\|\phi(x,y)\|_{L^1(\mathbb{R}^{d+3})}\frac{\chi_{\left\{\sqrt{k^2+q^2}\le 1\right\}}}{\sqrt{k^2+q^2}}.$$

This implies that

$$\left\|\frac{\tilde{\hat{\phi}}(k,q)}{\sqrt{k^2+q^2}}\chi_{\left\{\sqrt{k^2+q^2}\le 1\right\}}\right\|_{L^2(\mathbb{R}^{d+3})} \le$$

$$\le \frac{1}{(2\pi)^{\frac{d+3}{2}}}\frac{1}{1-I(u)}\|\phi(x,y)\|_{L^1(\mathbb{R}^{d+3})}\sqrt{\frac{|S^{d+3}|}{d+1}},$$

which is finite as assumed. Thus, $u(x,y) \in L^2(\mathbb{R}^{d+3})$ in the case of the theorem when $a = 0$.

Let us conclude the proof by addressing the case b) of the theorem. Let us apply the generalized Fourier transform (1.15) to both sides of problem (1.13) and derive

$$\tilde{\hat{u}}(k,q) = \frac{\tilde{\hat{\phi}}(k,q)}{\sqrt{k^2+q^2-a}}.$$

We introduce the set

$$B_\delta := \{(k,q)\in\mathbb{R}^{d+3} \mid a-\delta \le \sqrt{k^2+q^2} \le a+\delta\}, \quad 0<\delta<a. \tag{3.30}$$

Hence

$$\tilde{u}(k,q) = \frac{\tilde{\hat{\tilde{\phi}}}(k,q)}{\sqrt{k^2+q^2}-a}\chi_{B_\delta} + \frac{\tilde{\hat{\tilde{\phi}}}(k,q)}{\sqrt{k^2+q^2}-a}\chi_{B_\delta^c}. \tag{3.31}$$

Clearly, the second term in the right side of (3.31) can be estimated from above in the absolute value by $\frac{|\tilde{\hat{\tilde{\phi}}}(k,q)|}{\delta} \in L^2(\mathbb{R}^{d+3})$ due to the one of our assumptions. Evidently, we have the representation

$$\tilde{\hat{\tilde{\phi}}}(k,q) = \tilde{\hat{\tilde{f}}}(a,\sigma) + \int_a^{\sqrt{k^2+q^2}} \frac{\partial\tilde{\hat{\tilde{\phi}}}(s,\sigma)}{\partial s} ds.$$

This allows us to express the first term in the right side of (3.31) as

$$\frac{\tilde{\hat{\tilde{\phi}}}(a,\sigma)}{\sqrt{k^2+q^2}-a}\chi_{B_\delta} + \frac{\int_a^{\sqrt{k^2+q^2}} \frac{\partial\tilde{\hat{\tilde{\phi}}}(s,\sigma)}{\partial s} ds}{\sqrt{k^2+q^2}-a}\chi_{B_\delta}. \tag{3.32}$$

Apparently, we have the upper bound for the second term in (3.32) from above in the absolute value by

$$\|(\nabla_k+\nabla_q)\tilde{\hat{\tilde{\phi}}}(k,q)\|_{L^\infty(\mathbb{R}^{d+3})}\chi_{B_\delta} \in L^2(\mathbb{R}^{d+3}).$$

Note that under our assumptions $(\nabla_k+\nabla_q)\tilde{\hat{\tilde{\phi}}}(k,q) \in L^\infty(\mathbb{R}^{d+3})$ via Lemma 12 of [19]. It can be easily verified that, the first term in (3.32) is square integrable if and only if $\tilde{\hat{\tilde{\phi}}}(a,\sigma)$ vanishes, which is equivalent to orthogonality relation (1.16). ■

We conclude the article with establishing the solvability in the sense of sequences for our problem in the case of a free Laplacian added to a three dimensional Schrödinger operator.

Proof of Theorem 5. Suppose $u(x,y)$ and $u_n(x,y),\ n\in\mathbb{N}$ are the unique solutions of problems (1.13) and (1.17) in $H^1(\mathbb{R}^{d+3})$ with $a\ge 0$ respectively and it is known that $u_n(x,y)\to u(x,y)$ in $L^2(\mathbb{R}^{d+3})$ as $n\to\infty$. Then, it will can be shown that $u_n(x,y)\to u(x,y)$ in $H^1(\mathbb{R}^{d+3})$ as $n\to\infty$ as well. Indeed, from (1.13) and (1.17) we easily obtain that

$$L_U(u_n(x,y)-u(x,y)) = a(u_n(x,y)-u(x,y)) + [\phi_n(x,y)-\phi(x,y)].$$

Clearly, this yields

$$\|L_U(u_n(x,y)-u(x,y))\|_{L^2(\mathbb{R}^{d+3})} \le a\|u_n(x,y)-u(x,y)\|_{L^2(\mathbb{R}^{d+3})}+$$

$$+\|\phi_n(x,y)-\phi(x,y)\|_{L^2(\mathbb{R}^{d+3})}\to 0, \quad n\to\infty$$

due to our assumptions. Let us express

$$\|L_U(u_n(x,y)-u(x,y))\|^2_{L^2(\mathbb{R}^{d+3})}=\|\nabla(u_n(x,y)-u(x,y))\|^2_{L^2(\mathbb{R}^{d+3})}+$$

$$+\int_{\mathbb{R}^{d+3}}U(y)|u_n(x,y)-u(x,y)|^2dxdy,$$

where the scalar potential $U(y)$ is bounded via Assumption 1. Hence, in the equality above the left side along with the second term in the right side tend to zero as $n\to\infty$. This implies that $\nabla u_n(x,y)\to\nabla u(x,y)$ in $L^2(\mathbb{R}^{d+3})$ as $n\to\infty$, such that $u_n(x,y)\to u(x,y)$ in $H^1(\mathbb{R}^{d+3})$ as $n\to\infty$ as well.

In the case a) (1.13) and (1.17) admit unique solutions $u(x,y), u_n(x,y)$ belonging to $H^1(\mathbb{R}^{d+3})$ respectively with $n\in\mathbb{N}$ by means of the part a) of Theorem 4 above. We apply the generalized Fourier transform (1.15) to both sides of problems (1.13) and (1.17). This gives us

$$\tilde{\hat{u}}(k,q)=\frac{\tilde{\hat{\phi}}(k,q)}{\sqrt{k^2+q^2}}, \quad \tilde{\hat{u}}_n(k,q)=\frac{\tilde{\hat{\phi}}_n(k,q)}{\sqrt{k^2+q^2}}, \quad n\in\mathbb{N}.$$

Hence $\tilde{\hat{u}}_n(k,q)-\tilde{\hat{u}}(k,q)$ can be expressed as

$$\frac{\tilde{\hat{\phi}}_n(k,q)-\tilde{\hat{\phi}}(k,q)}{\sqrt{k^2+q^2}}\chi_{\left\{\sqrt{k^2+q^2}\le 1\right\}}+\frac{\tilde{\hat{\phi}}_n(k,q)-\tilde{\hat{\phi}}(k,q)}{\sqrt{k^2+q^2}}\chi_{\left\{\sqrt{k^2+q^2}>1\right\}}. \tag{3.33}$$

Obviously, the second term in (3.33) can be trivially estimated from above in the absolute value by $|\tilde{\hat{\phi}}_n(k,q)-\tilde{\hat{\phi}}(k,q)|$, such that

$$\left\|\frac{\tilde{\hat{\phi}}_n(k,q)-\tilde{\hat{\phi}}(k,q)}{\sqrt{k^2+q^2}}\chi_{\left\{\sqrt{k^2+q^2}>1\right\}}\right\|_{L^2(\mathbb{R}^{d+3})}\le\|\phi_n(x,y)-\phi(x,y)\|_{L^2(\mathbb{R}^{d+3})}\to 0$$

as $n\to\infty$ via the one of our assumptions. Let us obtain the upper bound in the the absolute value for the first term in (3.33) via the Corollary 2.2 of [21] by

$$\frac{1}{(2\pi)^{\frac{d+3}{2}}}\frac{1}{1-I(U)}\|\phi_n(x,y)-\phi(x,y)\|_{L^1(\mathbb{R}^{d+3})}\frac{\chi_{\left\{\sqrt{k^2+q^2}\le 1\right\}}}{\sqrt{k^2+q^2}},$$

such that

$$\left\|\frac{\tilde{\hat{\phi}}_n(k,q)-\tilde{\hat{f}}(k,q)}{\sqrt{k^2+q^2}}\chi_{\left\{\sqrt{k^2+q^2}\le 1\right\}}\right\|_{L^2(\mathbb{R}^{d+3})}\le$$

$$\le\frac{1}{(2\pi)^{\frac{d+3}{2}}}\frac{1}{1-I(U)}\|\phi_n(x,y)-\phi(x,y)\|_{L^1(\mathbb{R}^{d+3})}\frac{\sqrt{|S^{d+3}|}}{d+1}\to 0, \quad n\to\infty$$

via the one of our assumptions. Therefore, $u_n(x,y) \to u(x,y)$ in $L^2(\mathbb{R}^{d+3})$ as $n \to \infty$ when the parameter $a=0$.

Finally, let us proceed to the proof of the part b) of the theorem. For each $n \in \mathbb{N}$ problem (1.17) has a unique solution $u_n(x,y) \in H^1(\mathbb{R}^{d+3})$ via the result of the part b) of Theorem 4 above. By means of (1.16) along with Corollary 2.2 of [21], we estimate for $(k,q) \in S_a^{d+3}$

$$\left|\left(\phi(x,y), \frac{e^{ikx}}{(2\pi)^{\frac{d}{2}}}\eta_q(y)\right)_{L^2(\mathbb{R}^{d+3})}\right| = \left|\left(\phi(x,y)-\phi_n(x,y), \frac{e^{ikx}}{(2\pi)^{\frac{d}{2}}}\eta_q(y)\right)_{L^2(\mathbb{R}^{d+3})}\right| \leq$$

$$\leq \frac{1}{(2\pi)^{\frac{d+3}{2}}}\frac{1}{1-I(U)}\|\phi_n(x,y)-\phi(x,y)\|_{L^1(\mathbb{R}^{d+3})} \to 0, \quad n \to \infty.$$

Note that under our assumptions $\phi_n(x,y) \to \phi(x,y)$ in $L^1(\mathbb{R}^{d+3})$ via the elementary argument on p.116 of [26]. Thus, we arrive at

$$\left(\phi(x,y), \frac{e^{ikx}}{(2\pi)^{\frac{d}{2}}}\eta_q(y)\right)_{L^2(\mathbb{R}^{d+3})} = 0, \quad (k,q) \in S_a^{d+3}. \tag{3.34}$$

Therefore, problem (1.13) has a unique solution $u(x,y) \in H^1(\mathbb{R}^{d+3})$ via the result of the part b) of Theorem 4 above. Let us apply the generalized Fourier transform (1.15) to both sides of equations (1.13) and (1.17). This yields

$$\tilde{\hat{u}}_n(k,q) - \tilde{\hat{u}}(k,q) = \frac{\tilde{\hat{\phi}}_n(k,q) - \tilde{\hat{\phi}}(k,q)}{\sqrt{k^2+q^2-a}}\chi_{B_\delta} + \frac{\tilde{\hat{\phi}}_n(k,q) - \tilde{\hat{\phi}}(k,q)}{\sqrt{k^2+q^2-a}}\chi_{B_\delta^c} \tag{3.35}$$

with the set B_δ defined in (3.30). Evidently, the second term in the right side of (3.35) can be estimated from above in the absolute value by $\dfrac{|\tilde{\hat{\phi}}_n(k,q) - \tilde{\hat{\phi}}(k,q)|}{\delta}$, such that

$$\left\|\frac{\tilde{\hat{\phi}}_n(k,q) - \tilde{\hat{\phi}}(k,q)}{\sqrt{k^2+q^2-a}}\chi_{B_\delta^c}\right\|_{L^2(\mathbb{R}^{d+3})} \leq \frac{\|\phi_n(x,y)-\phi(x,y)\|_{L^2(\mathbb{R}^{d+3})}}{\delta} \to 0, \quad n \to \infty$$

due to the one of our assumptions. Orthogonality relations (1.16) and (3.34) imply that

$$\tilde{\hat{\phi}}(a,\sigma) = 0, \quad \tilde{\hat{\phi}}_n(a,\sigma) = 0, \quad n \in \mathbb{N},$$

such that

$$\tilde{\hat{\phi}}(k,q) = \int_a^{\sqrt{k^2+q^2}} \frac{\partial \tilde{\hat{\phi}}(s,\sigma)}{\partial s}ds, \quad \tilde{\hat{\phi}}_n(k,q) = \int_a^{\sqrt{k^2+q^2}} \frac{\partial \tilde{\hat{f}}_n(s,\sigma)}{\partial s}ds, \quad n \in \mathbb{N}.$$

This allows us to express the first term in the right side of (3.35) as

$$\frac{\int_a^{\sqrt{k^2+q^2}}\left[\frac{\partial\tilde{\hat{\phi}}_n(s,\sigma)}{\partial s}-\frac{\partial\tilde{\hat{\phi}}(s,\sigma)}{\partial s}\right]ds}{\sqrt{k^2+q^2}-a}\chi_{B_\delta}. \tag{3.36}$$

Apparently, (3.36) can be bounded from above in the absolute value by

$$\|(\nabla_k+\nabla_q)(\tilde{\hat{\phi}}_n(k,q)-\tilde{\hat{\phi}}(k,q))\|_{L^\infty(\mathbb{R}^{d+3})}\chi_{B_\delta},$$

which us to estimate the $L^2(\mathbb{R}^{d+3})$ norm of (3.36) from above by

$$C\|(\nabla_k+\nabla_q)(\tilde{\hat{\phi}}_n(k,q)-\tilde{\hat{\phi}}(k,q))\|_{L^\infty(\mathbb{R}^{d+3})}\to 0, \quad n\to\infty$$

by virtue of the part b) of Lemma 5 of [26] under the given assumptions. This proves that $u_n(x,y)\to u(x,y)$ in $L^2(\mathbb{R}^{d+3})$ as $n\to\infty$. ∎

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