BIRKHOFF NORMAL FORM AND LONG TIME EXISTENCE FOR PERIODIC GRAVITY WATER WAVES

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Abstract. We consider the gravity water waves system with a periodic one-dimensional interface in infinite depth, and prove a rigorous reduction of these equations to Bikhoff normal form up to degree four. This proves a conjecture of Zakharov-Dyachenko [62] based on the formal Birkhoff integrability of the water waves Hamiltonian truncated at order four. As a consequence, we also obtain a long-time stability result: periodic perturbations of a flat interface that are of size ε in a sufficiently smooth Sobolev space lead to solutions that remain regular and small up to times of order ε^{-3}.

Main difficulties in the proof are the quasilinear nature of the equations, the presence of small divisors arising from near-resonances, and non-trivial resonant four-waves interactions, the so-called Benjamin-Feir resonances. The main ingredients that we use are: (1) various reductions to constant coefficient operators through flow conjugation techniques; (2) the verification of key algebraic properties of the gravity water waves system which imply the integrability of the equations at non-negative orders; (3) smoothing procedures and Poincaré-Birkhoff normal form transformations; (4) a normal form identification argument that allows us to handle Benajamin-Feir resonances by comparing with the formal computations of [62, 22, 30, 20].

Contents

1. Introduction

We consider an incompressible and irrotational perfect fluid, under the action of gravity, occupying at time t a two dimensional domain with infinite depth, periodic in the horizontal variable, given by

$$\mathcal{D}_\eta := \big\{(x,y) \in \mathbb{T}\times\mathbb{R}\,;\ -\infty < y < \eta(t,x)\big\}, \quad \mathbb{T} := \mathbb{R}/(2\pi\mathbb{Z})\,, \tag{1.1}$$

where η is a smooth enough function. The velocity field in the time dependent domain $\mathcal{D}_\eta$ is the gradient of a harmonic function Φ, called the velocity potential. The time-evolution of the fluid is determined by a system of equations for the two functions $(t,x) \to \eta(t,x)$, $(t,x,y) \to \Phi(t,x,y)$.

This research was supported by PRIN 2015 "Variational methods, with applications to problems in mathematical physics and geometry". The third author was supported in part by Start-up grants from Princeton University and the University of Toronto, and NSERC grant RGPIN-201.

Following Zakharov [61] and Craig-Sulem [21] we denote $\psi(t,x) = \Phi(t,x,\eta(t,x))$ the restriction of the velocity potential to the free interface. Given the shape $\eta(t,x)$ of the domain $\mathcal{D}_\eta$ and the Dirichlet value $\psi(t,x)$ of the velocity potential at the top boundary, one can recover $\Phi(t,x,y)$ as the unique solution of the elliptic problem

$$\Delta\Phi = 0 \text{ in } \mathcal{D}_\eta\,, \quad \partial_y\Phi \to 0 \text{ as } y \to -\infty\,, \quad \Phi = \psi \text{ on } \{y = \eta(t,x)\}. \tag{1.2}$$

The (η,ψ) variables then satisfy the gravity water waves system

$$\begin{cases} \partial_t\eta = G(\eta)\psi \\ \partial_t\psi = -g\eta - \dfrac{1}{2}\psi_x^2 + \dfrac{1}{2}\dfrac{(\eta_x\psi_x + G(\eta)\psi)^2}{1+\eta_x^2} \end{cases} \tag{1.3}$$

where $G(\eta)\psi$ is the Dirichlet-Neumann operator

$$G(\eta)\psi := \sqrt{1+\eta_x^2}(\partial_n\Phi)|_{y=\eta(t,x)} = (\partial_y\Phi - \eta_x\partial_x\Phi)(t,x,\eta(t,x)) \tag{1.4}$$

and n is the outward unit normal at the free interface $y = \eta(t,x)$. $G(\eta)$ is a pseudo-differential operator with principal symbol $|D|$, self-adjoint with respect to the L^2 scalar product, positive-semidefinite, and its kernel contains only the constant functions. Without loss of generality, we set the gravity constant to $g = 1$.

It was first observed by Zakharov [61] that (1.3) are the Hamiltonian system

$$\partial_t\eta = \nabla_\psi H(\eta,\psi)\,, \quad \partial_t\psi = -\nabla_\eta H(\eta,\psi)\,, \tag{1.5}$$

where ∇ denotes the L^2-gradient, with Hamiltonian

$$H(\eta,\psi) := \frac{1}{2}\int_{\mathbb{T}} \psi\, G(\eta)\psi\, dx + \frac{1}{2}\int_{\mathbb{T}} \eta^2\, dx \tag{1.6}$$

given by the sum of the kinetic and potential energy of the fluid. Recall that the Poisson bracket between functions $H(\eta,\psi), F(\eta,\psi)$ is defined as

$$\{F,H\} = \int_{\mathbb{T}} \big(\nabla_\eta H \nabla_\psi F - \nabla_\psi H \nabla_\eta F\big) dx\,. \tag{1.7}$$

Note that the "mass" $\int_{\mathbb{T}} \eta\, dx$ is a prime integral of (1.3) and, with no loss of generality, we can fix it to zero by shifting the y coordinate. Moreover (1.3) is invariant under spatial translations and Noether's theorem implies that the momentum $\int_{\mathbb{T}} \eta_x(x)\psi(x)\, dx$ is a prime integral of (1.5).

Let $H^s(\mathbb{T}) := H^s$, $s \in \mathbb{R}$, be the Sobolev spaces of 2π-periodic functions of x. The natural phase space of (1.3) is

$$(\eta,\psi) \in H_0^s(\mathbb{T}) \times \dot{H}^s(\mathbb{T}) \tag{1.8}$$

where $\dot{H}^s(\mathbb{T}) := H^s(\mathbb{T})/_\sim$ is the homogeneous Sobolev space obtained by the equivalence relation $\psi_1(x) \sim \psi_2(x)$ if and only if $\psi_1(x) - \psi_2(x) = c$ is a constant[1], and $H_0^s(\mathbb{T})$ is the subspace of $H^s(\mathbb{T})$ of zero average functions. Moreover, since the space averages $\widehat{\eta}_0(t) := \frac{1}{2\pi}\int_{\mathbb{T}} \eta(t,x)\, dx$, $\widehat{\psi}_0(t) := \frac{1}{2\pi}\int_{\mathbb{T}} \psi(t,x)\, dx$ evolve according to the decoupled equations[2]

$$\partial_t\widehat{\eta}_0(t) = 0\,, \quad \partial_t\widehat{\psi}_0(t) = -g\widehat{\eta}_0(t)\,, \tag{1.9}$$

[1]The fact that $\psi \in \dot{H}^s$ is coherent with the fact that only the velocity field $\nabla_{x,y}\Phi$ has physical meaning, and the velocity potential Φ is defined up to a constant. For simplicity of notation we denote the equivalence class $[\psi]$ by ψ and, since the quotient map induces an isometry of $\dot{H}^s(\mathbb{T})$ onto $H_0^s(\mathbb{T})$, we will conveniently identify ψ with a function with zero average.

[2]Since the ocean has infinite depth, if Φ solves (1.2), then $\Phi_c(x,y) := \Phi(x,y-c)$ solves the same problem in $\mathcal{D}_{\eta+c}$ assuming the Dirichlet datum ψ at the free boundary $\eta + c$. Therefore $G(\eta+c) = G(\eta)$, $\forall c \in \mathbb{R}$, and $\int_{\mathbb{T}} \nabla_\eta K\, dx = 0$ where $K := \frac{1}{2}\int_{\mathbb{T}} \psi G(\eta)\psi\, dx$ denotes the kinetic energy.

we may restrict, with no loss of generality, to the invariant subspace

$$\int_{\mathbb{T}} \eta \, dx = \int_{\mathbb{T}} \psi \, dx = 0 \, .$$

The main result of this paper (Theorem 1.1) proves a conjecture of Zakharov-Dyachenko [62] and Craig-Worfolk [22], on the approximate integrability of the water waves system (1.3). More precisely, we show that (1.3) can be conjugated, via a bounded and invertible transformation in a neighborhood of the origin in phase space, to its Birkhoff normal form, up to order 4. This latter was formally computed in [62, 22], see also [20], and, remarkably, shown to be integrable. As a consequence, we obtain a long-time stability result (Theorem 1.2) for small periodic perturbations of flat interfaces: periodic perturbations that are initially ε-close to the flat equilibrium in a sufficiently regular Sobolev space, lead to solutions that remain regular and small for times of order ε^{-3}.

While in recent years several results have been obtained for quasilinear equations set in an Euclidean space $\mathbb{R}^d$, fewer results are available in the periodic setting, or on other compact manifolds. In this context, the achievement of extended stability results through rigorous reductions to high-order Birkhoff normal forms should be seen as a key step to understand the global dynamics of evolution PDEs in non-dispersive settings.

1.1. **Main results.** We denote the horizontal and vertical components of the velocity field at the free interface by

$$V = V(\eta, \psi) := (\partial_x \Phi)(x, \eta(x)) = \psi_x - \eta_x B \, , \tag{1.10}$$

$$B = B(\eta, \psi) := (\partial_y \Phi)(x, \eta(x)) = \frac{G(\eta)\psi + \eta_x \psi_x}{1 + \eta_x^2} \, , \tag{1.11}$$

and the "good unknown" of Alinhac

$$\omega := \psi - Op^{\mathrm{BW}}(B(\eta, \psi))\eta \, , \tag{1.12}$$

as introduced in Alazard-Metivier [6] (see Definition 2.4 for the definition of the paradifferential operator Op^{BW}).

To state our first main result concerning the rigorous reduction to Birkhoff normal form of the system (1.3), let us assume that, for N large enough and some $T > 0$, we have a classical solution

$$(\eta, \psi) \in C^0([-T, T]; H^{N+\frac{1}{4}} \times H^{N+\frac{1}{4}}) \tag{1.13}$$

of the Cauchy problem for (1.3) with the initial height satisfying

$$\int_{\mathbb{T}} \eta(0, x) \, dx = 0 \, . \tag{1.14}$$

The existence of such a solution, at least for small enough T, is guaranteed by the local well-posedness Theorem of Alazard-Burq-Zuily [3] (see Theorem 1.3 below) under the regularity assumption $(\eta, \psi, V, B)(0) \in X^{N-\frac{1}{4}}$ where we denote

$$X^s := H^{s+\frac{1}{2}} \times H^{s+\frac{1}{2}} \times H^s \times H^s \, . \tag{1.15}$$

Defining the complex scalar unknown

$$u := \frac{1}{\sqrt{2}} |D|^{-\frac{1}{4}} \eta + \frac{\mathrm{i}}{\sqrt{2}} |D|^{\frac{1}{4}} \omega \, , \tag{1.16}$$

we deduce, by (1.13), that $u \in C^0([-T, T]; H^N)$, and u solves an evolution equation of the form

$$\partial_t u + \mathrm{i}\omega(D) u = M_{\geq 2}(u, \overline{u}) \, , \qquad \omega(k) := \sqrt{|k|} \, , \tag{1.17}$$

where $M_{\geq 2}(u,\overline{u})$ is a fully nonlinear vector field which contains up to first order derivatives of u. Moreover, since the zero average condition (1.14) is preserved by the flow of (1.3), it follows that

$$\int_{\mathbb{T}} u(t,x)\,dx = 0\,, \qquad \forall\, t\in[-T,T]\,. \tag{1.18}$$

This is our first main result.

Theorem 1.1. (Birkhoff normal form) *Let u be defined as in* (1.16)*, with ω as in* (1.12)*, for (η,ψ) solution of* (1.3) *satisfying* (1.13)-(1.14)*. There exist $N \gg K \gg 1$ and $0<\overline{\varepsilon}\ll 1$, such that, if*

$$\sup_{t\in[-T,T]} \sum_{k=0}^{K} \|\partial_t^k u(t)\|_{\dot{H}^{N-k}} \leq \overline{\varepsilon}\,, \tag{1.19}$$

then there exist a bounded and invertible transformation $\mathfrak{B}=\mathfrak{B}(u)$ of $\dot{H}^N$, which depends (nonlinearly) on u, and a constant $C:=C(N)>0$ such that

$$\|\mathfrak{B}(u)\|_{\mathcal{L}(\dot{H}^N,\dot{H}^N)} + \|(\mathfrak{B}(u))^{-1}\|_{\mathcal{L}(\dot{H}^N,\dot{H}^N)} \leq 1 + C\|u\|_{\dot{H}^N}\,, \tag{1.20}$$

and the variable $z:=\mathfrak{B}(u)u$ satisfies the equation

$$\partial_t z = -\mathrm{i}\partial_{\overline{z}} H_{ZD}(z,\overline{z}) + \mathcal{X}^+_{\geq 4} \tag{1.21}$$

where:

(1) *the Hamiltonian H_{ZD} has the form*

$$H_{ZD} = H^{(2)}_{ZD} + H^{(4)}_{ZD}\,, \qquad H^{(2)}_{ZD}(z,\overline{z}) := \frac{1}{2}\int_{\mathbb{T}} \big||D|^{\frac14} z\big|^2\,dx\,, \tag{1.22}$$

with

$$\begin{aligned} H^{(4)}_{ZD}(z,\overline{z}) := {}& \frac{1}{4\pi}\sum_{k\in\mathbb{Z}} |k|^3\big(|z_k|^4 - 2|z_k|^2|z_{-k}|^2\big) \\ &+ \frac{1}{\pi}\sum_{\substack{k_1,k_2\in\mathbb{Z},\,\mathrm{sign}(k_1)=\mathrm{sign}(k_2)\\ |k_2|<|k_1|}} |k_1||k_2|^2\big(-|z_{-k_1}|^2|z_{k_2}|^2 + |z_{k_1}|^2|z_{k_2}|^2\big) \end{aligned} \tag{1.23}$$

where z_k denotes the k-th Fourier coefficient of the function z, see (2.3)*.*

(2) *$\mathcal{X}^+_{\geq 4} := \mathcal{X}^+_{\geq 4}(u,\overline{u},z,\overline{z})$ is a quartic nonlinear term satisfying, for some $C:=C(N)>0$, the "energy estimate"*

$$\mathrm{Re}\int_{\mathbb{T}} |D|^N \mathcal{X}^+_{\geq 4}\cdot\overline{|D|^N z}\,dx \leq C\|z\|^5_{\dot{H}^N}\,. \tag{1.24}$$

The main point of Theorem 1.1 is the construction of the *bounded* and *invertible* transformation $\mathfrak{B}(u)$ in (1.20) which recasts the water waves system (1.3) (in the form of the equation (1.17) satisfied by u) into the equation (1.21)-(1.24). Purely formal, i.e. unbounded, non-invertible, transformations mapping the Hamiltonian (1.6) to the Hamiltonian (1.22), up to higher order degrees of homogeneity, were previously exhibited by Zakharov-Dyachenko [62] (hence our notation H_{ZD}), Craig-Worfolk [22], and Craig-Sulem [20].

The main consequence of Theorem 1.1 is to rigorously relate the flow of the full water waves system (1.3) to the flow of the system (1.21), which is made by the explicit Hamiltonian component H_{ZD} plus remainders of higher homogeneity. These remainders are under full control thanks to the energy estimates (1.24). With simple calculations one can verify that the Hamiltonian H_{ZD} is *integrable*, as observed in [62, 22], and its flow preserves all the Sobolev norms; see Theorem 1.4. Thus, as a consequence of Theorem 1.1, we obtain the following long-time existence result.

Theorem 1.2. (Long-time existence) *There exists $s_0 > 0$ such that, for all $s \geq s_0$, there is $\varepsilon_0 > 0$ such that, for any initial data (η_0, ψ_0) satisfying (recall* (1.15)*)*

$$\|(\eta_0, \psi_0, V_0, B_0)\|_{X^s} \leq \varepsilon \leq \varepsilon_0 \,, \quad \int_{\mathbb{T}} \eta_0(x) dx = 0 \,, \tag{1.25}$$

where $V_0 := V(\eta_0, \psi_0)$, $B_0 := B(\eta_0, \psi_0)$ are defined by (1.10)*-*(1.11)*, the following holds: there exist constants $c > 0$, $C > 0$ and a unique classical solution $(\eta, \psi, V, B) \in C^0([-T_\varepsilon, T_\varepsilon], X^s)$ of the water waves system* (1.3) *with initial condition $(\eta, \psi)(0) = (\eta_0, \psi_0)$ with*

$$T_\varepsilon \geq c\varepsilon^{-3} \,, \tag{1.26}$$

satisfying

$$\sup_{[-T_\varepsilon, T_\varepsilon]} \big(\|(\eta, \psi)\|_{H^s \times H^s} + \|(V, B)\|_{H^{s-1} \times H^{s-1}} \big) \leq C\varepsilon \,, \qquad \int_{\mathbb{T}} \eta(t, x) dx = 0 \,. \tag{1.27}$$

The main conclusion of the above theorem is the existence time T_ε of order $O(\varepsilon^{-3})$. This goes well beyond the time of $O(\varepsilon^{-1})$ which is guaranteed by the local existence theory. It also extends past the natural time scale of $O(\varepsilon^{-2})$ which one expects for non-resonant equations, and that has indeed been achieved for the system (1.3) in the works of Wu [58], Ionescu-Pusateri [43] and Hunter-Ifrim-Tataru [37]. To our knowledge, this is the first ε^{-3} existence result for water waves, or quasilinear systems, in absence of external parameters. The regularity index in Theorem 1.2 is a large number s_0 which we did not try to optimize. By a more careful analysis and some adjustments to our setting for the paradifferential calculus, one could likely set $s_0 = 30$.

Before discussing the literature on long-time existence results and normal forms, we briefly describe some of the key points of this paper, and refer to Subsection 1.3.2 for a longer explanation of our strategy.

- The long-time existence Theorem 1.2 is obtained by a different mechanism compared to all previous works in the literature, such as [58, 41, 37]. It relies on a complete conjugation of the water waves vector field (1.3) to its Birkhoff normal form up to order 4, Theorem 1.1, and not on the use of energies as in [23, 25, 41, 37, 39].
- The gravity water waves system (1.3) presents a family of non-trivial quartic resonances, the so-called Benjamin-Feir resonances, and there are no external parameters which can be used to modulate the dispersion relation to avoid higher order resonances.
- The nonlinear Birkhoff normal form transformation of Theorem 1.1 is constructed by composing several paradifferential flow conjugations. Since the gravity water waves dispersion relation $\omega(k) = \sqrt{|k|}$ is sublinear (in particular it is of lower order compared to the first order quasilinear transport term in the equation) the Birkhoff normal form reduction procedure is very different from [13], where the dispersion relation $\sim |k|^{3/2}$ is superlinear. However, we still employ the paradifferential framework developed in [13] as it readily provides us with a paralinearization of the Dirichlet-Neumann map with multilinear expansions, and several tools for conjugations via paradifferential flows.
- Besides the resonant interactions, one also needs to pay attention to near resonances which can prevent the boundedness of the Poincaré-Birkhoff normal form transformations. This is addressed by performing iterative diagonalization and smoothing procedures.
- Our transformations are non-symplectic and the final resonant Poincaré-Birkhoff normal form system is not a priori explicit. Then, an important step in our proof is a normal form uniqueness argument, that relies on the absence of three-waves resonances, and allows us to identify our system with the Hamiltonian equations associated to the Zakharov-Dyachenko-Craig-Worfolk Hamiltonian H_{ZD} in (1.22)-(1.23), up to degree 4 of homogeneity.

We have chosen to formulate our long-time existence result using the original symplectic variables (η, ψ) as well as the velocity components (V, B) in (1.10)-(1.11) consistently with the formulation of the local existence theorem of [3], that we reproduce below.

Theorem 1.3 (Local existence [3]). *Let $s > 3/2$ and consider (η_0, ψ_0) such that $(\eta_0, \psi_0, V_0, B_0)$ is in X^s, see (1.15). Then the following holds:*

(1) *there exists $T_{\rm loc} > 0$ such that the Cauchy problem for (1.3) with initial data (η_0, ψ_0) has a unique solution $(\eta, \psi) \in C^0([0, T_{\rm loc}], H^{s+\frac{1}{2}} \times H^{s+\frac{1}{2}})$ with $(V, B) \in C^0([0, T_{\rm loc}], H^s \times H^s)$;*

(2) *let T_* be the maximal time of existence of the solution $(\eta, \psi) \in C^0([0, T_{\rm loc}], H^{s+\frac{1}{2}} \times H^{s+\frac{1}{2}})$. If, for some $T_0 > 0$,*
$$\sup_{[0,T_0]} \|(\eta, \psi, V, B)(t)\|_{X^5} < +\infty \tag{1.28}$$
then $T_0 < T_*$ and $\sup_{[0,T_0]} \|(\eta, \psi, V, B)(t)\|_{X^s} < +\infty$.

Part (1) of Theorem 1.3 is the local existence result [3, Theorem 1.2], stated in the case of the torus $\mathbb{T}$, for a fluid in infinite depth. The result is based on energy methods for hyperbolic symmetrizable quasi-linear systems, which are the same in $\mathbb{T}^d$ and in $\mathbb{R}^d$. A more precise version, which implies also the continuation criterion in (2), is Theorem 1.2 of De Poyferré [29]. By time-reversibility, the solutions of (1.3) are defined in a symmetric interval $[-T, T]$. Note that the system (4.2) which we derive in Proposition 4.1 admits energy estimates. Therefore, one could also prove a local existence result based on this, implementing an iterative scheme as in [31].

1.2. **Literature.**

We now present some known results on the well-posedness and normal form theory for the water waves equations.

Local well-posedness. Early results on the local well-posedness of the water waves system include those by Nalimov [49], Yosihara [60], and Craig [19], which deal with the case of small perturbations of a flat interface. It was then proved by Wu [56, 57] that local-in-time solutions can be constructed with initial data of arbitrary size in Sobolev spaces, in the irrotational case. The question of local well-posedness of the water waves and free boundary Euler equations has then been addressed by many authors, see for example [15, 17, 47, 7, 45, 18, 51, 16, 2, 3]. We refer to [43, Section 2] for a longer discussion. Due to the cited contributions, the local well-posedness theory is presently well-understood: in a variety of different scenarios, for sufficiently nice initial data, it is possible to prove the existence of classical smooth solutions on a small time interval that depends on the size of the initial data (and the arc-chord constant of the initial interface). In particular, for data which are ε close to a flat interface, solutions exist and stay regular for times of order ε^{-1}.

Long-time regularity in the Euclidean case. In the Euclidean case, i.e. when the horizontal variable $x \in \mathbb{R}^d$, it is also possible to construct global-in-time solutions for the water waves equations. The main mechanism used in these cases is dispersion which, combined with localization (decay at spatial infinity), transfers the decay of linear solutions to the nonlinear problem, and gives control for long times.

For 3-dimensional fluids (2d interfaces), the first global well-posedness results were proved by Germain-Masmoudi-Shatah [33] and Wu [59] for gravity water waves, and for the capillary problem ($g = 0$, $\kappa > 0$, κ being the surface tension coefficient when capillarity at the interface is included in the system) by Germain-Masmoudi-Shatah [34]. The more difficult question of global regularity for the gravity-capillary water waves ($g > 0$, $\kappa > 0$) has been recently solved by Deng-Ionescu-Pausader-Pusateri [28]. For the case of a finite flat bottom see the works of Wang [54, 55].

For 2-dimensional fluids (1d interfaces), the first long-time existence result is due to Wu [58], who proved an almost-global existence result for gravity water waves. This was improved to global regularity by Ionescu-Pusateri [40], Alazard-Delort [4, 5], Hunter and Ifrim-Tataru [37, 38]. See also the improvements by Wang [53]. For the capillary problem in 2d, global regularity was proved by Ionescu-Pusateri [42], see also [41], and by Ifrim-Tataru [39]. We refer to [43] for a more extensive list of references.

Long-time existence on Tori: Normal Forms. In the case of the torus, $x \in \mathbb{T}^d$, there are no obvious dispersive effects that help to control solutions for long times. In addition, the quasilinear nature of the equations and the lack of conserved quantities which control high Sobolev norms, prevent the effective use of semilinear techniques.

An important tool that can be used to extend the lifespan of solutions for quasilinear equations is normal form theory. To explain the idea, let us consider a generic evolution equation of the form

$$\partial_t u + \mathrm{i}\omega(D)u = Q(u,\overline{u})\,, \quad u(t=0)=u_0\,, \quad \|u_0\|_{H^N} \le \varepsilon\,, \tag{1.29}$$

where $\omega = \omega(D)$ is a real-valued Fourier multiplier, and Q is a quadratic nonlinearity, semi- or quasi-linear, which depends on $(u,\overline{u})$ and their derivatives. In the case of (1.3) the dispersion relation is $\omega(k) = \sqrt{|k|}$. An energy estimate for (1.29) of the form $\frac{d}{dt}E(t) \lesssim \|u(t)\|_{H^N} E(t)$, where $E(t) \approx \|u(t)\|_{H^N}^2$, allows the construction of local solutions on time scales of $O(\varepsilon^{-1})$.

To extend the time of existence one can try to obtain a *quartic energy inequality* of the form

$$|E(t)-E(0)| \lesssim \int_0^t \|u(\tau)\|_{H^N}^2 E(\tau)\, d\tau\,. \tag{1.30}$$

This will then give existence for times of $O(\varepsilon^{-2})$. For water waves, such inequalities have been proven in [58, 52, 40, 4, 37] for the system (1.3), and also in the case of pure capillarity [42, 39], and gravity over a flat bottom [35]. Similar results were obtained in [26] for the Klein-Gordon equation on $\mathbb{T}^d$, which corresponds to the dispersion relation $\omega(k) = \sqrt{|k|^2+m^2}$ in (1.29). Although some delicate analysis is needed in the case of quasilinear PDEs, the possibility of proving an inequality of the form (1.30) relies on the absence of *3-waves resonances*, that is, non-zero integers (n_1,n_2,n_3) solving

$$\sigma_1\omega(n_1)+\sigma_2\omega(n_2)+\sigma_3\omega(n_3)=0\,, \qquad \sigma_1 n_1+\sigma_2 n_2+\sigma_3 n_3 = 0\,, \tag{1.31}$$

for $\sigma_j \in \{+,-\}$. In order to further extend the existence time, one can try to upgrade (1.30) to an estimate like

$$|E(t)-E(0)| \lesssim \int_0^t \|u(\tau)\|_{H^N}^3 E(\tau)\, d\tau\,. \tag{1.32}$$

At a formal level, this is possible in the absence of *4-waves resonances*, that is, non-trivial integer solutions of

$$\varphi_{\sigma_1\sigma_2\sigma_3\sigma_4}(n_1,n_2,n_3,n_4) := \sum_{j=1}^4 \sigma_j\omega(n_j) = 0\,, \qquad \sum_{j=1}^4 \sigma_j n_j = 0\,. \tag{1.33}$$

Here, by "trivial" solutions we mean those 4-tuples where the frequencies $n_1,\dots,n_4$ appear in pairs with corresponding opposite signs. These unavoidable resonances can then often be handled by exploiting the Hamiltonian/reversibile structure of the equation to show that they do not contribute to the energy inequality.

The condition on the absence of non-trivial solutions to (1.33) is, however, not satisfied in many cases, and it does not hold true for the gravity water waves system (1.3). See the expressions (1.38) of the nontrivial solutions of (1.33). Nevertheless, (1.32) could still follow from the more precise (and quantitative) inequality

$$\left|\frac{q_{\sigma_1\dots\sigma_4}(n_1,n_2,n_3,n_4)}{\varphi_{\sigma_1\dots\sigma_4}(n_1,n_2,n_3,n_4)}\right| \lesssim (\text{third largest frequency among } \langle n_1\rangle,\langle n_2\rangle,\langle n_3\rangle,\langle n_4\rangle)^p \tag{1.34}$$

for some $p>0$, where $q_{\sigma_1\dots\sigma_4}$ is an appropriate symbol determined by the nonlinearity Q. Note that for (1.34) to hold it is necessary that the symbol $q_{\sigma_1\dots\sigma_4}(n_1,n_2,n_3,n_4)$ vanishes at all the sites of the non-trivial 4 resonances. Verifying (1.34) can be a difficult task, because of complicated algebra, losses of derivatives from the nonlinearity, and the presence of near-resonant interactions. To our knowledge, such a condition has not been verified for (1.3) or any other quasilinear system.

As we will explain in detail in Subsection 1.3.2 below, our approach to the question of long-time existence is very different from the works cited above. We will actually prove a quintic energy inequality of the form (1.32) as a consequence of a much more precise analysis of the equations, and the reduction of (1.3) to its integrable Birkhoff normal form up to order 4.

Existence results for longer times can sometimes be obtained when the dispersion relation $\omega(k)$ in (1.29) depends in a non-degenerate way on some additional *parameter*. In these cases it is possible to verify algebraic conditions such as (1.34) and its higher order analogues for almost all values of these parameters. Trivial resonances can again be handled using the the Hamiltonian (or reversible) structure of the equations. In this direction we mention the works of Bambusi [10], Delort-Szeftel [27], Bambusi-Delort-Grebért-Szeftel [11], and Bambusi-Grebért [12] which developed normal form theory for Hamiltonian semilinear PDEs. In the context of quasilinear PDEs, Delort [24, 25] obtained an ε^{-M} existence result for arbitrary M, for almost all mass parameters m for Hamiltonian Klein-Gordon equations on spheres.

For water waves, the only extended stability result proven so far, is that of Berti-Delort [13] who obtained an ε^{-M} existence result for 1d periodic gravity-capillary waves with depth h, corresponding to $\omega(k) = \sqrt{\tanh(h|k|)(g|k| + \kappa|k|^3)}$, for almost all values of the surface tension parameter κ. This work introduced a general procedure for dealing with normal forms of quasi-linear PDEs, performing a paradifferential regularization of the unbounded operators up to smoothing remainders, before starting Birkhoff normal form reductions to eliminate the homogeneous components of the vector field of low degree in u. We also refer to Feola-Iandoli [32] where this strategy is applied to fully nonlinear reversible Schrödinger equations.

Quasi-periodic solutions. We finally mention that global in time quasi-periodic solutions for 1d space periodic water waves equations have been recently constructed, using KAM techniques combined with a systematic use of pseudo-differential calculus, in Berti-Montalto [14] for gravity-capillary waves, using κ as a parameter (see [1] for periodic solutions), and in Baldi-Berti-Haus-Montalto [9] for pure gravity waves in finite depth (see [50], [44] for periodic solutions) using the depth or the wavelength as a parameter. For the construction of quasi-periodic solutions using the "initial conditions" as parameters we refer to Baldi-Berti-Montalto [8] for quasilinear perturbations of KdV.

1.3. **The Zakharov-Dyachenko conjecture and our strategy.** In this subsection we first recall the calculations of [62, 22, 30, 20] concerning the formal integrability, up to order four, of the pure gravity water waves Hamiltonian (1.6) in infinite depth. We then discuss the strategy of proof of Theorem 1.1 which rigorously justifies this integrability.

1.3.1. *The formal Birkhoff normal form of Zakharov-Dyachenko* [62]. Consider the Hamiltonian H in (1.6) on the phase space (1.8). Introduce the complex variable

$$u := \frac{1}{\sqrt{2}}|D|^{-\frac{1}{4}}\eta + \frac{\mathrm{i}}{\sqrt{2}}|D|^{\frac{1}{4}}\psi\,, \tag{1.35}$$

and let $H_{\mathbb{C}}$ be the Hamiltonian expressed in $(u, \overline{u})$. By a Taylor expansion of the Dirichlet-Neumann operator for small η, see for example [21], one can expand $H = H_{\mathbb{C}}^{(2)} + H_{\mathbb{C}}^{(3)} + \cdots$ where $H_{\mathbb{C}}^{(\ell)}$ are ℓ-homogeneous in $(u, \overline{u})$, see (7.8)-(7.9)-(7.10). Notice that in this Taylor expansion there is a priori no control on the boundedness of the Hamiltonian vector fields associated to $H_{\mathbb{C}}^{(\ell)}$, $\ell = 3, 4, \ldots$

Applying the usual Birkhoff normal form procedure for Hamiltonian systems (see Subsection 7.1), it is possible to find a formal symplectic transformation Φ such that

$$H \circ \Phi = H_{\mathbb{C}}^{(2)} + H_{ZD}^{(4)} + \cdots \tag{1.36}$$

where: (1) all terms of homogeneity 3 have been eliminated due to the absence of 3-waves resonant interactions, that is, non-zero integer solutions of (1.31), and (2) the term $H_{ZD}^{(4)}$ is supported only on Birkhoff resonant quadruples, i.e.

$$H_{ZD}^{(4)} = \sum_{\substack{\sigma_1 n_1+\sigma_2 n_2+\sigma_3 n_3+\sigma_4 n_4=0\\ \sigma_1\omega(n_1)+\sigma_2\omega(n_2)+\sigma_3\omega(n_3)+\sigma_4\omega(n_4)=0}} H_{n_1,n_2,n_3,n_4}^{\sigma_1,\sigma_2,\sigma_3,\sigma_4} u_{n_1}^{\sigma_1} u_{n_2}^{\sigma_2} u_{n_3}^{\sigma_3} u_{n_4}^{\sigma_4} \tag{1.37}$$

where $H_{n_1,n_2,n_3,n_4}^{\sigma_1,\sigma_2,\sigma_3,\sigma_4} \in \mathbb{C}$, $\omega(n) = \sqrt{|n|}$ and $n_1, n_2, n_3, n_4 \in \mathbb{Z}\setminus\{0\}$. As shown in [62], there are many solutions to the constraints for the sum in (1.37). For example, if $\sigma_1 = \sigma_3 = 1 = -\sigma_2 = -\sigma_4$, and up to permutations, there are trivial solutions of the form (k, k, j, j), which give rise to benign integrable monomials $|u_k|^2|u_j|^2$ and the two parameter family of solutions, called Benjamin-Feir resonances,

$$\bigcup_{\lambda\in\mathbb{Z}\setminus\{0\}, b\in\mathbb{N}} \Big\{ n_1 = -\lambda b^2,\, n_2 = \lambda(b+1)^2\,,\, n_3 = \lambda(b^2+b+1)^2,\, n_4 = \lambda(b+1)^2 b^2 \Big\}\,. \tag{1.38}$$

As a consequence, one could expect, a priori, the presence in (1.37) of non-integrable monomials of the form $u_{-\lambda b^2}\overline{u_{\lambda(b+1)^2}}u_{\lambda(b^2+b+1)^2}\overline{u_{\lambda(b+1)^2b^2}}$ and their complex conjugates. The striking property proved in [62], see also [22, 20], is that the coefficients $H_{n_1,n_2,n_3,n_4}^{\sigma_1,\sigma_2,\sigma_3,\sigma_4}$ in (1.37) which are supported on Benjamin-Feir resonances are actually zero. The consequence of this "null condition" of the gravity water waves system in infinite depth is the following remarkable result:

Theorem 1.4. (Formal integrability of the water waves Hamiltonian (1.6) **at order four** [62, 30, 22, 20]**)** *The Hamiltonian* $H_{ZD}^{(4)}$ *in* (1.37) *has the form* (1.23). *The Hamiltonian* $H_{ZD} = H_{ZD}^{(2)} + H_{ZD}^{(4)}$ *in* (1.22) *is integrable and it can be written in action-angle variables as* (7.19). *The Hamiltonian vector field generated by* H_{ZD}, *explicitly written in* (7.21) *and* (7.22), *possesses the actions* $|u_n|^2$, $n \in \mathbb{Z}\setminus\{0\}$, *as prime integrals. In particular, the flow of* H_{ZD} *preserves all Sobolev norms.*

We refer the reader to Subsection 7.1 for further details on the structure of the Hamiltonian H_{ZD}. Unfortunately, this striking result is a purely formal calculation because the transformation Φ in (1.36) is not bounded and invertible, and there is no control on the higher order remainder terms. Thus, no actual relation can be established between the flow of H (which is well-posed for short times) and that of $H \circ \Phi$.

1.3.2. *Strategy for the proof of Theorem 1.1.* We now describe how the water-waves system (1.3) can be conjugated, through finitely many well-defined, *bounded* and *invertible* transformations, to the Hamiltonian equation (1.21),

$$\partial_t z = -\mathrm{i}|D|^{1/2} z - \mathrm{i}\partial_{\overline{z}} H_{ZD}^{(4)}(z, \overline{z}) + \mathcal{X}_{\geq 4}^{+}\,,$$

where $H_{ZD}^{(4)}$ is the Hamiltonian (1.23) and the quartic *remainder* $\mathcal{X}_{\geq 4}^{+}$ *admits energy estimates* in the sense of (1.24).

Step 1: Diagonalization up to smoothing remainders. We begin our analysis by paralinearizing the water-waves system (1.3), writing it as a system in the complex variable

$$U := (u, \overline{u}), \qquad u := \frac{1}{\sqrt{2}}|D|^{-\frac14}\eta + \frac{\mathrm{i}}{\sqrt{2}}|D|^{\frac14}\omega\,,$$

where ω is the "good-unknown" defined in (1.12). The good unknown ω has been introduced by Alazard-Metivier [6] and systematically used in the works on the local existence theory of Alazard-Burq-Zuily [2, 3] to prove energy estimates; see also Alazard-Delort [4, 5]. In this paper we use the paralinearization results proved in [13], collected in Proposition 3.1, which, in addition, provide expansions in homogeneous components in η, ω of the paralinearized system. The precise form of the system satisfied by the complex variable U is given by (3.8) in Proposition 3.3.

Our first task is to perform a diagonalization in $(u, \overline{u})$ of this system up to smoothing remainders. We remark that the highest order quasilinear transport operator in the system (3.8) is already diagonal. Hence, as a first step in Section 4.1 we diagonalize the sub-principal operator (which is of order $1/2$), as one would do to obtain local-in-time energy estimates. We then use an iterative descent procedure to diagonalize the operators of order 0, $-1/2$, and so on, up to a large negative order. The outcome of this procedure is described in Proposition 4.1, in which we obtain that (3.8) is reduced to the system (4.2).

The main reason for this "super" diagonalization procedure, inspired by [13], is that, when combined with a reduction to constant coefficients (Step 2 below), it allows us to handle all the losses of derivatives that arise from quasilinear terms and small divisors when performing Poincaré-Birkhoff normal form reductions (Step 3 below).

Step 2: Reduction to constant coefficients and Poincaré-Birkhoff normal forms. In Section 5 we reduce all the para-differential operators in the diagonalized system (4.2) to constant-in-x coefficients, which are integrable in the sense of Definition 5.1, up to smoothing remainders of homogeneity 2 and 3, and higher order contributions that admit energy estimates of the form (1.24). The most delicate reductions concern the quasilinear components in the right-hand side of (4.2): the highest order fully nonlinear transport term $\mathrm{i}Op^{\mathrm{BW}}(V\xi)$ and the quasilinear dispersive term $\mathrm{i}Op^{\mathrm{BW}}((1+a^{(0)})|\xi|^{\frac{1}{2}})$.

Let us briefly describe how to deal with the transport term. Roughly speaking, at the highest order, system (4.2) (which we represent using only its first equation, the second one being the complex conjugate) looks like

$$\partial_t w = -\mathrm{i}Op^{\mathrm{BW}}(V\xi)w + \cdots \tag{1.39}$$

where $V = V(u)$ is a real-valued function that depends on u, hence on x and t, and vanishes at $u = 0$. Our aim is to transform (1.39) into an equation of the form

$$\partial_t v = -\mathrm{i}Op^{\mathrm{BW}}(\widetilde{V}\xi)v + \cdots \qquad \text{where} \qquad \widetilde{V} = \zeta(u) + O(u^3) \tag{1.40}$$

is a real valued function independent of x up to cubic order in u. To do this we consider an auxiliary flow Φ^θ obtained by solving

$$\partial_\theta \Phi^\theta = \mathcal{A}\Phi^\theta, \quad \Phi^{\theta=0} = \mathrm{Id}\,, \tag{1.41}$$

where $\mathcal{A} = \mathrm{i}Op^{\mathrm{BW}}(\beta(u)\xi)$ is a para-differential operator, with $\beta(u)$ a real-valued function to be determined depending on the solution u, and possibly on θ. Since β is real valued, (1.41) is a paradifferential transport equation which is well-posed in the auxiliary time θ, and gives rise to a bounded and invertible flow Φ^θ, $\theta \in [0,1]$. The conjugation through the flow $\Phi^{\theta=1}$ corresponds to a paradifferential change of variables which is approximately given by the paracomposition operator associated to the diffeomorphism $x \mapsto x + \beta(u(x,t))$ of $\mathbb{T}$.

We then define a new variable through the time-1 flow of (1.41),

$$v := \Phi^{\theta=1} w\,.$$

Conjugating (1.39) through the flow $\Phi^{\theta=1}$ one obtains (see Lemma A.1)

$$\partial_t v = -\mathrm{i}Op^{\mathrm{BW}}(V\xi)v - [\partial_t, \mathcal{A}]v + \cdots = -\mathrm{i}Op^{\mathrm{BW}}\big((V(u) + \partial_t\beta(u))\xi\big)v + \cdots \tag{1.42}$$

where "$\cdots$" denote paradifferential operators of order less than 1, or terms satisfying the energy estimates (1.24). Notice that the contribution at the highest order 1 comes from the conjugation of ∂_t because the dispersion relation $-\mathrm{i}|D|^{1/2}$ has sublinear growth. For this reason, all our transformations are very different with respect to those performed in [13] for the gravity-capillary equations where the dispersion relation $\sim -\mathrm{i}|D|^{3/2}$ is superlinear. In Appendix A.2 we provide the general transformation rules of a paradifferential operator under the flow generated by a paradifferential equation like (1.41). In particular, a key feature is that paradifferential operators are transformed into paradifferential ones with symbols which can be algorithmically computed.

In light of (1.42) we look for β solving

$$\partial_t \beta(u) + V(u) = \zeta(u) + O(u^3)\,, \tag{1.43}$$

where $\zeta(u)$ is constant-in-x. However, in general it is only possible to obtain

$$\partial_t \beta(u) + V(u) = \sum_{n\in\mathbb{Z}\setminus\{0\}} (\mathtt{V}_2^{(1)})^{+-}_{n,n} |u_n|^2 + \sum_{n\in\mathbb{Z}\setminus\{0\}} (\mathtt{V}_2^{(1)})^{+-}_{n,-n} u_n \overline{u_{-n}} e^{\mathrm{i}2nx} + O(u^3)\,,$$

where $(\mathtt{V}_2^{(1)})^{+-}_{n_1 n_2}$ are some coefficients depending on the function V. We then verify the essential cancellation $(\mathtt{V}_2^{(1)})^{+-}_{n,-n} \equiv 0$, thus reducing the equation (1.42) to the desired form (1.40). More specifically, $\zeta(u)$ has the "integrable" form

$$\zeta(u) = \frac{1}{\pi} \sum_{n\in\mathbb{Z}\setminus\{0\}} n|n||u_n|^2\,,$$

and (1.40) is given in Fourier by

$$\dot v_n = -\frac{\mathrm{i}}{\pi}\Big(\sum_{j\in\mathbb{Z}\setminus\{0\}} j|j||u_j|^2\Big) n v_n + \cdots \tag{1.44}$$

which (substitute $u_j = v_j + \cdots$) is composed only by Birkhoff resonant cubic vector field monomials.

Remark 1.5. *While we do verify explicitly several key cancellations, such as the one leading to* (1.43)*, some, but not all, of them can be derived as a consequence of the following invariance properties of the water waves system* (1.3)*:*
(i) The water waves vector field $X(\eta,\psi)$ is reversible with respect to the involution

$$S : \begin{bmatrix} \eta(x) \\ \psi(x) \end{bmatrix} \mapsto \begin{bmatrix} \eta(-x) \\ -\psi(-x) \end{bmatrix}, \quad \text{i.e.} \quad X\circ S = -S\circ X\,; \tag{1.45}$$

(ii) X is even-to-even, i.e. maps even functions into even functions.

The reduction described above is performed in Subsection 5.1 in two separate steps corresponding to the degrees of homogeneity one and two in u. Similar arguments can be used to reduce to constant coefficients – and in Birkhoff normal form – the modified dispersive term $\mathrm{i}(1+\mathtt{a}_2)|\xi|^{1/2}$. Actually, thanks to additional algebraic cancellations, which appear to be intrinsic to the water waves system (1.3), it turns out that the new dispersive term is exactly $-\mathrm{i}|D|^{\frac12}$, up to lower order symbols. The transformation which is used for the conjugation is the flow (thus bounded and invertible) generated by a paradifferential "semi-Fourier integral operator" as (1.41) with generator $\mathcal{A} = \mathrm{i}Op^{\mathrm{BW}}(\beta(u)|\xi|^{\frac12})$ for a suitable real $\beta(u)$.

All lower order symbols can also be reduced to constant coefficients in x – and in Poincaré-Birkhoff normal form – using flows generated by Banach space ODEs. Eventually we obtain the system (5.4), which is in Poincaré-Birkhoff-normal form up to cubic degree in u, and up to a smoothing remainder and admissible symbols which satisfy energy estimates. We say that (5.4) is in Poincaré-Birkhoff normal form, and not just Birkhoff, because it is not Hamiltonian, since we performed non-symplectic transformations.

Step 3: Poincaré-Birkhoff normal form reductions. By the previous transformations we have obtained a system of the form

$$\partial_t z = -\zeta(z)\partial_x z - \mathrm{i}|D|^{\frac12} z + r_{-1/2}(z;D)[z] + R(z) + \mathcal{X}_{\geq 4} \tag{1.46}$$

where $r_{-1/2}$ is a constant-coefficient integrable symbol of order $-1/2$, up to some very regular nonlinear term $R(z)$, plus an admissible remainder term $\mathcal{X}_{\geq 4}$ of higher homogeneity satisfying energy estimates as (1.24). Our next step, in Section 6, is to apply Poincaré-Birkhoff normal form transformations to eliminate all non-resonant quadratic and cubic nonlinear terms in the smoothing remainder R. Thanks to these normal forms transformations the new system becomes (Proposition 6.2)

$$\partial_t z = -\zeta(z)\partial_x z - \mathrm{i}|D|^{\frac{1}{2}} z + r_{-1/2}(z;D)[z] + R^{\mathrm{res}}(z) + \mathcal{X}_{\geq 4} \tag{1.47}$$

where $R^{\mathrm{res}}(z)$ is a cubic term of the form

$$R^{\mathrm{res}}(z) = \sum_{\substack{\sigma_1 n_1+\sigma_2 n_2+\sigma_3 n_3=n \\ \sigma_1\omega(n_1)+\sigma_2\omega(n_2)+\sigma_3\omega(n_3)=\omega(n)}} c^{\sigma_1\sigma_2,\sigma_3}_{n_1,n_2,n_3}\, z^{\sigma_1}_{n_1} z^{\sigma_2}_{n_2} z^{\sigma_3}_{n_3} e^{\mathrm{i}nx} \tag{1.48}$$

with coefficients $c^{\sigma_1\sigma_2\sigma_3}_{n_1,n_2,n_3} \in \mathbb{C}$ (compare with (1.37)). Solutions to the constraints in (1.48) are of two types:

(a) *Trivial Resonances*: These occur when, say $(\sigma_1,\sigma_2,\sigma_3) = (+,-,+)$, one has $n_1 = n_2$ and $n = n_3$ (or permutations) producing resonant cubic terms of the form $c_{n_1,n}|z_{n_1}|^2 z_n e^{\mathrm{i}nx}$.

(b) *Benjamin-Feir resonances*: These occur when three of the frequencies (n_1,n_2,n_3,n_4) have the same sign and, in the case of $(\sigma_1,\sigma_2) = (+,-)$ say, are given by the two-parameter family in (1.38).

In the case of Hamiltonian systems – or, more in general, in the presence of other algebraic structures – one can expect that the trivially resonant terms will not impact the dynamics. Notice however the following difficulty: we have performed non-symplectic transformations so that the Hamiltonian nature of (1.47) is lost. In addition, the presence of 4-waves resonances, such as the Benjamin-Feir, is a strong obstruction to prove bounded dynamics for times of the order ε^{-3}. One may expect, in analogy with Theorem 1.4, to be able to check by direct computations that the coefficients $c^{\sigma_1\sigma_2\sigma_3}_{n_1,n_2,n_3}$ in (1.48) vanish on the Benjamin-Feir resonances. However, after, having performed all the reductions described before, this computation seems rather involved. As we describe in Step 4 below, in this paper we will prove such a property by an indirect uniqueness argument of the cubic Poincaré-Birkhoff normal form.

Before moving to this last step let us comment on the issue of *small divisors*. To perform the above Birkhoff normal form reduction of the smoothing terms, we need to deal with near-resonances. Indeed, our normal form transformation is generated by a flow as in (1.41) where, roughly speaking, the coefficients of the operator $\mathcal{A}$ are obtained through division by the phase $\sigma_1\omega(n_1)+\sigma_2\omega(n_2)+\sigma_3\omega(n_3)-\omega(n)$. This becomes dangerous if it degenerates rapidly close to the resonances. For example, if $\sigma_1 = 1 = \sigma_3$, $\sigma_2 = -1$, and $n_1 = k$, $n_2 = -k$, $n_3 = j$, $n = j + 2k$, with $j \gg k$ we get $|\omega(n_1) - \omega(n_2) + \omega(n_3) - \omega(n)| \approx j^{-1/2}$. Dividing by this expression then causes a loss of (at least) a $1/2$ derivative. In our proof this issue is overcome thanks to the fact that the smoothing remainders $R(z)$ can tolerate losses of derivatives.

Step 4: Normal form identification. In our last main step in Subsection 7.2 we prove that the cubic terms in (1.47)-(1.48) coincide with the Hamiltonian vector field generated by the quartic Hamiltonian in (1.23)

$$-\zeta(z)\partial_x z + r_{-1/2}(z;D)[z] + R^{\mathrm{res}}(z) = -\mathrm{i}\partial_{\bar z} H^{(4)}_{ZD}\,. \tag{1.49}$$

This implies in particular that $R^{\mathrm{res}}(z)$ is supported only on trivial resonances. To obtain (1.49) we use a normal form identification argument which relies on the uniqueness of solutions of the quadratic homological equation (7.43). The final outcome is that the equation (1.47), that we have obtained through the bounded and invertible transformations described in Steps 1–3, coincides up to quartic terms, with the Hamiltonian vector field generated by the Hamiltonian (1.22) formally derived by Zakharov-Dyachenko-Craig-Worfolk-Sulem. Notice that this argument also proves that the cubic vector field of the Poincaré-Birkhoff normal form (1.47) is Hamiltonian, which was not known a priori since we have performed non-symplectic transformations. This algebraic identification can be seen as philosophically similar to Moser's indirect proof of the convergence of the Lindsted series for a KAM torus [48]: Moser rigorously proves the existence of quasi-periodic solutions, which are analytic in ε, and then shows, a posteriori, that their Taylor expansions in ε coincide with the formal Lindsted power series.

Acknowledgements. We thank M. Procesi and W. Craig for stimulating discussions on the topic.

2. Functional setting and para-differential calculus

In this section we introduce our notation and recall several results on para-differential calculus, mostly following Chapter 3 of the monograph [13]. We find convenient the use of this set-up to obtain our initial paralinearization of the water waves equations (1.3) with multilinear expansions, as stated in Proposition 3.1, and several tools for conjugations via paradifferential flows which are contained in Appendix A.2.

Given an interval $I \subset \mathbb{R}$ symmetric with respect to $t=0$ and $s \in \mathbb{R}$ we define the space

$$C_*^K(I, \dot{H}^s(\mathbb{T}, \mathbb{C}^2)) := \bigcap_{k=0}^{K} C^k\big(I; \dot{H}^{s-k}(\mathbb{T}; \mathbb{C}^2)\big),$$

endowed with the norm

$$\sup_{t\in I} \|U(t,\cdot)\|_{K,s} \quad \text{where} \quad \|U(t,\cdot)\|_{K,s} := \sum_{k=0}^{K} \|\partial_t^k U(t,\cdot)\|_{\dot{H}^{s-k}}. \tag{2.1}$$

We denote by $C_{*\mathbb{R}}^K(I, \dot{H}^s(\mathbb{T}, \mathbb{C}^2))$ the space of functions U in $C_*^K(I, \dot{H}^s(\mathbb{T}, \mathbb{C}^2))$ such that $U = \big[\begin{smallmatrix} u \\ \bar{u} \end{smallmatrix}\big]$. Given $r>0$ we set

$$B_s^K(I; r) := \Big\{ U \in C_*^K(I, \dot{H}^s(\mathbb{T}; \mathbb{C}^2)) : \sup_{t\in I} \|U(t,\cdot)\|_{K,s} < r \Big\}. \tag{2.2}$$

With similar meaning we denote $C_*^K(I; \dot{H}^s(\mathbb{T}; \mathbb{C}))$. We expand a 2π-periodic function $u(x)$, with zero average in x (which is identified with u in the homogeneous space), in Fourier series as

$$u(x) = \sum_{n\in\mathbb{Z}\setminus\{0\}} \widehat{u}(n) \frac{e^{\mathrm{i}nx}}{\sqrt{2\pi}}, \qquad \widehat{u}(n) := \frac{1}{\sqrt{2\pi}} \int_{\mathbb{T}} u(x) e^{-\mathrm{i}nx}\, dx. \tag{2.3}$$

We also use the notation

$$u_n^+ := u_n := \widehat{u}(n) \qquad \text{and} \qquad u_n^- := \overline{u_n} := \overline{\widehat{u}(n)}. \tag{2.4}$$

For $n \in \mathbb{N}^* := \mathbb{N} \setminus \{0\}$ we denote by Π_n the orthogonal projector from $L^2(\mathbb{T}; \mathbb{C})$ to the subspace spanned by $\{e^{\mathrm{i}nx}, e^{-\mathrm{i}nx}\}$, i.e.

$$(\Pi_n u)(x) := \widehat{u}(n) \frac{e^{\mathrm{i}nx}}{\sqrt{2\pi}} + \widehat{u}(-n) \frac{e^{-\mathrm{i}nx}}{\sqrt{2\pi}}, \tag{2.5}$$

and we denote by Π_n also the corresponding projector in $L^2(\mathbb{T}, \mathbb{C}^2)$. If $\mathcal{U} = (U_1, \dots, U_p)$ is a p-tuple of functions, $\vec{n} = (n_1, \dots, n_p) \in (\mathbb{N}^*)^p$, we set

$$\Pi_{\vec{n}} \mathcal{U} := (\Pi_{n_1} U_1, \dots, \Pi_{n_p} U_p). \tag{2.6}$$

In this paper we deal with vector fields X which satisfy the *x-translation invariance* property

$$X \circ \tau_\theta = \tau_\theta \circ X, \quad \forall\, \theta \in \mathbb{R}, \tag{2.7}$$

where

$$\tau_\theta : u(x) \mapsto (\tau_\theta u)(x) := u(x+\theta). \tag{2.8}$$

Para-differential operators. We first give the definition of the classes of symbols that we are going to use, collecting Definitions 3.1, 3.2 and 3.4 in [13]. Roughly speaking, the class $\widetilde{\Gamma}^m_p$ contains homogeneous symbols of order m and homogeneity p in U, while the class $\Gamma^m_{K,K',p}$ contains non-homogeneous symbols of order m which vanish at degree at least p in U, and that are $(K-K')$-times differentiable in t.

Definition 2.1. (Classes of symbols) *Let $m \in \mathbb{R}$, $p, N \in \mathbb{N}$, $p \leq N$, $K' \leq K$ in $\mathbb{N}$, $r > 0$.*

(i) p**-homogeneous symbols.** *We denote by $\widetilde{\Gamma}^m_p$ the space of symmetric p-linear maps from $(\dot{H}^\infty(\mathbb{T};\mathbb{C}^2))^p$ to the space of C^∞ functions of $(x,\xi) \in \mathbb{T}\times\mathbb{R}$, $\mathcal{U} \to ((x,\xi) \to a(\mathcal{U};x,\xi))$, satisfying the following. There is $\mu > 0$ and, for any $\alpha, \beta \in \mathbb{N}$, there is $C > 0$ such that*

$$|\partial_x^\alpha \partial_\xi^\beta a(\Pi_{\vec{n}}\mathcal{U};x,\xi)| \leq C|\vec{n}|^{\mu+\alpha}\langle\xi\rangle^{m-\beta}\prod_{j=1}^{p}\|\Pi_{n_j}U_j\|_{L^2} \tag{2.9}$$

for any $\mathcal{U} = (U_1,\dots,U_p)$ in $(\dot{H}^\infty(\mathbb{T};\mathbb{C}^2))^p$, and $\vec{n} = (n_1,\dots,n_p) \in (\mathbb{N}^*)^p$. Moreover we assume that, if for some $(n_0,\dots,n_p) \in \mathbb{N}\times(\mathbb{N}^*)^p$,

$$\Pi_{n_0} a(\Pi_{n_1}U_1,\dots,\Pi_{n_p}U_p;\cdot) \neq 0\,, \tag{2.10}$$

then there exists a choice of signs $\sigma_0,\dots,\sigma_p \in \{-1,1\}$ such that $\sum_{j=0}^p \sigma_j n_j = 0$. For $p = 0$ we denote by $\widetilde{\Gamma}^m_0$ the space of constant coefficients symbols $\xi \mapsto a(\xi)$ which satisfy (2.9) with $\alpha = 0$ and the right hand side replaced by $C\langle\xi\rangle^{m-\beta}$. In addition we require the translation invariance property

$$a(\tau_\theta\mathcal{U};x,\xi) = a(\mathcal{U};x+\theta,\xi)\,, \quad \forall\theta\in\mathbb{R}\,. \tag{2.11}$$

(ii) **Non-homogeneous symbols.** *Let $p \geq 1$. We denote by $\Gamma^m_{K,K',p}[r]$ the space of functions $(U;t,x,\xi) \mapsto a(U;t,x,\xi)$, defined for $U \in B^K_{s_0}(I;r)$, for some large enough s_0, with complex values such that for any $0 \leq k \leq K-K'$, any $\sigma \geq s_0$, there are $C > 0$, $0 < r(\sigma) < r$ and for any $U \in B^K_{s_0}(I;r(\sigma)) \cap C^{k+K'}_*(I,\dot{H}^\sigma(\mathbb{T};\mathbb{C}^2))$ and any $\alpha,\beta \in \mathbb{N}$, with $\alpha \leq \sigma - s_0$*

$$|\partial_t^k\partial_x^\alpha\partial_\xi^\beta a(U;t,x,\xi)| \leq C\langle\xi\rangle^{m-\beta}\|U\|^{p-1}_{k+K',s_0}\|U\|_{k+K',\sigma}\,. \tag{2.12}$$

(iii) **Symbols.** *We denote by $\Sigma\Gamma^m_{K,K',p}[r,N]$ the space of functions $(U,t,x,\xi) \to a(U;t,x,\xi)$ such that there are homogeneous symbols $a_q \in \widetilde{\Gamma}^m_q$ for $q = p,\dots,N-1$ and a non-homogeneous symbol $a_N \in \Gamma^m_{K,K',N}[r]$ such that*

$$a(U;t,x,\xi) = \sum_{q=p}^{N-1} a_q(U,\dots,U;x,\xi) + a_N(U;t,x,\xi)\,. \tag{2.13}$$

We denote by $\Sigma\Gamma^m_{K,K',p}[r,N] \otimes \mathcal{M}_2(\mathbb{C})$ the space 2×2 matrices whose entries are symbols in $\Sigma\Gamma^m_{K,K',p}[r,N]$.

Remark 2.2. *The translation invariance property (2.11) means that the dependence with respect to the variable x of the symbol $a(\mathcal{U};x,\xi)$ enters only through the function $\mathcal{U}(x)$, and it implies (2.10).*

Notice that

$$\begin{aligned} a \in \widetilde{\Gamma}^m_p,\ \ b \in \widetilde{\Gamma}^{m'}_q \quad &\Rightarrow \quad ab \in \widetilde{\Gamma}^{m+m'}_{p+q},\ \partial_x a \in \widetilde{\Gamma}^m_p\,,\ \partial_\xi a \in \widetilde{\Gamma}^{m-1}_p\,;\\ a \in \Gamma^m_{K,K',p}[r],\ K'+1 \leq K \quad &\Rightarrow \quad \partial_t a \in \Gamma^m_{K,K'+1,p}[r],\ \partial_x a \in \Gamma^m_{K,K',p}[r]\,,\ \partial_\xi a \in \Gamma^{m-1}_{K,K',p}[r]\,;\\ a \in \Gamma^m_{K,K',p}[r], b \in \Gamma^{m'}_{K,K',q}[r] \quad &\Rightarrow \quad ab \in \Gamma^{m+m'}_{K,K',p+q}[r]\\ a(\mathcal{U};\cdot) \in \widetilde{\Gamma}^m_p \quad &\Rightarrow \quad a(U,\dots,U;\cdot) \in \Gamma^m_{K,0,p}[r]\,,\ \forall r > 0\,. \end{aligned} \tag{2.14}$$

Throughout this paper we will systematically use the following expansions, which are a consequence of (2.11) and $u \in \dot H^\infty(\mathbb{T};\mathbb{C})$. If $\mathtt{a}_1 \in \widetilde\Gamma^m_1$ then

$$\mathtt{a}_1(U;x,\xi) = \frac{1}{\sqrt{2\pi}} \sum_{n\in\mathbb{Z}\setminus\{0\},\sigma=\pm} (\mathtt{a}_1)^\sigma_n(\xi) u^\sigma_n e^{\mathrm{i}\sigma n x}\,, \tag{2.15}$$

for some $(\mathtt{a}_1)^\sigma_n(\xi) \in \mathbb{C}$, and, if $\mathtt{a}_2 \in \widetilde\Gamma^m_2$ then

$$\mathtt{a}_2(U;x,\xi) = \sum_{\substack{n_1,n_2\in\mathbb{Z}\setminus\{0\}\\ \sigma=\pm}} (\mathtt{a}_2)^{\sigma\sigma}_{n_1,n_2}(\xi) u^\sigma_{n_1} u^\sigma_{n_2} \frac{e^{\mathrm{i}\sigma(n_1+n_2)x}}{2\pi} + \sum_{n_1,n_2\in\mathbb{Z}\setminus\{0\}} (\mathtt{a}_2)^{+-}_{n_1,n_2}(\xi) u_{n_1}\overline{u_{n_2}} \frac{e^{\mathrm{i}(n_1-n_2)x}}{2\pi} \tag{2.16}$$

for some $(\mathtt{a}_2)^{\sigma\sigma'}_{n_1,n_2}(\xi) \in \mathbb{C}$ with $\sigma,\sigma' = \pm$.

We also define the following classes of functions in analogy with our classes of symbols.

Definition 2.3. (Functions) *Fix $N\in\mathbb{N}$, $p\in\mathbb{N}$ with $p\le N$, $K,K'\in\mathbb{N}$ with $K'\le K$, $r>0$. We denote by $\widetilde{\mathcal{F}}_p$, resp. $\mathcal{F}_{K,K',p}[r]$, $\Sigma\mathcal{F}_p[r,N]$, the subspace of $\widetilde\Gamma^0_p$, resp. $\Gamma^0_p[r]$, resp. $\Sigma\Gamma^0_p[r,N]$, made of those symbols which are independent of ξ. We write $\widetilde{\mathcal{F}}^{\mathbb{R}}_p$, resp. $\mathcal{F}^{\mathbb{R}}_{K,K',p}[r]$, $\Sigma\mathcal{F}^{\mathbb{R}}_p[r,N]$, to denote functions in $\widetilde{\mathcal{F}}_p$, resp. $\mathcal{F}_{K,K',p}[r]$, $\Sigma\mathcal{F}_p[r,N]$, which are real valued.*

Note that functions $\mathtt{a}_1\in\widetilde{\mathcal{F}}_1$, $\mathtt{a}_2\in\widetilde{\mathcal{F}}_2$ expanded as in (2.15), (2.16) are real valued if and only if

$$\overline{(\mathtt{a}_1)^+_n} = (\mathtt{a}_1)^-_n\,, \quad \overline{(\mathtt{a}_2)^{++}_{n_1,n_2}} = (\mathtt{a}_2)^{--}_{n_1,n_2}\,, \quad \overline{(\mathtt{a}_2)^{+-}_{n_1,n_2}} = (\mathtt{a}_2)^{+-}_{n_2,n_1}\,. \tag{2.17}$$

Paradifferential quantization. Given $p\in\mathbb{N}$ we consider smooth functions $\chi_p\in C^\infty(\mathbb{R}^p\times\mathbb{R};\mathbb{R})$ and $\chi\in C^\infty(\mathbb{R}\times\mathbb{R};\mathbb{R})$, even with respect to each of their arguments, satisfying, for some $0<\delta\ll 1$,

$$\operatorname{supp}\chi_p \subset \{(\xi',\xi)\in\mathbb{R}^p\times\mathbb{R}; |\xi'|\le\delta\langle\xi\rangle\}\,, \qquad \chi_p(\xi',\xi)\equiv 1 \ \text{ for } \ |\xi'|\le\delta\langle\xi\rangle/2\,, \tag{2.18}$$

$$\operatorname{supp}\chi \subset \{(\xi',\xi)\in\mathbb{R}\times\mathbb{R}; |\xi'|\le\delta\langle\xi\rangle\}\,, \qquad \chi(\xi',\xi)\equiv 1 \ \text{ for } \ |\xi'|\le\delta\langle\xi\rangle/2\,. \tag{2.19}$$

For $p=0$ we set $\chi_0\equiv 1$. We assume moreover that

$$\begin{aligned} &|\partial^\alpha_\xi\partial^\beta_{\xi'}\chi_p(\xi',\xi)| \le C_{\alpha,\beta}\langle\xi\rangle^{-\alpha-|\beta|}, \qquad \forall\alpha\in\mathbb{N},\ \beta\in\mathbb{N}^p\,,\\ &|\partial^\alpha_\xi\partial^\beta_{\xi'}\chi(\xi',\xi)| \le C_{\alpha,\beta}\langle\xi\rangle^{-\alpha-\beta}, \qquad \forall\alpha,\ \beta\in\mathbb{N}\,. \end{aligned} \tag{2.20}$$

A function satisfying the above condition is $\chi(\xi',\xi) := \widetilde\chi(\xi'/\langle\xi\rangle)$ where $\widetilde\chi$ is a function in $C^\infty_0(\mathbb{R};\mathbb{R})$ having a small enough support and equal to one in a neighborhood of zero.

Definition 2.4. (Bony-Weyl quantization) *If a is a symbol in $\widetilde\Gamma^m_p$, respectively in $\Gamma^m_{K,K',p}[r]$, we define its* Weyl *quantization as the operator acting on a 2π-periodic function $u(x)$ (written as in (2.3)) as*

$$Op^W(a)u = \frac{1}{\sqrt{2\pi}}\sum_{k\in\mathbb{Z}}\Big(\sum_{j\in\mathbb{Z}}\widehat a\big(k-j,\frac{k+j}{2}\big)\widehat u(j)\Big)\frac{e^{\mathrm{i}kx}}{\sqrt{2\pi}} \tag{2.21}$$

where $\widehat a(k,\xi)$ is the $k^{th}-$Fourier coefficient of the $2\pi-$periodic function $x\mapsto a(x,\xi)$.
We set, using notation (2.6),

$$a_{\chi_p}(\mathcal{U};x,\xi) = \sum_{\vec n\in\mathbb{N}^p}\chi_p(\vec n,\xi)\,a(\Pi_{\vec n}\mathcal{U};x,\xi), \quad a_\chi(U;t,x,\xi) = \frac{1}{2\pi}\int_{\mathbb{T}}\chi(\xi',\xi)\,\widehat a(U;t,\xi',\xi)e^{\mathrm{i}\xi' x}d\xi'\,,$$

where in the last equality $\widehat a$ stands for the Fourier transform with respect to the x variable. Then we define the Bony-Weyl *quantization of a as*

$$Op^{\mathrm{BW}}(a(\mathcal{U};\cdot)) = Op^W(a_{\chi_p}(\mathcal{U};\cdot)), \qquad Op^{\mathrm{BW}}(a(U;t,\cdot)) = Op^W(a_\chi(U;t,\cdot))\,. \tag{2.22}$$

If a is a symbol in $\Sigma\Gamma^m_{K,K',p}[r,N]$, that we decompose as in (2.13)*, we define its* Bony-Weyl *quantization*

$$Op^{\rm BW}(a(U;t,\cdot)) = \sum_{q=p}^{N-1} Op^{\rm BW}(a_q(U,\dots,U;\cdot)) + Op^{\rm BW}(a_N(U;t,\cdot))\,.$$

• By the translation invariance property (2.11), we have

$$Op^{\rm BW}(a_q(\tau_\theta U,\dots,\tau_\theta U;\cdot,\xi))[\tau_\theta V] = \tau_\theta\big(Op^{\rm BW}(a_q(U,\dots,U;\cdot,\xi))[V]\big)\,. \tag{2.23}$$

• The operator $Op^{\rm BW}(a)$ acts on homogeneous spaces of functions, see Proposition 2.6.
• The action of $Op^{\rm BW}(a)$ on homogeneous spaces only depends on the values of the symbol $a = a(U;t,x,\xi)$ (or $a(\mathcal{U};t,x,\xi)$) for $|\xi|\geq 1$. Therefore, we may identify two symbols $a(U;t,x,\xi)$ and $b(U;t,x,\xi)$ if they agree for $|\xi|\geq 1/2$. In particular, whenever we encounter a symbol that is not smooth at $\xi=0$, such as, for example, $a = g(x)|\xi|^m$ for $m\in\mathbb{R}\setminus\{0\}$, or ${\rm sign}(\xi)$, we will consider its smoothed out version $\chi(\xi)a$, where $\chi\in\mathcal{C}^\infty(\mathbb{R};\mathbb{R})$ is an even and positive cut-off function satisfying

$$\chi(\xi)=0 \ \text{ if } \ |\xi|\leq\frac{1}{8}\,, \quad \chi(x)=1 \ \text{ if } \ |\xi|>\frac{1}{4}, \quad \partial_\xi\chi(\xi)>0 \quad \forall\xi\in\big(\frac{1}{8},\frac{1}{4}\big)\,. \tag{2.24}$$

• If a is a homogeneous symbol, the two definitions of quantization in (2.22), differ by a smoothing operator that we introduce in Definition 2.5 below.

Definition 2.4 is independent of the cut-off functions χ_p, χ satisfying (2.18)-(2.20) up to smoothing operators that we define below (see Definition 3.7 in [13]). Roughly speaking, the class $\widetilde{\mathcal{R}}^{-\rho}_p$ contains smoothing operators which gain ρ derivatives and are homogeneous of degree p in U, while the class $\mathcal{R}^{-\rho}_{K,K',p}$ contains non-homogeneous ρ-smoothing operators which vanish at degree at least p in U, and are $(K-K')$-times differentiable in t.

Given $(n_1,\dots,n_{p+1})\in\mathbb{N}^{p+1}$ we denote by $\max_2(n_1,\dots,n_{p+1})$ the second largest among the integers $n_1,\dots,n_{p+1}$.

Definition 2.5. (Classes of smoothing operators) *Let $K'\leq K\in\mathbb{N}$, $N\in\mathbb{N}$ with $N\geq 1$, $\mu\in\mathbb{R}$, $\rho\geq 0$ and $r>0$.*

(i) **p-homogeneous smoothing operators.** *We denote by $\widetilde{\mathcal{R}}^{-\rho}_p$ the space of $(p+1)$-linear maps R from the space $(\dot H^\infty(\mathbb{T};\mathbb{C}^2))^p\times\dot H^\infty(\mathbb{T};\mathbb{C})$ to the space $\dot H^\infty(\mathbb{T};\mathbb{C})$ symmetric in $(U_1,\dots,U_p)$, of the form $(U_1,\dots,U_{p+1})\to R(U_1,\dots,U_p)U_{p+1}$ that satisfy the following. There are $\mu\geq 0$, $C>0$ such that*

$$\|\Pi_{n_0}R(\Pi_{\vec n}\mathcal{U})\Pi_{n_{p+1}}U_{p+1}\|_{L^2}\leq C\frac{\max_2(n_1,\dots,n_{p+1})^{\rho+\mu}}{\max(n_1,\dots,n_{p+1})^\rho}\prod_{j=1}^{p+1}\|\Pi_{n_j}U_j\|_{L^2}$$

for any $\mathcal{U}=(U_1,\dots,U_p)\in(\dot H^\infty(\mathbb{T};\mathbb{C}^2))^p$, any $U_{p+1}\in\dot H^\infty(\mathbb{T};\mathbb{C})$, any $\vec n=(n_1,\dots,n_p)\in(\mathbb{N}^*)^p$, any $n_0,n_{p+1}\in\mathbb{N}^*$. Moreover, if

$$\Pi_{n_0}R(\Pi_{n_1}U_1,\dots,\Pi_{n_p}U_p)\Pi_{n_{p+1}}U_{p+1}\neq 0\,, \tag{2.25}$$

then there is a choice of signs $\sigma_0,\dots,\sigma_{p+1}\in\{\pm1\}$ such that $\sum_{j=0}^{p+1}\sigma_j n_j=0$. In addition we require the translation invariance property

$$R(\tau_\theta\mathcal{U})[\tau_\theta U_{p+1}] = \tau_\theta\big(R(\mathcal{U})U_{p+1}\big)\,, \quad \forall\theta\in\mathbb{R}\,. \tag{2.26}$$

(ii) **Non-homogeneous smoothing operators.** *We denote by $\mathcal{R}^{-\rho}_{K,K',N}[r]$ the space of maps $(V,U)\mapsto R(V)U$ defined on $B^K_{s_0}(I;r)\times C^K_*(I,\dot H^{s_0}(\mathbb{T},\mathbb{C}))$ which are linear in the variable U and such that the following holds true. For any $s\geq s_0$ there exist a constant $C>0$ and*

$r(s) \in]0, r[$ such that for any $V \in B^K_{s_0}(I;r) \cap C^K_*(I, \dot H^s(\mathbb{T},\mathbb{C}^2))$, any $U \in C^K_*(I, \dot H^s(\mathbb{T},\mathbb{C}))$, any $0 \le k \le K-K'$ and any $t \in I$, we have

$$\|\partial_t^k \left(R(V)U\right)(t,\cdot)\|_{\dot H^{s-k+\rho}} \le \sum_{k'+k''=k} C\Big(\|U\|_{k'',s}\|V\|^N_{k'+K',s_0} + \|U\|_{k'',s_0}\|V\|^{N-1}_{k'+K',s_0}\|V\|_{k'+K',s}\Big). \tag{2.27}$$

(iii) **Smoothing operators.** *We denote by $\Sigma\mathcal{R}^{-\rho}_{K,K',p}[r,N]$ the space of maps $(V,t,U) \to R(V,t)U$ that may be written as*

$$R(V;t)U = \sum_{q=p}^{N-1} R_q(V,\dots,V)U + R_N(V;t)U$$

for some R_q in $\widetilde{\mathcal{R}}^{-\rho}_q$, $q = p,\dots,N-1$ and R_N in $\mathcal{R}^{-\rho}_{K,K',N}[r]$.

We denote by $\Sigma\mathcal{R}^{-\rho}_{K,K',p}[r,N] \otimes \mathcal{M}_2(\mathbb{C})$ the space of 2×2 matrices whose entries are in $\Sigma\mathcal{R}^{-\rho}_{K,K',p}[r,N]$.

- If R is in $\widetilde{\mathcal{R}}^{-\rho}_p$ then $(V,U) \mapsto R(V,\dots,V)U$ is in $\mathcal{R}^{-\rho}_{K,0,p}[r]$, i.e. (2.27) holds with $N \rightsquigarrow p$, $K'=0$.
- If $R_i \in \Sigma\mathcal{R}^{-\rho}_{K,K',p_i}[r,N]$, $i=1,2$, then the composition $R_1 \circ R_2$ is in $\Sigma\mathcal{R}^{-\rho}_{K,K',p_1+p_2}[r,N]$.

The next proposition states boundedness properties on Sobolev spaces of the paradifferential operators (see Proposition 3.8 in [13]).

Proposition 2.6. (Action of para-differential operator) *Let $r>0$, $m \in \mathbb{R}$, $p \in \mathbb{N}$, $K' \le K \in \mathbb{N}$. Then:*

(i) There is $s_0 > 0$ such that for any symbol $a \in \widetilde{\Gamma}^m_p$, there is a constant $C>0$, depending only on s and on (2.9) with $\alpha=\beta=0$, such that for any $\mathcal{U} = (U_1,\dots,U_p)$

$$\|Op^{\mathrm{BW}}(a(\mathcal{U};\cdot))U_{p+1}\|_{\dot H^{s-m}} \le C\prod_{j=1}^p \|U_j\|_{\dot H^{s_0}}\|U_{p+1}\|_{\dot H^s}\,, \tag{2.28}$$

for $p \ge 1$, while for $p=0$ the (2.28) holds by replacing the right hand side with $C\|U_{p+1}\|_{\dot H^s}$.

(ii) There is $s_0>0$ such that for any symbol $a \in \Gamma^m_{K,K',p}[r]$ there is a constant $C>0$, depending only on s,r and (2.12) with $0 \le \alpha \le 2$, $\beta = 0$, such that, for any $t \in I$, any $0 \le k \le K-K'$,

$$\|Op^{\mathrm{BW}}(\partial_t^k a(U;t,\cdot))\|_{\mathcal{L}(\dot H^s,\dot H^{s-m})} \le C\|U\|^p_{k+K',s_0}\,.$$

- If $a \in \Sigma\Gamma^m_{K,K',p}[r,N]$ with $m \le 0$ and $p \ge 1$, then $Op^{\mathrm{BW}}(a(V;t,\cdot))U$ is in $\Sigma\mathcal{R}^m_{K,K',p}[r,N]$.

Below we deal with classes of operators without keeping track of the number of lost derivatives in a precise way (see Definition 3.9 in [13]). The class $\widetilde{\mathcal{M}}^m_p$ denotes multilinear maps that lose m derivatives and are p-homogeneous in U, while the class $\mathcal{M}^m_{K,K',p}$ contains non-homogeneous maps which lose m derivatives, vanish at degree at least p in U, and are $(K-K')$-times differentiable in t.

Definition 2.7. (Classes of maps) *Let $p,N \in \mathbb{N}$, with $p \le N$, $N \ge 1$, $K,K' \in \mathbb{N}$ with $K' \le K$ and $m \ge 0$.*

(i) **p-homogeneous maps.** *We denote by $\widetilde{\mathcal{M}}^m_p$ the space of $(p+1)$-linear maps M from the space $(\dot H^\infty(\mathbb{T};\mathbb{C}^2))^p \times \dot H^\infty(\mathbb{T};\mathbb{C})$ to the space $\dot H^\infty(\mathbb{T};\mathbb{C})$ which are symmetric in $(U_1,\dots,U_p)$, of the form $(U_1,\dots,U_{p+1}) \to M(U_1,\dots,U_p)U_{p+1}$ and that satisfy the following. There is $C>0$ such that*

$$\|\Pi_{n_0} M(\Pi_{\vec n}\mathcal{U})\Pi_{n_{p+1}}U_{p+1}\|_{L^2} \le C(n_0+n_1+\dots+n_{p+1})^m \prod_{j=1}^{p+1}\|\Pi_{n_j}U_j\|_{L^2}$$

for any $\mathcal{U} = (U_1, \ldots, U_p) \in (\dot{H}^\infty(\mathbb{T};\mathbb{C}^2))^p$*, any* $U_{p+1} \in \dot{H}^\infty(\mathbb{T};\mathbb{C})$*, any* $\vec{n} = (n_1, \ldots, n_p) \in (\mathbb{N}^*)^p$*, any* $n_0, n_{p+1} \in \mathbb{N}^*$*. Moreover the properties* (2.25)-(2.26) *hold.*

(ii) **Non-homogeneous maps.** *We denote by* $\mathcal{M}^m_{K,K',N}[r]$ *the space of maps* $(V,u) \mapsto M(V)U$ *defined on* $B^K_{s_0}(I;r) \times C^K_*(I, \dot{H}^{s_0}(\mathbb{T},\mathbb{C}))$ *which are linear in the variable* U *and such that the following holds true. For any* $s \geq s_0$ *there exist a constant* $C > 0$ *and* $r(s) \in]0,r[$ *such that for any* $V \in B^K_{s_0}(I;r) \cap C^K_*(I, \dot{H}^s(\mathbb{T},\mathbb{C}^2))$*, any* $U \in C^K_*(I, \dot{H}^s(\mathbb{T},\mathbb{C}))$*, any* $0 \leq k \leq K - K'$ *and any* $t \in I$*, we have* $\|\partial_t^k \left(M(V)U\right)(t,\cdot)\|_{\dot{H}^{s-k-m}}$ *is bounded by the right hand side of* (2.27).

(iii) **Maps.** *We denote by* $\Sigma\mathcal{M}^m_{K,K',p}[r,N]$ *the space of maps* $(V,t,U) \to M(V,t)U$ *that may be written as*

$$M(V;t)U = \sum_{q=p}^{N-1} M_q(V,\ldots,V)U + M_N(V;t)U$$

for some M_q *in* $\widetilde{\mathcal{M}}^m_q$*,* $q = p, \ldots, N-1$ *and* M_N *in* $\mathcal{M}^m_{K,K',N}[r]$*. Finally we set* $\widetilde{\mathcal{M}}_p := \cup_{m\geq 0}\widetilde{\mathcal{M}}^m_p$*,* $\mathcal{M}_{K,K',p}[r] := \cup_{m\geq 0}\mathcal{M}^m_{K,K',p}[r]$ *and* $\Sigma\mathcal{M}_{K,K',p}[r,N] := \cup_{m\geq 0}\Sigma\mathcal{M}^m_{K,K',p}[r]$.

We denote by $\Sigma\mathcal{M}^m_{K,K',p}[r,N] \otimes \mathcal{M}_2(\mathbb{C})$ *the space of* 2×2 *matrices whose entries are maps in the class* $\Sigma\mathcal{M}^m_{K,K',p}[r,N]$*. We also set* $\Sigma\mathcal{M}_{K,K',p}[r,N] \otimes \mathcal{M}_2(\mathbb{C}) = \cup_{m\in\mathbb{R}}\Sigma\mathcal{M}^m_{K,K',p}[r,N] \otimes \mathcal{M}_2(\mathbb{C})$.

- If M is in $\widetilde{\mathcal{M}}^m_p$, $p \geq N$, then $(V,U) \to M(V,\ldots,V)U$ is in $\mathcal{M}^m_{K,0,N}[r]$.
- If $a \in \Sigma\Gamma^m_{K,K',p}[r,N]$ for $p \geq 1$, then $Op^{\mathrm{BW}}(a(V;t,\cdot))U$ is in $\Sigma\mathcal{M}^{m'}_{K,K',p}[r,N]$ for some $m' \geq m$.
- Any $R \in \Sigma\mathcal{R}^{-\rho}_{K,K',p}[r,N]$ defines an element of $\Sigma\mathcal{M}^m_{K,K',p}[r,N]$ for some $m \geq 0$.
- If $M \in \Sigma\mathcal{M}_{K,K_1',p}[r,N]$ and $\tilde{M} \in \Sigma\mathcal{M}_{K,K_2',1}[r,N-p]$, then $(V,t,U) \to M(V + \tilde{M}(V;t)V;t)[U]$ is in $\Sigma\mathcal{M}_{K,K_1'+K_2',p}[r,N]$.
- If $M \in \Sigma\mathcal{M}^m_{K,K',p}[r,N]$ and $\tilde{M} \in \Sigma\mathcal{M}^{m'}_{K,K',q}[r,N]$, then $M(U;t)\circ\tilde{M}(U;t)$ is in $\Sigma\mathcal{M}^{m+m'}_{K,K',p+q}[r,N]$.

Notice that, given $M_1 \in \widetilde{\mathcal{M}}_1$, the property (2.26) implies that

$$M_1(U)U = \frac{1}{2\pi}\sum_{\substack{n_1,n_2\in\mathbb{Z}\setminus\{0\},\\ \sigma=\pm}} (M_2)^{\sigma\sigma}_{n_1,n_2} u^\sigma_{n_1} u^\sigma_{n_2} e^{\mathrm{i}\sigma(n_1+n_2)x} + \frac{1}{2\pi}\sum_{n_1,n_2\in\mathbb{Z}\setminus\{0\}} (M_2)^{+-}_{n_1,n_2} u_{n_1}\overline{u_{n_2}} e^{\mathrm{i}(n_1-n_2)x} \tag{2.29}$$

for some coefficients $(M_2)^{\sigma\sigma'}_{n_1,n_2} \in \mathbb{C}$ with $\sigma,\sigma' = \pm$ and $n_1,n_2 \in \mathbb{Z}\setminus\{0\}$.

Composition theorems. Let

$$\sigma(D_x, D_\xi, D_y, D_\eta) := D_\xi D_y - D_x D_\eta$$

where $D_x := \frac{1}{\mathrm{i}}\partial_x$ and D_ξ, D_y, D_η are similarly defined.

Definition 2.8. (Asymptotic expansion of composition symbol) *Let* $K' \leq K, \rho, p, q$ *be in* $\mathbb{N}$*,* $m, m' \in \mathbb{R}$*,* $r > 0$*. Consider* $a \in \Sigma\Gamma^m_{K,K',p}[r,N]$ *and* $b \in \Sigma\Gamma^{m'}_{K,K',q}[r,N]$*. For* U *in* $B^K_\sigma(I;r)$ *we define, for* $\rho < \sigma - s_0$*, the symbol*

$$(a\#_\rho b)(U;t,x,\xi) := \sum_{k=0}^{\rho}\frac{1}{k!}\left(\frac{\mathrm{i}}{2}\sigma(D_x,D_\xi,D_y,D_\eta)\right)^k \Big[a(U;t,x,\xi)b(U;t,y,\eta)\Big]_{|x=y,\xi=\eta} \tag{2.30}$$

modulo symbols in $\Sigma\Gamma^{m+m'-\rho}_{K,K',p+q}[r,N]$.

- By (2.14) the symbol $a\#_\rho b$ belongs to $\Sigma\Gamma^{m+m'}_{K,K',p+q}[r,N]$.
- We have the expansion $a\#_\rho b = ab + \frac{1}{2\mathrm{i}}\{a,b\} + \cdots$, up to a symbol in $\Sigma\Gamma^{m+m'-2}_{K,K',p+q}[r,N]$, where

$$\{a,b\} := \partial_\xi a \partial_x b - \partial_x a \partial_\xi b$$

denotes the Poisson bracket.

Proposition 2.9. (Composition of Bony-Weyl operators) *Let $K' \le K, \rho, p, q$ be in $\mathbb{N}$, $m, m' \in \mathbb{R}$, $r > 0$. Consider $a \in \Sigma\Gamma^m_{K,K',p}[r,N]$ and $b \in \Sigma\Gamma^{m'}_{K,K',q}[r,N]$. Then*

$$R(U) := Op^{\rm BW}(a(U;t,x,\xi)) \circ Op^{\rm BW}(b(U;t,x,\xi)) - Op^{\rm BW}\big((a\#_\rho b)(U;t,x,\xi)\big)$$

is a non-homogeneous smoothing remainder in $\Sigma\mathcal{R}^{-\rho+m+m'}_{K,K',p+q}[r,N]$.

Proof. See Propositions 3.12 and 3.15 in [13]. Let us justify that the homogeneous components of $R(U)$ satisfy the translation invariance property (2.26). The homogeneous components of the symbols a and b (that for simplicity we still denote by a, b) satisfy (2.11). Then, by (2.23), the composed operator $Op^{\rm BW}(a(U;\cdot,\xi)) \circ Op^{\rm BW}(b(U;\cdot,\xi))$ satisfies (2.23) as well. In addition also the symbol $a\#_\rho b$ defined in (2.30) satisfies (2.11), and therefore $Op^{\rm BW}((a\#_\rho b)(U;\cdot,\xi))$ satisfies (2.23). Thus the homogeneous components of $R(U)$ satisfy (2.26) by difference. □

• As proved in the remark after the proof of Proposition 3.12 in [13], the remainder obtained by the composition of paradifferential operators in Proposition 2.9 has actually better estimates than (2.27), i.e. it is bounded from $\dot H^s$ to $\dot H^{s+\rho-(m+m')}$ for *any* s, with operator norm bounded by $\|U\|^{p+q}_{K,s_0}$.

Proposition 2.10. (Compositions) *Let $m, m', m'' \in \mathbb{R}$, $K, K', N, p_1, p_2, p_3, \rho \in \mathbb{N}$ with $K' \le K$, $p_1 + p_2 < N$, $\rho \ge 0$ and $r > 0$. Let $a \in \Sigma\Gamma^m_{K,K',p_1}[r,N]$, $R \in \Sigma\mathcal{R}^{-\rho}_{K,K',p_2}[r,N]$ and $M \in \Sigma\mathcal{M}^{m''}_{K,K',p_3}[r,N]$. Then*

- (i) *$R(U;t) \circ Op^{\rm BW}(a(U;t,x,\xi))$, $Op^{\rm BW}(a(U;t,x,\xi)) \circ R(U;t)$ are in $\Sigma\mathcal{R}^{-\rho+m}_{K,K',p_1+p_2}[r,N]$.*
- (ii) *$R(U;t) \circ M(U;t)$ and $M(U;t) \circ R(U;t)$ are smoothing operators in $\Sigma\mathcal{R}^{-\rho+m''}_{K,K',p_2+p_3}[r,N]$.*
- (iii) *If $R_2 \in \widetilde{\mathcal{R}}^{-\rho}_{p_2}$ then $R_2(U, M(U;t)U;t)$ belongs to $\Sigma\mathcal{R}^{-\rho+m''}_{K,K',p_2+p_3}[r,N]$.*
- (iv) *Let c be in $\widetilde\Gamma^m_p$, $p \in \mathbb{N}$. Then*

$$U \to c_M(U;t,x,\xi) := c(U,\dots,U,M(U;t)U;t,x,\xi)$$

  is in $\Sigma\Gamma^m_{K,K',p+p_3}[r,N]$. If the symbol c is independent of ξ (i.e. c is in $\widetilde{\mathcal{F}}_p$), so is the symbol c_M (thus it is a function in $\Sigma\mathcal{F}_{K,K',p+p_3}[r,N]$). Moreover if c is a symbol in $\Gamma^m_{K,K',N}[r]$ then the symbol c_M is in $\Gamma^m_{K,K',N}[r]$.
- (v) *$Op^{\rm BW}(c(U,\dots,U,W;t,x,\xi))_{|W=M(U;t)U} = Op^{\rm BW}(b(U;t,x,\xi)) + R(U;t)$ where*

$$b(U;t,x,\xi) := c(U,\dots,U,M(U;t)U;t,x,\xi)$$

  and $R(U;t)$ is in $\Sigma\mathcal{R}^{-\rho}_{K,K',p+p_1}[r,N]$.

Proof. See Proposition 3.16, 3.17, 3.18 in [13]. The translation invariance properties for the composed operators and symbols in items (i)-(v) follow as in the proof of Proposition 2.9. □

Real-to-real operators. Given a linear operator $R(U)[\cdot]$ acting on $\mathbb{C}^2$ (it may be a smoothing operator in $\Sigma\mathcal{R}^{-\rho}_{K,K',1}$ or a map in $\Sigma\mathcal{M}_{K,K',1}$) we associate the linear operator defined by the relation

$$\overline{R}(U)[V] := \overline{R(U)[\overline{V}]}\,, \quad \forall V \in \mathbb{C}^2\,. \tag{2.31}$$

We say that a matrix of operators acting in $\mathbb{C}^2$ is *real-to-real*, if it has the form

$$R(U) = \begin{pmatrix} R_1(U) & R_2(U) \\ \overline{R_2}(U) & \overline{R_1}(U) \end{pmatrix}\,. \tag{2.32}$$

Notice that

• if $R(U)$ is a real-to-real matrix of operators then, given $V = \left[\begin{smallmatrix} v \\ \bar{v} \end{smallmatrix}\right]$, the vector $Z := R(U)[V]$ has the form $Z = \left[\begin{smallmatrix} z \\ \bar{z} \end{smallmatrix}\right]$, namely the second component is the complex conjugated of the first one.

• If a matrix of symbols $A(U; x, \xi)$, in some class $\Sigma\Gamma^m_{K,K',1} \otimes \mathcal{M}_2(\mathbb{C})$, has the form

$$A(U; x, \xi) = \begin{pmatrix} a(U; x, \xi) & b(U; x, \xi) \\ \overline{b(U; x, -\xi)} & \overline{a(U; x, -\xi)} \end{pmatrix} \tag{2.33}$$

then the matrix of operators $Op^{\mathrm{BW}}(A(U; x, \xi))$ is real-to-real.

Notation.

- To simplify the notation, we will often omit the dependence on the time t from the symbols, smoothing remainders and maps, writing $a(U; x, \xi)$, $R(U)$, $M(U)$ instead of $a(U; t, x, \xi)$, $R(U; t)$, $M(U; t)$. Moreover, given a symbol in $\Sigma\Gamma^m_{K,K',p}$ we may omit to write its dependence on U, writing $b(x, \xi)$ instead of $b(U; x, \xi)$, when this does not cause confusion.
- Since in the rest of the paper we only need to control expansions in degrees of homogeneity of symbols, smoothing operators and maps, up to cubic terms $O(u^3)$, we fix once and for all $N = 3$. We will omit the dependence on r and $N = 3$ in the class of symbols, writing $\Sigma\Gamma^m_{K,K',p}$, instead of $\Sigma\Gamma^m_{K,K',p}[r, 3]$, and similarly for smoothing operators and maps.
- $A \lesssim_s B$ means $A \leq C(s)B$ where $C(s) > 0$ is a constant depending on $s \in \mathbb{R}$.
- In this paper we will deal with parameters

$$s \geq s_0 \gg K \gg \rho \gg 1 .$$

The order of regularization $\rho \gg 1$ will be chosen large enough to control the loss of derivatives coming from the small divisors in the two steps of Birkhoff normal form, see Section 6. More precisely $\rho \sim N_0$ where N_0 is the exponent appearing in (6.17). This requires to develop para-differenatial calculus for functions U in $\dot{H}^{s_0}$ with $s_0 \gg \rho$. In order to transform the water waves equations (1.3) into a paradifferential system plus a ρ-smoothing remainder we perform several para-differential changes of variables for solutions $U(t)$ which are K-times differentiable in time with $\partial_t^k U \in \dot{H}^{s-k}$, $0 \leq k \leq K$. Since each of the conjugations performed in Section 4 consumes one time derivative, we need to require $K \gg \rho$, more precisely $K \sim 2\rho$, see Proposition 4.1. We then require that the Sobolev exponents satisfy $s \geq s_0 \gg K$.

3. Complex form of the water waves equations

3.1. **Paralinearization and complex variables.** Following [3, 5], we begin by writing the water waves system (1.3) using the good-unknown (1.12)

$$\omega = \psi - Op^{\mathrm{BW}}(B(\eta, \psi))\eta$$

where $B(\eta, \psi)$ is the real valued function introduced in (1.11). The water-waves equations (1.3), written in the new coordinates

$$\begin{bmatrix} \eta \\ \omega \end{bmatrix} = \mathcal{G}\begin{bmatrix} \eta \\ \psi \end{bmatrix} := \begin{bmatrix} \eta \\ \psi - Op^{\mathrm{BW}}(B(\eta,\psi))\eta \end{bmatrix}, \tag{3.1}$$

assume the following paralinearized form derived in [13].

Proposition 3.1. **(Water-waves equations in (η, ω) variables)** *Let $I = [-T, T]$ with $T > 0$. Let $K \in \mathbb{N}^*$ and $\rho \gg 1$. There exists $s_0 > 0$ such that, for any $s \geq s_0$, for all $0 < r \leq r_0(s)$ small enough, if $(\eta, \psi) \in B^K_s(I; r)$ solves (1.3), then*

$$\partial_t \eta = |D|\omega + Op^{\mathrm{BW}}\big(-\mathrm{i}V\xi - \frac{V_x}{2}\big)\eta + Op^{\mathrm{BW}}(b_{-1}(\eta; \cdot))\omega + R_1(\eta, \omega)\omega + R'_1(\eta, \omega)\eta \tag{3.2}$$

$$\partial_t \omega = -\eta + Op^{\mathrm{BW}}(-\mathrm{i}V\xi + \frac{V_x}{2})\omega - Op^{\mathrm{BW}}(\partial_t B + V B_x)\eta + R'_2(\eta, \omega)\omega + R''_2(\eta, \omega)\eta \tag{3.3}$$

where the functions V, B defined in (1.10)-(1.11) *are in* $\Sigma\mathcal{F}^{\mathbb{R}}_{K,0,1}$*, the symbol* $b_{-1}(\eta;\cdot)$ *belongs to* $\Sigma\Gamma^{-1}_{K,0,1}$*, and the smoothing operators* R_1*,* R_1'*,* R_2*,* R_2' *are in* $\Sigma\mathcal{R}^{-\rho}_{K,0,1}$*. The vector field in the right hand side of* (3.2)-(3.3) *is* x*-translation invariant, i.e.* (2.7) *holds.*

Proof. The proof of this proposition follows from the computations in [13] in the absence of capillarity and specified in the case of infinite depth. In particular, the explicit expression of the symbols in (3.2) follows by developing the computations in Proposition 7.5 and Chapter 8.2 in [13]. Notice that this expansion agrees with the paralinearization of the Dirichlet-Neumann operator in Theorem 2.12 in [6], in the case of dimension 1 and using the Bony-Weyl quantization. The equation (3.3) follows by developing the computations in Proposition 7.6 in [13].

The Dirichlet-Neumann operator satisfies the translation invariance property $G(\tau_\theta\eta)[\tau_\theta\psi] = \tau_\theta G(\eta)[\psi]$. Hence the functions V, B defined in (1.10)-(1.11) satisfy the x-invariance property (2.11) as well, and so do $V_x, B_x, \partial_t B$. The symbol $b_{-1}(\eta;\cdot)$ satisfies the x-invariance property (2.11) checking the construction in [13] (its x-dependence enters only through (η,ω)). Moreover, since $B(\tau_\theta\eta,\tau_\theta\psi)(x) = B(\eta,\psi)(x+\theta)$, we get

$$Op^{\mathrm{BW}}(B(\tau_\theta\eta,\tau_\theta\psi))[\tau_\theta\eta](x) = Op^{\mathrm{BW}}(B(\eta,\psi))[\eta](x+\theta)\,, \quad \forall\theta\in\mathbb{R}\,,$$

and therefore the good-unknown transformation $\mathcal{G}$ defined in (3.1) satisfies $\mathcal{G}\circ\tau_\theta = \tau_\theta\circ\mathcal{G}$, where τ_θ is the translation operator in (2.8). This implies that the whole vector field in the right hand side of (3.2)-(3.3) satisfies the x-invariance property and therefore the smoothing remainders satisfy (2.26) by difference. □

In Subsection 3.2 we will provide explicit expansions for the symbols of non-negative order in (3.2)-(3.3) in linear and quadratic degrees of homogeneity.

Remark 3.2. (Expansion of the Dirichlet-Neumann operator)

(i) *Substituting* (3.1) *in the right hand side of* (3.2)*, which is equal to* $G(\eta)\psi$*, we have, using the remarks under Definition 2.7, that* $G(\eta)-|D|$ *is a map in* $\Sigma\mathcal{M}_{K,0,1}$ *and*

$$G(\eta)\psi = |D|\psi + \widetilde{M}_1(\eta)\psi + \widetilde{M}_2(\eta)\psi + \widetilde{M}_{\geq 3}(\eta)\psi \tag{3.4}$$

for some maps $\widetilde{M}_1\in\widetilde{\mathcal{M}}_1$*,* $\widetilde{M}_2\in\widetilde{\mathcal{M}}_2$ *and* $\widetilde{M}_{\geq3}\in\mathcal{M}_{K,0,3}$*.*

(ii) *The Dirichlet-Neumann operator admits a Taylor expansion (see e.g. formula (2.5) of* [20]*) of the form*

$$G(\eta)\psi = |D|\psi + G_1(\eta)\psi + G_2(\eta)\psi + G_{\geq3}(\eta)\psi \tag{3.5}$$

where, $D := \frac{1}{\mathrm{i}}\partial_x$*,*

$$\begin{aligned} G_1(\eta) &:= -\partial_x\eta\partial_x - |D|\eta|D| \\ G_2(\eta) &:= -\frac{1}{2}\Big(D^2\eta^2|D| + |D|\eta^2D^2 - 2|D|\eta|D|\eta|D|\Big) \end{aligned} \tag{3.6}$$

and where $G_{\geq3}$ *collects all the terms with homogeneity in* η *greater than* 2*. We then see that the quadratic and cubic components of the expansions* (3.5) *and* (3.4) *coincide, namely* $G_1 = \widetilde{M}_1$ *and* $G_2 = \widetilde{M}_2$*. It follows that* $G_{\geq3}$ *is in* $\mathcal{M}_{K,0,3}$*.*

(iii) *Performing the paralinearization of* $G_1(\eta)\psi$ *and* $G_2(\eta)\psi$ *in* (3.6)*, one obtains the expansion*

$$G(\eta)\psi = |D|\omega + Op^{\mathrm{BW}}\big(-\mathrm{i}\mathtt{V}_{\leq2}\xi - \frac{1}{2}(\mathtt{V}_{\leq2})_x\big)\eta + \text{cubic terms}$$

up to smoothing operators, where $\mathtt{V}_{\leq2} = \psi_x - \eta_x(|D|\psi)$ *contains the linear and quadratic components of the function* V *in* (1.10)*. This formula agrees with* (3.2) *showing that the symbol* b_{-1} *is zero (at least) at cubic degree of homogeneity.*

We now write the equations (3.2)-(3.3) in terms of the complex variable u defined by, see (1.16),

$$u := \frac{1}{\sqrt{2}}|D|^{-\frac{1}{4}}\eta + \frac{\mathrm{i}}{\sqrt{2}}|D|^{\frac{1}{4}}\omega\,. \tag{3.7}$$

Proposition 3.3. (Water-waves equations in complex variables) *Let $K \in \mathbb{N}^*$ and $\rho \gg 1$. There exists $s_0 > 0$ such that, for any $s \geq s_0$, for all $0 < r \leq r_0(s)$ small enough, if (η, ω) solves (3.2)-(3.3) and $U := \left[\begin{smallmatrix} u \\ \overline{u} \end{smallmatrix}\right]$ with u defined in (3.7) belongs to $B^K_s(I; r)$, then U solves*

$$\partial_t U = Op^{\mathrm{BW}}\big(\mathrm{i}A_1(U;x)\xi + \mathrm{i}A_{1/2}(U;x)|\xi|^{\frac{1}{2}} + A_0(U;x) + A_{-1}(U;x,\xi)\big)U + R(U)U \tag{3.8}$$

where

$$A_1(U;x) := \begin{pmatrix} -V(U;x) & 0 \\ 0 & -V(U;x) \end{pmatrix} \tag{3.9}$$

$$A_{1/2}(U;x) := \begin{pmatrix} -(1+a(U;x)) & -a(U;x) \\ a(U;x) & 1+a(U;x) \end{pmatrix}, \qquad a := \frac{1}{2}(\partial_t B + V B_x)\,, \tag{3.10}$$

$$A_0(U;x) := -\frac{1}{4}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} V_x(U;x)\,, \tag{3.11}$$

A_{-1} is a matrix of symbols in $\Sigma\Gamma^{-1}_{K,1,1} \otimes \mathcal{M}_2(\mathbb{C})$, and $R(U)$ is a matrix of smoothing operators belonging to $\Sigma\mathcal{R}^{-\rho}_{K,1,1} \otimes \mathcal{M}_2(\mathbb{C})$. The vector field in the right hand side of (3.8) is x-invariant *and it is* real-to-real *according to (2.32), i.e. the second equation for $\overline{u}$ is the complex conjugated of the first equation for u.*

Proof. We first rewrite (3.2)-(3.3) as the system

$$\partial_t \left[\begin{smallmatrix} \eta \\ \omega \end{smallmatrix}\right] = Op^{\mathrm{BW}}\left(\left[\begin{smallmatrix} -\mathrm{i}V\xi - \frac{V_x}{2} & |\xi| + b_{-1} \\ -(1+a_0) & -\mathrm{i}V\xi + \frac{V_x}{2} \end{smallmatrix}\right]\right)\left[\begin{smallmatrix} \eta \\ \omega \end{smallmatrix}\right] + R(\eta,\omega)\left[\begin{smallmatrix} \eta \\ \omega \end{smallmatrix}\right], \qquad R \in \Sigma\mathcal{R}^{-\rho}_{K,0,1} \otimes \mathcal{M}_2(\mathbb{C})\,, \tag{3.12}$$

where the function $a_0 := \partial_t B + V B_x$ is in $\Sigma\mathcal{F}^{\mathbb{R}}_{K,1,1}$. We now symmetrize (3.12) at the highest order, applying the change of variable

$$\left[\begin{smallmatrix} \eta \\ \omega \end{smallmatrix}\right] := \left[\begin{smallmatrix} |D|^{1/4} & 0 \\ 0 & |D|^{-1/4} \end{smallmatrix}\right]\left[\begin{smallmatrix} \tilde{\eta} \\ \tilde{\omega} \end{smallmatrix}\right]. \tag{3.13}$$

The conjugated system is, by Propositions 2.9 and 2.10,

$$\partial_t \left[\begin{smallmatrix} \tilde{\eta} \\ \tilde{\omega} \end{smallmatrix}\right] = Op^{\mathrm{BW}}\left(\left[\begin{smallmatrix} |\xi|^{-1/4} & 0 \\ 0 & |\xi|^{1/4} \end{smallmatrix}\right]\#_\rho\left[\begin{smallmatrix} -\mathrm{i}V\xi - \frac{V_x}{2} & |\xi| + b_{-1} \\ -(1+a_0) & -\mathrm{i}V\xi + \frac{V_x}{2} \end{smallmatrix}\right]\#_\rho\left[\begin{smallmatrix} |\xi|^{1/4} & 0 \\ 0 & |\xi|^{-1/4} \end{smallmatrix}\right]\right)\left[\begin{smallmatrix} \tilde{\eta} \\ \tilde{\omega} \end{smallmatrix}\right] + R(\tilde{\eta},\tilde{\omega})\left[\begin{smallmatrix} \tilde{\eta} \\ \tilde{\omega} \end{smallmatrix}\right] \tag{3.14}$$

for a new smoothing remainder R in $\Sigma\mathcal{R}^{-\rho+1}_{K,1,1} \otimes \mathcal{M}_2(\mathbb{C})$. Recalling (2.30) we expand in decreasing orders the symbols in (3.14).

Diagonal symbols. Up to a symbol in $\Sigma\Gamma^{-1}_{K,0,1}$ we have (using Proposition 2.9 and formula (2.30))

$$\begin{aligned} |\xi|^{-1/4}\#_\rho(-\mathrm{i}V\xi - \frac{V_x}{2})\#_\rho|\xi|^{1/4} &= -\mathrm{i}V\xi - \frac{V_x}{4}\,, \\ |\xi|^{1/4}\#_\rho(-\mathrm{i}V\xi + \frac{V_x}{2})\#_\rho|\xi|^{-1/4} &= -\mathrm{i}V\xi + \frac{V_x}{4}\,. \end{aligned} \tag{3.15}$$

Off-diagonal symbols. Up to a symbol in $\Sigma\Gamma^{-3/2}_{K,0,1}$ we get (using Proposition 2.9 and formula (2.30))

$$|\xi|^{-1/4}\#_\rho(|\xi| + b_{-1})\#_\rho|\xi|^{-1/4} = |\xi|^{1/2} \tag{3.16}$$

(recall that b_{-1} is in $\Sigma\Gamma^{-1}_{K,0,1}$) and, up to a symbol in $\Sigma\Gamma^{-3/2}_{K,1,1}$, we have

$$-|\xi|^{1/4}\#_\rho(1+a_0)\#_\rho|\xi|^{1/4} = -(1+a_0)|\xi|^{1/2}. \tag{3.17}$$

The expansion (3.15), (3.16), (3.17) imply that the system (3.14) has the form

$$\partial_t \left[\begin{smallmatrix} \tilde\eta \\ \tilde\omega \end{smallmatrix}\right] = Op^{\rm BW}\Big(\left[\begin{smallmatrix} -{\rm i}V\xi - \frac{V_x}{4} & |\xi|^{1/2} \\ -(1+a_0)|\xi|^{1/2} & -{\rm i}V\xi + \frac{V_x}{4} \end{smallmatrix}\right] + A_{-1} \Big) \left[\begin{smallmatrix} \tilde\eta \\ \tilde\omega \end{smallmatrix}\right] + R(\tilde\eta,\tilde\omega)\left[\begin{smallmatrix} \tilde\eta \\ \tilde\omega \end{smallmatrix}\right] \tag{3.18}$$

where A_{-1} is a matrix of symbols in $\Sigma\Gamma^{-1}_{K,1,1}\otimes\mathcal{M}_2(\mathbb{C})$ and R is in $\Sigma\mathcal{R}^{-\rho+1}_{K,1,1}\otimes\mathcal{M}_2(\mathbb{C})$.

Finally we write (3.18) in the complex variable (3.7), i.e. recalling (3.13),

$$\left[\begin{smallmatrix} u \\ \overline{u} \end{smallmatrix}\right] := \frac{1}{\sqrt{2}}\left[\begin{smallmatrix} 1 & {\rm i} \\ 1 & -{\rm i} \end{smallmatrix}\right]\left[\begin{smallmatrix} \tilde\eta \\ \tilde\omega \end{smallmatrix}\right], \quad \text{with inverse} \quad \left[\begin{smallmatrix} \tilde\eta \\ \tilde\omega \end{smallmatrix}\right] := \frac{1}{\sqrt{2}}\left[\begin{smallmatrix} 1 & 1 \\ -{\rm i} & {\rm i} \end{smallmatrix}\right]\left[\begin{smallmatrix} u \\ \overline{u} \end{smallmatrix}\right],$$

and we deduce (3.8) with matrices as in (3.9), (3.10), (3.11) and a new matrix of symbols A_{-1} in $\Sigma\Gamma^{-1}_{K,1,1}\otimes\mathcal{M}_2(\mathbb{C})$ and a new smoothing operator $R(U)$ in $\Sigma\mathcal{R}^{-\rho}_{K,1,1}\otimes\mathcal{M}_2(\mathbb{C})$, renaming $\rho-1$ as ρ. Finally, since the Fourier multiplier transformation (3.7) trivially commutes with the translation operators τ_θ, the water waves vector field in (3.8) is x-invariant as the water waves vector field (3.2)-(3.3). ☐

In some instances we will write the water waves system (3.8) as

$$\partial_t U = -{\rm i}\Omega U + \mathbf{M}(U)[U], \quad \Omega := \left[\begin{smallmatrix} |D|^{\frac12} & 0 \\ 0 & -|D|^{\frac12} \end{smallmatrix}\right], \tag{3.19}$$

where $\mathbf{M}(U)$ is a real-to-real matrix of maps in $\Sigma\mathcal{M}^{m_1}_{K,1,1}\otimes\mathcal{M}_2(\mathbb{C})$ for some $m_1>0$, see the Remarks after Definition 2.7. We will also write system (3.19) in Fourier basis as

$$\dot u_n = -{\rm i}\omega_n u_n + {\rm i}(F_2(U) + F_{\geq 3}(U))_n, \quad n\in\mathbb{Z}\setminus\{0\}, \tag{3.20}$$

where $F_2(U) = M_1(U)[U]$ is the quadratic component of the water-waves vector field and $F_{\geq3}(U)$ collects all the cubic terms (the second equation of (3.19) for $\overline{u}$ is just the complex conjugated of the one for u). Using the x-invariance property, the vector field $F_2(U)$ can be expanded as

$$F_2(U) = \sum_{n_1,n_2\in\mathbb{Z}\setminus\{0\},\sigma=\pm} (F_2)^{\sigma\sigma}_{n_1,n_2} u^\sigma_{n_1}u^\sigma_{n_2}\frac{e^{{\rm i}\sigma(n_1+n_2)x}}{2\pi} + \sum_{n_1,n_2\in\mathbb{Z}\setminus\{0\},} (F_2)^{+-}_{n_1,n_2} u_{n_1}\overline{u_{n_2}}\frac{e^{{\rm i}(n_1-n_2)x}}{2\pi} \tag{3.21}$$

with coefficients $(F_2)^{\sigma\sigma'}_{n_1,n_2}$ in $\mathbb{C}$. We provide the explicit expression of ${\rm i}F_2(U)$ in (3.40).

3.2. **Homogeneity expansions.** By the expansion of the Dirichlet-Neumann operator in Remark 3.2, we get the quadratic approximation of the water waves equations (1.3),

$$\begin{cases} \partial_t\eta = |D|\psi - \partial_x(\eta\partial_x\psi) - |D|(\eta|D|\psi), \\ \partial_t\psi = -\eta - \frac12\psi_x^2 + \frac12(|D|\psi)^2, \end{cases} \tag{3.22}$$

up to functions in $\mathcal{F}^{\mathbb{R}}_{K,1,3}$. In this section, using this expansion, we compute explicitly the quadratic vector field ${\rm i}F_2(U)$ in (3.20), and the homogeneous expansions up to cubic terms of the functions V and a appearing in (3.9)-(3.11). We write

$$V = \mathtt{V}_1 + \mathtt{V}_2 + \mathtt{V}_{\geq3}, \qquad \mathtt{V}_j\in\widetilde{\mathcal{F}}^{\mathbb{R}}_j,\ j=1,2,\ \ \mathtt{V}_{\geq3}\in\mathcal{F}^{\mathbb{R}}_{K,0,3}, \tag{3.23}$$

$$a = \mathtt{a}_1 + \mathtt{a}_2 + \mathtt{a}_{\geq3} \qquad \mathtt{a}_j\in\widetilde{\mathcal{F}}^{\mathbb{R}}_j,\ j=1,2,\ \ \mathtt{a}_{\geq3}\in\mathcal{F}^{\mathbb{R}}_{K,1,3}. \tag{3.24}$$

In the following it is useful to note that the relation (3.7) has inverse

$$\eta := \frac{1}{\sqrt2}|D|^{\frac14}(u+\overline{u}), \quad \omega = \frac{1}{{\rm i}\sqrt2}|D|^{-\frac14}(u-\overline{u}). \tag{3.25}$$

We have the following Lemma.

Lemma 3.4. (Expansion of V) *The function V defined in* (1.10) *admits the expansion*

$$V = \omega_x + \partial_x\big(Op^{\rm BW}(|D|\omega)\eta\big) - (|D|\omega)\eta_x + \mathtt{V}_{\geq 3} \tag{3.26}$$

where $\mathtt{V}_{\geq 3}$ is a function in $\mathcal{F}^{\mathbb{R}}_{K,0,3}$. Thus, in the complex variable u in (3.7),(3.25), *we have*

$$\mathtt{V}_1 = \frac{1}{\mathrm{i}\sqrt{2}}\partial_x|D|^{-\frac14}(u-\overline{u}) \tag{3.27}$$

$$\mathtt{V}_2 = \frac{1}{2\mathrm{i}}\partial_x\Big(Op^{\rm BW}\big(|D|^{\frac34}(u-\overline{u})\big)\big[|D|^{\frac14}(u+\overline{u})\big]\Big) - \frac{1}{2\mathrm{i}}\big(|D|^{\frac34}(u-\overline{u})\big)\big(\partial_x|D|^{\frac14}(u+\overline{u})\big)\,. \tag{3.28}$$

Proof. By (1.11) and using the expansion (3.5), we deduce $B = |D|\psi$ up to a quadratic function in $\mathcal{F}^{\mathbb{R}}_{K,0,2}$. As a consequence, by (1.10) and (3.1), we have

$$V = \psi_x - B\eta_x = (\omega + Op^{\rm BW}(B)\eta)_x - B\eta_x = \omega_x + \partial_x\big(Op^{\rm BW}(|D|\psi)\eta\big) - (|D|\psi)\eta_x$$

up to a function in $\mathcal{F}^{\mathbb{R}}_{K,0,3}$. Since $\psi = \omega$ plus a quadratic function in $\mathcal{F}^{\mathbb{R}}_{K,0,2}$ (see (3.1)) we get (3.26). □

Lemma 3.5. (Expansion of $\partial_t B$) *Let B the function defined in* (1.11). *Then*

$$\partial_t B = -|D|\eta - \eta|D|^2\eta + |D|(\eta|D|\eta) + |D|\Big(-\frac12\omega_x^2 - \frac12(|D|\omega)^2\Big) + (|D|\omega)(|D|^2\omega) \tag{3.29}$$

plus a cubic function in $\mathcal{F}^{\mathbb{R}}_{K,1,3}$.

Proof. Recalling (1.11), and using (3.5), we have to compute the expansion of

$$\partial_t B = \frac{\partial_t(G(\eta)\psi + \eta_x\psi_x)}{1+\eta_x^2} - \frac{(G(\eta)\psi+\eta_x\psi_x)2\eta_x(\eta_t)_x}{(1+\eta_x^2)^2} = \partial_t(G(\eta)\psi) + (\partial_t\eta)_x\psi_x + \eta_x(\partial_t\psi)_x \tag{3.30}$$

plus a cubic function in $\mathcal{F}^{\mathbb{R}}_{K,1,3}$. By (3.22) the second plus the third terms in (3.30) have the expansion

$$(\partial_t\eta)_x\psi_x + \eta_x(\partial_t\psi)_x = (|D|\psi_x)\psi_x - \eta_x^2\,, \tag{3.31}$$

up to a function in $\mathcal{F}^{\mathbb{R}}_{K,1,3}$. For the first term in (3.30) we use the "shape derivative" formula (see e.g. [46])

$$G'(\eta)[\widehat{\eta}]\psi = \lim_{\epsilon\to 0}\frac{1}{\epsilon}\{G(\eta+\epsilon\widehat{\eta})\psi - G(\eta)\psi\} = -G(\eta)(B\widehat{\eta}) - \partial_x(V\widehat{\eta}) \tag{3.32}$$

where $V = \psi_x - B\eta_x$ is defined in (1.10). Then, up to functions in $\mathcal{F}^{\mathbb{R}}_{K,1,3}$, we have

$$\begin{aligned}\partial_t(G(\eta)\psi) = G'(\eta)[\eta_t,\psi] + G(\eta)(\partial_t\psi) &\overset{(3.32)}{=} -G(\eta)(B\eta_t) - \partial_x(V\eta_t) + G(\eta)(\partial_t\psi)\\ &\overset{(1.10),(1.11),(3.22)}{=} -|D|((|D|\psi)^2) - \partial_x(\psi_x(|D|\psi))\\ &\quad + |D|\big(-\eta - \frac12\psi_x^2 + \frac12(|D|\psi)^2\big) + \partial_x(\eta\eta_x) + |D|(\eta|D|\eta)\,.\end{aligned} \tag{3.33}$$

Finally, by (3.30), (3.31), (3.33), we obtain, after simplification,

$$\partial_t B = -|D|\eta - \frac12|D|((|D|\psi)^2) - \frac12|D|\psi_x^2 + |D|(\eta|D|\eta) + \eta\eta_{xx} - \psi_{xx}|D|\psi \tag{3.34}$$

plus a cubic function $\mathcal{F}^{\mathbb{R}}_{K,1,3}$. Since $\partial_{xx} = -|D|^2$ and $\psi = \omega$ plus a quadratic function in $\mathcal{F}^{\mathbb{R}}_{K,1,2}$, we have that (3.34) implies (3.29). □

We now expand the function $a = \frac12(\partial_t B + VB_x)$ which appears in (3.10).

Lemma 3.6. (Expansion of a) *We have*

$$a=-\frac{1}{2}|D|\eta-\frac{\eta}{2}(|D|^2\eta)+\frac{1}{2}|D|(\eta|D|\eta)-\frac{1}{4}|D|\Big(\omega_x^2+(|D|\omega)^2\Big)+\frac{1}{2}(|D|\omega)(|D|^2\omega)+\frac{1}{2}\omega_x(\partial_x|D|\omega)$$

plus a cubic function in $\mathcal{F}^{\mathbb{R}}_{K,0,3}$.

Proof. By (3.26) and (1.11) we have that $a=\frac{1}{2}(\partial_t B+VB_x)=\frac{1}{2}\partial_t B+\frac{1}{2}\omega_x(|D|\psi_x)$ plus a cubic function in $\mathcal{F}^{\mathbb{R}}_{K,1,3}$. Hence (3.29) implies the lemma. □

We Fourier develop the functions $\mathtt{a}_1, \mathtt{V}_1, \mathtt{a}_2, \mathtt{V}_2$, as in (2.15), (2.16).

Lemma 3.7. (Coefficients of $\mathtt{V}_1$ and $\mathtt{V}_2$) *The coefficients of the functions* $\mathtt{V}_1$ *and* $\mathtt{V}_2$ *in* (3.27)-(3.28) *are, for all* $n\in\mathbb{Z}\setminus\{0\}$,

$$(\mathtt{V}_1)_n^+=(\mathtt{V}_1)_n^-=\frac{1}{\sqrt{2}}n|n|^{-1/4}\,,\qquad (\mathtt{V}_2)^{+-}_{n,n}=n|n|\,,\quad (\mathtt{V}_2)^{+-}_{n,-n}=0\,. \tag{3.35}$$

Proof. By (3.27) (recalling (2.3)) we have

$$\mathtt{V}_1=\frac{1}{\mathrm{i}\sqrt{2}}\partial_x|D|^{-1/4}(u-\overline{u})=\frac{1}{\sqrt{2\pi}}\frac{1}{\sqrt{2}}\sum_{n\neq 0}n|n|^{-1/4}u_ne^{\mathrm{i}nx}+n|n|^{-1/4}\overline{u_n}e^{-\mathrm{i}nx}$$

which implies the expressions for $(\mathtt{V}_1)_n^{\pm}$ in (3.35). By (3.28) an explicit computation using Definition 2.4 of the Bony-Weyl quantitation (and (2.21)) shows that

$$(\mathtt{V}_2)^{+-}_{n_1,n_2}=\frac{1}{2}(n_1-n_2)\chi_1(n_1,n_2)\Big(|n_1|^{\frac{3}{4}}|n_2|^{\frac{1}{4}}-|n_2|^{\frac{3}{4}}|n_1|^{\frac{1}{4}}\Big)+\frac{1}{2}\Big(n_2|n_1|^{\frac{3}{4}}|n_2|^{\frac{1}{4}}+n_1|n_2|^{\frac{3}{4}}|n_1|^{\frac{1}{4}}\Big)$$

where χ_1 is a cut-off function as in (2.18) with $p=1$ (even in all its arguments). This formula implies the expressions for $(\mathtt{V}_2)_n^{\pm}$ in (3.35). □

We now compute the coefficients of the linear and quadratic component of the function a in (3.10).

Lemma 3.8. (Coefficients of $\mathtt{a}_1$ and $\mathtt{a}_2$) *The coefficients of the functions* $\mathtt{a}_1$ *and* $\mathtt{a}_2$ *in* (3.24) *satisfy*

$$(\mathtt{a}_1)_n^+=(\mathtt{a}_1)_n^-=-\frac{1}{2\sqrt{2}}|n|^{5/4}\,,\qquad (\mathtt{a}_2)^{+-}_{n,n}=\frac{1}{2}|n|^{5/2}\,,\quad \forall n\in\mathbb{Z}\setminus\{0\}\,. \tag{3.36}$$

Proof. By Lemma 3.6 we have $\mathtt{a}_1=-\frac{1}{2}|D|\eta\overset{(3.25)}{=}-\frac{1}{2\sqrt{2}}|D|^{5/4}(u+\overline{u})$ and the formulas for $(\mathtt{a}_1)_n^{\pm}$ in (3.36) follow. We next compute the coefficients of $(\mathtt{a}_2)^{+-}_{n,n}$. We remark that the terms with the operator $|D|$ in front do not contribute to $(\mathtt{a}_2)^{+-}_{n,n}$, because

$$|D|\Big(\sum_{n_1,n_2}m^{+-}_{n_1,n_2}u_{n_1}\overline{u_{n_2}}e^{\mathrm{i}(n_1-n_2)x}\Big)=\sum_{n_1,n_2}|n_1-n_2|m^{+-}_{n_1,n_2}u_{n_1}\overline{u_{n_2}}e^{\mathrm{i}(n_1-n_2)x}\,,$$

whose coefficients vanish for $n_1=n_2$. Thus we have to consider the bilinear contribution in the variables $u,\overline{u}$ defined in (3.7), coming from the terms

$$-\frac{1}{2}\eta(|D|^2\eta)=-\frac{1}{4}\big(|D|^{1/4}(u+\overline{u})\big)\big(|D|^{9/4}(u+\overline{u})\big) \tag{3.37}$$

$$\frac{1}{2}(|D|\omega)(|D|^2\omega)=-\frac{1}{4}\big(|D|^{3/4}(u-\overline{u})\big)\big(|D|^{7/4}(u-\overline{u})\big) \tag{3.38}$$

$$\frac{1}{2}\omega_x(\partial_x|D|\omega)=-\frac{1}{4}\big(\partial_x|D|^{-1/4}(u-\overline{u})\big)\big(\partial_x|D|^{3/4}(u-\overline{u})\big)\,. \tag{3.39}$$

The contribution from (3.37)-(3.39) is $(\mathtt{a}_2)^{+-}_{n,n}=\frac{1}{2}|n|^{5/2}$ proving the second formula in (3.36). □

It turns out that $(\mathtt{a}_2)^{+-}_{n,-n} = |n|^{5/2}$ but we do not use this information in the paper.

Lemma 3.9. (Quadratic water waves vector field $\mathrm{i}F_2(U)$**)** *The quadratic water waves vector field* $\mathrm{i}F_2(U)$ *in* (3.20) *is*

$$
\begin{aligned}
\mathrm{i}F_2(U) = \frac{1}{\sqrt{2}}|D|^{-\frac14}\Big(|D|Op^{\mathrm{BW}}(|D|\omega)\eta - \partial_x(\eta\partial_x\omega) - |D|(\eta|D|\omega)\Big) \\
+ \frac{\mathrm{i}}{\sqrt{2}}|D|^{\frac14}\Big(-\frac12\omega_x^2 + \frac12(|D|\omega)^2 + Op^{\mathrm{BW}}(|D|\eta)\eta - Op^{\mathrm{BW}}(|D|\omega)|D|\omega\Big)
\end{aligned}
\tag{3.40}
$$

expressing (η,ω) *in terms of* $(u,\overline{u})$ *as in* (3.25)*. The coefficients* $(F_2)^{+-}_{n_1,n_2}$ *defined in* (3.21) *satisfy*

$$
(F_2)^{+-}_{n,-n} = (F_2)^{+-}_{-n,n} = 2^{-\frac14}|n|^{\frac74}\,. \tag{3.41}
$$

Proof. By (1.11) (recalling (3.5)) we have the expansion $\omega = \psi - Op^{\mathrm{BW}}(|D|\psi)\eta$, up to a cubic function in $\mathcal{F}^{\mathbb{R}}_{K,0,3}$, of the good unknown in (1.12). Then the equations in (3.22) reads

$$
\begin{aligned}
\partial_t\eta &= |D|\omega + |D|Op^{\mathrm{BW}}(|D|\psi)\eta - \partial_x(\eta\partial_x\omega) - |D|(\eta|D|\omega)\,, \\
\partial_t\omega &= \psi_t - Op^{\mathrm{BW}}(|D|\psi_t)\eta - Op^{\mathrm{BW}}(|D|\psi)\eta_t \\
&= -\eta - \frac12\omega_x^2 + \frac12(|D|\omega)^2 + Op^{\mathrm{BW}}(|D|\eta)\eta - Op^{\mathrm{BW}}(|D|\omega)[|D|\omega]
\end{aligned}
$$

up to cubic functions in $\mathcal{F}^{\mathbb{R}}_{K,1,3}$. In the complex variable u defined in (3.7) we obtain the equation $u_t = -\mathrm{i}|D|^{\frac12}u + \mathrm{i}F_2(U)$ with $\mathrm{i}F_2(U)$ defined in (3.40). Expressing in (3.40) the variables (η,ω) in terms of $(u,\overline{u})$ as in (3.25) and passing to the Fourier coordinates, we derive (3.41) by a direct calculus similar to those in Lemmata 3.7 and 3.8. ☐

Remark 3.10. *The only property of the coefficients* $(F_2)^{+-}_{n,-n}$ *in* (3.41) *that we are going to use is that* $(F_2)^{+-}_{n,-n} = \overline{(F_2)^{+-}_{-n,n}}$*, see the proof of Lemma 5.5. This property could be also derived by the reversibility and even-to-even property of the water waves system* (3.8)*, or* (3.20)*, which are preserved by the good unknown transformation. The involution* S *in* (1.45) *reads* $u(x)\mapsto\overline{u}(-x)$*, and in the Fourier basis* $(u_j)\mapsto(\overline{u_j})$*.*

4. Block-diagonalization

The goal of this section is to transform the water waves system (3.8) into the system (4.2) below which is block-diagonal in the variables $(u,\overline{u})$, modulo a smoothing operator $R(U)$.

Proposition 4.1. (Block-Diagonalization) *Let* $\rho \gg 1$ *and* $K \ge K' := 2\rho+2$*. There exists* $s_0>0$ *such that, for any* $s\ge s_0$*, for all* $0<r\le r_0(s)$ *small enough, and any solution* $U\in B^K_s(I;r)$ *of* (3.8)*, the following holds:*

(i) *there is a map* $\mathbf{\Psi}^\theta_{diag}(U)$*,* $\theta\in[0,1]$*, satisfying, for some* $C=C(s,r,K)>0$*,*

$$
\|\partial_t^k\mathbf{\Psi}^\theta_{diag}(U)[V]\|_{\dot H^{s-k}} + \|\partial_t^k(\mathbf{\Psi}^\theta_{diag}(U))^{-1}[V]\|_{\dot H^{s-k}} \le \big(1 + C\|U\|_{K,s_0}\big)\|V\|_{k,s}\,, \tag{4.1}
$$

for any $0\le k\le K-K'$ *and any* $V=\left[\begin{smallmatrix} v\\ \overline{v}\end{smallmatrix}\right]$ *in* $C^{K-K'}_{*\mathbb{R}}(I,\dot H^s(\mathbb{T};\mathbb{C}^2))$*,* $\theta\in[0,1]$*;*

(ii) *the function* $W := (\mathbf{\Psi}^\theta_{diag}(U)U)_{|\theta=1}$ *solves the system*

$$
\partial_t W = Op^{\mathrm{BW}}\begin{pmatrix} d(U;x,\xi)+r_{-1/2}(U;x,\xi) & 0 \\ 0 & \overline{d(U;x,-\xi)}+\overline{r_{-1/2}(U;x,-\xi)}\end{pmatrix}W + R(U)[W] \tag{4.2}
$$

where $d(U;x,\xi)$ *is a symbol of the form*

$$
d(U;x,\xi) := -\mathrm{i}V(U;x)\xi - \mathrm{i}(1+a^{(0)}(U;x))|\xi|^{1/2} \tag{4.3}
$$

where $a^{(0)}$ is a function in $\Sigma\mathcal{F}^{\mathbb{R}}_{K,1,1}$, $r_{-1/2}(U;x,\xi)$ is a symbol in $\Sigma\Gamma^{-1/2}_{K,2\rho+2,1}$, and $R(U)$ is a real-to-real matrix of smoothing operators in $\Sigma\mathcal{R}^{-\rho}_{K,2\rho+2,1}\otimes\mathcal{M}_2(\mathbb{C})$. The function $a^{(0)}$ has the expansion

$$a^{(0)} = \mathtt{a}_1 + \mathtt{a}_2^{(0)} + \mathtt{a}_{\geq 3}^{(0)}\,, \quad \mathtt{a}_2^{(0)} := \mathtt{a}_2 - \frac{1}{2}\mathtt{a}_1^2 \in \widetilde{\mathcal{F}}_2^{\mathbb{R}}\,, \tag{4.4}$$

where $\mathtt{a}_1$ and $\mathtt{a}_2$ are defined in (3.24).

Proposition 4.1 is proved applying a sequence a transformations which iteratively block-diagonalize (3.8) in decreasing orders. In Subsection 4.1 we block-diagonalize (3.8) at the order 1/2 and in Subsection 4.2 we perform the block-diagonalization until the negative order $-\rho$.

4.1. **Block-Diagonalization at order** $1/2$**.** The aim of this subsection is to diagonalize the matrix of symbols $A_{1/2}(U;x)|\xi|^{1/2}$ in (3.8), up to a matrix of symbols of order 0. We apply a parametrix argument conjugating the system (3.8) with a paradifferential operator whose principal matrix symbol is

$$C := \begin{pmatrix} f & g \\ g & f \end{pmatrix}, \quad f(U;x) := \frac{1+a+\lambda_+}{\sqrt{(1+a+\lambda_+)^2 - a^2}}, \quad g(U;x) := \frac{-a}{\sqrt{(1+a+\lambda_+)^2 - a^2}}\,, \tag{4.5}$$

where

$$\lambda_\pm = \lambda_\pm(U;x) := \pm\sqrt{(1+a)^2 - a^2} \tag{4.6}$$

are the eigenvalues of $A_{1/2}$. We have

$$\det(C) = f^2 - g^2 = 1\,, \quad C^{-1} = \begin{pmatrix} f & -g \\ -g & f \end{pmatrix}, \tag{4.7}$$

and

$$C^{-1}A_{1/2}C = \begin{pmatrix} -\lambda_+ & 0 \\ 0 & \lambda_+ \end{pmatrix} = \begin{pmatrix} -(1+a^{(0)}) & 0 \\ 0 & 1+a^{(0)} \end{pmatrix}, \quad a^{(0)} := \lambda_+ - 1 \in \Sigma\mathcal{F}^{\mathbb{R}}_{K,1,1}\,. \tag{4.8}$$

Lemma 4.2. *There exists a function $m_{-1}(U;x)$ in $\Sigma\mathcal{F}_{K,1,1}$ such that, the flow*

$$\partial_\theta \mathbf{\Psi}^\theta_{-1}(U) = Op^{\mathrm{BW}}(M_{-1})\mathbf{\Psi}^\theta_{-1}(U),\ \mathbf{\Psi}^0_{-1}(U) = \mathrm{Id}\,, \quad M_{-1} := \begin{pmatrix} 0 & m_{-1}(U;x) \\ \overline{m_{-1}(U;x)} & 0 \end{pmatrix}, \tag{4.9}$$

has the form

$$(\mathbf{\Psi}^\theta_{-1}(U))_{|\theta=1} = Op^{\mathrm{BW}}(C^{-1}) + R(U), \quad R(U) \in \Sigma\mathcal{R}^{-\rho}_{K,1,1}\otimes\mathcal{M}_2(\mathbb{C})\,, \tag{4.10}$$

$$(\mathbf{\Psi}^\theta_{-1}(U))^{-1}_{|\theta=1} = Op^{\mathrm{BW}}(C) + Q(U), \quad Q(U) \in \Sigma\mathcal{R}^{-\rho}_{K,1,1}\otimes\mathcal{M}_2(\mathbb{C})\,. \tag{4.11}$$

Moreover, if U solves (3.8)*, then the function*

$$W_0 := (\mathbf{\Psi}^\theta_{-1}(U))_{|\theta=1}U \tag{4.12}$$

solves the system

$$\partial_t W_0 = Op^{\mathrm{BW}}\Big(\begin{pmatrix} d(U;x,\xi) & 0 \\ 0 & \overline{d(U;x,-\xi)} \end{pmatrix} + A^{(0)}\Big)W_0 + R^{(0)}(U)W_0 \tag{4.13}$$

where $d(U;x,\xi)$ is the symbol in (4.3) *with $a^{(0)}(U;x)$ defined in* (4.8)*, a matrix of symbols*

$$A^{(0)} := \begin{pmatrix} c_0(U;x,\xi) & b_0(U;x,\xi) \\ \overline{b_0(U;x,-\xi)} & \overline{c_0(U;x,-\xi)} \end{pmatrix}, \ c_0 \in \Sigma\Gamma^{-\frac{1}{2}}_{K,2,1},\ \ b_0 \in \Sigma\Gamma^0_{K,2,1}\,, \tag{4.14}$$

and a real-to-real matrix of smoothing operators $R^{(0)}(U)$ in $\Sigma\mathcal{R}^{-\rho}_{K,2,1}\otimes\mathcal{M}_2(\mathbb{C})$. Moreover the function $a^{(0)}$ has the expansion (4.4).

Proof. We prove (4.10)-(4.11) in Appendix A.4. We conjugate (3.8) with the flow $(\mathbf{\Psi}^{\theta}_{-1}(U))_{|\theta=1}$ using formula (A.2) in Lemma A.1. By Proposition 2.10 we deduce that, if U solves (3.8), then

$$\begin{aligned}\partial_t W_0 &\overset{(4.10),(4.11)}{=} \partial_t Op^{\rm BW}(C^{-1})Op^{\rm BW}(C)W_0 \\ &\quad + Op^{\rm BW}(C^{-1})Op^{\rm BW}\big(\mathrm{i}A_1\xi + \mathrm{i}A_{1/2}|\xi|^{\frac{1}{2}} + A_0 + A_{-1}\big)Op^{\rm BW}(C)W_0\end{aligned}$$

up to a matrix of smoothing operators in $\Sigma\mathcal{R}^{-\rho+1}_{K,2,1}\otimes\mathcal{M}_2(\mathbb{C})$. Moreover Proposition 2.9 imply that

$$\partial_t W_0 = Op^{\rm BW}\Big(\partial_t C^{-1}\#_\rho C + C^{-1}\#_\rho\big(\mathrm{i}A_1\xi + \mathrm{i}A_{1/2}|\xi|^{\frac{1}{2}} + A_0 + A_{-1}\big)\#_\rho C\Big)W_0 \tag{4.15}$$

up to terms in $\Sigma\mathcal{R}^{-\rho+1}_{K,2,1}\otimes\mathcal{M}_2(\mathbb{C})$. We now prove that (4.15) has the form (4.13). By (4.5), (4.7) we have

$$(\partial_t C^{-1})\#_\rho C = \begin{pmatrix}(\partial_t f)f-(\partial_t g)g & (\partial_t f)g-(\partial_t g)f\\ (\partial_t f)g-(\partial_t g)f & (\partial_t f)f-(\partial_t g)g\end{pmatrix} = \begin{pmatrix}0 & (\partial_t f)g-(\partial_t g)f\\ (\partial_t f)g-(\partial_t g)f & 0\end{pmatrix} \tag{4.16}$$

because differentiating $f^2-g^2=1$ we get $(\partial_t f)f-(\partial_t g)g=0$.

By (3.9), using symbolic calculus and $f^2-g^2=1$ (see (4.7)), we obtain the exact expansion

$$C^{-1}\#_\rho(\mathrm{i}A_1\xi)\#_\rho C \;=\begin{pmatrix}-\mathrm{i}V\xi & V(f_x g-g_x f)\\ V(f_x g-g_x f) & -\mathrm{i}V\xi\end{pmatrix}. \tag{4.17}$$

By (4.8) we have

$$C^{-1}\#_\rho(\mathrm{i}A_{1/2}|\xi|^{\frac{1}{2}})\#_\rho C = \mathrm{i}\begin{pmatrix}-(1+a^{(0)})|\xi|^{\frac{1}{2}} & 0\\ 0 & (1+a^{(0)})|\xi|^{\frac{1}{2}}\end{pmatrix} \tag{4.18}$$

modulo a matrix of symbols $\Sigma\Gamma^{-\frac{1}{2}}_{K,1,1}\otimes\mathcal{M}_2(\mathbb{C})$. Moreover, recalling (3.11), we have the paraproduct expansion

$$C^{-1}\#_\rho A_0\#_\rho C = A_0 = -\frac{1}{4}\left[\begin{smallmatrix}0&1\\1&0\end{smallmatrix}\right]V_x \tag{4.19}$$

and finally, since A_{-1} is in $\Sigma\Gamma^{-1}_{K,1,1}\otimes\mathcal{M}_2(\mathbb{C})$ we deduce

$$C^{-1}\#_\rho A_{-1}\#_\rho C \in \Sigma\Gamma^{-1}_{K,1,1}\otimes\mathcal{M}_2(\mathbb{C})\,. \tag{4.20}$$

Formulas (4.15)-(4.20) imply (4.13), (4.14), with a remainder $R^{(0)}(U)$ in $\Sigma\mathcal{R}^{-\rho}_{K,2,1}\otimes\mathcal{M}_2(\mathbb{C})$, renaming $\rho-1$ as ρ. Finally, by (4.8), (4.6) and (3.24) we get the expansion (4.4). ☐

4.2. **Block-Diagonalization at negative orders.** The aim of this subsection is to iteratively block-diagonalize the system (4.13) (which is yet block-diagonal at the orders 1 and 1/2) into (4.2).

Lemma 4.3. *For $j=0,\dots,2\rho$, there are*

- *paradifferential operators of the form*

$$\mathcal{Y}^{(j)}(U) := Op^{\rm BW}\begin{pmatrix}d(U;x,\xi) & 0\\ 0 & \overline{d(U;x,-\xi)}\end{pmatrix} + Op^{\rm BW}(A^{(j)}) \tag{4.21}$$

where $d(U;x,\xi)$ is the symbol defined in Lemma 4.2, $A^{(j)}$ is a matrix of symbols of the form

$$A^{(j)} = \begin{pmatrix}c_j(U;x,\xi) & b_j(U;x,\xi)\\ \overline{b_j(U;x,-\xi)} & \overline{c_j(U;x,-\xi)}\end{pmatrix},\quad c_j\in\Sigma\Gamma^{-\frac{1}{2}}_{K,j+2,1},\quad b_j\in\Sigma\Gamma^{-\frac{j}{2}}_{K,j+2,1}\,, \tag{4.22}$$

- *a real-to-real matrix of smoothing operators $R^{(j)}(U)$ in $\Sigma\mathcal{R}^{-\rho}_{K,j+2,1}\otimes\mathcal{M}_2(\mathbb{C})$, such that, if W_j, $j=0,\dots,2\rho-1$, solves*

$$\partial_t W_j = \big(\mathcal{Y}^{(j)}(U)+R^{(j)}(U)\big)W_j,\quad W_j := \left[\begin{smallmatrix}w_j\\ \overline{w_j}\end{smallmatrix}\right], \tag{4.23}$$

then

$$W_{j+1} := (\mathbf{\Psi}_j^{\theta}(U)W_j)_{|\theta=1} \tag{4.24}$$

where $\mathbf{\Psi}_j^{\theta}(U)$ *is the flow at time* $\theta\in[0,1]$ *of*

$$\partial_\theta \mathbf{\Psi}_j^{\theta}(U) = \mathrm{i}Op^{\mathrm{BW}}(M_j(U;x,\xi))\mathbf{\Psi}_j^{\theta}(U)\,, \quad \Psi_j^0(U)=\mathrm{Id}\,, \tag{4.25}$$

with

$$M_j(U;x,\xi) := \begin{pmatrix} 0 & -\mathrm{i}m_j(U;x,\xi) \\ -\mathrm{i}\overline{m_j(U;x,-\xi)} & 0\end{pmatrix},\ m_j = \frac{-\chi(\xi)b_j(U;x,\xi)}{2\mathrm{i}(1+a^{(0)}(U;x))|\xi|^{\frac12}} \in \Sigma\Gamma^{-\frac{j+1}{2}}_{K,j+2,1}\,, \tag{4.26}$$

and χ *defined in* (2.24)*, satisfies a system of the form* (4.23) *with* $j+1$ *instead of* j*.*

Proof. The proof proceeds by induction.

Inizialization. System (4.13) is (4.23) for $j=0$ where the paradifferential operator $\mathcal{Y}^{(0)}(U)$ has the form (4.21) with the matrix of symbols $A^{(0)}$ defined in Lemma 4.2.

Iteration. We now argue by induction. Suppose that W_j solves system (4.23) with operators $\mathcal{Y}^{(j)}(U)$ of the form (4.21)-(4.22) and smoothing operators $R^{(j)}(U)$ in $\Sigma\mathcal{R}^{-\rho}_{K,j+2,1}\otimes\mathcal{M}_2(\mathbb{C})$. Let us study the system solved by the function W_{j+1} defined in (4.24). Notice that the symbols of the matrix M_j defined in (4.26) have negative order for any $j\geq 0$. By formula (A.2) the conjugated system has the form

$$\partial_t W_{j+1} = Op^{\mathrm{BW}}\big((\partial_t\mathbf{\Psi}_j^1(U))\mathbf{\Psi}_j^{-1}(U) + \mathbf{\Psi}_j^1(U)\mathcal{Y}^{(j)}(U)\mathbf{\Psi}_j^{-1}(U)\big)W_{j+1} \tag{4.27}$$

up to a smoothing operator in $\Sigma\mathcal{R}^{-\rho}_{K,j+2,1}\otimes\mathcal{M}_2(\mathbb{C})$. Moreover the operator $(\partial_t\mathbf{\Psi}_j^1(U))\mathbf{\Psi}_j^{-1}(U)$ admits the Lie expansion in (A.4) specified for $\mathbf{A} := Op^{\mathrm{BW}}(M_j(U))$. We recall (see (2.30)) that

$$M_j\#_\rho\partial_t M_j - \partial_t M_j\#_\rho M_j = \{M_j,\partial_t M_j\}\in\Sigma\Gamma^{-(j+1)-1}_{K,j+3,2}$$

up to a symbol in $\Sigma\Gamma^{-(j+1)-3}_{K,j+3,2}$. By Proposition 2.9 we have that $\mathrm{Ad}_{\mathrm{i}Op^{\mathrm{BW}}(M_j)}[\mathrm{i}Op^{\mathrm{BW}}(\partial_t M_j)]$ is a paradifferential operator with symbol in $\Sigma\Gamma^{-(j+1)-1}_{K,j+3,2}\otimes\mathcal{M}_2(\mathbb{C})$ plus a smoothing remainder in $\Sigma\mathcal{R}^{-\rho}_{K,j+3,2}\otimes\mathcal{M}_2(\mathbb{C})$. As a consequence we deduce, for $k\geq 2$,

$$\mathrm{Ad}^k_{\mathrm{i}Op^{\mathrm{BW}}(M_j)}[\mathrm{i}Op^{\mathrm{BW}}(\partial_t M_j)] = Op^{\mathrm{BW}}(B_k)+R_k, \qquad B_k\in\Gamma^{-\frac{j+1}{2}(k+1)-k}_{K,j+3,k+1}\otimes\mathcal{M}_2(\mathbb{C})\,,$$

and $R_k\in\mathcal{R}^{-\rho}_{K,j+3,k+1}\otimes\mathcal{M}_2(\mathbb{C})$. By taking L large enough with respect to ρ, we obtain that $(\partial_t\mathbf{\Psi}_j^1(U))\mathbf{\Psi}_j^{-1}(U)$ is a paradifferential operator with symbol in $\Sigma\Gamma^{-\frac{j+1}{2}}_{K,j+3,1}\otimes\mathcal{M}_2(\mathbb{C})$ plus a smoothing operator in $\Sigma\mathcal{R}^{-\rho}_{K,j+3,1}\otimes\mathcal{M}_2(\mathbb{C})$. We now want to apply the expansion (A.3) with $\mathbf{A} := Op^{\mathrm{BW}}(M_j(U))$ and $X := \mathcal{Y}^{(j)}$ in order to study the second summand in (4.27). We claim that

$$\mathbf{\Psi}_j^1(U)\mathcal{Y}^{(j)}(U)\mathbf{\Psi}_j^{-1}(U) = Op^{\mathrm{BW}}(\mathcal{Y}^{(j)}(U)) + [Op^{\mathrm{BW}}(\mathrm{i}M_j(U)),\mathcal{Y}^{(j)}(U)] \tag{4.28}$$

plus a paradifferential operator with symbol in $\Sigma\Gamma^{-\frac{j+1}{2}}_{K,j+2,1}\otimes\mathcal{M}_2(\mathbb{C})$ and a smoothing operator in $\Sigma\mathcal{R}^{-\rho}_{K,j+2,1}\otimes\mathcal{M}_2(\mathbb{C})$. We first give the expansion of $[Op^{\mathrm{BW}}(\mathrm{i}M_j(U)),\mathcal{Y}^{(j)}(U)]$ using the expression of $\mathcal{Y}^{(j)}(U)$ in (4.21). We have

$$\begin{aligned}&\Big[Op^{\mathrm{BW}}(\mathrm{i}M_j(U)), Op^{\mathrm{BW}}\Big(\Big[\begin{smallmatrix} d(U;x,\xi) & 0\\ 0 & \overline{d(U;x,-\xi)}\end{smallmatrix}\Big]\Big)\Big] := Op^{\mathrm{BW}}\Big(\Big[\begin{smallmatrix} 0 & p_j(U;x,\xi)\\ \overline{p_j(U;x,-\xi)} & 0\end{smallmatrix}\Big]\Big)\\ &p_j := 2\mathrm{i}m_j(U;x,\xi)(1+a^{(0)}(U;x))|\xi|^{\frac12}\end{aligned} \tag{4.29}$$

up to a symbol in $\Sigma\Gamma^{-\frac{j+1}{2}}_{K,j+2,1}\otimes\mathcal{M}_2(\mathbb{C})$. Moreover, since $A^{(j)}$, is a matrix of symbols of order $-1/2$, for $j\geq 1$, respectively 0 for $j=0$ (see (4.22)), we have that

$$\big[Op^{\rm BW}({\rm i}M_j), Op^{\rm BW}(A^{(j)})\big]\in\begin{cases}\Sigma\Gamma^{-\frac{j+2}{2}}_{K,j+2,1}\otimes\mathcal{M}_2(\mathbb{C}) & \text{for } j\geq 1\\ \Sigma\Gamma^{-\frac{1}{2}}_{K,2,1}\otimes\mathcal{M}_2(\mathbb{C}) & \text{for } j=0\end{cases} \tag{4.30}$$

up to a smoothing operator in $\Sigma\mathcal{R}^{-\rho}_{K,j+2,1}\otimes\mathcal{M}_2(\mathbb{C})$. It follows that the off-diagonal symbols of order $-j/2$ in (4.28) are of the form $\big[\begin{smallmatrix}0 & q_j(U;x,\xi)\\ \overline{q_j(U;x,-\xi)} & 0\end{smallmatrix}\big]$ with

$$q_j(U;x,\xi)\stackrel{(4.29)}{:=} b_j(U;x,\xi)+2{\rm i}m_j(U;x,\xi)(1+a^{(0)})|\xi|^{\frac{1}{2}}\,. \tag{4.31}$$

By the definition of χ in (2.24) and the remark under Definition 2.6, the operator $Op^{\rm BW}((1-\chi(\xi))b_j(U;x,\xi))$ is in $\Sigma\mathcal{R}^{-\rho}_{K,j+2,1}\otimes\mathcal{M}_2(\mathbb{C})$ for any $\rho\geq 0$. Moreover, by the choice of $m_j(U;x,\xi)$ in (4.26) we have that

$$\chi(\xi)b_j(U;x,\xi)+2{\rm i}m_j(U;x,\xi)(1+a^{(0)})|\xi|^{\frac{1}{2}}=0\,.$$

This implies that $[{\rm i}Op^{\rm BW}(M_j),\mathcal{Y}^{(j)}(U)]$ is a paradifferential operator with symbol in $\Sigma\Gamma^{-\frac{j+1}{2}}_{K,j+2,1}\otimes\mathcal{M}_2(\mathbb{C})$ plus a remainder in $\Sigma\mathcal{R}^{-\rho}_{K,j+2,1}\otimes\mathcal{M}_2(\mathbb{C})$. Now, using Proposition 2.9, we deduce, for $k\geq 2$,

$${\rm Ad}^k_{{\rm i}Op^{\rm BW}(M_j)}[\mathcal{Y}^{(j)}(U)]=Op^{\rm BW}(\widetilde{B}_k)+\widetilde{R}_k,\qquad \widetilde{B}_k\in\Gamma^{-\frac{j+1}{2}k}_{K,j+2,k+1}\otimes\mathcal{M}_2(\mathbb{C})\,,$$

where $\widetilde{R}_k$ is in $\mathcal{R}^{-\rho}_{K,j+2,k+1}\otimes\mathcal{M}_2(\mathbb{C})$. Using formula (A.3) with L large enough an the estimates of flow in (4.25) (see Lemma A.2) one obtains the claim in (4.28). We conclude that (4.24) solves a system of the form

$$\partial_t W_{j+1}=Op^{\rm BW}\big(\big[\begin{smallmatrix}d(U;x,\xi) & 0\\ 0 & \overline{d(U;x,-\xi)}\end{smallmatrix}\big]\big)W_{j+1}+Op^{\rm BW}(A^{(j+1)})W_{j+1}+R^{(j+1)}(U)W_{j+1}$$

for some matrix of symbol $A^{(j+1)}$ of the form (4.22) with $j\rightsquigarrow j+1$ and smoothing operators $R^{(j+1)}(U)$ in $\Sigma\mathcal{R}^{-\rho}_{K,j+3,1}\otimes\mathcal{M}_2(\mathbb{C})$. □

Proof of Proposition 4.1. For $\theta\in[0,1]$ we define

$$\mathbf{\Psi}^{\theta}_{diag}(U):=\mathbf{\Psi}^{\theta}_{2\rho-1}(U)\circ\cdots\circ\mathbf{\Psi}^{\theta}_{0}(U)\circ\mathbf{\Psi}^{\theta}_{-1}(U) \tag{4.32}$$

where the maps $\mathbf{\Psi}^{\theta}_{-1}(U)$ and $\mathbf{\Psi}^{\theta}_{j}(U)$, $j=0,1,\ldots,2\rho-1$ are defined respectively in (4.12), (4.24). The bound (4.1) follows by Lemma A.2. Lemmata 4.2, 4.3 imply that if U solves (3.8) then the function $W:=W_{2\rho}=(\mathbf{\Psi}^{\theta}_{diag}(U)U)_{|\theta=1}$ solves the system (4.23) with $j=2\rho$ which is (4.2) with $r_{-1/2}:=c_{2\rho}$ and

$$R(U):=Op^{\rm BW}\Big(\big[\begin{smallmatrix}0 & b_{2\rho}(U;x,\xi)\\ \overline{b_{2\rho}(U;x,-\xi)} & 0\end{smallmatrix}\big]\Big)+R^{(2\rho)}(U)\,,\quad b_{2\rho}\in\Sigma\Gamma^{-\rho}_{K,2\rho+2,1}\,,$$

which is a smoothing operator in $\Sigma\mathcal{R}^{-\rho}_{K,2\rho+2,1}\otimes\mathcal{M}_2(\mathbb{C})$ by the remark below Proposition 2.6. The expansion (4.4) is proved in Lemma 4.2. □

5. Reductions to constant coefficients

The aim of this section is to conjugate (4.2) to a system in which the symbols of the paradifferential operators are constant in the spatial variable x and are "*integrable*" according to Definition 5.1 below, up to symbols which are "*admissible*" according to Definition 5.2.

Definition 5.1. (Integrable symbol) *A homogeneous symbol f in $\widetilde{\Gamma}^m_2$ is* integrable *if it is independent of x and it has the form*

$$f(U;x,\xi) = f(U;\xi) := \frac{1}{2\pi}\sum_{n\in\mathbb{Z}\setminus\{0\}} f^{+-}_{n,n}(\xi)|u_n|^2, \qquad f^{+-}_{n,n}(\xi)\in\mathbb{C}, \quad n\in\mathbb{Z}\setminus\{0\}. \tag{5.1}$$

Definition 5.2. (Admissible symbol) *A non-homogeneous symbol $H_{\geq 3}$ in $\Gamma^1_{K,K',3}$ is* admissible *if it has the form*

$$H_{\geq 3}(U;x,\xi) := \mathrm{i}\alpha_{\geq 3}(U;x)\xi + \mathrm{i}\beta_{\geq 3}(U;x)|\xi|^{\frac12} + \gamma_{\geq 3}(U;x,\xi) \tag{5.2}$$

with real valued functions $\alpha_{\geq 3}(U;x), \beta_{\geq 3}(U;x)$ in $\mathcal{F}^{\mathbb{R}}_{K,K',3}$ and a symbol $\gamma_{\geq 3}(U;x,\xi)$ in $\Gamma^0_{K,K',3}$. A matrix of symbols $\mathbf{H}_{\geq 3}$ in $\Gamma^1_{K,K',3}\otimes\mathcal{M}_2(\mathbb{C})$ is admissible if it has the form

$$\mathbf{H}_{\geq 3}(U;x,\xi) = \begin{bmatrix} H_{\geq 3}(U;x,\xi) & 0 \\ 0 & \overline{H_{\geq 3}(U;x,-\xi)} \end{bmatrix} \tag{5.3}$$

for a scalar admissible symbol $H_{\geq 3}$.

The relevance of Definition 5.2 is explained in the next remark.

Remark 5.3. *An equation of the form $\partial_t v = Op^{\mathrm{BW}}(H_{\geq 3}(U;x,\xi))[v]$, where $H_{\geq 3}(U;x,\xi)$ is an admissible symbol in $\Gamma^1_{K,K',3}$, admits an energy estimate of the form*

$$\partial_t \|v(t,\cdot)\|^2_{\dot H^s} \lesssim_s \|U(t,\cdot)\|^3_{K,s_0}\|v(t,\cdot)\|^2_{\dot H^s}$$

for $s\geq s_0 \gg 1$, see Lemma 7.5. For this reason vector fields of this form are "admissible" to prove existence of solutions up to times $O(\varepsilon^{-3})$.

The main result of this section is the following.

Proposition 5.4. (Integrability of water waves at cubic degree up to smoothing remainders) *Fix $\rho>0$ arbitrary and $K\geq K' := 2\rho+2$. There exists $s_0>0$ such that, for any $s\geq s_0$, for all $0<r\leq r_0(s)$ small enough, and any solution $U\in B^K_s(I;r)$ of (3.8), there is a family of nonlinear maps $\mathbf{F}^\theta(U)$, $\theta\in[0,1]$, such that the function $Z:=\mathbf{F}^1(U)$ solves the system*

$$\partial_t Z = -\mathrm{i}\Omega Z + Op^{\mathrm{BW}}\big(-\mathrm{i}\mathtt{D}(U;\xi) + \mathtt{H}_{\geq 3}\big)Z + \mathtt{R}(U)[Z] \tag{5.4}$$

where Ω is defined in (3.19) and

- *the symbol $\mathtt{D}(U;\xi)$ has the form*

$$\mathtt{D}(U;\xi) := \begin{bmatrix} \zeta(U)\xi + \mathcal{D}_{-1/2}(U;\xi) & 0 \\ 0 & \overline{\zeta(U)\xi - \mathcal{D}_{-1/2}(U;-\xi)} \end{bmatrix}, \qquad \zeta(U) := \frac{1}{\pi}\sum_{n\in\mathbb{Z}\setminus\{0\}} n|n||u_n|^2, \tag{5.5}$$

  with an integrable symbol $\mathcal{D}_{-1/2}(U;\xi)\in\widetilde{\Gamma}^{-\frac12}_2$ (see Definition 5.1);
- *the matrix of symbols $\mathtt{H}_{\geq 3}\in\Gamma^1_{K,K',3}\otimes\mathcal{M}_2(\mathbb{C})$ is admissible (see Definition 5.2);*
- *$\mathtt{R}(U)$ is a real-to-real matrix of smoothing operators in $\Sigma\mathcal{R}^{-\rho+4m}_{K,K',1}\otimes\mathcal{M}_2(\mathbb{C})$ for some $m>0$.*
- *The family of transformations has the form*

$$\mathbf{F}^\theta(U) := \mathfrak{F}^\theta(U)[U] \tag{5.6}$$

with $\mathfrak{F}^{\theta}(U)$ *real-to-real, bounded and invertible, and there is a constant* $C = C(s,r,K)$*, such that,* $\forall\, 0 \le k \le K-K'$*, for any* $V \in C^{K-K'}_{*\mathbb{R}}(I; \dot{H}^s(\mathbb{T};\mathbb{C}^2))$*,*

$$\|\partial_t^k \mathfrak{F}^{\theta}(U)[V]\|_{\dot{H}^{s-k}} + \|\partial_t^k (\mathfrak{F}^{\theta}(U))^{-1}[V]\|_{\dot{H}^{s-k}} \le \|V\|_{k,s}(1 + C\|U\|_{K,s_0})\,, \tag{5.7}$$

uniformly in $\theta \in [0,1]$*.*

The proof of Proposition 5.4 above is divided into several steps in Subsections 5.1-5.3 below. We combine these steps in Subsection 5.4.

5.1. **Integrability at order 1.** By Proposition 4.1 we have obtained, writing only the first line of the system (4.2)-(4.3),

$$\partial_t w = Op^{\mathrm{BW}}\big(-\mathrm{i}V(U;x)\xi - \mathrm{i}(1 + a^{(0)}(U;x))|\xi|^{1/2} + r_{-1/2}\big)w + R(U)[W] \tag{5.8}$$

where $R(U)$ is a 1×2 matrix of smoothing operators in $\Sigma\mathcal{R}^{-\rho}_{K,K',1}$ with $K' = 2\rho + 2$ and $W = \left[\begin{smallmatrix} w \\ \overline{w}\end{smallmatrix}\right]$. The second component of system (4.2) is the complex conjugated of the first one. Expanding in degrees of homogeneity the symbol

$$r_{-1/2} = \mathtt{r}_1 + \mathtt{r}_2 + \mathtt{r}_{\ge 3}\,, \quad \mathtt{r}_1 \in \widetilde{\Gamma}_1^{-\frac{1}{2}}\,,\ \mathtt{r}_2 \in \widetilde{\Gamma}_2^{-\frac{1}{2}}\,,\ \mathtt{r}_{\ge 3} \in \Gamma^{-\frac{1}{2}}_{K,K',3}\,,$$

recalling (3.23) and item (ii) in Proposition 4.1, we rewrite (5.8) as

$$\partial_t w = Op^{\mathrm{BW}}\big(-\mathrm{i}(\mathtt{V}_1 + \mathtt{V}_2)\xi - \mathrm{i}\big(1 + \mathtt{a}_1 + \mathtt{a}_2^{(0)}\big)|\xi|^{1/2} + \mathtt{r}_1 + \mathtt{r}_2 + H_{\ge 3}\big)w + R(U)[W] \tag{5.9}$$

where $H_{\ge 3}$ is an *admissible* symbol according to Definition 5.2.

5.1.1. *Elimination of the linear symbol of the transport.* The goal of this subsection is to eliminate the transport operator $Op^{\mathrm{BW}}(-\mathrm{i}\mathtt{V}_1\xi)$ in (5.9). With this aim we conjugate the equation (5.9) under the flow

$$\partial_\theta \Phi_1^{\theta}(U) = \mathrm{i}Op^{\mathrm{BW}}(b(U;\theta,x)\xi)\Phi_1^{\theta}(U)\,, \quad \Phi_1^0(U) = \mathrm{Id}\,, \tag{5.10}$$

with

$$b(U;\theta,x) := \frac{\beta(U;x)}{1 + \theta\beta_x(U;x)}\,, \tag{5.11}$$

where $\beta(U;x)$ is a real valued function in $\widetilde{\mathcal{F}}_1^{\mathbb{R}}$ of the same form of $\mathtt{V}_1(U;x)$, i.e.

$$\beta(U;x) = \frac{1}{\sqrt{2\pi}} \sum_{n\in\mathbb{Z}\setminus\{0\}} \beta_n^+ u_n e^{\mathrm{i}nx} + \beta_n^- \overline{u_n} e^{-\mathrm{i}nx}\,. \tag{5.12}$$

The function $\beta(U;x)$ is real if a condition like (2.17) holds, i.e.

$$\overline{\beta_n^+} = \beta_n^-\,. \tag{5.13}$$

The flow of the transport equation (5.11) is well posed by Lemma A.2. We introduce the new variable

$$V_1 := \left[\begin{smallmatrix} v_1 \\ \overline{v_1}\end{smallmatrix}\right] = \big(\mathbf{\Phi}_1^{\theta}(U)[W]\big)_{|\theta=1} = \begin{pmatrix} \Phi_1^{\theta}(U)[w] \\ \overline{\Phi_1^{\theta}(U)[\overline{w}]}\end{pmatrix}_{|\theta=1}\,, \qquad W := \left[\begin{smallmatrix} w \\ \overline{w}\end{smallmatrix}\right], \tag{5.14}$$

where the operator $\overline{\Phi_1^{\theta}(U)}[\cdot]$ is defined as in (2.31).

Lemma 5.5. *Define* $\beta \in \widetilde{\mathcal{F}}_1^{\mathbb{R}}$ *in* (5.12) *with coefficients*

$$\beta_n^+ := -\frac{(\mathtt{V}_1)_n^+}{\mathrm{i}\omega_n} = \frac{\mathrm{i}n}{\sqrt{2}|n|^{\frac{3}{4}}}\,, \quad \beta_n^- := \frac{(\mathtt{V}_1)_n^-}{\mathrm{i}\omega_n} = -\frac{\mathrm{i}n}{\sqrt{2}|n|^{\frac{3}{4}}}\,, \quad n \ne 0\,, \tag{5.15}$$

and $(\beta)_0^{\sigma} := 0$*,* $\sigma = \pm$*. Then, if* w *solves* (5.9)*, the function* v_1 *defined in* (5.14) *solves*

$$\partial_t v_1 = Op^{\mathrm{BW}}\big(-\mathrm{i}\mathtt{V}_2^{(1)}\xi - \mathrm{i}\big(1 + \mathtt{a}_2^{(1)}\big)|\xi|^{\frac{1}{2}} + \mathtt{r}_1^{(1)} + \mathtt{r}_2^{(1)} + H_{\ge 3}^{(1)}\big)v_1 + R^{(1)}(U)[V_1] \tag{5.16}$$

where:

- $\mathtt{V}_2^{(1)} \in \widetilde{\mathcal{F}}_2^{\mathbb{R}}$ *and its coefficients (according to the expansion* (2.16)*) satisfy*
$$(\mathtt{V}_2^{(1)})_{n,n}^{+-} = 2n|n|\,, \quad (\mathtt{V}_2^{(1)})_{n,-n}^{+-} = 0\,; \tag{5.17}$$
- $\mathtt{a}_2^{(1)} \in \widetilde{\mathcal{F}}_2^{\mathbb{R}}$ *and its coefficients satisfy*
$$(\mathtt{a}_2^{(1)})_{n,n}^{+-} = 0\,; \tag{5.18}$$
- $\mathtt{r}_1^{(1)} \in \widetilde{\Gamma}_1^{-\frac{1}{2}}$ *and* $\mathtt{r}_2^{(1)} \in \widetilde{\Gamma}_2^{-\frac{1}{2}}$*;*
- $H_{\geq 3}^{(1)} \in \Gamma_{K,K',3}^{1}$ *is an admissible symbol, and* $R^{(1)}(U)$ *belongs to* $\Sigma\mathcal{R}_{K,K',1}^{-\rho}$.

Notice that the procedure that eliminates the linear term of the transport in (5.16), that is, the contribution with degree of homogeneity 1 to the coefficient of ξ, automatically also eliminates the contribution with degree of homogeneity 1 to the coefficient of the symbol of order 1/2.

Proof of Lemma 5.5. Conjugation under the flow in (5.10)*.* We use Lemmata A.4 and A.5.

Step 1. We apply Lemma A.5 with β in $\widetilde{\mathcal{F}}_1^{\mathbb{R}} \subset \mathcal{F}_{K,0,1}^{\mathbb{R}}$ by the fourth remark in (2.14). Then Lemma A.5 implies that
$$\partial_t \Phi_1^1(U)(\Phi_1^1(U))^{-1} = Op^{\mathrm{BW}}\big(\mathrm{i}(\beta_t - \beta_x\beta_t)\xi + H_{\geq 3}\big) + R(U)$$
where $H_{\geq 3} := \mathrm{i}g_{\geq 3}\xi$ is an admissible symbol in $\Gamma_{K,1,3}^{1}$ and $R(U)$ belongs to $\Sigma\mathcal{R}_{K,1,1}^{-\rho}$.

Step 2. We apply Lemma A.4 with $a = -\mathrm{i}V\xi$. Thus (A.15)-(A.17) imply that, noting that $a_\Phi^{(1)} = 0$,
$$\Phi_1^1(U)Op^{\mathrm{BW}}(-\mathrm{i}V\xi)(\Phi_1^1(U))^{-1} = Op^{\mathrm{BW}}\big(-\mathrm{i}V(x+\beta(x))(1+\tilde{\beta}_y(y))_{|y=x+\beta(x)}\xi\big) + R(U)$$
where $x = y + \tilde{\beta}(y)$ denotes the inverse diffeomorphism of $y = x + \beta(x)$ and $R(U)$ is a smoothing operator in $\Sigma\mathcal{R}_{K,K',2}^{-\rho+1}$. By (A.18) we deduce the expansion
$$\Phi_1^1(U)Op^{\mathrm{BW}}(-\mathrm{i}V\xi)(\Phi_1^1(U))^{-1} = Op^{\mathrm{BW}}\big(-\mathrm{i}(\mathtt{V}_1+\mathtt{V}_2)\xi + \mathrm{i}(\mathtt{V}_1\beta_x - (\mathtt{V}_1)_x\beta)\xi + H_{\geq 3}\big) + R(U)$$
where $H_{\geq 3} \in \Gamma_{K,K',3}^{1}$ is an admissible symbol and $R(U)$ belongs to $\Sigma\mathcal{R}_{K,K',2}^{-\rho+1}$.

Step 3. By Lemma A.4 we have that, up to a smoothing remainder in $\Sigma\mathcal{R}_{K,K',1}^{-\rho+\frac{1}{2}}$,
$$-\Phi_1^1(U)Op^{\mathrm{BW}}\big(\mathrm{i}\big(1+\mathtt{a}_1+\mathtt{a}_2^{(0)}\big)|\xi|^{1/2}\big)(\Phi_1^1(U))^{-1} = -Op^{\mathrm{BW}}(a_\Phi^{(0)} + a_\Phi^{(1)})$$
where $a_\Phi^{(0)} \in \Sigma\Gamma_{K,K',0}^{\frac{1}{2}}$ is given by (A.17) and $a_\Phi^{(1)} \in \Sigma\Gamma_{K,K',1}^{-\frac{3}{2}}$. By (A.18) we have the expansion
$$\begin{aligned} a_\Phi^{(0)} &= -\mathrm{i}\big(1+\mathtt{a}_1+\mathtt{a}_2^{(0)}\big)|\xi|^{1/2} + \big\{\beta\xi, -\mathrm{i}\big(1+\mathtt{a}_1\big)|\xi|^{1/2}\big\} \\ &\quad + \frac{1}{2}\Big(\big\{\beta\xi, \big\{\beta\xi, -\mathrm{i}|\xi|^{1/2}\big\}\big\} - \big\{\beta\beta_x\xi, -\mathrm{i}|\xi|^{1/2}\big\}\Big) + \text{admissible symbol} \end{aligned}$$
and a direct computation gives
$$\begin{aligned} &-\Phi_1^1(U)Op^{\mathrm{BW}}\big(\mathrm{i}\big(1+\mathtt{a}_1+\mathtt{a}_2^{(0)}\big)|\xi|^{1/2}\big)(\Phi_1^1(U))^{-1} = \\ &\qquad Op^{\mathrm{BW}}\Big(-\mathrm{i}\big(1+\mathtt{a}_1 - \frac{\beta_x}{2} + \mathtt{a}_2^{(0)} + (\mathtt{a}_1)_x\beta - \frac{1}{2}\beta_x\mathtt{a}_1 + \frac{3}{8}\beta_x^2\big)|\xi|^{1/2} + r + H_{\geq 3}\Big) + R(U) \end{aligned}$$
where $r \in \Sigma\Gamma_{K,K',1}^{-\frac{3}{2}}$, $H_{\geq 3} \in \Gamma_{K,K',3}^{\frac{1}{2}}$ is an admissible symbol and $R(U)$ is in $\Sigma\mathcal{R}_{K,K',1}^{-\rho+\frac{1}{2}}$.

Step 4. By Lemma A.4 the conjugated operator
$$\Phi_1^1(U)Op^{\mathrm{BW}}\big(\mathtt{r}_1+\mathtt{r}_2+H_{\geq 3}\big)(\Phi_1^1(U))^{-1} = Op^{\mathrm{BW}}\big(\mathtt{r}_1^{(1)}+\mathtt{r}_2^{(1)}+H'_{\geq 3}\big) + R(U)$$

where $\mathtt{r}_1^{(1)} \in \widetilde{\Gamma}_1^{-\frac{1}{2}}$, $\mathtt{r}_2^{(1)} \in \widetilde{\Gamma}_2^{-\frac{1}{2}}$, a new admissible symbol $H'_{\geq 3} \in \Gamma^1_{K,K',3}$, and a smoothing remainder $R(U)$ in $\Sigma\mathcal{R}^{-\rho+1}_{K,K',1}$.

STEP 5. By Lemma A.2-(*ii*) we write $\Phi^1(U) = \mathrm{Id} + M(U)$ with $M(U)$ in $\Sigma\mathcal{M}^m_{K,K',1}$ for some $m > 0$. Hence, using Proposition 2.10-(ii), we have that the conjugated of the operator $R(U)$ in (5.9) is a smoothing remainder in $\Sigma\mathcal{R}^{-\rho+2m}_{K,K',1}$. In conclusion we get that if w solves (5.9) then v_1 defined in (5.14) satisfies

$$\begin{aligned}\partial_t v_1 &= \mathrm{i}Op^{\mathrm{BW}}\big(\big(-\mathtt{V}_1 + \partial_t\beta\big)\xi + \big(-\mathtt{V}_2 + (\mathtt{V}_1\beta_x - (\mathtt{V}_1)_x\beta) - \beta_x\beta_t\big)\xi\big)v_1 \\ &\quad + \mathrm{i}Op^{\mathrm{BW}}\Big(-|\xi|^{\frac12} - (\mathtt{a}_1 - \frac{\beta_x}{2})|\xi|^{\frac12} - \big(\mathtt{a}_2^{(0)} + (\mathtt{a}_1)_x\beta - \frac12\beta_x\mathtt{a}_1 + \frac38\beta_x^2\big)|\xi|^{\frac12}\Big)v_1 \\ &\quad + Op^{\mathrm{BW}}\big(\mathtt{r}_1^{(1)} + \mathtt{r}_2^{(1)}\big)v_1 + Op^{\mathrm{BW}}(H_{\geq 3})v_1 + R^{(1)}(U)[V_1]\end{aligned} \tag{5.19}$$

where $\mathtt{r}_1^{(1)} \in \widetilde{\Gamma}_1^{-\frac{1}{2}}$, $\mathtt{r}_2^{(1)} \in \widetilde{\Gamma}_2^{-\frac{1}{2}}$, $H_{\geq 3} \in \Gamma^1_{K,K',3}$ is admissible according to Definition 5.2 and $R^{(1)}(U)$ is a 1×2 matrix of smoothing operators in $\Sigma\mathcal{R}^{-\rho}_{K,K',1}$ (renaming $\rho - 2m$ as ρ).

Choice of β. Recall that the coefficients $\beta_n^{\pm}$ defined in (5.15) satisfy (5.13) and the function $\beta(U;x)$ is real. Using (3.20) we get

$$\partial_t\beta(U;x) = \frac{1}{\sqrt{2\pi}}\sum_{n\in\mathbb{Z}\setminus\{0\}}(-\mathrm{i}\omega_n)\beta_n^+ e^{\mathrm{i}nx}u_n + \mathrm{i}\omega_n\beta_n^- e^{-\mathrm{i}nx}\overline{u_n} + \mathtt{h}_2 + \mathtt{h}_{\geq 3} \tag{5.20}$$

where $\mathtt{h}_2$, $\mathtt{h}_{\geq 3}$ are defined as

$$\begin{aligned}\mathtt{h}_2 &:= \frac{1}{\sqrt{2\pi}}\sum_{n\in\mathbb{Z}\setminus\{0\}}\beta_n^+\mathrm{i}[F_2(U)]_n e^{\mathrm{i}nx} - \beta_n^-\mathrm{i}\overline{[F_2(U)]_n}e^{-\mathrm{i}nx}\,, \\ \mathtt{h}_{\geq 3} &:= \frac{1}{\sqrt{2\pi}}\sum_{n\in\mathbb{Z}\setminus\{0\}}\beta_n^+\mathrm{i}[F_{\geq 3}(U)]_n e^{\mathrm{i}nx} - \beta_n^-\mathrm{i}\overline{[F_{\geq 3}(U)]_n}e^{-\mathrm{i}nx}\,.\end{aligned} \tag{5.21}$$

By (5.20) and (5.15) we deduce that

$$-\mathtt{V}_1 + \partial_t\beta = \mathtt{h}_2 + \mathtt{h}_{\geq 3}. \tag{5.22}$$

By (5.13) the functions $\mathtt{h}_2$ and $\mathtt{h}_{\geq 3}$ are real. Moreover $\mathtt{h}_2 \in \widetilde{\mathcal{F}}_2^{\mathbb{R}}$ and $\mathtt{h}_{\geq 3} \in \mathcal{F}^{\mathbb{R}}_{K,1,3}$ by item (iv) of Proposition 2.10 and the fact that $F_2(U) + F_{\geq 3}(U) = M(U)[U]$ for some M in $\Sigma\mathcal{M}_{K,1,1}$, see (3.19).

The new equation. From (5.22) and the first line of (5.19) we deduce that $\mathtt{V}_2^{(1)}$ in (5.16) is given by

$$-\mathtt{V}_2^{(1)} := \mathtt{h}_2 - \mathtt{V}_2 - (\mathtt{V}_1)_x\beta, \tag{5.23}$$

having used $(\mathtt{V}_1 - \partial_t\beta)\beta_x \in \mathcal{F}^{\mathbb{R}}_{K,1,3}$. From the second line of (5.19) we deduce that $\mathtt{a}_2^{(1)}$ in (5.16) is given by

$$\mathtt{a}_2^{(1)} := \mathtt{a}_2^{(0)} + (\mathtt{a}_1)_x\beta - \frac12\beta_x\mathtt{a}_1 + \frac38\beta_x^2 \in \widetilde{\mathcal{F}}_2^{\mathbb{R}} \tag{5.24}$$

having noted that the function $\mathtt{a}_1 - \frac{\beta_x}{2} = 0$ by (3.36) and (5.15).

Let us prove (5.17). By (5.23) and (5.15) we have

$$((\mathtt{V}_2)^{(1)})^{+-}_{n_1,n_2} = -(\mathtt{h}_2)^{+-}_{n_1,n_2} + (\mathtt{V}_2)^{+-}_{n_1,n_2} + \mathrm{i}(n_1+n_2)\big((\mathtt{V}_1)^+_{n_1}\beta^-_{n_2} - (\mathtt{V}_1)^-_{n_2}\beta^+_{n_1}\big)\,. \tag{5.25}$$

The coefficients $(\mathtt{h}_2)^{+-}_{n_1,n_2}$ associated to $\mathtt{h}_2$ defined in (5.21) are

$$(\mathtt{h}_2)^{+-}_{n_1,n_2} = \mathrm{i}\beta^+_{n_1-n_2}(F_2)^{+-}_{n_1,n_2} - \mathrm{i}\beta^-_{-(n_1-n_2)}\overline{(F_2)^{+-}_{n_2,n_1}}$$

with $(F_2)^{+-}_{n_1,n_2}$ defined by (3.20)-(3.21). We claim that

$$(\mathtt{h}_2)^{+-}_{n,n}=0, \qquad (\mathtt{h}_2)^{+-}_{n,-n}=0\,. \tag{5.26}$$

The first identity in (5.26) is trivial since the coefficients β_n^σ in (5.15) are zero for $n=0$. To prove the second identity in (5.26) we compute by (5.21) and (3.41)

$$(\mathtt{h}_2)^{+-}_{n,-n}=\mathrm{i}(F_2)^{+-}_{n,-n}\big(\beta^+_{2n}-\beta^-_{-2n}\big)=0$$

in view of (5.15). From (5.25), (5.26), (3.35) we then obtain $(\mathtt{V}_2^{(1)})^{+-}_{n,-n}=0$. By (5.25), (5.26), (5.15), (3.35) we get $(\mathtt{V}_2^{(1)})^{+-}_{n,n}=2n|n|$.

To conclude we prove (5.18). From (5.24) we calculate

$$(\mathtt{a}_2^{(1)})^{+-}_{n,n}=(\mathtt{a}_2^{(0)})^{+-}_{n,n}+\mathrm{i}n(\mathtt{a}_1)^+_n\beta^-_n-\mathrm{i}n(\mathtt{a}_1)^-_n\beta^+_n-\frac{\mathrm{i}n}{2}\big(\beta^+_n(\mathtt{a}_1)^-_n-\beta^-_n(\mathtt{a}_1)^+_n\big)+\frac{3}{4}\beta^+_n\beta^-_n n^2$$

where β_n^σ are defined in (5.15). By (4.4) we have $(\mathtt{a}_2^{(0)})^{+-}_{n_1,n_2}=(\mathtt{a}_2)^{+-}_{n_1,n_2}-(\mathtt{a}_1)^+_{n_1}(\mathtt{a}_1)^-_{n_2}$ and one can check directly using the formulas (3.36) and (5.15), that $(\mathtt{a}_2^{(1)})^{+-}_{n,n}=0$. □

5.1.2. *Reduction of the quadratic symbol of the transport.* The aim of this section is to reduce the transport operator $-\mathrm{i}Op^{\mathrm{BW}}(\mathtt{V}_2^{(1)}(U;x)\xi)$ in (5.16) into the "integrable" one $-\mathrm{i}Op^{\mathrm{BW}}(\zeta(U)\xi)$ where $\zeta(U)$ is the function, constant in x, defined in (5.5). To do this we conjugate the equation (5.16) under the flow of the transport equation

$$\partial_\theta\Phi_2^\theta(U)=\mathrm{i}Op^{\mathrm{BW}}(b_2(U;\theta,x)\xi)\Phi_2^\theta(U)\,,\quad \Phi_2^0(U)=\mathrm{Id}\,, \tag{5.27}$$

where b_2 is defined as in (5.11) in terms of a real valued function $\beta_2(U;x)\in\widetilde{\mathcal{F}}_2^{\mathbb{R}}$. The flow in (5.27) is well-posed by Lemma A.2. We then define the new variable

$$V_2:=\big[\begin{smallmatrix}v_2\\ \overline{v_2}\end{smallmatrix}\big]=\big(\boldsymbol{\Phi}_2^\theta(U)[V_1]\big)_{|\theta=1}:=\begin{pmatrix}\Phi_2^\theta(U)[v_1]\\ \overline{\Phi_2^\theta(U)[\overline{v_1}]}\end{pmatrix}_{|\theta=1} \tag{5.28}$$

where $\overline{\Phi_2^\theta(U)}$ is defined as in (2.31).

Lemma 5.6. *Define $\beta_2\in\widetilde{\mathcal{F}}_2^{\mathbb{R}}$ with coefficients for $n_1,n_2\in\mathbb{Z}\setminus\{0\}$,*

$$(\beta_2)^{\sigma\sigma}_{n_1,n_2}:=\frac{-(\mathtt{V}_2^{(1)})^{\sigma\sigma}_{n_1,n_2}}{\mathrm{i}\sigma(\omega_{n_1}+\omega_{n_2})}\,,\ \sigma=\pm\,,\qquad (\beta_2)^{+-}_{n_1,n_2}:=\frac{-(\mathtt{V}_2^{(1)})^{+-}_{n_1,n_2}}{\mathrm{i}(\omega_{n_1}-\omega_{n_2})}\,,\ n_1\neq\pm n_2\,, \tag{5.29}$$

and $(\beta_2)^{\sigma\sigma}_{0,0}:=0$, $(\beta_2)^{+-}_{n,\sigma n}:=0$, $\sigma=\pm$, where $\mathtt{V}_2^{(1)}$ is the real-valued function defined in Lemma 5.5. If v_1 solves (5.16) then the function v_2 in (5.28) solves

$$\partial_t v_2=Op^{\mathrm{BW}}\big(-\mathrm{i}\zeta(U)\xi-\mathrm{i}(1+\mathtt{a}_2^{(2)})|\xi|^{\frac12}+\mathtt{r}_1^{(1)}+\mathtt{r}_2^{(2)}+H^{(2)}_{\geq3}\big)v_2+R^{(2)}(U)[V_2] \tag{5.30}$$

where:

- *$\zeta(U)\in\widetilde{\mathcal{F}}_2^{\mathbb{R}}$ is the integrable function defined in (5.5);*
- *$\mathtt{a}_2^{(2)}\in\widetilde{\mathcal{F}}_2^{\mathbb{R}}$ satisfies*

$$\mathtt{a}_2^{(2)}:=\mathtt{a}_2^{(1)}-\frac12(\beta_2)_x\,,\qquad (\mathtt{a}_2^{(2)})^{+-}_{n,n}=0\,; \tag{5.31}$$

- *$\mathtt{r}_1^{(1)}\in\widetilde{\Gamma}_1^{-\frac12}$ is the same symbol in (5.16), and $\mathtt{r}_2^{(2)}\in\widetilde{\Gamma}_2^{-\frac12}$;*
- *$H^{(2)}_{\geq3}\in\Gamma^1_{K,K',3}$ is an admissible symbol, and $R^{(2)}(U)$ is a 1×2 matrix of smoothing operators in $\Sigma\mathcal{R}^{-\rho}_{K,K',1}$.*

Proof. The function β_2 is real valued since the coefficients $(\mathtt{V}_2)^{\sigma\sigma'}_{n_1,n_2}$ of the real function $\mathtt{V}_2^{(1)}$ in (5.24) satisfy (2.17). In order to conjugate (5.16) under the map Φ_2^θ in (5.28) we apply Lemmata A.4 and A.5. By (A.18) and (A.21), and since β_2 is quadratic in u, the only quadratic contributions are $Op^{\mathrm{BW}}(\{\beta_2\xi, -\mathrm{i}|\xi|^{\frac12}\})v_2 + \mathrm{i}Op^{\mathrm{BW}}(\partial_t\beta_2\xi)v_2$, implying

$$\partial_t v_2 = Op^{\mathrm{BW}}\big(\mathrm{i}(-\mathtt{V}_2^{(1)} + \partial_t\beta_2)\xi + \mathrm{i}\frac{(\beta_2)_x}{2}|\xi|^{\frac12} - \mathrm{i}(1+\mathtt{a}_2^{(1)})|\xi|^{\frac12} + \mathtt{r}_1^{(1)} + \widetilde{\mathtt{r}}_2^{(1)} + H^{(1)}_{\geq 3}\big)v_2 + R(U)[V_2] \tag{5.32}$$

where $\widetilde{\mathtt{r}}_2^{(1)}$ is a symbol in $\widetilde{\Gamma}_2^{-\frac12}$, $H^{(1)}_{\geq3} \in \Gamma^1_{K,K',3}$ is a new admissible symbol, and $R(U)$ is a 1×2 matrix of smoothing operators in $\Sigma\mathcal{R}^{-\rho}_{K,K',1}$ (by renaming $\rho - 2m$ as ρ).

By the choice of β_2 in (5.29), using (3.20) and reasoning as in the proof of Lemma 5.5 we have

$$-\mathtt{V}_2^{(1)} + \partial_t\beta_2 = -\Big(\frac{1}{2\pi}\sum_{n\in\mathbb{Z}\setminus\{0\}} (\mathtt{V}_2^{(1)})^{+-}_{n,n}|u_n|^2 + (\mathtt{V}_2^{(1)})^{+-}_{n,-n}u_n\overline{u_{-n}}e^{\mathrm{i}2nx}\Big) + f_{\geq3}$$

where $f_{\geq3}$ is a function in $\mathcal{F}^{\mathbb{R}}_{K,1,3}$. Therefore, by (5.17), we get

$$-\mathtt{V}_2^{(1)} + \partial_t\beta_2 = -\zeta(U) + f_{\geq3} \tag{5.33}$$

with $\zeta(U)$ defined in (5.5). System (5.32) and (5.33) imply (5.30) where $\mathtt{a}_2^{(2)}$ is the function defined in (5.31). Recalling (5.18) we deduce that $(\mathtt{a}_2^{(2)})^{+-}_{n,n} = 0$. □

Remark 5.7. *For a cubic vector field of the form $Op^{\mathrm{BW}}(\mathrm{i}\mathtt{V}_2(U;x)\xi)[u]$ with a real valued function $\mathtt{V}_2$ in $\widetilde{\mathcal{F}}_2^{\mathbb{R}}$ the reversibility and even-to-even properties imply $(\mathtt{V}_2)^{+-}_{n,-n} = 0$, $(\mathtt{V}_2)^{+-}_{n,n} = -(\mathtt{V}_2)^{+-}_{-n,-n}$, $(\mathtt{V}_2)^{+-}_{n,n} \in \mathbb{R}$, in agreement with (5.17). We remark that the cancellation $(\mathtt{V}_2)^{+-}_{n,-n} = 0$ is essential for the whole argument to work. Notice also that on the subspace of even functions u one has $\zeta(U) = 0$.*

5.2. **Integrability at order** $1/2$ **and** 0. The first aim of this subsection is to reduce the operator $-\mathrm{i}Op^{\mathrm{BW}}\big(\mathtt{a}_2^{(2)}(U;x)|\xi|^{1/2}\big)$ in (5.30) to an integrable one. It actually turns out that, thanks to (5.31), we reduce it to the Fourier multiplier $-\mathrm{i}|D|^{1/2}$, see (5.47). This is done in two steps. In 5.2.1 we apply a transformation which is a paradifferential "semi-Fourier integral operator", generated as the flow of (5.34). Then, in Subsection 5.2.2 we apply the para-differential version of a torus diffeomorphism which is "almost" time independent, see (5.43)-(5.44). Eventually we deal with the operators of order 0 in Subsection 5.2.3.

5.2.1. *Elimination of the time dependence at order* $1/2$ *up to* $O(u^3)$. We conjugate (5.30) under the flow

$$\partial_\theta\Phi_3^\theta(U) = \mathrm{i}Op^{\mathrm{BW}}(\beta_3(U;x)|\xi|^{\frac12})\Phi_3^\theta(U), \quad \Phi_3^0(U) = \mathrm{Id}, \tag{5.34}$$

where $\beta_3(U;x) \in \widetilde{\mathcal{F}}_2^{\mathbb{R}}$ is a real valued function. We set

$$V_3 := \big[\begin{smallmatrix} v_3 \\ \overline{v_3}\end{smallmatrix}\big] = \big(\mathbf{\Phi}_3^\theta(U)[V_2]\big)_{|\theta=1} = \begin{pmatrix}\Phi_3^\theta(U)[v_2] \\ \overline{\Phi_3^\theta(U)}[\overline{v_2}]\end{pmatrix}_{|\theta=1} \tag{5.35}$$

where $\overline{\Phi_3^\theta(U)}$ is defined as in (2.31).

Lemma 5.8. *Define $\beta_3 \in \widetilde{\mathcal{F}}_2^{\mathbb{R}}$ with coefficients*

$$(\beta_3)^{\sigma\sigma}_{n_1,n_2} := \frac{-(\mathtt{a}_2^{(2)})^{\sigma\sigma}_{n_1,n_2}}{\mathrm{i}\sigma(\omega_{n_1}+\omega_{n_2})}, \quad \sigma=\pm, \quad (\beta_3)^{+-}_{n_1,n_2} := \frac{-(\mathtt{a}_2^{(2)})^{+-}_{n_1,n_2}}{\mathrm{i}(\omega_{n_1}-\omega_{n_2})}, \; n_1 \neq \pm n_2, \tag{5.36}$$

and $(\beta_3)^{\sigma\sigma}_{0,0} := 0$, $(\beta_3)^{+-}_{n,\sigma n} := 0$, $\sigma=\pm$, where $\mathtt{a}_2^{(2)}$ is defined in (5.31). If v_2 solves (5.30) then

$$\partial_t v_3 = Op^{\mathrm{BW}}\big(-\mathrm{i}\zeta(U)\xi - \mathrm{i}(1+\mathtt{a}_2^{(3)})|\xi|^{\frac12} + \mathrm{i}\mathtt{b}_2^{(3)}\mathrm{sign}(\xi) + \mathtt{r}_1^{(1)} + \mathtt{r}_2^{(3)} + H^{(3)}_{\geq3}\big)v_3 + R^{(3)}(U)[V_3] \tag{5.37}$$

where

$$\mathtt{a}_2^{(3)} := \frac{1}{2\pi}\sum_{n\in\mathbb{Z}\setminus\{0\}}(\mathtt{a}_2^{(2)})_{n,-n}^{+-}u_n\overline{u_{-n}}e^{\mathrm{i}2nx}, \qquad \mathtt{b}_2^{(3)} := \frac{1}{2}(\beta_3)_x\,, \tag{5.38}$$

$\mathtt{r}_1^{(1)} \in \widetilde{\Gamma}_1^{-\frac{1}{2}}$ *is the same symbol in* (5.16), $\mathtt{r}_2^{(3)} \in \widetilde{\Gamma}_2^{-\frac{1}{2}}$, $H_{\geq 3}^{(3)} \in \Gamma^1_{K,K',3}$ *is admissible, and* $R^{(3)}(U)$ *is a* 1×2 *matrix of smoothing operators in* $\Sigma\mathcal{R}^{-\rho}_{K,K',1}$. *Moreover*

$$(\mathtt{b}_2^{(3)})_{n,n}^{+-} = (\mathtt{b}_2^{(3)})_{n,-n}^{+-} = 0\,. \tag{5.39}$$

Proof. By (5.36) and (2.17) we deduce that β_3 is a real function. To conjugate system (5.30) we apply Lemmata A.6 and A.7 with $m \rightsquigarrow 1/2$ and $m' \rightsquigarrow 1$. The only new contribution at quadratic degree of homogeneity and positive order is $Op^{\mathrm{BW}}(\{\beta_3|\xi|^{\frac{1}{2}}, -\mathrm{i}|\xi|^{\frac{1}{2}}\})$ and $\mathrm{i}Op^{\mathrm{BW}}(\partial_t\beta_3|\xi|^{\frac{1}{2}})$. Then we have

$$\partial_t v_3 = Op^{\mathrm{BW}}\Big(-\mathrm{i}\zeta(U)\xi - \mathrm{i}(1+\mathtt{a}_2^{(2)}-\partial_t\beta_3)|\xi|^{\frac{1}{2}} + \mathrm{i}\frac{(\beta_3)_x}{2}\mathrm{sign}(\xi) + \mathtt{r}_1^{(1)} + \widetilde{\mathtt{r}}_2^{(3)} + H_{\geq 3}\Big)v_3 + R(U)[V_3] \tag{5.40}$$

where $\widetilde{\mathtt{r}}_2^{(3)} \in \widetilde{\Gamma}_2^{-\frac{1}{2}}$, the symbol $H_{\geq 3} \in \Gamma^1_{K,K',3}$ is admissible and $R(U)$ is a 1×2 matrix of smoothing operators in $\Sigma\mathcal{R}^{-\rho}_{K,K',1}$. By (5.36) and (3.20) we have

$$-\mathrm{i}\mathtt{a}_2^{(2)} + \mathrm{i}\partial_t\beta_3 = -\mathrm{i}\sum_{n\in\mathbb{Z}\setminus\{0\}}(\mathtt{a}_2^{(2)})_{n,n}^{+-}|u_n|^2 + (\mathtt{a}_2^{(2)})_{n,-n}^{+-}u_n\overline{u_{-n}}e^{\mathrm{i}2nx} \tag{5.41}$$

up to a function $f_{\geq 3}$ in $\mathcal{F}^{\mathbb{R}}_{K,1,3}$. The conjugation of the remainder $R^{(2)}(U)$ in (5.30) is performed as in Step 5 of Lemma 5.5. In conclusion, (5.40)-(5.41) and the vanishing of the coefficients (5.31) imply (5.37)-(5.38). Finally, (5.39) follows from $(\mathtt{b}_2^{(3)})_{n_1,n_2}^{+-} = \frac{1}{2}(\beta_3)_{n_1,n_2}^{+-}(\mathrm{i}n_1 - \mathrm{i}n_2)$. □

Remark 5.9. *The cancellation* $(\mathtt{a}_2^{(2)})_{n,n}^{+-} = 0$ *in* (5.31) *does not follow from the properties of reality, parity and reversibility of the water waves equations. This appears to be an intrinsic property of the gravity water waves system* (1.3) *and is, of course, in agreement with the normal form identification of Section 7. We notice however that one would not need to prove this property for the sequel of the proof, since the symbol* $\mathrm{i}\sum_{n\in\mathbb{Z}\setminus\{0\}}(\mathtt{a}_2^{(2)})_{n,n}^{+-}|u_n|^2$ *is integrable and the coefficients* $(\mathtt{a}_2^{(2)})_{n,n}^{+-}$ *are real (by* (2.17) *and since the function* $\mathtt{a}_2^{(2)}$ *is real).*

5.2.2. *Elimination of the x-dependence at order* $1/2$ *up to* $O(u^3)$. The aim of this section is to cancel out the operator

$$-\mathrm{i}Op^{\mathrm{BW}}\Big(\frac{1}{2\pi}\sum_{n\in\mathbb{Z}}(\mathtt{a}_2^{(2)})_{n,-n}^{+-}u_n\overline{u_{-n}}e^{\mathrm{i}2nx}\Big) \tag{5.42}$$

arising by the non-integrable part of the function $\mathtt{a}_2^{(3)}(U;x)$ in (5.38). We argue in a way inspired by Section 12 in [9], noticing that the symbol in (5.42) is a prime integral up to cubic terms $O(u^3)$.

We conjugate (5.37) under the flow

$$\partial_\theta\Phi_4^\theta(U) = \mathrm{i}Op^{\mathrm{BW}}(b_4(U;\theta,x)\xi)\Phi_4^\theta(U)\,, \quad \Phi_4^0(U) = \mathrm{Id}\,, \tag{5.43}$$

where b_4 is defined as in (5.11) in terms of a real valued function $\beta_4(U;x) \in \widetilde{\mathcal{F}}_2^{\mathbb{R}}$ of the same form of the symbol in (5.42), i.e.

$$\beta_4(U;x) = \frac{1}{2\pi}\sum_{n\in\mathbb{Z}\setminus\{0\}}(\beta_4)_{n,-n}^{+-}u_n\overline{u_{-n}}e^{\mathrm{i}2nx}\,. \tag{5.44}$$

The flow in (5.43) is well-posed by Lemma A.2. We set

$$V_4 := \left[\begin{smallmatrix} v_4 \\ \overline{v_4} \end{smallmatrix}\right] = \big(\mathbf{\Phi}_4^\theta(U)[V_3]\big)_{|\theta=1} = \begin{pmatrix} \Phi_4^\theta(U)[v_3] \\ \overline{\Phi_4^\theta(U)}[\overline{v_3}] \end{pmatrix}_{|\theta=1} \tag{5.45}$$

where $\overline{\Phi_4^\theta(U)}$ is defined as in (2.31).

Lemma 5.10. *Define the function $\beta_4 \in \widetilde{\mathcal{F}}_2^{\mathbb{R}}$ as in (5.44) with coefficients*

$$(\beta_4)_{n,-n}^{+-} := \frac{(\mathtt{a}_2^{(2)})_{n,-n}^{+-}}{\mathrm{i}n}, \quad n \neq 0, \qquad (\beta_4)_{0,0}^{+-} := 0. \tag{5.46}$$

If v_3 solves (5.37) then

$$\partial_t v_4 = Op^{\mathrm{BW}}\big(-\mathrm{i}\zeta(U)\xi - \mathrm{i}|\xi|^{\frac{1}{2}} + \mathrm{i}\mathtt{b}_2^{(3)}\mathrm{sign}(\xi) + \mathtt{r}_1^{(1)} + \mathtt{r}_2^{(3)} + H_{\geq 3}^{(4)}\big)v_4 + R^{(4)}(U)[V_4] \tag{5.47}$$

where the symbols $\mathtt{b}_2^{(3)}$, $\mathtt{r}_1^{(1)}$, $\mathtt{r}_2^{(3)}$ are the same of equation (5.37), the symbol $H_{\geq 3}^{(4)} \in \Gamma^1_{K,K',3}$ is admissible and $R^{(4)}(U)$ is a 1×2 matrix of smoothing operators in $\Sigma\mathcal{R}^{-\rho}_{K,K',1}$.

Proof. In order to conjugate (5.37) we apply Lemmata A.4 and A.5. The contribution coming from the conjugation of ∂_t is $\mathrm{i}Op^{\mathrm{BW}}\big((\partial_t\beta_4)\xi\big)v_4$ plus a paradifferential operator with symbol $\mathrm{i}(-(\beta_4)_x(\beta_4)_t + g_{\geq 3})\xi$ (see (A.21)), which is admissible, and a smoothing remainder in $\Sigma\mathcal{R}^{-\rho}_{K,1,1}$. Recalling (3.20) we have

$$\begin{aligned} \frac{d}{dt}\sum_{n\in\mathbb{Z}\setminus\{0\}}(\beta_4)_{n,-n}^{+-}u_n\overline{u_{-n}}e^{\mathrm{i}2nx} &= \sum_{n\in\mathbb{Z}\setminus\{0\}}(\beta_4)_{n,-n}^{+-}\big(-\mathrm{i}\omega_n u_n\overline{u_{-n}} + u_n\mathrm{i}\omega_{-n}\overline{u_{-n}}\big)e^{\mathrm{i}2nx} + h_{\geq 3} \\ &= h_{\geq 3} \end{aligned} \tag{5.48}$$

because $\omega_{-n} = \omega_n$ and where, arguing as in the proof of Lemma 5.5, $h_{\geq 3}$ is a function in $\mathcal{F}^{\mathbb{R}}_{K,1,3}$. This implies that the function $\partial_t\beta_4$ is in $\mathcal{F}^{\mathbb{R}}_{K,1,3}$ and therefore $\mathrm{i}(\partial_t\beta_4)\xi$ is an admissible symbol.

Lemma A.4 implies that the conjugation of the spatial operator in (5.37) is a paradifferential operator with symbol

$$-\mathrm{i}\zeta(U)\xi - \mathrm{i}(1 + \mathtt{a}_2^{(3)})|\xi|^{\frac{1}{2}} + \{\beta_4\xi, -\mathrm{i}|\xi|^{1/2}\} + \mathrm{i}\mathtt{b}_2^{(3)}\mathrm{sign}(\xi) + \mathtt{r}_1^{(1)} + \mathtt{r}_2^{(3)} \tag{5.49}$$

plus a symbol in $\Sigma\Gamma^{-3/2}_{K,K',1}$ an admissible symbol and a smoothing operator in $\Sigma\mathcal{R}^{-\rho+1}_{K,1,1}$. Notice that $\{\beta_4\xi, -\mathrm{i}|\xi|^{\frac{1}{2}}\} = \frac{\mathrm{i}}{2}(\beta_4)_x|\xi|^{\frac{1}{2}}$ and that this equals $\mathtt{a}_2^{(3)}$ in view of the definitions of β_4 in (5.44) and (5.46), and of $\mathtt{a}_2^{(3)}$ in (5.38). It follows that the symbol in (5.49) reduces to

$$-\mathrm{i}\zeta(U)\xi - \mathrm{i}|\xi|^{\frac{1}{2}} + \mathrm{i}\mathtt{b}_2^{(3)}\mathrm{sign}(\xi) + \mathtt{r}_1^{(1)} + \mathtt{r}_2^{(3)}.$$

We have therefore obtained (5.47) (after slightly redefining ρ) as desired. □

5.2.3. *Integrability at order* 0. Our aim here is to eliminate in (5.47) the zero-th order paradifferential operator $Op^{\mathrm{BW}}\big(\mathrm{i}\mathtt{b}_2^{(3)}\mathrm{sign}(\xi)\big)$. We conjugate (5.47) with the flow

$$\partial_\theta\Phi_5^\theta(U) = Op^{\mathrm{BW}}(\mathrm{i}\beta_5(U;x)\,\mathrm{sign}(\xi))\Phi_5^\theta(U), \quad \Phi_5^0(U) = \mathrm{Id}, \tag{5.50}$$

where $\beta_5(U;x) \in \widetilde{\mathcal{F}}_2^{\mathbb{R}}$ is a real valued function. We introduce the variable

$$V_5 := \left[\begin{smallmatrix} v_5 \\ \overline{v_5} \end{smallmatrix}\right] = \big(\mathbf{\Phi}_5^\theta(U)[V_4]\big)_{|\theta=1} = \begin{pmatrix} \Phi_5^\theta(U)[v_4] \\ \overline{\Phi_5^\theta(U)}[\overline{v_4}] \end{pmatrix}_{|\theta=1} \tag{5.51}$$

where $\overline{\Phi_5^\theta(U)}$ is defined as in (2.31).

Lemma 5.11. *Define $\beta_5 \in \widetilde{\mathcal{F}}_2^{\mathbb{R}}$ (of the form (2.16)) with*

$$(\beta_5)_{n_1,n_2}^{\sigma\sigma} := \frac{(\mathtt{b}_2^{(3)})_{n_1,n_2}^{\sigma\sigma}}{\mathrm{i}\sigma(\omega_{n_1}+\omega_{n_2})}\,,\quad \sigma=\pm\,,\quad (\beta_5)_{n_1,n_2}^{+-} := \frac{(\mathtt{b}_2^{(3)})_{n_1,n_2}^{+-}}{\mathrm{i}(\omega_{n_1}-\omega_{n_2})}\,,\ n_1\neq \pm n_2\,, \tag{5.52}$$

and $(\beta_5)_{0,0}^{\sigma\sigma} := 0$, $(\beta_5)_{n,\sigma n}^{+-} := 0$, $\sigma = \pm$. If v_4 solves (5.47) then

$$\partial_t v_5 = Op^{\mathrm{BW}}\big(-\mathrm{i}\zeta(U)\xi - \mathrm{i}|\xi|^{\frac12} + \mathtt{r}_1^{(5)} + \mathtt{r}_2^{(5)} + H_{\geq 3}^{(5)}\big)v_5 + R^{(5)}(U)[V_5] \tag{5.53}$$

where $\mathtt{r}_1^{(5)} \in \widetilde{\Gamma}_1^{-\frac12}$, $\mathtt{r}_2^{(5)} \in \widetilde{\Gamma}_2^{-\frac12}$, the symbol $H_{\geq 3}^{(5)} \in \Gamma_{K,K',3}^{1}$ is admissible, and $R^{(5)}(U)$ is a 1×2 matrix of smoothing operators in $\Sigma\mathcal{R}_{K,K',1}^{-\rho}$.

Proof. To conjugate (5.47) we apply Lemmata A.6 and A.7. By (5.52) we have that

$$Op^{\mathrm{BW}}\big(\mathrm{i}(\mathtt{b}_2^{(3)} + \partial_t\beta_5)\mathrm{sign}(\xi)\big) = \mathrm{i}Op^{\mathrm{BW}}\big(\mathtt{b}_2^{(5)}\mathrm{sign}(\xi)\big),$$

up to symbols with degree of homogeneity greater than 3, and where

$$\mathtt{b}_2^{(5)}(U;x) := \frac{1}{2\pi}\sum_{n\in\mathbb{Z}\setminus\{0\}} (\mathtt{b}_2^{(3)})_{n,n}^{+-}|u_n|^2 + (\mathtt{b}_2^{(3)})_{n,-n}^{+-}u_n\overline{u_{-n}}e^{\mathrm{i}2nx} \overset{(5.39)}{=} 0\,.$$

The lemma is proved. □

Remark 5.12. *For a cubic vector field of the form $Op^{\mathrm{BW}}(\mathtt{c}(U;x) + \mathrm{i}\mathtt{b}(U;x)\mathrm{sign}(\xi))[u]$ with real valued functions $\mathtt{c},\mathtt{b}$ in $\widetilde{\mathcal{F}}_2^{\mathbb{R}}$ the reversibility and even-to-even properties imply $\mathtt{c}_{n,n}^{+-} = \mathtt{c}_{n,-n}^{+-} = \mathtt{b}_{n,-n}^{+-} = 0$, $\mathtt{b}_{n,n}^{+-} = -\mathtt{b}_{-n,-n}^{+-}$, $\mathtt{b}_{n,n}^{+-}\in\mathbb{R}$. The fact that actually $(\mathtt{b}_2^{(3)})_{n,n}^{+-} = 0$ as stated in (5.39) follows by other properties of the water waves equations, and, once again, is in agreement with the normal form identification of Section 7.2.*

In the following subsection we will be dealing with negative order operators, and will not need additional algebraic information about the coefficients and their vanishing.

5.3. **Integrability at negative orders.** In this section we algorithmically reduce the linear and quadratic symbols $\mathtt{r}_1^{(5)} + \mathtt{r}_2^{(5)}$ of order $-1/2$ in (5.53) into an integrable one, plus an admissible symbol.

Proposition 5.13. *For any $j = 0,\dots,2\rho-1$, there exist*

- *integrable symbols $\mathtt{p}_2^{(j)} \in \widetilde{\Gamma}_2^{-\frac12}$ (Definition 5.1), symbols*
$$\mathtt{q}^{(j)}(U;x,\xi) \in \Sigma\Gamma_{K,K',1}^{-m_j}\,,\quad m_j := \frac{j+1}{2}\,,$$
admissible symbols $H_{\geq 3}^{(j)}$ in $\Gamma_{K,K',3}^{1}$, and a 1×2 matrix of smoothing operators $R^{(j)}(U)$ in $\Sigma\mathcal{R}_{K,K',1}^{-\rho}$,
- *bounded maps $\boldsymbol{\Upsilon}_{j+1}^{\theta}(U)$, $\theta\in[0,1]$, defined as the compositions of three flows generated by paradifferential operators with symbols of order ≤ 0, (see (5.70) and (5.57), (5.61) and (5.67))*

such that: if z_j solves

$$\partial_t z_j = Op^{\mathrm{BW}}\big(-\mathrm{i}\zeta(U)\xi - \mathrm{i}|\xi|^{\frac12} + \mathtt{p}_2^{(j)}(U;\xi) + \mathtt{q}^{(j)}(U;x,\xi) + H_{\geq 3}^{(j)}\big)z_j + R^{(j)}(U)[Z_j]\,, \tag{5.54}$$

then the first component z_{j+1} of the vector defined by

$$Z_{j+1} = \big[\begin{smallmatrix} z_{j+1} \\ \overline{z_{j+1}}\end{smallmatrix}\big] := \big(\boldsymbol{\Upsilon}_{j+1}^{\theta}(U)\big)_{\theta=1}Z_j \tag{5.55}$$

solves an equation of the form (5.54) with $j+1$ instead of j.

The proof proceeds by induction.

Initialization. Notice that equation (5.53) has the form (5.54) with $j=0$, denoting $z_0 := v_5$, $\mathtt{p}_2^{(0)} := 0$, $\mathtt{q}^{(0)} := \mathtt{r}_1^{(5)} + \mathtt{r}_2^{(5)} \in \Sigma\Gamma^{-1/2}_{K,K',1}$, and renaming $H^{(0)}_{\geq 3}$ the admissible symbol $H^{(5)}_{\geq 3}$ in (5.53) and $R^{(0)}(U)$ the smoothing operator $R^{(5)}(U)$.

Iteration. The aim of the iterative procedure is to cancel out the symbol $\mathtt{q}^{(j)}$ up to a symbol of order $-m_j - 1/2$. This is done in two steps.

Step 1: Elimination of the linear symbols of negative order. We expand the symbol $\mathtt{q}^{(j)} = \mathtt{q}_1^{(j)} + \mathtt{q}_2^{(j)} + \cdots$ with $\mathtt{q}_l^{(j)} \in \Gamma_l^{-m_j}$, $l=1,2$. In order to eliminate the operator $Op^{\rm BW}(\mathtt{q}_1^{(j)}(U;x,\xi))$ in (5.54) we conjugate it by the flow

$$\partial_\theta \Phi^\theta_{\gamma^{(1)}_{j+1}}(U) = Op^{\rm BW}(\gamma^{(1)}_{j+1}(U;x,\xi))\Phi^\theta_{\gamma^{(1)}_{j+1}}(U)\,, \quad \Phi^0_{\gamma^{(1)}_{j+1}}(U) = \mathrm{Id}\,, \tag{5.56}$$

where $\gamma^{(1)}_{j+1}(U;x;\xi)$ is a symbol in $\widetilde{\Gamma}_1^{-m_j}$. The flow (5.56) is well posed because the order of $\gamma^{(1)}_{j+1}$ is negative. We introduce the new variable

$$\widetilde{Z}_{j+1} := \big[\begin{smallmatrix} \widetilde{z}_{j+1} \\ \overline{\widetilde{z}_{j+1}} \end{smallmatrix}\big] = \big(\mathcal{A}^\theta_{j+1,1}(U)[Z_j]\big)_{|\theta=1} = \begin{pmatrix} \Phi^\theta_{\gamma^{(1)}_{j+1}}(U)[z_j] \\ \overline{\Phi^\theta_{\gamma^{(1)}_{j+1}}(U)[\overline{z_j}]} \end{pmatrix}_{|\theta=1} \tag{5.57}$$

where the map $\overline{\Phi^\theta_{\gamma^{(1)}_{j+1}}(U)}$ is defined as in (2.31).

Lemma 5.14. *Define $\gamma^{(1)}_{j+1} \in \widetilde{\Gamma}_1^{-m_j}$ with coefficients*

$$(\gamma^{(1)}_{j+1})^+_n := \frac{(\mathtt{q}_1^{(j)})^+_n}{\mathrm{i}\omega_n}\,, \quad (\gamma_{j+1})^-_n := \frac{-(\mathtt{q}_1^{(j)})^-_n}{\mathrm{i}\omega_n}\,, \quad n \neq 0\,, \quad (\gamma^{(1)}_{j+1})^\sigma_0 := 0\,, \ \ \sigma = \pm\,. \tag{5.58}$$

If z_j solves (5.54) then

$$\begin{aligned} \partial_t \widetilde{z}_{j+1} = {} & Op^{\rm BW}\big(-\mathrm{i}\zeta(U)\xi - \mathrm{i}|\xi|^{\frac12} + \mathtt{p}_2^{(j)}(U;\xi) + \widetilde{\mathtt{q}}_2^{(j)}(U;x,\xi) + \widetilde{\mathtt{k}}_1^{(j)}(U;x,\xi) + \widetilde{\mathtt{k}}_2^{(j)}(U;x,\xi)\big)\widetilde{z}_{j+1} \\ & + Op^{\rm BW}\big(H^{(j)}_{\geq 3}\big)\widetilde{z}_{j+1} + R^{(j)}(U)[\widetilde{Z}_{j+1}] \end{aligned} \tag{5.59}$$

where $\mathtt{p}_2^{(j)}(U;\xi) \in \widetilde{\Gamma}_2^{-\frac12}$ is the same of (5.54),

$$\widetilde{\mathtt{q}}_2^{(j)} \in \widetilde{\Gamma}_2^{-m_j}\,, \quad \widetilde{\mathtt{k}}_1^{(j)} \in \widetilde{\Gamma}_1^{-m_j-\frac12}\,, \quad \widetilde{\mathtt{k}}_2^{(j)} \in \widetilde{\Gamma}_2^{-m_j-\frac12}\,,$$

the symbol $H^{(j)}_{\geq 3} \in \Gamma^1_{K,K'+1,3}$ is admissible and $R^{(j)}(U)$ is a 1×2 matrix of smoothing operators in $\Sigma\mathcal{R}^{-\rho}_{K,K',1}$.

Proof. In order to conjugate (5.54) we apply Lemmata A.6 and A.7. The only contributions at homogeneity degree 1 and order $-m_j$ are given by $Op^{\rm BW}(\mathtt{q}_1^{(j)} + \partial_t \gamma^{(1)}_{j+1})$ up to smoothing remainders. From the time contribution a symbol which has homogeneity 2 and order less or equal $-m_j - 1/2$ appears (see the term r_1 in (A.25) of Lemma A.7). By (5.58) and (3.20) we have that

$$\mathtt{q}_1^{(j)} + \partial_t \gamma^{(1)}_{j+1} = \mathtt{q}_{2,j} + \mathtt{q}_{\geq 3}\,, \quad \mathtt{q}_{2,j} \in \widetilde{\Gamma}_2^{-m_j}\,, \quad \mathtt{q}_{\geq 3} \in \Gamma^{-m_j}_{K,1,3}\,,$$

and we set $\widetilde{\mathtt{q}}_2^{(j)} := \mathtt{q}_2^{(j)} + \mathtt{q}_{2,j}$, and absorb $\mathtt{q}_{\geq 3}$ in the admissible symbol $H^{(j)}_{\geq 3}$. The contributions in (5.59) at order less or equal $-m_j - \frac12$, and homogeneity ≤ 2 come from the conjugation of the spatial operator $-\mathrm{i}|\xi|^{1/2}$. In particular, using formula (A.24), we can set $\widetilde{\mathtt{k}}_1^{(j)} := -\frac{\mathrm{i}}{2}(\gamma^{(1)}_{j+1})_x |\xi|^{-\frac12}\mathrm{sign}(\xi)$ and obtain (5.59) with some $\widetilde{\mathtt{k}}_2^{(j)}$ in $\Gamma_2^{-m_j - 1/2}$. □

Step 2: Reduction of the quadratic symbols of negative order. We now cancel out the symbol $\widetilde{\mathtt{q}}_2^{(j)}$ in (5.59), up to an integrable one and a lower order symbol. Following Section 5.2 we use two different transformations.

ELIMINATION OF THE TIME DEPENDENCE UP TO $O(u^3)$. We consider the flow generated by

$$\partial_\theta \Phi^\theta_{\gamma^{(2)}_{j+1}}(U) = Op^{\mathrm{BW}}(\gamma^{(2)}_{j+1}(U;x,\xi))\Phi^\theta_{\gamma^{(2)}_{j+1}}(U)\,, \quad \Phi^0_{\gamma^{(2)}_{j+1}}(U) = \mathrm{Id}\,, \tag{5.60}$$

where $\gamma^{(2)}_{j+1}(U;x;\xi)$ is a symbol in $\widetilde{\Gamma}_2^{-m_j}$. We introduce the new variable

$$\breve{Z}_{j+1} := \left[\begin{smallmatrix} \breve{z}_{j+1} \\ \overline{\breve{z}_{j+1}} \end{smallmatrix}\right] = \big(\mathcal{A}^\theta_{j+1,2}(U)[\widetilde{Z}_j]\big)_{|\theta=1} = \begin{pmatrix} \Phi^\theta_{\gamma^{(1)}_{j+1}}(U)[\widetilde{z}_j] \\ \overline{\Phi^\theta_{\gamma^{(1)}_{j+1}}(U)[\overline{\widetilde{z}_j}]} \end{pmatrix}_{|\theta=1} \tag{5.61}$$

where the map $\overline{\Phi^\theta_{\gamma^{(2)}_{j+1}}(U)}$ is defined as in (2.31).

Lemma 5.15. *Let $\gamma^{(2)}_{j+1}(U;x;\xi)$ be a symbol in $\widetilde{\Gamma}_2^{-m_j}$ of the form (2.16) with coefficients*

$$(\gamma^{(2)}_{j+1})^{\sigma\sigma}_{n_1,n_2} := \frac{(\widetilde{\mathtt{q}}_2^{(j)})^{\sigma\sigma}_{n_1,n_2}}{\mathrm{i}\sigma(\omega_{n_1}+\omega_{n_2})}\,, \quad \sigma=\pm\,, \quad (\gamma^{(2)}_{j+1})^{+-}_{n_1,n_2} := \frac{(\widetilde{\mathtt{q}}_2^{(j)})^{+-}_{n_1,n_2}}{\mathrm{i}(\omega_{n_1}-\omega_{n_2})}\,, \quad n_1 \neq \pm n_2\,. \tag{5.62}$$

If $\widetilde{z}_j$ solves (5.59) then

$$\begin{aligned} \partial_t \breve{z}_{j+1} &= Op^{\mathrm{BW}}\big(-\mathrm{i}|\xi|^{\frac12} - \mathrm{i}\zeta(U)\xi + \mathtt{p}_2^{(j)}(U;\xi)\big)\breve{z}_{j+1} \\ &\quad + Op^{\mathrm{BW}}\Big(\big(\sum_{n\in\mathbb{Z}\setminus\{0\}} (\widetilde{\mathtt{q}}_2^{(j)})^{+-}_{n,n}(\xi)|u_n|^2 + (\widetilde{\mathtt{q}}_2^{(j)})^{+-}_{n,-n}(\xi)u_n\overline{u_{-n}}e^{\mathrm{i}2nx}\big)\Big)\breve{z}_{j+1} \\ &\quad + Op^{\mathrm{BW}}\big(\breve{\mathtt{k}}_1^{(j)}(U;x,\xi) + \breve{\mathtt{k}}_2^{(j)}(U;x,\xi) + H^{(j)}_{\geq 3}\big)\breve{z}_{j+1} + R^{(j)}(U)[\breve{Z}_{j+1}] \end{aligned} \tag{5.63}$$

where $\breve{\mathtt{k}}_1^{(j)} \in \widetilde{\Gamma}_1^{-m_j-\frac12}$, $\breve{\mathtt{k}}_2^{(j)} \in \widetilde{\Gamma}_2^{-m_j-\frac12}$, the symbol $H^{(j)}_{\geq 3} \in \Gamma^1_{K,K',3}$ is admissible and $R^{(j)}(U)$ is a 1×2 matrix of smoothing operators in $\Sigma\mathcal{R}^{-\rho}_{K,K',1}$.

Proof. In order to conjugate (5.59) we apply Lemmata A.6 and A.7. The contributions at order $-m_j$ and degree 2 are given by $Op^{\mathrm{BW}}\big(\widetilde{\mathtt{q}}_2^{(j)} + \partial_t\gamma^{(2)}_{j+1}\big)$. All the other contributions have homogeneity greater or equal 3 and are admissible. By the choice of $\gamma^{(2)}_{j+1}$ in (5.62) we have

$$\widetilde{\mathtt{q}}_2^{(j)} + \partial_t \gamma^{(2)}_{j+1} = \frac{1}{2\pi}\sum_{n\in\mathbb{Z}\setminus\{0\}} (\widetilde{\mathtt{q}}_2^{(j)})^{+-}_{n,n}(\xi)|u_n|^2 + (\widetilde{\mathtt{q}}_2^{(j)})^{+-}_{n,-n}(\xi)u_n\overline{u_{-n}}e^{\mathrm{i}2nx}$$

up to a symbol in $\Gamma^{-m_j}_{K,1,3}$. □

ELIMINATION OF THE x-DEPENDENCE UP TO $O(u^3)$. In order to eliminate the non-integrable symbol

$$\frac{1}{2\pi}\sum_{n\in\mathbb{Z}\setminus\{0\}} (\widetilde{\mathtt{q}}_2^{(j)})^{+-}_{n,-n}(\xi)u_n\overline{u_{-n}}e^{\mathrm{i}2nx} \tag{5.64}$$

in (5.63) we follow the same strategy used in Subsection 5.2.2. We conjugate (5.63) by the flow

$$\partial_\theta \Phi^\theta_{\gamma^{(3)}_{j+1}}(U) = \mathrm{i}Op^{\mathrm{BW}}(\gamma^{(3)}_{j+1}(U;x,\xi))\Phi^\theta_{\gamma^{(3)}_{j+1}}(U)\,, \quad \Phi^0_{\gamma^{(3)}_{j+1}}(U) = \mathrm{Id}\,, \tag{5.65}$$

where $\gamma_{j+1}^{(3)}(U;x,\xi)$ is a symbol in $\widetilde{\Gamma}_2^{-m_j+\frac{1}{2}}$ of the same form (5.64), i.e.

$$\gamma_{j+1}^{(3)}(U;x,\xi) := \frac{1}{2\pi}\sum_{n\in\mathbb{Z}\setminus\{0\}} (\gamma_{j+1}^{(3)})_{n,-n}^{+-}(\xi) u_n \overline{u_{-n}} e^{\mathrm{i}2nx}\,. \tag{5.66}$$

We introduce the new variable

$$Z_{j+1} := \left[\begin{smallmatrix} z_{j+1} \\ \bar{z}_{j+1}\end{smallmatrix}\right] = \left(\mathcal{A}_{j+1,3}^{\theta}(U)[\breve{Z}_{j+1}]\right)_{|\theta=1} = \begin{pmatrix} \Phi^{\theta}_{\gamma_{j+1}^{(3)}}(U)[\breve{z}_{j+1}] \\ \overline{\Phi^{\theta}_{\gamma_{j+1}^{(3)}}(U)[\breve{z}_{j+1}]} \end{pmatrix}_{|\theta=1} \tag{5.67}$$

where the map $\overline{\Phi^{\theta}_{\gamma_{j+1}^{(3)}}(U)}$ is defined as in (2.31).

Lemma 5.16. *Define $\gamma_{j+1}^{(3)}$ in $\widetilde{\Gamma}_2^{-m_j+\frac{1}{2}}$ as in (5.66) with coefficients*

$$(\gamma_{j+1}^{(3)})_{n,-n}^{+-}(\xi) := |\xi|^{\frac{1}{2}}\mathrm{sign}(\xi)\frac{1}{n}(\widetilde{\mathtt{q}}_2^{(j)})_{n,-n}^{+-}(\xi)\,, \quad n\neq 0\,. \tag{5.68}$$

If $\breve{z}_j$ solves (5.63) then

$$\partial_t z_{j+1} = Op^{\mathrm{BW}}\big(-\mathrm{i}\zeta(U)\xi - \mathrm{i}|\xi|^{\frac{1}{2}} + \mathtt{p}_2^{(j+1)}(U;\xi) + \mathtt{q}^{(j+1)}(U;x,\xi) + H_{\geq 3}^{(j+1)}\big) z_{j+1} + R^{(j+1)}(U)[Z_{j+1}] \tag{5.69}$$

where $\mathtt{p}_2^{(j+1)}(U;\xi)$ is an integrable symbol in $\widetilde{\Gamma}_2^{-\frac{1}{2}}$, $\mathtt{q}^{(j+1)}(U;x,\xi)$ is in $\Sigma\Gamma_{K,K',1}^{-m_{j+1}}$, the symbol $H_{\geq 3}^{(j+1)} \in \Gamma_{K,K',3}^{1}$ is admissible, and $R^{(j+1)}(U)$ is a 1×2 matrix of smoothing operators in $\Sigma\mathcal{R}_{K,K',1}^{-\rho}$.

Proof. Reasoning as in (5.48), we have $\frac{d}{dt}\gamma_{j+1}^{(3)}(U;x,\xi) = 0$ up to a cubic symbol in $\Gamma_{K,1,3}^{-m_j+\frac{1}{2}}$. In order to conjugate (5.63) we apply Lemmata A.6 and A.7. The only contributions with homogeneity 2 and order $-m_j$ are

$$Op^{\mathrm{BW}}\Big(\frac{\mathrm{i}}{2}(\gamma_{j+1}^{(3)})_x|\xi|^{-\frac{1}{2}}\mathrm{sign}(\xi) + \frac{1}{2\pi}\sum_{n\in\mathbb{Z}\setminus\{0\}}(\widetilde{\mathtt{q}}_2^{(j)})_{n,n}^{+-}(\xi)|u_n|^2 + (\widetilde{\mathtt{q}}_2^{(j)})_{n,-n}^{+-}(\xi)u_n\overline{u_{-n}}e^{\mathrm{i}2nx}\Big)\,.$$

By the choice of $\gamma_{j+1}^{(3)}$ in (5.66), (5.68) we have

$$\frac{\mathrm{i}}{2}(\gamma_{j+1}^{(3)})_x|\xi|^{-\frac{1}{2}}\mathrm{sign}(\xi) + \frac{1}{2\pi}\sum_{n\in\mathbb{Z}\setminus\{0\}}(\widetilde{\mathtt{q}}_2^{(j)})_{n,-n}^{+-}(\xi)u_n\overline{u_{-n}}e^{\mathrm{i}2nx} = 0\,.$$

Then (5.69) follows with the new integrable symbol

$$\mathtt{p}_2^{(j+1)}(U;\xi) := \mathtt{p}_2^{(j)}(U;\xi) + \sum_{n\in\mathbb{Z}\setminus\{0\}}(\widetilde{\mathtt{q}}_2^{(j)})_{n,n}^{+-}(\xi)|u_n|^2$$

and a symbol $\mathtt{q}^{(j+1)}(U;x,\xi)$ in $\Sigma\Gamma_{K,K',1}^{-m_{j+1}}$ where $m_{j+1} = m_j + \frac{1}{2}$. □

Lemmata 5.14, 5.15, 5.16 imply Proposition 5.13 by defining the map

$$\Upsilon_{j+1}^{\theta}(U) := \mathcal{A}_{j+1,3}^{\theta}(U)\circ\mathcal{A}_{j+1,2}^{\theta}(U)\circ\mathcal{A}_{j+1,1}^{\theta}(U) \tag{5.70}$$

where $\mathcal{A}_{j+1,k}^{\theta}(U)$, for $k=1,2,3$, are defined respectively in (5.57), (5.61), (5.67).

5.4. **Proof of Proposition 5.4.** We set

$$\mathfrak{F}^{\theta}(U) := \mathbf{\Upsilon}^{\theta}_{fin}(U) \circ \mathbf{\Phi}^{\theta}_{5}(U) \circ \cdots \mathbf{\Phi}^{\theta}_{1}(U) \circ \mathbf{\Psi}^{\theta}_{diag}(U) \tag{5.71}$$

and $\mathbf{F}^{\theta}(U) := \mathfrak{F}^{\theta}(U)[U]$ as in (5.6), where $\mathbf{\Psi}^{\theta}_{diag}(U)$ is defined in Proposition 4.1, the maps $\mathbf{\Phi}^{\theta}_{j}(U)$, $j = 1, \dots, 5$ are given respectively in (5.14), (5.28), (5.35), (5.45), (5.51), and $\mathbf{\Upsilon}^{\theta}_{fin}(U) := \mathbf{\Upsilon}^{\theta}_{2\rho}(U) \circ \cdots \circ \mathbf{\Upsilon}^{\theta}_{1}(U)$ where $\mathbf{\Upsilon}^{\theta}_{j+1}(U)$, $j = 0, \dots, 2\rho - 1$, are defined in (5.70). Then, by the construction in Subsections 5.1-5.3, we have that $Z := (\mathbf{F}^{\theta}(U))_{\theta=1}$ solves the system (5.54) with $j = 2\rho - 1$ which has the form (5.4) with $\mathcal{D}_{-1/2}(U;\xi) \rightsquigarrow \mathtt{p}^{(2\rho-1)}_{2}(U;\xi)$, $\mathtt{H}_{\geq 3} \rightsquigarrow H^{(2\rho-1)}_{\geq 3}$ and $\mathtt{R}(U) \rightsquigarrow R^{(2\rho-1)}(U)$. The bounds (5.7) follow since $\mathfrak{F}^{\theta}(U)$ is the composition of maps constructed using Lemma A.2 (see bounds (A.11)).

6. Poincaré-Birkhoff Normal Forms

The aim of this section is to eliminate all the terms of the system (5.4) up to cubic degree of homogeneity which are not yet in Poincaré-Birkhoff normal form. Such terms appear only in the smoothing remainder $\mathtt{R}(U)[Z]$ that we write as

$$\mathtt{R}(U) = \mathtt{R}_1(U) + \mathtt{R}_2(U) + \mathtt{R}_{\geq 3}(U)\,, \quad \mathtt{R}_{\geq 3}(U) \in \mathcal{R}^{-\rho}_{K,K',3} \otimes \mathcal{M}_2(\mathbb{C})\,, \tag{6.1}$$

$$\mathtt{R}_i(U) = \begin{pmatrix} (\mathtt{R}_i(U))^{+}_{+} & (\mathtt{R}_i(U))^{-}_{+} \\ (\mathtt{R}_i(U))^{+}_{-} & (\mathtt{R}_i(U))^{-}_{-} \end{pmatrix}\,, \quad (\mathtt{R}_i(U))^{\sigma'}_{\sigma} \in \widetilde{\mathcal{R}}^{-\rho}_{i}\,, \quad (\mathtt{R}_i(U))^{\sigma'}_{\sigma} = \overline{(\mathtt{R}_i(U))^{-\sigma'}_{-\sigma}}\,, \tag{6.2}$$

for $\sigma, \sigma' = \pm$ and $i = 1, 2$. The third identity in (6.2) means that the matrix of operators $\mathtt{R}(U)$ is *real-to-real* (see (2.32)). For any $\sigma, \sigma' = \pm$ we expand

$$(\mathtt{R}_1(U))^{\sigma'}_{\sigma} = \sum_{\epsilon=\pm} (\mathtt{R}_{1,\epsilon}(U))^{\sigma'}_{\sigma}\,, \quad (\mathtt{R}_2(U))^{\sigma'}_{\sigma} = \sum_{\epsilon=\pm} (\mathtt{R}_{2,\epsilon,\epsilon}(U))^{\sigma'}_{\sigma} + (\mathtt{R}_{2,+,-}(U))^{\sigma'}_{\sigma}\,, \tag{6.3}$$

where $(\mathtt{R}_{1,\epsilon}(U))^{\sigma'}_{\sigma} \in \widetilde{\mathcal{R}}^{-\rho}_{1}$, $(\mathtt{R}_{2,\epsilon,\epsilon'}(U))^{\sigma'}_{\sigma} \in \widetilde{\mathcal{R}}^{-\rho}_{2}$ with $\epsilon, \epsilon' = \pm$, are the homogeneous smoothing operators

$$(\mathtt{R}_{1,\epsilon}(U))^{\sigma'}_{\sigma} z^{\sigma'} = \frac{1}{\sqrt{2\pi}} \sum_{j \in \mathbb{Z}\setminus\{0\}} \Big(\sum_{k \in \mathbb{Z}\setminus\{0\}} (\mathtt{R}_{1,\epsilon}(U))^{\sigma',k}_{\sigma,j} z^{\sigma'}_{k} \Big) e^{\mathrm{i}\sigma j x} \tag{6.4}$$

with entries

$$(\mathtt{R}_{1,\epsilon}(U))^{\sigma',k}_{\sigma,j} := \frac{1}{\sqrt{2\pi}} \sum_{\substack{n \in \mathbb{Z}\setminus\{0\} \\ \epsilon n + \sigma' k = \sigma j}} (\mathtt{r}_{1,\epsilon})^{\sigma,\sigma'}_{n,k} u^{\epsilon}_{n}\,, \quad j, k \in \mathbb{Z}\setminus\{0\}\,, \tag{6.5}$$

for suitable scalar coefficients $(\mathtt{r}_{1,\epsilon})^{\sigma,\sigma'}_{n,k} \in \mathbb{C}$, and

$$(\mathtt{R}_{2,\epsilon,\epsilon'}(U))^{\sigma'}_{\sigma} z^{\sigma'} = \frac{1}{\sqrt{2\pi}} \sum_{j \in \mathbb{Z}\setminus\{0\}} \Big(\sum_{k \in \mathbb{Z}\setminus\{0\}} (\mathtt{R}_{2,\epsilon,\epsilon'}(U))^{\sigma',k}_{\sigma,j} z^{\sigma'}_{k} \Big) e^{\mathrm{i}\sigma j x} \tag{6.6}$$

with entries

$$(\mathtt{R}_{2,\epsilon,\epsilon'}(U))^{\sigma',k}_{\sigma,j} := \frac{1}{2\pi} \sum_{\substack{n_1, n_2 \in \mathbb{Z}\setminus\{0\} \\ \epsilon n_1 + \epsilon' n_2 + \sigma' k = \sigma j}} (\mathtt{r}_{2,\epsilon,\epsilon'})^{\sigma,\sigma'}_{n_1,n_2,k} u^{\epsilon}_{n_1} u^{\epsilon'}_{n_2}\,, \quad j, k \in \mathbb{Z}\setminus\{0\}\,, \tag{6.7}$$

and suitable scalar coefficients $(\mathtt{r}_{2,\epsilon,\epsilon'})^{\sigma,\sigma'}_{n_1,n_2,k} \in \mathbb{C}$.

Definition 6.1. (Poincaré-Birkhoff Resonant smoothing operator) *Let* $\mathtt{R}(U)$ *be a real-to-real smoothing operator in* $\widetilde{\mathcal{R}}^{-\rho}_{2}\otimes\mathcal{M}_2(\mathbb{C})$ *with* $\rho\geq 0$ *and scalar coefficients* $(\mathtt{r}_{\epsilon,\epsilon'})^{\sigma,\sigma'}_{n_1,n_2,k}\in\mathbb{C}$ *defined as in* (6.7). *We denote by* $\mathtt{R}^{res}(U)$ *the real-to-real smoothing operator in* $\widetilde{\mathcal{R}}^{-\rho}_{2}\otimes\mathcal{M}_2(\mathbb{C})$ *with coefficients*

$$(\mathtt{R}^{res}_{\epsilon,\epsilon'}(U))^{\sigma',k}_{\sigma,j}:=\sum_{\substack{n_1,n_2\in\mathbb{Z}\setminus\{0\}\\ \epsilon n_1+\epsilon' n_2+\sigma' k-\sigma j=0\\ \epsilon\omega(n_1)+\epsilon'\omega(n_2)+\sigma'\omega(k)-\sigma\omega(j)=0}}(\mathtt{r}_{\epsilon,\epsilon'})^{\sigma,\sigma'}_{n_1,n_2,k}u^{\epsilon}_{n_1}u^{\epsilon'}_{n_2}\,,\quad j,k\in\mathbb{Z}\setminus\{0\}\,,\tag{6.8}$$

where we recall that $\omega(j)=|j|^{\frac12}$.

In Subsections 6.2.1 and 6.2.2 we will reduce the remainder $\mathtt{R}(U)$ in (6.1) to its Poincaré-Birkhoff resonant component. The key result of this section is the following.

Proposition 6.2. (Poincaré-Birkhoff normal form of the water waves at cubic degree) *There exists* $\rho_0>0$ *such that, for all* $\rho\geq\rho_0$, $K\geq K'=2\rho+2$, *there exists* $s_0>0$ *such that, for any* $s\geq s_0$, *for all* $0<r\leq r_0(s)$ *small enough, and any solution* $U\in B^K_s(I;r)$ *of the water waves system* (3.8), *there is a nonlinear map* $\mathbf{F}^\theta_T(U)$, $\theta\in[0,1]$, *of the form*

$$\mathbf{F}^\theta_T(U):=\mathfrak{C}^\theta(U)[U]\tag{6.9}$$

where $\mathfrak{C}^\theta(U)$ *is a real-to-real, bounded and invertible operator, such that the function* $Y:=\left[\begin{smallmatrix}y\\ \bar{y}\end{smallmatrix}\right]=\mathbf{F}^1_T(U)$ *solves*

$$\partial_t Y=-\mathrm{i}\Omega Y-\mathrm{i}Op^{\mathrm{BW}}(\mathtt{D}(Y;\xi))[Y]+\tilde{\mathtt{R}}^{res}(Y)[Y]+\mathcal{X}_{\geq 4}(U,Y)\tag{6.10}$$

where:

- Ω *is the diagonal matrix of Fourier multipliers defined in* (3.19), *and* $\mathtt{D}(Y;\xi)$ *is the diagonal matrix of integrable symbols* $\widetilde{\Gamma}^1_2\otimes\mathcal{M}_2(\mathbb{C})$ *defined in* (5.5);
- *the smoothing operator* $\tilde{\mathtt{R}}^{res}(Y)\in\widetilde{\mathcal{R}}^{-(\rho-\rho_0)}_2\otimes\mathcal{M}_2(\mathbb{C})$ *is Poincaré-Birkhoff resonant according to Definition 6.1;*
- $\mathcal{X}_{\geq 4}(U,Y)$ *has the form*
$$\mathcal{X}_{\geq 4}(U,Y)=Op^{\mathrm{BW}}(\mathfrak{H}_{\geq 3}(U;x,\xi))[Y]+\mathfrak{R}_{\geq 3}(U)[Y]\tag{6.11}$$
where $\mathfrak{H}_{\geq 3}(U;x,\xi)\in\Gamma^1_{K,K',3}\otimes\mathcal{M}_2(\mathbb{C})$ *is an admissible matrix of symbols (Definition 5.2) and* $\mathfrak{R}_{\geq 3}(U)$ *is a matrix of real-to-real smoothing operators in* $\mathcal{R}^{-(\rho-\rho_0)}_{K,K',3}\otimes\mathcal{M}_2(\mathbb{C})$.

Furthermore, the map $\mathbf{F}^\theta_T(U)$ *defined in* (6.9) *satisfies the following properties:*

(i) There is a constant C *depending on* s, r *and* K, *such that, for* $s\geq s_0$,

$$\|\partial_t^k\mathfrak{C}^\theta(U)[V]\|_{\dot{H}^{s-k}}+\|\partial_t(\mathfrak{C}^\theta(U))^{-1}[V]\|_{\dot{H}^{s-k}}\leq\|V\|_{k,s}(1+C\|U\|_{K,s_0})+C\|V\|_{k,s_0}\|U\|_{K,s}\,,\tag{6.12}$$

for any $0\leq k\leq K-K'$, $V\in C^{K-K'}_{*\mathbb{R}}(I;\dot{H}^s(\mathbb{T};\mathbb{C}^2))$ *and uniformly in* $\theta\in[0,1]$;

(ii) The function $Y=\mathbf{F}^\theta_T(U)_{|\theta=1}$ *satisfies*

$$C^{-1}\|U\|_{\dot{H}^s}\leq\|Y\|_{\dot{H}^s}\leq C\|U\|_{\dot{H}^s}\,.\tag{6.13}$$

(iii) The map $\mathbf{F}^\theta_T(U)$ *admits an expansion as*

$$\mathbf{F}^\theta_T(U)=U+\theta\big(M_1(U)[U]+M^{(1)}_2(U)[U]\big)+\theta^2M^{(2)}_2(U)[U]+M_{\geq 3}(\theta;U)[U]\,,$$

where $M_1(U)$ *is in* $\widetilde{\mathcal{M}}_1\otimes\mathcal{M}_2(\mathbb{C})$, *the maps* $M^{(1)}_2(U),M^{(2)}_2(U)$ *are in* $\widetilde{\mathcal{M}}_2\otimes\mathcal{M}_2(\mathbb{C})$, *and* $M_{\geq 3}(\theta;U)$ *is in* $\mathcal{M}_{K,K',3}\otimes\mathcal{M}_2(\mathbb{C})$ *with estimates uniform in* $\theta\in[0,1]$.

In the following subsection we provide lower bounds on the "small divisors" which appear in the Poincaré-Birkhoff reduction procedure. Then, in Subsection 6.2, we prove Proposition 6.2.

6.1. **Cubic and quartic wave interactions.** We study in this section the cubic and quartic resonances among the linear frequencies $\omega(n) = |n|^{\frac{1}{2}}$.

Proposition 6.3. (Non-resonance conditions) *There are constants* $\mathtt{c} > 0$ *and* $N_0 > 0$ *such that*

- **(cubic resonances)** *for any* $\sigma, \sigma' = \pm$ *and* $n_1, n_2, n_3 \in \mathbb{Z} \setminus \{0\}$ *satisfying*
$$n_1 + \sigma n_2 + \sigma' n_3 = 0 , \tag{6.14}$$
we have
$$|\omega(n_1) + \sigma\omega(n_2) + \sigma'\omega(n_3)| \geq \mathtt{c} . \tag{6.15}$$
- **(quartic resonances)** *For any* $\sigma, \sigma', \sigma'' = \pm$ *and* $n_1, n_2, n_3, n_4 \in \mathbb{Z} \setminus \{0\}$ *such that*
$$n_1 + \sigma n_2 + \sigma' n_3 + \sigma'' n_4 = 0 , \quad \omega(n_1) + \sigma\omega(n_2) + \sigma'\omega(n_3) + \sigma''\omega(n_4) \neq 0 , \tag{6.16}$$
we have
$$|\omega(n_1) + \sigma\omega(n_2) + \sigma'\omega(n_3) + \sigma''\omega(n_4)| \geq \mathtt{c} \max\{|n_1|, |n_2|, |n_3|, |n_4|\}^{-N_0} . \tag{6.17}$$

Proof. We first consider the cubic and then the quartic resonances.

Proof of (6.15). If $\sigma = \sigma' = +$ then the bound (6.15) is trivial. Assume $\sigma = +$ and $\sigma' = -$. By (6.14) we have that $|n_3| \leq |n_1| + |n_2|$ and therefore
$$\begin{aligned}|\sqrt{|n_1|} + \sqrt{|n_2|} - \sqrt{|n_3|}| &= \frac{||n_1| + |n_2| - |n_3| + 2\sqrt{|n_1||n_2|}|}{\sqrt{|n_1|} + \sqrt{|n_2|} + \sqrt{|n_3|}} \\ &\geq \frac{2\sqrt{|n_1||n_2|}}{\sqrt{|n_1|} + \sqrt{|n_2|} + \sqrt{|n_1| + |n_2|}} \geq \frac{2}{2 + \sqrt{2}}\end{aligned}$$
since $|n_1|, |n_2| \geq 1$. The bound (6.15) in the case $\sigma = -$ and $\sigma' = +$ is the same.

Proof of (6.17). The case $\sigma = \sigma' = \sigma'' = +$ is trivial. Assume $\sigma = \sigma' = +$ and $\sigma'' = -$. We have
$$|\omega(n_1) + \omega(n_2) + \omega(n_3) - \omega(n_4)| = \frac{||n_1| + |n_2| + |n_3| - |n_4| + 2\sqrt{|n_1 n_2|} + 2\sqrt{|n_2 n_3|} + 2\sqrt{|n_1 n_3|}|}{\omega(n_1) + \omega(n_2) + \omega(n_3) + \omega(n_4)} .$$
The first (momentum) condition in (6.16) implies that $|n_1| + |n_2| + |n_3| - |n_4| \geq 0$ and hence (6.17) follows (actually with $N_0 = 0$). It remains to study the case $\sigma = \sigma'' = -$ and $\sigma' = +$, i.e. we have to prove that the phase
$$\psi(n_1, n_2, n_3, n_4) := |n_1|^{\frac{1}{2}} - |n_2|^{\frac{1}{2}} + |n_3|^{\frac{1}{2}} - |n_4|^{\frac{1}{2}} \tag{6.18}$$
$$= \frac{|n_1| - |n_2| + |n_3| - |n_4| + 2\sqrt{|n_1 n_3|} - 2\sqrt{|n_2 n_4|}}{|n_1|^{\frac{1}{2}} + |n_2|^{\frac{1}{2}} + |n_3|^{\frac{1}{2}} + |n_4|^{\frac{1}{2}}} \tag{6.19}$$
satisfies (6.17). Notice that the first (momentum) equality in (6.16) becomes
$$n_1 - n_2 + n_3 - n_4 = 0 . \tag{6.20}$$
Let $|n_1| := \max\{|n_1|, |n_2|, |n_3|, |n_4|\}$ and assume, without loss of generality, that $n_1 > 0$ and $|n_2| \geq |n_4|$ (the phase (6.18) is symmetric in $|n_2|, |n_4|$). We consider different cases.

Case a) Assume that $n_1 = |n_2|$. Then by (6.18)
$$|\psi(n_1, n_2, n_3, n_4)| = \left||n_3|^{\frac{1}{2}} - |n_4|^{\frac{1}{2}}\right| = \frac{||n_3| - |n_4||}{|n_3|^{\frac{1}{2}} + |n_4|^{\frac{1}{2}}} .$$
Since $\psi \neq 0$ then $|n_3| - |n_4|$ is a non-zero integer and we get (6.17). Thus in the sequel we suppose
$$n_1 > |n_2| \geq |n_4| . \tag{6.21}$$
Case b) Assume that $|n_3| \geq |n_4|$. Then by (6.18) we get

$$\psi(n_1,n_2,n_3,n_4) \geq |n_1|^{\frac{1}{2}} - |n_2|^{\frac{1}{2}} = \frac{|n_1|-|n_2|}{|n_1|^{\frac{1}{2}}+|n_2|^{\frac{1}{2}}} \overset{(6.21)}{\geq} \frac{1}{|n_1|^{\frac{1}{2}}+|n_2|^{\frac{1}{2}}}$$

which implies (6.17). Thus in the sequel we suppose, in addition to (6.21), that

$$n_1 > |n_2| \geq |n_4| > |n_3| \,. \tag{6.22}$$

The case $n_2 < 0$ is not possible. Indeed, if $n_2 < 0$ then (6.20) implies $n_4 = n_1 + |n_2| + n_3 > n_1$ by (6.22) which is in contradiction with $n_1 > |n_4|$. Hence from now on we assume that

$$n_1 > n_2 \geq |n_4| > |n_3| > 0 \,. \tag{6.23}$$

Case c1) Assume that all the frequencies have all the same sign, i.e. $n_1 > n_2 \geq n_4 > n_3 > 0$. In this case, by (6.19)-(6.20), we get

$$|\psi(n_1,n_2,n_3,n_4)| = \frac{|2\sqrt{n_1n_3}-2\sqrt{n_2n_4}|}{|n_1|^{\frac{1}{2}}+|n_2|^{\frac{1}{2}}+|n_3|^{\frac{1}{2}}+|n_4|^{\frac{1}{2}}} \geq \frac{2}{\sum_{i=1}^4 |n_i|^{\frac{1}{2}}} \frac{|n_1n_3-n_2n_4|}{\sqrt{n_1n_3}+\sqrt{n_2n_4}} \,.$$

Since $\psi \neq 0$ we have $n_1n_3 \neq n_2n_4$, and therefore (6.17) follows.
Case c2) Assume now that two frequencies are positive and two are negative, i.e. $n_4 < n_3 < 0 < n_2 < n_1$. The momentum condition (6.20) becomes $n_1 - n_2 = -|n_4| + |n_3|$ and, since $n_1 > n_2$, then $|n_3| > |n_4|$ contradicting (6.23).
Case c3) Assume that three frequencies have the same sign and one has the opposite sign. By (6.20) and (6.23) we then have

$$n_1 > n_2 > n_4 > 0 > n_3 \,, \quad n_4 > |n_3| \,. \tag{6.24}$$

By (6.19) we get

$$\begin{aligned}\psi(n_1,n_2,n_3,n_4) &= \frac{n_1-n_2+|n_3|-n_4+2\sqrt{n_1|n_3|}-2\sqrt{n_2n_4}}{\sum_{i=1}^4 |n_i|^{\frac{1}{2}}} \\ &\overset{(6.20),(6.24)}{=} \frac{2}{\sum_{i=1}^4 |n_i|^{\frac{1}{2}}}\Big(|n_3| + \sqrt{n_1|n_3|} - \sqrt{n_2n_4}\Big) \\ &= \frac{2}{\sum_{i=1}^4 |n_i|^{\frac{1}{2}}} \frac{n_3^2 + n_1|n_3| - n_2n_4 + 2|n_3|\sqrt{n_1|n_3|}}{|n_3| + \sqrt{n_1|n_3|} + \sqrt{n_2n_4}} \,. \end{aligned} \tag{6.25}$$

If $n_2n_4 \leq n_1|n_3|$ then (6.25) implies the bound (6.17). If instead $n_2n_4 > n_1|n_3|$ we reason as follows. Notice that

$$B := n_3^2 + n_1|n_3| - n_2n_4 - 2|n_3|\sqrt{n_1|n_3|} \leq n_3^2 - 2|n_3|\sqrt{n_1|n_3|} \leq -|n_3|\sqrt{n_1|n_3|} \leq -1 \,,$$

in particular $B \neq 0$. Then we rationalize again (6.25) to obtain

$$\psi(n_1,n_2,n_3,n_4) = C \cdot A \cdot B^{-1}$$

where

$$A := (n_3^2 + n_1|n_3| - n_2n_4)^2 - 4|n_3|^3 n_1 \,, \quad C := \frac{2}{\sum_{i=1}^4 |n_i|^{\frac{1}{2}}} \frac{1}{|n_3| + \sqrt{n_1|n_3|} + \sqrt{n_2n_4}} \,.$$

Since $\psi \neq 0$ then A is a non zero integer and so $|\psi| \geq C|B|^{-1}$. Moreover $|B| \leq cn_1^2$, for some constant $c > 0$, and (6.17) follows. □

6.2. **Poincaré-Birkhoff reductions.** The proof of Proposition 6.2 is divided into two steps: in the first (Subsection 6.2.1) we eliminate all the quadratic terms in (5.4); in the second one (Subsection 6.2.2) we eliminate all the non resonant cubic terms.

6.2.1. *Elimination of the quadratic vector field.* In this section we cancel out the smoothing term $\mathtt{R}_1(U)$ in (6.1) of system (5.4). We conjugate (5.4) with the flow

$$\partial_\theta \mathcal{B}_1^\theta(U) = \mathtt{Q}_1(U)\mathcal{B}_1^\theta(U)\,, \quad \mathcal{B}_1^0(U) = \mathrm{Id}\,, \tag{6.26}$$

with $\mathtt{Q}_1(U) \in \widetilde{\mathcal{R}}_1^{-\rho} \otimes \mathcal{M}_2(\mathbb{C})$ of the same form of $\mathtt{R}_1(U)$ in (6.2)-(6.5), to be determined. We introduce the new variable

$$Y_1 := \left[\begin{smallmatrix} y_1 \\ \overline{y_1} \end{smallmatrix}\right] = \big(\mathcal{B}_1^\theta(U)[Z]\big)_{|\theta=1}\,. \tag{6.27}$$

Lemma 6.4. (First Poincaré-Birkhoff step) *Assume that $\mathtt{Q}_1(U) \in \widetilde{\mathcal{R}}_1^{-\rho} \otimes \mathcal{M}_2(\mathbb{C})$ solves the homological equation*

$$\mathtt{Q}_1(-\mathrm{i}\Omega U) + \big[\mathtt{Q}_1(U), -\mathrm{i}\Omega\big] + \mathtt{R}_1(U) = 0\,. \tag{6.28}$$

Then

$$\partial_t Y_1 = -\mathrm{i}\Omega Y_1 + Op^{\mathrm{BW}}(-\mathrm{i}\mathtt{D}(U;\xi) + \mathtt{H}_{\geq 3})[Y_1] + \big(\mathtt{R}_2^+(U) + \mathtt{R}_{\geq 3}^+(U)\big)[Y_1] \tag{6.29}$$

where Ω is defined in (3.19)*, $\mathtt{D}(U;\xi)$ in* (5.5)*, $\mathtt{H}_{\geq 3}$ is an admissible symbol in $\Gamma^1_{K,K',3} \otimes \mathcal{M}_2(\mathbb{C})$, and*

$$\mathtt{R}_2^+(U) \in \widetilde{\mathcal{R}}_2^{-\rho+m_1} \otimes \mathcal{M}_2(\mathbb{C})\,, \quad \mathtt{R}_{\geq 3}^+(U) \in \mathcal{R}_{K,K',3}^{-\rho+m_1} \otimes \mathcal{M}_2(\mathbb{C})\,,$$

with $m_1 \geq 1$ as in (3.19)*.*

Proof. To conjugate (5.4) we apply Lemma A.1 with $\mathtt{Q}_1(U) = \mathrm{i}\mathbf{A}(U)$. By (A.3) with $L = 1$ we have

$$\begin{aligned} -\mathrm{i}\mathcal{B}_1^1(U)\Omega(\mathcal{B}_1^1(U))^{-1} &= -\mathrm{i}\Omega + \big[\mathtt{Q}_1(U), -\mathrm{i}\Omega\big] \\ &\quad + \int_0^1 (1-\theta)\mathcal{B}_1^\theta(U)\big[\mathtt{Q}_1(U), \big[\mathtt{Q}_1(U), -\mathrm{i}\Omega\big]\big](\mathcal{B}_1^\theta(U))^{-1} d\theta\,. \end{aligned} \tag{6.30}$$

Using that $\mathtt{Q}_1(U)$ belongs to $\widetilde{\mathcal{R}}_1^{-\rho} \otimes \mathcal{M}_2(\mathbb{C})$ and applying Proposition 2.10, and Lemma A.3, the term in (6.30) is a smoothing operator in $\Sigma\mathcal{R}_{K,K',2}^{-\rho+\frac12} \otimes \mathcal{M}_2(\mathbb{C})$. Similarly we obtain

$$-\,\mathrm{i}\mathcal{B}_1^1(U)Op^{\mathrm{BW}}(\mathtt{D}(U;\xi))(\mathcal{B}_1^1(U))^{-1} = -\mathrm{i}Op^{\mathrm{BW}}(\mathtt{D}(U;\xi)) \tag{6.31}$$

up to a term in $\Sigma\mathcal{R}_{K,K',2}^{-\rho+1} \otimes \mathcal{M}_2(\mathbb{C})$, and

$$\mathcal{B}_1^1(U)Op^{\mathrm{BW}}(\mathtt{H}_{\geq 3})(\mathcal{B}_1^1(U))^{-1} = Op^{\mathrm{BW}}(\mathtt{H}_{\geq 3}) \tag{6.32}$$

up to a matrix of smoothing operators in $\Sigma\mathcal{R}_{K,K',2}^{-\rho+1} \otimes \mathcal{M}_2(\mathbb{C})$. Finally

$$\mathcal{B}_1^1(U)\big(\mathtt{R}_1(U) + \mathtt{R}_2(U) + \mathtt{R}_{\geq 3}(U)\big)(\mathcal{B}_1^1(U))^{-1} = \mathtt{R}_1(U) \tag{6.33}$$

plus a smoothing operator in $\Sigma\mathcal{R}_{K,K',2}^{-\rho+1} \otimes \mathcal{M}_2(\mathbb{C})$.

Next we consider the contribution coming from the conjugation of ∂_t. Applying formula (A.4) with $L = 2$, we get

$$\begin{aligned} \partial_t\mathcal{B}_1^1(U)(\mathcal{B}_1^1(U))^{-1} &= \partial_t\mathtt{Q}_1(U) + \frac12\big[\mathtt{Q}_1(U), \partial_t\mathtt{Q}_1(U)\big] \\ &\quad + \frac12\int_0^1 (1-\theta)^2\mathcal{B}_1^\theta(U)\big[\mathtt{Q}_1(U), \big[\mathtt{Q}_1(U), \partial_t\mathtt{Q}_1(U)\big]\big](\mathcal{B}_1^\theta(U))^{-1} d\theta\,. \end{aligned} \tag{6.34}$$

Recalling (3.19) we have

$$\partial_t\mathtt{Q}_1(U) = \mathtt{Q}_1(-\mathrm{i}\Omega U + \mathbf{M}(U)[U]) = \mathtt{Q}_1(-\mathrm{i}\Omega U) + \mathtt{Q}_1(\mathbf{M}(U)[U]) = \mathtt{Q}_1(-\mathrm{i}\Omega U) \tag{6.35}$$

up to a term in $\Sigma\mathcal{R}_{K,K',2}^{-\rho+m_1} \otimes \mathcal{M}_2(\mathbb{C})$, where we used the last two remarks under Definition 2.7. By (6.35) and the fact that $\mathtt{Q}_1(-\mathrm{i}\Omega U)$ is in $\widetilde{\mathcal{R}}_1^{-\rho+\frac12} \otimes \mathcal{M}_2(\mathbb{C})$ we have that the second line (6.34) belongs to $\Sigma\mathcal{R}_{K,K',2}^{-\rho+m_1} \otimes \mathcal{M}_2(\mathbb{C})$.

In conclusion, by (6.30), (6.31), (6.32), (6.33), (6.34) and the assumption that $\mathtt{Q}_1$ solves (6.28) we deduce (6.29). □

- **Notation.** Given $p \in \mathbb{N}$ we denote by $\max_2(|n_1|, |n_2|, \dots, |n_p|)$ and $\max(|n_1|, |n_2|, \dots, |n_p|)$ respectively the second largest and the largest among $|n_1|, \dots, |n_p|$.

We have the following lemma.

Lemma 6.5. *An operator* $\mathtt{R}_1(U)$ *of the form* (6.2)-(6.5) *belongs to* $\widetilde{\mathcal{R}}_1^{-\rho} \otimes \mathcal{M}_2(\mathbb{C})$ *if and only if, for some* $\mu > 0$,

$$|(\mathtt{r}_{1,\epsilon})_{n,k}^{\sigma,\sigma'}| \leq \frac{\max_2(|n|,|k|)^{\rho+\mu}}{\max(|n|,|k|)^{\rho}}, \quad \forall\, \epsilon, \sigma, \sigma' = \pm, \ \ n,k \in \mathbb{Z} \setminus \{0\}. \tag{6.36}$$

An operator $\mathtt{R}_2(U)$ *of the form* (6.2)-(6.3) *as in* (6.6)-(6.7) *belongs to* $\widetilde{\mathcal{R}}_2^{-\rho} \otimes \mathcal{M}_2(\mathbb{C})$ *if and only if, for some* $\mu > 0$,

$$|(\mathtt{r}_{2,\epsilon,\epsilon'})_{n_1,n_2,k}^{\sigma,\sigma'}| \leq \frac{\max_2(|n_1|,|n_2|,|k|)^{\rho+\mu}}{\max(|n_1|,|n_2|,|k|)^{\rho}}, \quad \forall\, \epsilon, \epsilon', \sigma, \sigma' = \pm, \ \ n_1, n_2, k \in \mathbb{Z} \setminus \{0\}. \tag{6.37}$$

Proof. By the definition of smoothing homogeneous operators given in Definition 2.5. □

We now solve the homological equation (6.28).

Lemma 6.6. (First homological equation) *The operator* $\mathtt{Q}_1$ *of the form* (6.2)-(6.5) *with coefficients*

$$(\mathtt{q}_{1,\epsilon})_{n,k}^{\sigma,\sigma'} := \frac{-(\mathtt{r}_{1,\epsilon})_{n,k}^{\sigma,\sigma'}}{\mathrm{i}\big(\sigma|j|^{\frac{1}{2}} - \sigma'|k|^{\frac{1}{2}} - \epsilon|n|^{\frac{1}{2}}\big)}, \quad \sigma j - \sigma' k - \epsilon n = 0, \tag{6.38}$$

with $\sigma, \sigma', \epsilon = \pm$, $j,n,k \in \mathbb{Z} \setminus \{0\}$ *solves the homological equation* (6.28) *and* $\mathtt{Q}_1$ *is in* $\widetilde{\mathcal{R}}_1^{-\rho} \otimes \mathcal{M}_2(\mathbb{C})$.

Proof. First notice that the coefficients in (6.38) are well-defined since $\sigma|j|^{\frac{1}{2}} - \sigma'|k|^{\frac{1}{2}} - \epsilon|n|^{\frac{1}{2}} \neq 0$ for any $\sigma, \sigma', \epsilon = \pm$, $n,k \in \mathbb{Z} \setminus \{0\}$, by Proposition 6.3, in particular (6.15). Moreover, by (6.15) and Lemma 6.5 we have

$$|(\mathtt{q}_{1,\epsilon})_{n,k}^{\sigma,\sigma'}| \leq \frac{\max_2(|n|,|k|)^{\rho+\mu}}{\max(|n|,|k|)^{\rho}},$$

and therefore the operator $\mathtt{Q}_1$ is in $\widetilde{\mathcal{R}}_1^{-\rho} \otimes \mathcal{M}_2(\mathbb{C})$.

Next, recalling (6.2) and (3.19), the homological equation (6.28) amounts to the equations

$$(\mathtt{Q}_1(-\mathrm{i}\Omega U))_{\sigma}^{\sigma'} - (\mathtt{Q}_1(U))_{\sigma}^{\sigma'} \sigma' \mathrm{i}|D|^{\frac{1}{2}} + \sigma \mathrm{i}|D|^{\frac{1}{2}} (\mathtt{Q}_1(U))_{\sigma}^{\sigma'} + (\mathtt{R}_1(U))_{\sigma}^{\sigma'} = 0, \ \forall \sigma, \sigma' = \pm,$$

and expanding $(\mathtt{Q}_1(U))_{\sigma}^{\sigma'}$ as in (6.3)-(6.5) with entries

$$(\mathtt{Q}_{1,\epsilon}(U))_{\sigma,j}^{\sigma',k} = \frac{1}{\sqrt{2\pi}} \sum_{\substack{n \in \mathbb{Z}\setminus\{0\} \\ \epsilon n + \sigma' k = \sigma j}} (\mathtt{q}_{1,\epsilon})_{n,k}^{\sigma,\sigma'} u_n^{\epsilon}, \quad j,k \in \mathbb{Z} \setminus \{0\}, \tag{6.39}$$

to the equations, for any $j,k \in \mathbb{Z} \setminus \{0\}$, $\epsilon = \pm$,

$$(\mathtt{Q}_{1,\epsilon}(-\mathrm{i}\Omega U))_{\sigma,j}^{\sigma',k} + (\mathtt{Q}_{1,\epsilon}(U))_{\sigma,j}^{\sigma',k} \big(\sigma \mathrm{i}|j|^{\frac{1}{2}} - \sigma' \mathrm{i}|k|^{\frac{1}{2}}\big) + (\mathtt{R}_{1,\epsilon}(U))_{\sigma,j}^{\sigma',k} = 0. \tag{6.40}$$

By (6.39) and (3.19) we have

$$(\mathtt{Q}_{1,\epsilon}(-\mathrm{i}\Omega U))_{\sigma,j}^{\sigma',k} = \frac{1}{\sqrt{2\pi}} \sum_{\substack{n \in \mathbb{Z}\setminus\{0\} \\ \epsilon n + \sigma' k = \sigma j}} (\mathtt{q}_{1,\epsilon})_{n,k}^{\sigma,\sigma'} (-\mathrm{i}\epsilon|n|^{\frac{1}{2}}) u_n^{\epsilon},$$

and (6.40) becomes, for $j,k,n\in\mathbb{Z}\setminus\{0\}$ and $\sigma,\sigma',\epsilon=\pm$ with $\epsilon n+\sigma' k=\sigma j$,

$$(\mathtt{q}_{1,\epsilon})^{\sigma,\sigma'}_{n,k}\,\mathrm{i}\big(\sigma|j|^{\frac12}-\sigma'|k|^{\frac12}-\epsilon|n|^{\frac12}\big)+(\mathtt{r}_{1,\epsilon})^{\sigma,\sigma'}_{n,k}=0\,,$$

which is solved by the coefficients $(\mathtt{q}_{1,\epsilon})^{\sigma,\sigma'}_{n,k}$ defined in (6.38). □

6.2.2. *Elimination of the cubic vector field.* In this section we reduce to Poincaré-Birkhoff normal form the smoothing term $\mathtt{R}_2^+(U)\in\widetilde{\mathcal{R}}_2^{-\rho+m_1}\otimes\mathcal{M}_2(\mathbb{C})$ in (6.29). We conjugate (6.29) with the flow

$$\partial_\theta\mathcal{B}_2^\theta(U)=\mathtt{Q}_2(U)\mathcal{B}_2^\theta(U)\,,\quad \mathcal{B}_2^0(U)=\mathrm{Id}\,, \tag{6.41}$$

where $\mathtt{Q}_2(U)$ is a matrix of smoothing operators in $\widetilde{\mathcal{R}}_2^{-\rho+N_0+m_1}\otimes\mathcal{M}_2(\mathbb{C})$ of the same form of $\mathtt{R}_2^+(U)$ to be determined. We introduce the new variable

$$Y_2:=\big[\begin{smallmatrix}y_2\\ \overline{y_2}\end{smallmatrix}\big]=\big(\mathcal{B}_2^\theta(U)[Y_1]\big)_{|\theta=1}\,. \tag{6.42}$$

- **Notation.** Given the operator $\mathtt{Q}_2(U)$ in (6.41), we denote by $\mathtt{Q}_2(-\mathrm{i}\Omega U)$ the operator of the form (6.2), (6.3), (6.6)-(6.7) with coefficients defined as

$$(\mathtt{Q}_{2,\epsilon,\epsilon'}(-\mathrm{i}\Omega U))^{\sigma',k}_{\sigma,j}=\frac{1}{2\pi}\sum_{\substack{n_1,n_2\in\mathbb{Z}\setminus\{0\}\\ \epsilon n_1+\epsilon' n_2+\sigma' k=\sigma j}}(\mathtt{q}_{2,\epsilon,\epsilon'})^{\sigma,\sigma'}_{n_1,n_2,k}\big(-\mathrm{i}\epsilon|n_1|^{\frac12}-\mathrm{i}\epsilon'|n_2|^{\frac12}\big)u^{\epsilon}_{n_1}u^{\epsilon'}_{n_2}\,. \tag{6.43}$$

Lemma 6.7. (Second Poincaré-Birkhoff step) *Assume that* $\mathtt{Q}_2(U)\in\widetilde{\mathcal{R}}_2^{-\rho+N_0+m_1}\otimes\mathcal{M}_2(\mathbb{C})$ *solves the homological equation*

$$\mathtt{Q}_2(-\mathrm{i}\Omega U)+\big[\mathtt{Q}_2(U),-\mathrm{i}\Omega\big]+\mathtt{R}_2^+(U)=(\mathtt{R}_2^+)^{res}(U)\,. \tag{6.44}$$

Then

$$\partial_t Y_2=-\mathrm{i}\Omega Y_2+Op^{\mathrm{BW}}(-\mathrm{i}\mathtt{D}(U;\xi)+\mathtt{H}_{\geq 3})[Y_2]+\big((\mathtt{R}_2^+)^{res}(U)+\mathtt{R}'_{\geq 3}(U)\big)[Y_2] \tag{6.45}$$

where Ω *is defined in* (3.19) *and* $\mathtt{D}(U;\xi)$ *in* (5.5), $\mathtt{H}_{\geq 3}$ *is an admissible symbol in* $\Gamma^1_{K,K',3}\otimes\mathcal{M}_2(\mathbb{C})$, $(\mathtt{R}_2^+)^{res}(U)$ *is a Poincaré-Birkhoff resonant smoothing operator according to Definition 6.1 in* $\widetilde{\mathcal{R}}_2^{-\rho+m_1}\otimes\mathcal{M}_2(\mathbb{C})$, *and* $\mathtt{R}'_{\geq 3}(U)$ *is a matrix of smoothing operators in* $\mathcal{R}^{-\rho+N_0+2m_1}_{K,K',3}\otimes\mathcal{M}_2(\mathbb{C})$ *with* $m_1\geq 1$ *as in* (3.19).

Proof. To conjugate system (6.29) we apply Lemma A.1 with $\mathtt{Q}_2(U)=\mathrm{i}\mathbf{A}(U)$. Applying formula (A.3) with $L=1$, the fact that $\mathtt{Q}_2(U)$ is a smoothing operator in $\widetilde{\mathcal{R}}_2^{-\rho+N_0+m_1}\otimes\mathcal{M}_2(\mathbb{C})$, Proposition 2.10 and Lemma A.3, we have that

$$\mathcal{B}_2^1(U)(-\mathrm{i}\Omega)(\mathcal{B}_2^1(U))^{-1}=-\mathrm{i}\Omega+\big[\mathtt{Q}_2(U),-\mathrm{i}\Omega\big]$$

plus a smoothing operator in $\mathcal{R}^{-\rho+N_0+m_1+1}_{K,K',3}\otimes\mathcal{M}_2(\mathbb{C})$. Similarly

$$\begin{aligned}&\mathcal{B}_2^1(U)\big(Op^{\mathrm{BW}}(-\mathrm{i}\mathtt{D}(U;\xi)+\mathtt{H}_{\geq 3})+\mathtt{R}_2^+(U)+\mathtt{R}^+_{\geq 3}(U)\big)(\mathcal{B}_2^1(U))^{-1}\\ &=Op^{\mathrm{BW}}\big(-\mathrm{i}\mathtt{D}(U;\xi)+\mathtt{H}_{\geq 3}\big)+\mathtt{R}_2^+(U)+\mathtt{R}^+_{\geq 3}(U)\end{aligned}$$

up to a smoothing operator in $\mathcal{R}^{-\rho+N_0+m_1+1}_{K,K',3}\otimes\mathcal{M}_2(\mathbb{C})$.

Next we consider the contribution coming from the conjugation of ∂_t. First, notice that, using equation (3.19),

$$\partial_t\mathtt{Q}_2(U)=\mathtt{Q}_2(\partial_t U)=\mathtt{Q}_2(-\mathrm{i}\Omega U) \tag{6.46}$$

(defined in (6.43)) up to a smoothing operator in $\mathcal{R}^{-\rho+N_0+2m_1}_{K,K',3}\otimes\mathcal{M}_2(\mathbb{C})$. The operator $\mathtt{Q}_2(-\mathrm{i}\Omega U)$ is in $\widetilde{\mathcal{R}}^{-\rho+N_0+m_1+\frac{1}{2}}_2\otimes\mathcal{M}_2(\mathbb{C})$. Then, applying formula (A.4) with $L=2$ we have

$$\begin{aligned}\partial_t\mathcal{B}^1_2(U)(\mathcal{B}^1_2(U))^{-1} &= \partial_t\mathtt{Q}_2(U)+\frac{1}{2}\big[\mathtt{Q}_2(U),\partial_t\mathtt{Q}_2(U)\big]\\ &\quad+\frac{1}{2}\int_0^1(1-\theta)^2\mathcal{B}^\theta_2(U)\big[\mathtt{Q}_2(U),\big[\mathtt{Q}_2(U),\partial_t\mathtt{Q}_2(U)\big]\big](\mathcal{B}^\theta_2(U))^{-1}d\theta \stackrel{(6.46)}{=} \mathtt{Q}_2(-\mathrm{i}\Omega U)\end{aligned}$$

up to a smoothing operator in $\mathcal{R}^{-\rho+N_0+2m_1}_{K,K',3}\otimes\mathcal{M}_2(\mathbb{C})$.

In conclusion $\mathtt{Q}_2(-\mathrm{i}\Omega U)+[\mathtt{Q}_2(U),-\mathrm{i}\Omega]$ $+\mathtt{R}^+_2(U)$ collects all the non integrable terms quadratic in U in the transformed system. Since $\mathtt{Q}_2$ solves (6.44) we conclude that Y_2 solves (6.45). ☐

We now solve the homological equation (6.44).

Lemma 6.8. (Second homological equation) *The operator* $\mathtt{Q}_2$ *of the form* (6.2)-(6.3), (6.6)-(6.7) *with coefficients*

$$(\mathtt{q}_{2,\epsilon,\epsilon'})^{\sigma,\sigma'}_{n_1,n_2,k}:=\begin{cases}\dfrac{-(\mathtt{r}^+_{2,\epsilon,\epsilon'})^{\sigma,\sigma'}_{n_1,n_2,k}}{\mathrm{i}(\sigma|j|^{\frac12}-\sigma'|k|^{\frac12}-\epsilon|n_1|^{\frac12}-\epsilon'|n_2|^{\frac12})}\,, & \sigma|j|^{\frac12}-\sigma'|k|^{\frac12}-\epsilon|n_1|^{\frac12}-\epsilon|n_2|^{\frac12}\neq0\\ 0 & \sigma|j|^{\frac12}-\sigma'|k|^{\frac12}-\epsilon|n_1|^{\frac12}-\epsilon|n_2|^{\frac12}=0\end{cases}\tag{6.47}$$

with $\sigma,\sigma',\epsilon,\epsilon'=\pm$, $n_1,n_2,k\in\mathbb{Z}\setminus\{0\}$, *satisfying* $\sigma j-\sigma'k-\epsilon n_1-\epsilon' n_2=0$, *solves the homological equation* (6.44). *We have that* $\mathtt{Q}_2$ *is in* $\widetilde{\mathcal{R}}^{-\rho+N_0+m_1}_2\otimes\mathcal{M}_2(\mathbb{C})$.

Proof. First notice that the coefficients in (6.47) are well-defined thanks to Proposition 6.3, in particular (6.17), and satisfy, using also $|j|\le|k|+|n_1|+|n_2|$,

$$|(\mathtt{q}_{2,\epsilon,\epsilon'})^{\sigma,\sigma'}_{n_1,n_2,k}|\le C|(\mathtt{r}^+_{2,\epsilon,\epsilon'})^{\sigma,\sigma'}_{n_1,n_2,k}|\max(|n_1|,|n_2|,|k|)^{N_0}\le C\frac{\max_2(|n_1|,|n_2|,|k|)^{\rho-m_1-N_0+\mu'}}{\max(|n_1|,|n_2|,|k|)^{\rho-m_1-N_0}}\tag{6.48}$$

with $\mu'=\mu+N_0$, because $(\mathtt{r}^+_{2,\epsilon,\epsilon'})^{\sigma,\sigma'}_{n_1,n_2,k}$ are the coefficients of a remainder in $\widetilde{\mathcal{R}}^{-\rho+m_1}_2\otimes\mathcal{M}_2(\mathbb{C})$, and so they satisfy the bound (6.37) with $\rho\rightsquigarrow\rho-m_1$. The estimate (6.48) and Lemma 6.5 imply that $\mathtt{Q}_2(U)$ belongs to the class $\widetilde{\mathcal{R}}^{-\rho+m_1+N_0}_2\otimes\mathcal{M}_2(\mathbb{C})$.

Next, the homological equation (6.44) amounts to, for any $\sigma,\sigma',\epsilon,\epsilon'=\pm$,

$$(\mathtt{Q}_{2,\epsilon,\epsilon'}(-\mathrm{i}\Omega U))^{\sigma',k}_{\sigma,j}+(\mathtt{Q}_{2,\epsilon,\epsilon'}(U))^{\sigma',k}_{\sigma,j}\big(\sigma\mathrm{i}|j|^{\frac12}-\sigma'\mathrm{i}|k|^{\frac12}\big)+(\mathtt{R}^+_{2,\epsilon,\epsilon'}(U))^{\sigma',k}_{\sigma,j}=\big((\mathtt{R}^+_{2,\epsilon,\epsilon'})^{res}(U)\big)^{\sigma',k}_{\sigma,j}\tag{6.49}$$

for any $j,k\in\mathbb{Z}\setminus\{0\}$. Recalling (6.43) and (6.8), the left hand side of (6.49) is given by

$$(\mathtt{q}_{2,\epsilon,\epsilon'})^{\sigma,\sigma'}_{n_1,n_2,k}\mathrm{i}\big(\sigma|j|^{\frac12}-\sigma'|k|^{\frac12}-\epsilon|n_1|^{\frac12}-\epsilon|n_2|^{\frac12}\big)+(\mathtt{r}^+_{2,\epsilon,\epsilon'})^{\sigma,\sigma'}_{n_1,n_2,k}\,,$$

for $j,k,n_1,n_2\in\mathbb{Z}\setminus\{0\}$, $\sigma,\sigma',\epsilon,\epsilon'=\pm$ and $\epsilon n_1+\epsilon' n_2+\sigma' k=\sigma j$. We deduce, recalling Definition 6.1, that the operator $\mathtt{Q}_2$ with coefficients $(\mathtt{q}_{2,\epsilon,\epsilon'})^{\sigma,\sigma'}_{n_1,n_2,k}$ defined in (6.47) solves the homological equation (6.44). ☐

We can now prove the main result of this section.

Proof of Proposition 6.2. Let Z be the function given by Proposition 5.4 which solves (5.4). We set

$$Y:=(\mathcal{B}^\theta(U))_{\theta=1}[Z]\,,\quad\text{where}\quad\mathcal{B}^\theta(U):=\mathcal{B}^\theta_2(U)\circ\mathcal{B}^\theta_1(U)\,,\quad\theta\in[0,1]\,,\tag{6.50}$$

and $\mathcal{B}^\theta_i(U)$, $i=1,2$, are the flow maps defined respectively in (6.26), (6.41) with generators $\mathtt{Q}_1(U)$, $\mathtt{Q}_2(U)$ defined respectively in Lemmata 6.4 and 6.7. Therefore the function Y defined in (6.50) solves the system (recall (6.45))

$$\partial_t Y=-\mathrm{i}\Omega Y+Op^{\mathrm{BW}}(-\mathrm{i}\mathtt{D}(U;\xi)+\mathtt{H}_{\ge3})[Y]+\tilde{\mathtt{R}}^{res}(U)[Y]+\mathtt{R}'_{\ge3}(U)[Y]\tag{6.51}$$

where Ω and $\mathtt{D}(U;\xi)$ are defined respectively in (3.19) and (5.5), the smoothing operator $\tilde{\mathtt{R}}^{res}(U) := (\mathtt{R}_2^+)^{res}(U)$ in $\widetilde{\mathcal{R}}_2^{-\rho+m_1} \otimes \mathcal{M}_2(\mathbb{C})$ (where $m_1 \geq 1$ is the loss in (3.19)) is Poincaré-Birkhoff resonant according to Definition 6.1, the symbol $\mathtt{H}_{\geq 3} \in \Gamma^1_{K,K',3} \otimes \mathcal{M}_2(\mathbb{C})$ is admissible, and $\mathtt{R}'_{\geq 3}(U)$ is in $\mathcal{R}^{-\rho+N_0+2m_1}_{K,K',3} \otimes \mathcal{M}_2(\mathbb{C})$ where the constant N_0 is defined by Proposition 6.3.

We set $\mathfrak{C}^\theta(U) := \mathcal{B}^\theta(U) \circ \mathfrak{F}^\theta(U)$ where $\mathcal{B}^\theta(U)$ is the map defined in (6.50) and $\mathfrak{F}^\theta(U)$ in (5.6). Then we define $\mathbf{F}^\theta_T(U) := \mathfrak{C}^\theta(U)[U]$ as in (6.9). The maps $\mathcal{B}^\theta_i$, $i = 1, 2$ are constructed as flows of smoothing remainders, hence, by Lemma A.3, they are well-defined and satisfy the bounds (A.14), (A.12). Then, since the map $\mathfrak{F}^\theta(U)$ satisfies (5.7), the composition map $\mathfrak{C}^\theta(U)$ satisfies (6.12) and (6.13). Moreover the map $\mathfrak{F}^\theta(U)$ is the composition of flows of para-differential operators (see its definition in (5.71)), hence, by Lemma A.2, it admits multilinear expansions as in (A.13). In the same way, by Lemma A.3 the map $\mathcal{B}^\theta(U)$ admits a multilinear expansion as in (A.13), and therefore $\mathbf{F}^\theta_T(U)$ admits an expansion like (A.13) as well, implying item *(iii)* of Proposition 6.2. Moreover

$$Y = (\mathbf{F}^\theta_T(U))_{|\theta=1} = U + \mathtt{M}(U)[U] \qquad \text{where} \qquad \mathtt{M}(U) \in \Sigma\mathcal{M}_{K,K',1} \otimes \mathcal{M}_2(\mathbb{C}) \,. \tag{6.52}$$

Then, substituting (6.52) in (6.51), we obtain (6.10)-(6.11) with

$$\mathfrak{H}_{\geq 3}(U;x,\xi) := -\mathrm{i}\big(\mathtt{D}(U;\xi) - \mathtt{D}(U+\mathtt{M}(U)[U];\xi)\big) + \mathtt{H}_{\geq 3}(U;x,\xi)\,, \tag{6.53}$$

$$\mathfrak{R}_{\geq 3}(U) := \tilde{\mathtt{R}}^{res}(U) - \tilde{\mathtt{R}}^{res}(U+\mathtt{M}(U)[U]) + \mathtt{R}'_{\geq 3}(U)\,. \tag{6.54}$$

Since the integrable symbol $\mathtt{D}(U;\xi)$ in (5.5) is homogeneous of degree 2, the quadratic terms in the right hand side of (6.53) cancel out and, by (6.52) and item *(iv)* of Proposition 2.10, we deduce that $\mathfrak{H}_{\geq 3}(U;x,\xi) \in \Gamma^1_{K,K',3} \otimes \mathcal{M}_2(\mathbb{C})$ is an admissible symbol. Similarly, since $\tilde{\mathtt{R}}^{res}(U)$ is a smoothing operator in $\widetilde{\mathcal{R}}_2^{-\rho+m_1} \otimes \mathcal{M}_2(\mathbb{C})$, we deduce, by (6.52) and item *(iii)* of Proposition 2.10, that $\mathfrak{R}_{\geq 3}(U)$ defined in (6.54) is a smoothing operator in $\Sigma\mathcal{R}^{-(\rho-\rho_0)}_{K,K',3} \otimes \mathcal{M}_2(\mathbb{C})$ where $\rho_0 := N_0 + 2m_1$. □

7. Long time existence

The system

$$\partial_t Y = -\mathrm{i}\Omega Y - \mathrm{i}Op^{\mathrm{BW}}\big(\mathtt{D}(Y;\xi)\big)[Y] + \tilde{\mathtt{R}}^{res}(Y)[Y]\,, \tag{7.1}$$

obtained retaining only the vector fields in (6.10) up to degree 3 of homogeneity, is in Poincaré-Birkhoff normal form. In Section 7.2 we will actually prove that this is uniquely determined and that (7.1) coincides with the Hamiltonian system generated by the fourth order Birkhoff normal form Hamiltonian H_{ZD} computed by a formal expansion in [62, 22, 30, 20], see Section 7.1. Such normal form is integrable and its corresponding Hamiltonian system preserves all Sobolev norms, see Theorem 1.4. The key new relevant information in Proposition 6.2 is that the quartic remainder in (6.11) satisfies energy estimates (see Lemma 7.5). This allows us to prove in Section 7.3 energy estimates for the whole system (6.10) and thus the long time existence result of Theorem 1.2.

7.1. **The formal Birkhoff normal form.** We introduce, as in formula (2.7) of [20], the complex symplectic variable

$$\begin{pmatrix} u \\ \overline{u} \end{pmatrix} = \Lambda \begin{pmatrix} \eta \\ \psi \end{pmatrix} := \frac{1}{\sqrt{2}} \begin{pmatrix} |D|^{-\frac14}\eta + \mathrm{i}|D|^{\frac14}\psi \\ |D|^{-\frac14}\eta - \mathrm{i}|D|^{\frac14}\psi \end{pmatrix}, \quad \begin{pmatrix} \eta \\ \psi \end{pmatrix} = \Lambda^{-1} \begin{pmatrix} u \\ \overline{u} \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} |D|^{\frac14}(u+\overline{u}) \\ -\mathrm{i}|D|^{-\frac14}(u-\overline{u}) \end{pmatrix}. \tag{7.2}$$

Compare this formula with (1.16) and recall that, in view of (1.18), we may disregard the zero frequency in what follows. In the new complex variables $(u, \overline{u})$, a vector field $X(\eta,\psi)$ becomes

$$X^{\mathbb{C}} := \Lambda^\star X := \Lambda X \Lambda^{-1}\,. \tag{7.3}$$

The push-forward acts naturally on the commutator of nonlinear vector fields (A.35), namely

$$\Lambda^\star [\![X, Y]\!] = [\![\Lambda^\star X, \Lambda^\star Y]\!] = [\![X^{\mathbb{C}}, Y^{\mathbb{C}}]\!] .$$

The Poisson bracket in (1.7) assumes the form

$$\{F, H\} = \frac{1}{\mathrm{i}} \sum_{k \in \mathbb{Z}\setminus\{0\}} \left(\partial_{u_k} H \partial_{\overline{u_k}} F - \partial_{\overline{u_k}} H \partial_{u_k} F \right) .$$

Given a Hamiltonian $F(\eta, \psi)$ we denote by $F_{\mathbb{C}} := F \circ \Lambda^{-1}$ the same Hamiltonian expressed in terms of the complex variables $(u, \overline{u})$. The associated Hamiltonian vector field $X_{F_{\mathbb{C}}}$ is

$$X_{F_{\mathbb{C}}} = \frac{1}{\sqrt{2\pi}} \sum_{k \in \mathbb{Z}\setminus\{0\}, \sigma = \pm} \begin{pmatrix} -\mathrm{i}\partial_{\overline{u_k}} F_{\mathbb{C}}\, e^{\mathrm{i}kx} \\ \mathrm{i}\partial_{u_k} F_{\mathbb{C}}\, e^{-\mathrm{i}kx} \end{pmatrix} , \tag{7.4}$$

that we also identify, using the standard vector field notation, with

$$X_{F_{\mathbb{C}}} = \sum_{k \in \mathbb{Z}\setminus\{0\}, \sigma = \pm} -\mathrm{i}\sigma \partial_{u_k^{-\sigma}} F_{\mathbb{C}}\, \partial_{u_k^\sigma} . \tag{7.5}$$

Note that, if X_F is the Hamiltonian vector field of F in the real variables, then, using (7.3), we have

$$X_F^{\mathbb{C}} := \Lambda^\star X_F = X_{F_{\mathbb{C}}} , \quad F_{\mathbb{C}} := F \circ \Lambda^{-1} , \tag{7.6}$$

and

$$[\![X_H^{\mathbb{C}}, X_K^{\mathbb{C}}]\!] = X_{\{H,K\}}^{\mathbb{C}} = X_{\{H_{\mathbb{C}}, K_{\mathbb{C}}\}} . \tag{7.7}$$

We now describe the formal Birkhoff normal form procedure performed in [62, 30, 22, 20]. One first expands the water waves Hamiltonian (1.6), written in the complex variables $(u, \overline{u})$, in degrees of homogeneity

$$H_{\mathbb{C}} := H \circ \Lambda^{-1} = H_{\mathbb{C}}^{(2)} + H_{\mathbb{C}}^{(3)} + H_{\mathbb{C}}^{(4)} + H_{\mathbb{C}}^{(\geq 5)}, \qquad H_{\mathbb{C}}^{(2)} = \sum_{j \in \mathbb{Z}\setminus\{0\}} \omega_j u_j \overline{u_j} , \quad \omega_j := \sqrt{|j|} , \tag{7.8}$$

where

$$H_{\mathbb{C}}^{(3)} = \sum_{\sigma_1 j_1 + \sigma_2 j_2 + \sigma_3 j_3 = 0} H_{j_1, j_2, j_3}^{\sigma_1, \sigma_2, \sigma_3} u_{j_1}^{\sigma_1} u_{j_2}^{\sigma_2} u_{j_3}^{\sigma_3} , \tag{7.9}$$

$$H_{\mathbb{C}}^{(4)} = \sum_{\sigma_1 j_1 + \sigma_2 j_2 + \sigma_3 j_3 + \sigma_4 j_4 = 0} H_{j_1, j_2, j_3, j_4}^{\sigma_1, \sigma_2, \sigma_3, \sigma_4} u_{j_1}^{\sigma_1} u_{j_2}^{\sigma_2} u_{j_3}^{\sigma_3} u_{j_4}^{\sigma_4} , \tag{7.10}$$

can be explicitly computed. The Hamiltonian $H_{\mathbb{C}}^{(\geq 5)}$ collects all the monomials of homogeneity greater or equal 5. The Hamiltonians $H_{\mathbb{C}}^{(3)}$, $H_{\mathbb{C}}^{(4)}$ are real valued if and only if their coefficients satisfy

$$\overline{H_{j_1, j_2, j_3}^{\sigma_1, \sigma_2, \sigma_3}} = H_{j_1, j_2, j_3}^{-\sigma_1, -\sigma_2, -\sigma_3}, \qquad \overline{H_{j_1, j_2, j_3, j_4}^{\sigma_1, \sigma_2, \sigma_3, \sigma_4}} = H_{j_1, j_2, j_3, j_4}^{-\sigma_1, -\sigma_2, -\sigma_3, -\sigma_4}. \tag{7.11}$$

STEP 1. ELIMINATION OF CUBIC HAMILTONIAN. One looks for a symplectic transformation $\Phi^{(3)}$ as the (formal) time 1 flow generated by a cubic real Hamiltonian $F_{\mathbb{C}}^{(3)}$ of the form (7.9). Then a Lie expansion gives

$$H_{\mathbb{C}} \circ \Phi^{(3)} = H_{\mathbb{C}}^{(2)} + \{F_{\mathbb{C}}^{(3)}, H_{\mathbb{C}}^{(2)}\} + H_{\mathbb{C}}^{(3)} + H_{\mathbb{C}}^{(4)} + \frac{1}{2}\{F_{\mathbb{C}}^{(3)}, \{F_{\mathbb{C}}^{(3)}, H_{\mathbb{C}}^{(2)}\}\} + \{F_{\mathbb{C}}^{(3)}, H_{\mathbb{C}}^{(3)}\} + \cdots \tag{7.12}$$

up to terms of quintic degree. The cohomological equation

$$H_{\mathbb{C}}^{(3)} + \{F_{\mathbb{C}}^{(3)}, H_{\mathbb{C}}^{(2)}\} = 0 \tag{7.13}$$

has a unique solution since

$$\{u_{j_1}^{\sigma_1} u_{j_2}^{\sigma_2} u_{j_3}^{\sigma_3}, H_{\mathbb{C}}^{(2)}\} = -\mathrm{i}(\sigma_1 \omega(j_1) + \sigma_2 \omega(j_2) + \sigma_3 \omega(j_3)) u_{j_1}^{\sigma_1} u_{j_2}^{\sigma_2} u_{j_3}^{\sigma_3} ,$$

and the system

$$\sigma_1 j_1 + \sigma_2 j_2 + \sigma_3 j_3 = 0\,, \qquad \sigma_1\omega(j_1) + \sigma_2\omega(j_2) + \sigma_3\omega(j_3) = 0\,, \tag{7.14}$$

has no integer solutions, see Proposition 6.3. Hence, defining the cubic real valued Hamiltonian (see (7.11))

$$F^{(3)}_{\mathbb{C}} = \sum_{\sigma_1 j_1+\sigma_2 j_2+\sigma_3 j_3=0} \frac{H^{\sigma_1,\sigma_2,\sigma_3}_{j_1,j_2,j_3}}{\mathrm{i}(\sigma_1\omega(j_1)+\sigma_2\omega(j_2)+\sigma_3\omega(j_3))} u^{\sigma_1}_{j_1} u^{\sigma_2}_{j_2} u^{\sigma_3}_{j_3}\,,$$

the Hamiltonian in (7.12) reduces to

$$H_{\mathbb{C}} \circ \Phi^{(3)} = H^{(2)}_{\mathbb{C}} + H^{(4)}_{\mathbb{C}} + \frac{1}{2}\{F^{(3)}_{\mathbb{C}}, H^{(3)}_{\mathbb{C}}\} + \text{quintic terms}\,. \tag{7.15}$$

Step 2. Normalization of the quartic Hamiltonian. Similarly, one can finds a symplectic transformation $\Phi^{(4)}$, defined as the (formal) time 1 flow generated by a real quartic Hamiltonian $F^{(4)}_{\mathbb{C}}$ of the form (7.10), such that

$$H_{\mathbb{C}} \circ \Phi^{(3)} \circ \Phi^{(4)} = H^{(2)}_{\mathbb{C}} + \Pi_{\ker}\big(H^{(4)}_{\mathbb{C}} + \frac{1}{2}\{F^{(3)}_{\mathbb{C}}, H^{(3)}_{\mathbb{C}}\}\big) + \text{quintic terms}\,, \tag{7.16}$$

where, given a quartic monomial $u^{\sigma_1}_{j_1} u^{\sigma_2}_{j_2} u^{\sigma_3}_{j_3} u^{\sigma_4}_{j_4}$ satisfying $\sigma_1 j_1 + \sigma_2 j_2 + \sigma_3 j_3 + \sigma_4 j_4 = 0$, we define

$$\Pi_{\ker}\Big(u^{\sigma_1}_{j_1} u^{\sigma_2}_{j_2} u^{\sigma_3}_{j_3} u^{\sigma_4}_{j_4}\Big) := \begin{cases} u^{\sigma_1}_{j_1} u^{\sigma_2}_{j_2} u^{\sigma_3}_{j_3} u^{\sigma_4}_{j_4} & \text{if } \sigma_1\omega(j_1)+\sigma_2\omega(j_2)+\sigma_3\omega(j_3)+\sigma_4\omega(j_4)=0 \\ 0 & \text{otherwise}\,. \end{cases} \tag{7.17}$$

The fourth order (formal) Birkhoff normal form Hamiltonian in (7.16), that is,

$$H_{ZD} = H^{(2)}_{ZD} + H^{(4)}_{ZD}\,, \qquad H^{(2)}_{ZD} := H^{(2)}_{\mathbb{C}}\,, \quad H^{(4)}_{ZD} := \Pi_{\ker}\big(H^{(4)}_{\mathbb{C}} + \frac{1}{2}\{F^{(3)}_{\mathbb{C}}, H^{(3)}_{\mathbb{C}}\}\big)\,, \tag{7.18}$$

has been computed explicitly in [62, 22, 30, 20], and it is completely integrable. In [22] this is expressed as

$$H_{ZD} = \sum_{k>0} \Big(2\omega_k I_1(k) - \frac{k^3}{2\pi}(I_1^2(k) - 3I_2^2(k))\Big) + \frac{4}{\pi}\sum_{0<k<l} k^2 l I_2(k) I_2(l) \tag{7.19}$$

with actions

$$I_1(k) := \frac{z_k\overline{z_k} + z_{-k}\overline{z_{-k}}}{2}\,, \quad I_2(k) := \frac{z_k\overline{z_k} - \overline{z_{-k}}z_{-k}}{2}\,, \tag{7.20}$$

where z_k denote the Fourier coefficients of $z = \frac{1}{\sqrt{2}}|D|^{-\frac14}\eta + \frac{\mathrm{i}}{\sqrt{2}}|D|^{\frac14}\psi$ defined in (7.2). When expressed in terms of the complex variables $(z_k, \overline{z_k})$, the Hamiltonian H_{ZD} is given by $H^{(2)}_{ZD} + H^{(4)}_{ZD}$ as in (1.22)-(1.23).

The associated Hamiltonian system is (see (7.5))

$$\dot z_n = -\mathrm{i}\omega_n z_n + \frac{\mathrm{i}}{\pi}\sum_{\substack{|k_4|<|n|,\\ -\mathrm{sign}(n)=\mathrm{sign}(k_4)}} |n||k_4|^2|z_{k_4}|^2 z_n - \frac{\mathrm{i}}{\pi}\sum_{\substack{|k_4|<|n|,\\ \mathrm{sign}(n)=\mathrm{sign}(k_4)}} |n||k_4|^2|z_{k_4}|^2 z_n + [R(z)]_n \tag{7.21}$$

where

$$[R(z)]_n := -\frac{\mathrm{i}}{2\pi}|n|^3\big(|z_n|^2 - 2|z_{-n}|^2\big)z_n + \frac{\mathrm{i}}{\pi}\sum_{\substack{|n|<|k_1|,\\ \mathrm{sign}(k_1)=\mathrm{sign}(n)}} |k_1||n|^2 z_n\big(|z_{-k_1}|^2 - |z_{k_1}|^2\big)\,. \tag{7.22}$$

Notice in particular that $|z_n|^2$ are prime integrals, as stated in Theorem 1.4.

Although it is not necessary for the paper, for completeness we compare explicitly the structure of the normal form vector field (7.21)-(7.22) with (7.1).

Lemma 7.1. *The Hamiltonian system* (7.21)-(7.22) *has the form*

$$\dot z_n = -\mathrm{i}\omega_n z_n - \frac{\mathrm{i}}{\pi}\Big(\sum_{|j|<\epsilon|n|} j|j||z_j|^2\Big) n z_n + [\mathfrak{R}(z)]_n \tag{7.23}$$

where $\mathfrak{R}(z)$ *is a smoothing vector field in the sense that, for any* $0 \le \rho \le 2s-3$,

$$\|\mathfrak{R}(z)\|_{s+\rho} \le C(s)\|z\|_s^3\,, \tag{7.24}$$

where for a sequence $a = \{a_j\}_{j\in\mathbb{Z}}$ *we define* $\|a\|_s^2 := \sum_{j\in\mathbb{Z}}\langle j\rangle^{2s}|a_j|^2$. *Notice that*

$$-\sum_n \frac{\mathrm{i}}{\pi}\Big(\sum_{|j|<\epsilon|n|} j|j||z_j|^2\Big) n z_n \frac{1}{\sqrt{2\pi}} e^{\mathrm{i}nx} = Op^{\mathrm{BW}}(-\mathrm{i}\zeta(Z)\xi) z \tag{7.25}$$

where $\zeta(Z)$ *is defined in* (5.5).

Proof. Notice that

$$\begin{aligned}
&\sum_{|k_4|<|n|,-\mathrm{sign}(n)=\mathrm{sign}(k_4)} |n||k_4|^2|z_{k_4}|^2 z_n - \sum_{|k_4|<|n|,\mathrm{sign}(n)=\mathrm{sign}(k_4)} |n||k_4|^2|z_{k_4}|^2 z_n \\
&= -\sum_{|k_4|<|n|,-\mathrm{sign}(n)=\mathrm{sign}(k_4)} n\,\mathrm{sign}(k_4)|k_4|^2|z_{k_4}|^2 z_n - \sum_{|k_4|<|n|,\mathrm{sign}(n)=\mathrm{sign}(k_4)} n \mathrm{sign}(k_4)|k_4|^2|z_{k_4}|^2 z_n \\
&= -n\Big(\sum_{|k_4|<|n|} \mathrm{sign}(k_4)|k_4|^2|z_{k_4}|^2\Big) z_n = -n\Big(\sum_{|j|<|n|} j|j||z_j|^2\Big) z_n\,.
\end{aligned} \tag{7.26}$$

Then (7.26) allows to write (7.21) as (7.23) where

$$[\mathfrak{R}(z)]_n := -\frac{\mathrm{i}}{\pi}\Big(\sum_{\epsilon|n|\le|j|<|n|} j|j||z_j|^2\Big) n z_n + [R(z)]_n\,.$$

The vector $\mathfrak{R}(z)$ satisfies the (super) smoothing estimate (7.24) since it contains three high comparable frequencies. □

Notice that (7.25) is the transport paradifferential operator of order 1 in (6.10) and (5.5). Notice also that $\zeta(Z)$ in (5.5) vanishes on the subspace of functions even in x, coherently with the fact that the Hamiltonian in the second line of (1.23) vanishes as well on even functions. We also remark that (7.23) does not contain paradifferential operators at non-negative orders, in agreement with the form of the cubic terms in the Poincaré-Birkhoff normal form in (6.10) and (5.5). In the next section we actually prove that these have to coincide.

7.2. **Normal form identification.**

In Sections 3-6 we have transformed the water waves system (1.3) into (6.10), whose cubic component (7.1) is in Poincaré-Birkhoff normal form. All the conjugation maps that we have used have an expansion in homogeneous components up to degree 4. In this section we identify the cubic monomials left in the Poincaré-Birkhoff normal form (7.1). The main result is the following.

Proposition 7.2. (Identification of normal forms) *The cubic vector field component in* (6.10), *i.e.*

$$\mathcal{X}_{Res}(Y) := -\mathrm{i}Op^{\mathrm{BW}}\big(\mathtt{D}(Y;\xi)\big)[Y] + \tilde{\mathtt{R}}^{res}(Y)[Y]\,, \tag{7.27}$$

coincides with the Hamiltonian vector field

$$\mathcal{X}_{Res} = X_{\Pi_{\ker}(H^{(4)}_{\mathbb{C}} + \frac12\{F^{(3)}_{\mathbb{C}}, H^{(3)}_{\mathbb{C}}\})} = X_{H^{(4)}_{ZD}} \tag{7.28}$$

where the Hamiltonians $H^{(l)}_{\mathbb{C}}$, $l = 3,4$, *are defined in* (7.8), $F^{(3)}_{\mathbb{C}}$ *is the unique solution of* (7.13), *and* $\Pi_{\ker}$ *is defined in* (7.17).

The rest of the section is devoted to the proof of Proposition 7.2, which is based on a uniqueness argument for the Poincaré-Birkhoff normal form up to quartic remainders. The idea is the following. We first expand the water waves Hamiltonian vector field in (1.3),(1.5) in degrees of homogeneity

$$X_H = X_1 + X_2 + X_3 + X_{\geq 4} \qquad \text{where} \qquad X_1 := X_{H^{(2)}}\,,\ X_2 := X_{H^{(3)}}\,,\ X_3 := X_{H^{(4)}}\,, \tag{7.29}$$

and $X_{\geq 4}$ collects of the higher order terms. Then, in order to identify the cubic monomial vector fields in (7.27) we express the transformed system (6.10), obtained conjugating (1.3) via the good-unknown transformation $\mathcal{G}$ in (3.1) and $\mathbf{F}^1_T$ in Proposition 6.2, by a Lie commutator expansion up to terms of homogeneity at least 4. See Lemma A.10. Notice that the quadratic and cubic terms in (7.27) may arise by only the conjugation of $X_1+X_2+X_3$ under the homogeneous components up to cubic terms of the paradifferential transformations $\mathcal{G}$ and $\mathbf{F}^1_T$. Then, after some algebraic manipulation, we obtain the formulas (7.40)-(7.42). Since the adjoint operator $\mathrm{Ad}_{X^{\mathbb{C}}_{H^{(2)}}} := [\,\cdot\,, X^{\mathbb{C}}_{H^{(2)}}]$ acting on quadratic monomial vector fields satisfying the momentum conservation property is injective and surjective we then obtain the identity (7.44), and can eventually deduce (7.28).

- **Notation.** We use the Lie expansion (A.37) induced by a time-dependent vector field S, which contains quadratic and cubic terms. Given a homogeneous vector field X, we denote by $\Phi^\star_S X$ the induced (formal) push forward

$$\Phi^\star_S X = X + [\![S, X]\!]_{|\theta=0} + \frac{1}{2}[\![S, [\![S, X]\!]]\!]_{|\theta=0} + \frac{1}{2}[\![\partial_\theta S_{|\theta=0}, X]\!] + \cdots \tag{7.30}$$

where $[\![\cdot,\cdot]\!]$ is the non linear commutator defined in (A.35).

Step 1. The good unknown change of variable $\mathcal{G}$ in (3.1). We first provide the Lie expansion up to degree four of the vector field in (3.2)-(3.3), which is obtained by transforming the water waves vector field $X_1 + X_2 + X_3$ in (7.29) under the nonlinear map $\mathcal{G}$ in (3.1).

We first note that $\mathcal{G}(\eta,\psi) = (\Phi^\theta(\eta,\psi))_{\theta=1}$ where $\Phi^\theta(\eta,\psi) := (\eta, \psi - \theta Op^{\mathrm{BW}}(B(\eta,\psi))\eta)$, $\theta \in [0,1]$. Since $B(\eta,\psi)$ is a function in $\Sigma\mathcal{F}^{\mathbb{R}}_{K,0,1}$ we have, using the remarks under Definition 2.7, that the map $\Phi^\theta(\eta,\psi)$ has the form (A.26) in which U denotes the real variables (η,ψ), plus a map in $\mathcal{M}_{K,0,3} \otimes \mathcal{M}_2(\mathbb{C})$. By Lemma A.9 we regard the inverse of the map $\mathcal{G}_{\leq 3}$, obtained approximating $\mathcal{G}$ up to quartic remainders, as the (formal) time one flow of a non-autonomous vector field of the form

$$S(\theta) := \mathtt{S}_2 + \theta \mathtt{S}_3 \quad \text{ where } \quad \mathtt{S}_2 := S_1(\eta,\psi)\left[\begin{smallmatrix}\eta\\ \psi\end{smallmatrix}\right], \quad \mathtt{S}_3 := S_2(\eta,\psi)\left[\begin{smallmatrix}\eta\\ \psi\end{smallmatrix}\right], \tag{7.31}$$

where $S_1(\eta,\psi)$ is in $\widetilde{\mathcal{M}}_1 \otimes \mathcal{M}_2(\mathbb{C})$ and $S_2(\eta,\psi)$ is in $\widetilde{\mathcal{M}}_2 \otimes \mathcal{M}_2(\mathbb{C})$. By (7.29), (7.30) and (7.31), we get

$$\Phi^\star_S(X_1 + X_2 + X_3) = X_1 + X_{2,1} + X_{3,1} + \cdots \tag{7.32}$$

where

$$X_{2,1} := X_2 + [\![\mathtt{S}_2, X_1]\!]\,, \quad X_{3,1} := X_3 + [\![\mathtt{S}_2, X_2]\!] + \frac{1}{2}[\![\mathtt{S}_2, [\![\mathtt{S}_2, X_1]\!]]\!] + \frac{1}{2}[\![\mathtt{S}_3, X_1]\!]\,. \tag{7.33}$$

Complex coordinates Λ in (7.2). In the complex coordinates (7.2), the vector field (7.32) reads, recalling the notation (7.3),

$$\Lambda^\star\Phi^\star_S(X_1 + X_2 + X_3) = \Lambda^\star X_1 + \Lambda^\star X_{2,1} + \Lambda^\star X_{3,1} + \cdots = X^{\mathbb{C}}_1 + X^{\mathbb{C}}_{2,1} + X^{\mathbb{C}}_{3,1} + \cdots \tag{7.34}$$

where $X^{\mathbb{C}}_1$ is the linear Hamiltonian vector field $X^{\mathbb{C}}_1 = X^{\mathbb{C}}_{H^{(2)}} = -\mathrm{i}\sum_{j,\sigma} \sigma\omega_j u^\sigma_j \partial_{u^\sigma_j}$.

Step 2. The transformation $\mathbf{F}^1_T$ in Proposition 6.2. We consider the nonlinear map $(\mathbf{F}^1_T)_{\leq 3}$ obtained retaining only the terms of the map $\mathbf{F}^1_T := (\mathbf{F}^\theta_T)_{|\theta=1}$ up to quartic remainders. The approximate inverse of the map $(\mathbf{F}_T^1)_{\leq 3}$ provided by Lemma A.8, can be regarded, by Lemma A.9, as the (formal) approximate time-one flow of a non-autonomous vector field

$$T(\theta) := \mathtt{T}_2 + \theta \mathtt{T}_3 \quad \text{where} \quad \mathtt{T}_2(U) := T_1(U)[U]\,, \quad \mathtt{T}_3(U) := T_2(U)[U]\,,$$

for some $T_1(U)$ in $\widetilde{\mathcal{M}}_1 \otimes \mathcal{M}_2(\mathbb{C})$ and $T_2(U) \in \widetilde{\mathcal{M}}_2 \otimes \mathcal{M}_2(\mathbb{C})$. We transform the system obtained retaining only the components $X_1^{\mathbb{C}} + X_{2,1}^{\mathbb{C}} + X_{3,1}^{\mathbb{C}}$ in (7.34). By (7.30) we get

$$\Phi_T^\star \Lambda^\star \Phi_S^\star (X_1 + X_2 + X_3) = X_1^{\mathbb{C}} + X_{2,2}^{\mathbb{C}} + X_{3,2}^{\mathbb{C}} + \cdots \tag{7.35}$$

where

$$X_{2,2}^{\mathbb{C}} := X_{2,1}^{\mathbb{C}} + [\![\mathtt{T}_2, X_1^{\mathbb{C}}]\!]\,, \quad X_{3,2}^{\mathbb{C}} := X_{3,1}^{\mathbb{C}} + [\![\mathtt{T}_2, X_{2,1}^{\mathbb{C}}]\!] + \frac{1}{2}[\![\mathtt{T}_2, [\![\mathtt{T}_2, X_1^{\mathbb{C}}]\!]]\!] + \frac{1}{2}[\![\mathtt{T}_3, X_1^{\mathbb{C}}]\!]$$

and, recalling the expressions of $X_{2,1}, X_{3,1}$ in (7.33), the quadratic and the cubic components of the vector field (7.35) are given by

$$X_{2,2}^{\mathbb{C}} = X_2^{\mathbb{C}} + [\![\mathtt{S}_2^{\mathbb{C}} + \mathtt{T}_2, X_1^{\mathbb{C}}]\!] \tag{7.36}$$

and

$$\begin{aligned} X_{3,2}^{\mathbb{C}} = X_3^{\mathbb{C}} + [\![\mathtt{S}_2^{\mathbb{C}} + \mathtt{T}_2, X_2^{\mathbb{C}}]\!] + \frac{1}{2}[\![\mathtt{S}_2^{\mathbb{C}}, [\![\mathtt{S}_2^{\mathbb{C}}, X_1^{\mathbb{C}}]\!]]\!] + [\![\mathtt{T}_2, [\![\mathtt{S}_2^{\mathbb{C}}, X_1^{\mathbb{C}}]\!]]\!] + \frac{1}{2}[\![\mathtt{T}_2, [\![\mathtt{T}_2, X_1^{\mathbb{C}}]\!]]\!] \\ + \frac{1}{2}[\![\mathtt{S}_3^{\mathbb{C}} + \mathtt{T}_3, X_1^{\mathbb{C}}]\!]\,. \end{aligned} \tag{7.37}$$

Lemma 7.3. *Given vector fields X, Y, Z we have the identity*

$$\frac{1}{2}[\![Y, [\![Y, X]\!]]\!] + [\![Z, [\![Y, X]\!]]\!] + \frac{1}{2}[\![Z, [\![Z, X]\!]]\!] = \frac{1}{2}[\![Y + Z, [\![Y + Z, X]\!]]\!] + \frac{1}{2}[\![[\![Y + Z, Y]\!], X]\!]\,. \tag{7.38}$$

Proof. Use the Jacobi identity $[\![X, [\![Y, Z]\!]]\!] + [\![Z, [\![X, Y]\!]]\!] + [\![Y, [\![Z, X]\!]]\!] = 0$. $\square$

Using the identity (7.38), the term $X_{3,2}^{\mathbb{C}}$ in (7.37) is

$$X_{3,2}^{\mathbb{C}} = X_3^{\mathbb{C}} + [\![\mathtt{S}_2^{\mathbb{C}} + \mathtt{T}_2, X_2^{\mathbb{C}}]\!] + \frac{1}{2}[\![\mathtt{S}_2^{\mathbb{C}} + \mathtt{T}_2, [\![\mathtt{S}_2^{\mathbb{C}} + \mathtt{T}_2, X_1^{\mathbb{C}}]\!]]\!] + \frac{1}{2}[\![[\![\mathtt{S}_2^{\mathbb{C}} + \mathtt{T}_2, \mathtt{S}_2^{\mathbb{C}}]\!] + \mathtt{S}_3^{\mathbb{C}} + \mathtt{T}_3, X_1^{\mathbb{C}}]\!]\,. \tag{7.39}$$

Step 3. Identification of quadratic and cubic vector fields. The vector field $\Phi_T^\star \Lambda^\star \Phi_S^\star (X_1 + X_2 + X_3)$ in (7.35) is the vector field in the right hand side of (7.1), up to quartic remainders. Thus, recalling the expression of the quadratic, respectively cubic, vector field in (7.36), respectively (7.39), the expansion (7.29), formula (7.6), and the definition of $\mathcal{X}_{Res}$ in (7.27), we have the identification order by order:

$$X_1^{\mathbb{C}}(Y) = -\mathrm{i}\Omega Y \tag{7.40}$$

$$X_{H^{(3)}}^{\mathbb{C}} + [\![\mathtt{S}_2^{\mathbb{C}} + \mathtt{T}_2, X_{H^{(2)}}^{\mathbb{C}}]\!] = 0 \tag{7.41}$$

$$X_{H^{(4)}}^{\mathbb{C}} + [\![\mathtt{S}_2^{\mathbb{C}} + \mathtt{T}_2, X_2^{\mathbb{C}}]\!] + \frac{1}{2}[\![\mathtt{S}_2^{\mathbb{C}} + \mathtt{T}_2, [\![\mathtt{S}_2^{\mathbb{C}} + \mathtt{T}_2, X_1^{\mathbb{C}}]\!]]\!] + \frac{1}{2}[\![[\![\mathtt{S}_2^{\mathbb{C}} + \mathtt{T}_2, \mathtt{S}_2^{\mathbb{C}}]\!] + \mathtt{S}_3^{\mathbb{C}} + \mathtt{T}_3, X_1^{\mathbb{C}}]\!] = \mathcal{X}_{Res}\,. \tag{7.42}$$

Quadratic vector fields. Since $F_{\mathbb{C}}^{(3)}$ solves (7.13), by (7.7), we have

$$X_{H^{(3)}}^{\mathbb{C}} + [\![X_{F_{\mathbb{C}}^{(3)}}, X_{H_{\mathbb{C}}^{(2)}}]\!] = 0\,. \tag{7.43}$$

Subtracting (7.41) and (7.43), and since $X_{H_{\mathbb{C}}^{(2)}} = X_{H^{(2)}}^{\mathbb{C}}$, we deduce

$$[\![\mathtt{S}_2^{\mathbb{C}} + \mathtt{T}_2 - X_{F_{\mathbb{C}}^{(3)}}, X_{H^{(2)}}^{\mathbb{C}}]\!] = 0\,.$$

The adjoint operator $\mathrm{Ad}_{X_{H^{(2)}}^{\mathbb{C}}} := [\,\cdot\,, X_{H^{(2)}}^{\mathbb{C}}]$ acting on quadratic monomial vector fields $u_{j_1}^{\sigma_1} u_{j_2}^{\sigma_2} \partial_{u_j^\sigma}$ satisfying the momentum conservation property $\sigma j = \sigma_1 j_1 + \sigma_2 j_2$, is injective and surjective. Indeed we have that $[\![u_{j_1}^{\sigma_1}u_{j_2}^{\sigma_2}\partial_{u_j^\sigma}, X^{\mathbb{C}}_{H^{(2)}}]\!] = \mathrm{i}(\sigma\omega(j)-\sigma_1\omega(j_1)-\sigma_2\omega(j_2))u_{j_1}^{\sigma_1}u_{j_2}^{\sigma_2}\partial_{u_j^\sigma}$ and the system (7.14) has no solutions. As a consequence

$$\mathtt{S}_2^{\mathbb{C}} + \mathtt{T}_2 = X_{F^{(3)}_{\mathbb{C}}} . \tag{7.44}$$

Cubic vector fields. The vector field $\mathcal{X}_{Res}$ defined in (7.27) is in Poincaré-Birkhoff normal form, since the symbol $\mathtt{D}(Y;\xi)$ is *integrable* (Definition 5.1) and $\tilde{\mathtt{R}}^{res}(U)$ is *Birkhoff resonant* (Definition 6.1). Therefore, defining the linear operator $\Pi_{\ker}$ acting on a cubic monomial vector field $u_{j_1}^{\sigma_1}u_{j_2}^{\sigma_2}u_{j_3}^{\sigma_3}\partial_{u_j^\sigma}$ as

$$\Pi_{\ker}\Big(u_{j_1}^{\sigma_1}u_{j_2}^{\sigma_2}u_{j_3}^{\sigma_3}\partial_{u_j^\sigma}\Big) := \begin{cases} u_{j_1}^{\sigma_1}u_{j_2}^{\sigma_2}u_{j_3}^{\sigma_3}\partial_{u_j^\sigma} & \text{if } -\sigma\omega(j)+\sigma_1\omega(j_1)+\sigma_2\omega(j_2)+\sigma_3\omega(j_3)=0 \\ 0 & \text{otherwise}, \end{cases} \tag{7.45}$$

we have

$$\Pi_{\ker}(\mathcal{X}_{Res}) = \mathcal{X}_{Res} . \tag{7.46}$$

In addition, since $[\![u_{j_1}^{\sigma_1}u_{j_2}^{\sigma_2}u_{j_3}^{\sigma_3}\partial_{u_j^\sigma}, X^{\mathbb{C}}_{H^{(2)}}]\!] = \mathrm{i}(\sigma\omega(j)-\sigma_1\omega(j_1)-\sigma_2\omega(j_2)-\sigma_3\omega(j_3))u_{j_1}^{\sigma_1}u_{j_2}^{\sigma_2}u_{j_3}^{\sigma_3}\partial_{u_j^\sigma}$ we deduce that, for any cubic vector field G_3,

$$\Pi_{\mathrm{Ker}}[\![G_3, X^{\mathbb{C}}_{H^{(2)}}]\!] = 0 . \tag{7.47}$$

We can then calculate

$$\begin{aligned}
\mathcal{X}_{Res} &\overset{(7.46)}{=} \Pi_{\ker}(\mathcal{X}_{Res}) \overset{(7.42),(7.47)}{=} \Pi_{\ker}\Big(X^{\mathbb{C}}_{H^{(4)}} + [\![\mathtt{S}_2^{\mathbb{C}}+\mathtt{T}_2, X_2^{\mathbb{C}}]\!] + \frac{1}{2}[\![\mathtt{S}_2^{\mathbb{C}}+\mathtt{T}_2, [\![\mathtt{S}_2^{\mathbb{C}}+\mathtt{T}_2, X_1^{\mathbb{C}}]\!]]\!]\Big) \\
&\overset{(7.44),(7.29)}{=} \Pi_{\ker}\Big(X^{\mathbb{C}}_{H^{(4)}} + [\![X_{F^{(3)}_{\mathbb{C}}}, X^{\mathbb{C}}_{H^{(3)}}]\!] + \frac{1}{2}[\![X_{F^{(3)}_{\mathbb{C}}}, [\![X_{F^{(3)}_{\mathbb{C}}}, X_{H^{(2)}_{\mathbb{C}}}]\!]]\!]\Big) \\
&\overset{(7.6),(7.7)}{=} \Pi_{\ker}\Big(X_{H^{(4)}_{\mathbb{C}} + \{F^{(3)}_{\mathbb{C}}, H^{(3)}_{\mathbb{C}}\} + \frac{1}{2}\{F^{(3)}_{\mathbb{C}}, \{F^{(3)}_{\mathbb{C}}, H^{(2)}_{\mathbb{C}}\}\}}\Big) \\
&\overset{(7.13)}{=} \Pi_{\ker}\big(X_{H^{(4)}_{\mathbb{C}} + \frac{1}{2}\{F^{(3)}_{\mathbb{C}}, H^{(3)}_{\mathbb{C}}\}}\big) \\
&\overset{(7.45),(7.17)}{=} X_{\Pi_{\ker}(H^{(4)}_{\mathbb{C}} + \frac{1}{2}\{F^{(3)}_{\mathbb{C}}, H^{(3)}_{\mathbb{C}}\})}
\end{aligned}$$

which is (7.28); the second identity follows by the definition of $H^{(4)}_{ZD}$ in (7.18).

7.3. Energy estimate and proof of Theorem 1.1.

We first prove the following lemma.

Lemma 7.4. *Let $K \in \mathbb{N}^*$. There is $s_0 > 0$ such that, for any $s \geq s_0$, for all $0 < r \leq r_0(s)$ small enough, if U belongs to $B_s^K(I;r)$ and solves (3.8), then there is a constant $C_{s,K} > 0$ such that*

$$\|\partial_t^k U(t,\cdot)\|_{\dot{H}^{s-k}} \leq C_{s,K}\|U(t,\cdot)\|_{\dot{H}^s} , \quad \forall\, 0 \leq k \leq K . \tag{7.48}$$

In particular the norm $\|U(t,\cdot)\|_{K,s}$ defined in (2.1) is equivalent to the norm $\|U(t,\cdot)\|_{\dot{H}^s}$.

Proof. For $k = 0$ the estimate (7.48) is trivial. We are going to estimate $\partial_t^k U$ by (3.8). Since the matrix of symbols $\mathrm{i}A_1(U;x)\xi + \mathrm{i}A_{1/2}(U;x)|\xi|^{\frac{1}{2}} + A_0(U;x,\xi) + A_{-1/2}(U;x,\xi)$ in (3.8) belongs to $\Sigma\Gamma^1_{K,1,0} \otimes \mathcal{M}_2(\mathbb{C})$ and the smoothing operator $R(U)$ is in $\Sigma\mathcal{R}^{-\rho}_{K,1,1} \otimes \mathcal{M}_2(\mathbb{C})$, applying Proposition 2.6-(*ii*) (with $K' = 1$, $k = 0$), the estimate (2.27) for $R(U)$ (with $K' = 1$, $k = 0$, $N = 1$), and recalling (2.1), we deduce, for $s \geq s_0$ large enough,

$$\begin{aligned}
\|\partial_t U(t,\cdot)\|_{\dot{H}^{s-1}} \lesssim_s\ & \|U(t,\cdot)\|_{\dot{H}^s}\big(1 + \|U(t,\cdot)\|_{\dot{H}^{s_0}} + \|\partial_t U(t,\cdot)\|_{\dot{H}^{s_0-1}}\big) \\
& + \|\partial_t U(t,\cdot)\|_{\dot{H}^{s-1}}\|U(t,\cdot)\|_{\dot{H}^{s_0}} .
\end{aligned} \tag{7.49}$$

Evaluating (7.49) at $s = s_0$ and since $\|U(t,\cdot)\|_{\dot{H}^{s_0}}$ is small, we get $\|\partial_t U(t,\cdot)\|_{\dot{H}^{s_0-1}} \lesssim_s \|U(t,\cdot)\|_{\dot{H}^{s_0}}$. The latter inequality and (7.49) imply (7.48) for $k = 1$, for any $s \geq s_0$. Differentiating in t the system (3.8) and arguing by induction on k, one proves similarly (7.48) for any $k \geq 2$. □

We now prove the following energy estimate.

Lemma 7.5. (Energy estimate) *Under the same assumptions of Proposition 6.2 the vector field* $\mathcal{X}_{\geq 4}(U,Y) = \Big[\begin{smallmatrix}\mathcal{X}^+_{\geq 4}(U,Y) \\ \overline{\mathcal{X}^+_{\geq 4}(U,Y)}\end{smallmatrix}\Big]$ *in (6.11) satisfies, for any $t \in I$, the energy estimate*

$$\operatorname{Re}\int_{\mathbb{T}} |D|^s \mathcal{X}^+_{\geq 4}(U,Y) \cdot \overline{|D|^s y}\, dx \lesssim_s \|y\|^5_{\dot H^s}\,. \tag{7.50}$$

Proof. By (6.11) and (5.3), we have that $\mathcal{X}^+_{\geq 4}(U,Y) = Op^{\rm BW}(H_{\geq 3})[y] + \mathfrak{R}^+_{\geq 3}(U)[Y]$ where $H_{\geq 3}$ is an admissible symbol as in (5.2) that we write

$$H_{\geq 3} = h^+_{\geq 3}(U;x,\xi) + \gamma_{\geq 3}(U;x,\xi)\,, \quad h^+_{\geq 3}(U;x,\xi) := {\rm i}\alpha_{\geq 3}(U;x)\xi + {\rm i}\beta_{\geq 3}(U;x)|\xi|^{\frac12}\,, \tag{7.51}$$

and $\mathfrak{R}^+_{\geq 3}(U)$ denotes the first row of $\mathfrak{R}_{\geq 3}$. Then the left hand side of (7.50) is equal to

$$\frac12\big(|D|^s y, |D|^s Op^{\rm BW}(h^+_{\geq 3})[y]\big)_{L^2} + \frac12\big(|D|^s Op^{\rm BW}(h^+_{\geq 3})[y], |D|^s y\big)_{L^2} \tag{7.52}$$

$$+ \operatorname{Re}\int_{\mathbb{T}} |D|^s Op^{\rm BW}(\gamma_{\geq 3})[y]\cdot \overline{|D|^s y}\, dx + \operatorname{Re}\int_{\mathbb{T}} |D|^s \mathfrak{R}^+_{\geq 3}(U)[Y]\cdot\overline{|D|^s y}\, dx\,. \tag{7.53}$$

Since $\gamma_{\geq 3} \in \Gamma^0_{K,K',3}$ and $\mathfrak{R}^+_{\geq 3}(U)$ is a 1×2 matrix of smoothing operators in $\dot{\mathcal{R}}^0_{K,K',3}$, Cauchy-Schwarz inequality, Proposition 2.6 and (2.27) imply that

$$|(7.53)| \lesssim_s \|y(t,\cdot)\|^2_{\dot H^s}\|U(t,\cdot)\|^3_{K,s}\,. \tag{7.54}$$

We now prove that (7.52) satisfies the same bound. Since the symbol $h^+_{\geq 3}$ has positive order we decompose it according to

$$(7.52) = \big(|D|^s y, |D|^s(\mathcal{H}_{\geq 3} + \mathcal{H}^*_{\geq 3})y\big)_{L^2} + \big(|D|^s y, [\mathcal{H}^*_{\geq 3}, |D|^s]y\big)_{L^2} + \big([|D|^s, \mathcal{H}_{\geq 3}]y, |D|^s y\big)_{L^2} \tag{7.55}$$

where $\mathcal{H}_{\geq 3} := Op^{\rm BW}(h^+_{\geq 3}(U;x,\xi))$ and $\mathcal{H}^*_{\geq 3} = Op^{\rm BW}(\overline{h^+_{\geq 3}(U;x,\xi)})$ is its adjoint with respect to the L^2-scalar product. Recalling (7.51) and that the functions $\alpha_{\geq 3}(U;x)$, $\beta_{\geq 3}(U;x)$ are real we have

$$\mathcal{H}_{\geq 3} + \mathcal{H}^*_{\geq 3} = Op^{\rm BW}\big(h^+_{\geq 3} + \overline{h^+_{\geq 3}}\big) = 0 \tag{7.56}$$

Furthermore, by Proposition 2.9 and the remark below it, the commutators $[\mathcal{H}^*_{\geq 3}, |D|^s]$, $[|D|^s, \mathcal{H}_{\geq 3}]$ are paradifferential operators with symbol in $\Gamma^s_{K,K',3}$, up to a bounded operator in $\mathcal{L}(\dot H^s, \dot H^0)$ with operator norm bounded by $\|U\|^3_{K,s_0}$. Then applying Proposition 2.6 we get

$$\big|\big(|D|^s y, [\mathcal{H}^*_{\geq 3}, |D|^s]y\big)_{L^2}\big| + \big|\big([|D|^s, \mathcal{H}_{\geq 3}]y, |D|^s y\big)_{L^2}\big| \lesssim_s \|y(t,\cdot)\|^2_{\dot H^s}\|U(t,\cdot)\|^3_{K,s}\,. \tag{7.57}$$

In conclusion, by (7.54), (7.55), (7.56), (7.57), and using Lemma 7.4 we deduce

$$\operatorname{Re}\int_{\mathbb{T}} |D|^s \mathcal{X}^+_{\geq 4}(U,Y)\cdot\overline{|D|^s y}\, dx \lesssim_s \|y(t,\cdot)\|^2_{\dot H^s}\|U(t,\cdot)\|^3_{\dot H^s} \lesssim_s \|y(t,\cdot)\|^5_{\dot H^s}$$

by (6.13), proving the energy estimate (7.50). □

We can now prove Theorem 1.1.

Proof of Theorem 1.1. By (1.19), the function $U = \big[\begin{smallmatrix} u \\ \bar u\end{smallmatrix}\big]$, where u is the variable defined in (1.16) and ω in (1.12), belongs to the ball $B^K_N(I;r)$ (recall (2.2)) with $r = \bar\varepsilon \ll 1$ and $I = [-T,T]$. By Proposition 3.3 the function U solves system (3.8). Then we apply Poincaré-Birkhoff Proposition 6.2 with $s \rightsquigarrow N \gg K \geq 2\rho + 2 \geq 2\rho_0 + 2$. The map $\mathbf{F}^1_T(U) = \mathfrak{C}^1(U)[U]$ in (6.9) transforms the water waves system (3.8) into (6.10), which, thanks to Proposition 7.2, is expressed in terms of the Zakharov-Dyachenko Hamiltonian H_{ZD} in (1.22), as

$$\partial_t Y = X_{H_{ZD}}(Y) + \mathcal{X}_{\geq 4}(U,Y)\,.$$

Renaming $y \rightsquigarrow z$ and recalling (7.4), the first component of the above system is the equation (1.21), and denoting $\mathfrak{B}(u)u$ the first component of $\mathfrak{C}^1(U)[U]$. The bound (1.20) follows by (6.12) with $s \rightsquigarrow N$ and $k = 0$, and Lemma 7.4. The energy estimate (1.24) is proved in Lemma 7.5. □

7.4. **Proof of Theorem 1.2.** The next bootstrap Proposition 7.6 is the main ingredient for the proof of the long time existence Theorem 1.2. Proposition 7.6 is a consequence of Theorem 1.1 and the integrability of the fourth order Hamiltonian $H^{(4)}_{ZD}$ in (1.23).

By time reversibility we may, without loss of generality, only look at positive times $t > 0$.

Proposition 7.6. (Main bootstrap) *Fix the constants* $\bar{\varepsilon}, K, N$ *as in Theorem 1.1 and let the function* $u \in C^0([0,T]; H^N)$ *be defined as in* (1.16)*, with* ω *in* (1.12) *and* (η, ψ) *solution of* (1.3) *satisfying* (1.13)*,* (1.14)*. The function* u *satisfies* (1.18)*. Then there exists* $c_0 > 0$ *such that, for any* $0 < \varepsilon_1 \le \bar{\varepsilon}$*, if*

$$\|u(0)\|_{H^N} \le c_0\varepsilon_1 , \qquad \sup_{t\in[0,T]} \sum_{k=0}^{K} \|\partial_t^k u(t)\|_{H^{N-k}} \le \varepsilon_1 , \qquad T \le c_0 \varepsilon_1^{-3} , \tag{7.58}$$

then we have the improved bound

$$\sup_{t\in[0,T]} \sum_{k=0}^{K} \|\partial_t^k u(t)\|_{H^{N-k}} \le \frac{\varepsilon_1}{2} . \tag{7.59}$$

Proof. In view of (7.58) the smallness condition (1.19) holds and we can apply Theorem 1.1 obtaining the new variable $z = \mathfrak{B}(u)u$ satisfying the equation (1.21)-(1.24). The integrability of $H^{(4)}_{ZD}$ in Theorem 1.4 (see the Hamiltonian system written in (7.21)-(7.22)) gives

$$\mathrm{Re} \int_{\mathbb{T}} |D|^N \big(\mathrm{i}\partial_{\bar{z}} H^{(4)}_{ZD}\big) \cdot \overline{|D|^N z}\, dx = 0 .$$

From this, (1.21) and (1.24) we obtain the energy estimate

$$\frac{d}{dt} \|z(t)\|^2_{\dot{H}^N} \lesssim_N \|z(t)\|^5_{\dot{H}^N} . \tag{7.60}$$

Using (1.20) and (7.58) we deduce that, for all $0 \le t \le T$,

$$\|u(t)\|^2_{\dot{H}^N} \lesssim_N \|z(t)\|^2_{\dot{H}^N} \overset{(7.60)}{\lesssim_N} \|z(0)\|^2_{\dot{H}^N} + \int_0^t \|z(\tau)\|^5_{\dot{H}^N}\, d\tau \le C\|u(0)\|^2_{\dot{H}^N} + C\int_0^t \|u(\tau)\|^5_{\dot{H}^N}\, d\tau$$

for some $C = C(N) > 0$. Then, by the a priori assumption (7.58) we get, for all $0 \le t \le T \le c_0\varepsilon_1^{-3}$,

$$\|u(t)\|^2_{\dot{H}^N} \le C c_0^2 \varepsilon_1^2 + C\, T\, \varepsilon_1^5 \le \varepsilon_1^2 (C c_0^2 + C c_0) . \tag{7.61}$$

The desired conclusion (7.59) on the norms $C_t^k H_x^{N-k}$ follows by Lemma 7.4, (7.61), and recalling that $\int_{\mathbb{T}} u(t,x) dx = 0$, choosing c_0 small enough depending on N. □

We now prove the long-time existence Theorem 1.2, by Theorem 1.1 and Proposition 7.6.

Step 1: Local existence. Let $s > 3/2$. By the assumption (1.25), Theorem 1.3 guarantees the existence of a time $T_{\mathrm{loc}} > 0$ and a unique classical solution $(\eta, \psi) \in C^0([0, T_{\mathrm{loc}}]; H^{s+\frac12} \times H^{s+\frac12})$ of (1.3), with initial data as in (1.25), such that

$$\sup_{t\in[0,T_{\mathrm{loc}}]} \|(\eta, \psi, V, B)(t)\|_{X^s} \le C\varepsilon , \qquad \int_{\mathbb{T}} \eta(t,x)\, dx = 0 . \tag{7.62}$$

Step 2: Preliminary estimates in high norms. We now show that, for any $K>0$, if $s\geq K+\sigma_0$, for some σ_0 large enough, and ε is small enough, then the time derivatives $(\partial_t^k\eta,\partial_t^k\psi)$, $k=0,\dots,K$, satisfy, for all $t\in[0,T_{\rm loc}]$,

$$\|\partial_t^k\eta\|_{H^{s+\frac12-k}}+\|\partial_t^k\psi\|_{H^{s+\frac12-k}}\lesssim_s\|\eta\|_{H^{s+\frac12}}+\|\psi\|_{H^{s+\frac12}}\lesssim_s\varepsilon\,. \tag{7.63}$$

One argues by induction on k. For $k=0$ the second estimate in (7.63) is (7.62). Assume that (7.63) holds for any $0\leq j\leq k-1\leq K-1$, $k\geq1$. By differentiating in t the water waves system (1.3) we get

$$\partial_t^k\eta=\partial_t^{k-1}\big(G(\eta)\psi\big)\,,\quad \partial_t^k\psi=\partial_t^{k-1}\big(\mathcal{F}(\eta,\eta_x,\psi_x,G(\eta)\psi)\big)\,,\quad k=1,\dots,K\,, \tag{7.64}$$

where $\mathcal{F}$ is an analytic function vanishing at the origin. Then, using that $G(\eta)\psi$ is expressed from the right hand side of (3.2), Proposition 2.6, (2.27) and the inductive hypothesis, we get

$$\|\partial_t^{k-1}(G(\eta)\psi)\|_{H^{s+\frac12-k}}\lesssim_s\sum_{k'\leq k-1}\|\partial_t^{k'}\psi\|_{H^{s+\frac12-k+1}}+\|\partial_t^{k'}\eta\|_{H^{s+\frac12-k+1}}\lesssim_s\|\eta\|_{H^{s+\frac12}}+\|\psi\|_{H^{s+\frac12}}\,.$$

This implies, in view of the first equation in (7.64), that $\|\partial_t^k\eta\|_{H^{s+\frac12-k}}$ is bounded as in (7.63). To estimate $\|\partial_t^k\psi\|_{H^{s+\frac12-k}}$, we use the second equation in (7.64), the inductive estimates for $(\partial_t^j\eta,\partial_t^j\psi)$, $0\leq j\leq k-1$, the previous bound on $\|\partial_t^{k-1}(G(\eta)\psi)\|_{H^{s+\frac12-k}}$ and the fact that for $s\geq K+\sigma_0$ the space $H^{s+\frac12-K}$ is an algebra.

Step 3: A priori estimate for the basic diagonal complex variable. We now look at the complex variable

$$u=\frac{1}{\sqrt2}|D|^{-\frac14}\eta+\frac{\mathrm{i}}{\sqrt2}|D|^{\frac14}\omega \tag{7.65}$$

defined in (1.16) where $\omega=\psi-Op^{\rm BW}(B(\eta,\psi))\eta$ is the good unknown defined in (1.12). Since the function B is in $\Sigma\mathcal{F}^{\mathbb{R}}_{K,0,1}$ (Proposition 3.1), we deduce, applying Proposition 2.6 for $s\geq s_0$ large enough, that ω is in $C^0([0,T_{\rm loc}];\dot H^{s+\frac12})$, and so

$$u\in C^0([0,T_{\rm loc}];H^N)\,,\qquad N:=s+\frac14\,. \tag{7.66}$$

Moreover, using (1.25), (7.62)-(7.63), we estimate $\|\partial_t^k u\|_{H^{N-k}}$, $k=0,\dots,K$ for $N\gg K$, by

$$\begin{aligned}\|\partial_t^k u\|_{H^{N-k}}&\overset{(7.65)}{\lesssim_N}\|\partial_t^k\eta\|_{H^{N-\frac14-k}}+\|\partial_t^k\omega\|_{H^{N+\frac14-k}}\\&\overset{(1.12),\text{Prop.2.6}}{\lesssim_N}\|\partial_t^k\eta\|_{H^{N+\frac14-k}}+\|\partial_t^k\psi\|_{H^{N+\frac14-k}}\overset{(7.63)}{\lesssim_N}\varepsilon\,,\end{aligned}$$

for any $t\in[0,T_{\rm loc}]$. In conclusion, there is $C_1=C_1(N)>0$ such that

$$\|u(0)\|_{H^N}\leq2\varepsilon\,,\qquad\sup_{t\in[0,T_{\rm loc}]}\sum_{k=0}^K\|\partial_t^k u(t)\|_{H^{N-k}}\leq C_1\varepsilon\,,\qquad\int_{\mathbb{T}}u(t,x)\,dx=0\,. \tag{7.67}$$

Step 4: Bootstrap argument and continuation criterion. With $\bar\varepsilon,K,N$ given by Theorem 1.1, and c_0 by Proposition 7.6, we choose ε_0 in (1.25) small enough so that, for $0<\varepsilon\leq\varepsilon_0$ we have $2\varepsilon\leq c_0\bar\varepsilon$, $C_1\varepsilon\leq\bar\varepsilon$ where C_1 is the constant in (7.67). Moreover we take $s\geq s_0$ large enough in such a way that (7.66)-(7.67) hold with N given by Theorem 1.1. Hence the first two assumptions in (7.58) hold with $\varepsilon_1=\varepsilon\max\{2c_0^{-1},C_1\}$ on the time interval $[0,T_{\rm loc}]$. Then Proposition 7.6 and a standard bootstrap argument guarantee that $u(t)$ can be extended up to a time

$$T_\varepsilon:=c_0\varepsilon_1^{-3}\,,$$

consistently with the existence time (1.26) of the statement, and that

$$\sup_{[0,T_\varepsilon]} \|u(t)\|_{H^N} \le \varepsilon_1 \,, \qquad \int_{\mathbb{T}} u(t,x)\,dx = 0\,. \tag{7.68}$$

Finally, we prove that the solution of (1.3) satisfies (1.27) and that $(\eta,\psi,V,B)(t)$ takes values in X^s for all $t \in [0,T_\varepsilon]$. Expressing (η,ω) in terms of $u,\overline{u}$ as in (3.25), we deduce by (7.68) that

$$\sup_{[0,T_\varepsilon]} \left(\|\eta(t)\|_{H^s} + \|\omega(t)\|_{H^{s+\frac12}}\right) \lesssim_s \varepsilon\,. \tag{7.69}$$

Then we estimate

$$\sup_{[0,T_\varepsilon]} \|\psi(t)\|_{H^s} \lesssim_s \varepsilon \tag{7.70}$$

by (1.12), (7.69) and Proposition 2.6, and

$$\sup_{[0,T_\varepsilon]} \|(V,B)(t)\|_{H^{s-1}\times H^{s-1}} \lesssim_s \varepsilon\,, \tag{7.71}$$

using (3.2) for $G(\eta)\psi$. The estimates (7.69)-(7.71) imply (1.27) and, in particular, that

$$\sup_{[0,T_\varepsilon]} \|(\eta,\psi,V,B)(t)\|_{X^{s-1}} \lesssim_s \varepsilon$$

thus guaranteeing (1.28), for $s-1 \ge 5$, on the time interval $[0,T_\varepsilon]$. The continuation criterion in Theorem 1.3-(2) implies that the solution (η,ψ,V,B) is in $C^0([0,T],X^s)$ for $T \ge T_\varepsilon$. □

Appendix A. Flows and conjugations

In this Appendix we study the conjugation rules of a vector field under flow maps.

A.1. **Conjugation rules.** We first give this simple lemma that we use in sections 4 and 6.

Lemma A.1. *For $U = \left[\begin{smallmatrix} u \\ \overline{u}\end{smallmatrix}\right]$ consider a system $\partial_t U = X(U)U$ with $X(U)$ in $\Sigma\mathcal{M}_{K,K',0}\otimes\mathcal{M}_2(\mathbb{C})$ and let $\mathbf{\Phi}^\theta(U)$ be the flow of*

$$\partial_\theta \mathbf{\Phi}^\theta(U) = \mathrm{i}\mathbf{A}(U)\mathbf{\Phi}^\theta(U)\,, \quad \mathbf{\Phi}^0(U) = \mathrm{Id}\,, \tag{A.1}$$

where $\mathbf{A} := \mathbf{A}(U)$ is in $\Sigma\mathcal{R}^0_{K,K',1}\otimes\mathcal{M}_2(\mathbb{C})$. Under the change of variable $V := (\mathbf{\Phi}^\theta(U))_{\theta=1}U$, the new system becomes

$$\partial_t V = X^+(U)V\,, \qquad X^+(U) := (\partial_t\mathbf{\Phi}^1(U))(\mathbf{\Phi}^1(U))^{-1} + \mathbf{\Phi}^1(U)X(U)(\mathbf{\Phi}^1(U))^{-1}\,. \tag{A.2}$$

The operator $X^+(U)$ is in $\Sigma\mathcal{M}_{K,K'+1,0}\otimes\mathcal{M}_2(\mathbb{C})$ and, setting $\mathrm{A}_{\mathrm{i}\mathbf{A}}[X] := [\mathrm{i}\mathbf{A},X]$, it admits the Lie expansion

$$\mathbf{\Phi}^1(U)X(U)(\mathbf{\Phi}^1(U))^{-1} = X + \sum_{q=1}^{L}\frac{1}{q!}\mathrm{Ad}^q_{\mathrm{i}\mathbf{A}}[X] + \frac{1}{L!}\int_0^1 (1-\theta)^L \mathbf{\Phi}^\theta(U)\mathrm{Ad}^{L+1}_{\mathrm{i}\mathbf{A}}[X](\mathbf{\Phi}^\theta(U))^{-1}d\theta \tag{A.3}$$

$$\begin{aligned}(\partial_t\mathbf{\Phi}^1(U))(\mathbf{\Phi}^1(U))^{-1} &= \mathrm{i}\partial_t\mathbf{A} + \sum_{q=2}^{L}\frac{1}{q!}\mathrm{Ad}^{q-1}_{\mathrm{i}\mathbf{A}}[\mathrm{i}\partial_t\mathbf{A}] \\ &\quad + \frac{1}{L!}\int_0^1 (1-\theta)^L\mathbf{\Phi}^\theta(U)\mathrm{Ad}^L_{\mathrm{i}\mathbf{A}}[\mathrm{i}\partial_t\mathbf{A}](\mathbf{\Phi}^\theta(U))^{-1}d\theta\,.\end{aligned} \tag{A.4}$$

Proof. The expression (A.2) follows by an explicit computation. In order to prove (A.3) notice that the vector field $P(\theta) := \mathbf{\Phi}^\theta(U)X(U)(\mathbf{\Phi}^\theta(U))^{-1}$ satisfies the Heisenberg equation

$$\partial_\theta P(\theta) = [\mathrm{i}\mathbf{A}, P(\theta)] = \mathrm{i}\mathbf{A}P(\theta) - P(\theta)\mathrm{i}\mathbf{A}\,, \qquad P(0) = X(U)\,.$$

Since the vector field $\mathbf{A}$ is independent of θ we also have

$$\partial_\theta P(\theta) = \mathbf{\Phi}^\theta(U)\mathrm{Ad}_{\mathrm{i}\mathbf{A}}[X]\mathbf{\Phi}^\theta(U)^{-1}\,.$$

Then (A.3) follows by a Taylor expansion. To prove (A.4) we reason as follows. We have that

$$\mathbf{\Phi}^1(U)\circ\partial_t\circ(\mathbf{\Phi}^1(U))^{-1} = \partial_t + \mathbf{\Phi}^1(U)\big[\partial_t(\mathbf{\Phi}^1(U))^{-1}\big] = \partial_t - (\partial_t\mathbf{\Phi}^1(U))(\mathbf{\Phi}^1(U))^{-1}. \tag{A.5}$$

Moreover, using the expansion (A.3), we have

$$\mathbf{\Phi}^1(U)\circ\partial_t\circ(\mathbf{\Phi}^1(U))^{-1} = \partial_t - \sum_{q=1}^{L}\frac{1}{q!}\mathrm{Ad}_{\mathrm{i}\mathbf{A}}^{q-1}[\mathrm{i}\partial_t\mathbf{A}] - \frac{1}{L!}\int_0^1(1-\theta)^L\mathbf{\Phi}^\theta(U)\mathrm{Ad}_{\mathrm{i}\mathbf{A}}^{L}[\mathrm{i}\partial_t\mathbf{A}](\mathbf{\Phi}^\theta(U))^{-1}d\theta$$

which, together with (A.5), implies (A.4).

By Taylor expanding $\mathbf{\Phi}^1(U)$ using (A.1), we derive that $\mathbf{\Phi}^1(U) - \mathrm{Id}$ is in $\Sigma\mathcal{M}_{K,K',1}\otimes\mathcal{M}_2(\mathbb{C})$. The translation invariance property (2.26) of the homogeneous components of $\mathbf{\Phi}^1(U)$ follows since the generator $\mathbf{A}(U)$ satisfies (2.26). Then, the operator $X^+(U)$ in (A.2) belongs to $\Sigma\mathcal{M}_{K,K'+1,0}\otimes\mathcal{M}_2(\mathbb{C})$ by Proposition 2.10. Let us justify the translation invariance property of the homogeneous components of $X^+(U)$. Denoting by $\mathbf{\Phi}^1_{\le 2}(U)$ the sum of its homogeneous components of degree less or equal to 2 we have that, for any $\vartheta\in\mathbb{R}$, $\tau_\vartheta\mathbf{\Phi}^1_{\le2}(U) = \mathbf{\Phi}^1_{\le2}(\tau_\vartheta U)\tau_\vartheta$, and so

$$\tau_\vartheta d_U\mathbf{\Phi}^1_{\le2}(U)[\widehat{H}] = d_U\mathbf{\Phi}^1_{\le2}(\tau_\vartheta U)[\tau_\vartheta\widehat{H}]\tau_\vartheta\,. \tag{A.6}$$

Then

$$\tau_\vartheta(\partial_t\mathbf{\Phi}^1_{\le2}(U)) = \tau_\vartheta\big(d_U\mathbf{\Phi}^1_{\le2}(U)[X(U)U]\big) \overset{(A.6)}{=} d_U\mathbf{\Phi}^1_{\le2}(\tau_\vartheta U)[\tau_\vartheta X(U)U]\tau_\vartheta = (\partial_t\mathbf{\Phi}^1_{\le2}(\tau_\vartheta U))\tau_\vartheta$$

using the translation invariance of $X(U)U$. By composition we deduce that the homogeneous components of $X^+(U)$ in (A.2) satisfy (2.26). □

In the next subsection we analyze how paradifferential operators change under the flow maps generated by paradifferential operators.

A.2. **Conjugation of paradifferential operators via flows.** We consider the flow equation

$$\partial_\theta\Phi^\theta = \mathrm{i}Op^{\mathrm{BW}}(f(\theta,U;x,\xi))\Phi^\theta, \quad \Phi^0 = \mathrm{Id}\,, \tag{A.7}$$

where f is a symbol assuming one of the following forms:

$$f(\theta,U;x,\xi) := b(\theta,U;x)\xi := \frac{\beta(U;x)}{1+\theta\beta_x(U;x)}\xi\,, \quad \beta(U;x)\in\Sigma\mathcal{F}^{\mathbb{R}}_{K,K',1}\,, \tag{A.8}$$

$$f(\theta,U;x,\xi) := f(U;x,\xi) := \beta(U;x)|\xi|^{\frac12}, \qquad \beta(U;x)\in\Sigma\mathcal{F}^{\mathbb{R}}_{K,K',1}\,, \tag{A.9}$$

$$f(\theta,U;x,\xi) := f(U;x,\xi)\in\Sigma\Gamma^m_{K,K',1}\,, \qquad m\le 0\,. \tag{A.10}$$

Notice that (A.7) with f as in (A.8) is a para-differential transport equation. This is used in Section 5.1 and Subsection 5.2.2. Flows with f as in (A.9) are used in Section 5.2 and with f as in (A.10) in Subsection 5.2.3 and Section 5.3.

Lemma A.2. (Linear flows generated by a para-differential operator) *Assume that f has the form* (A.8) *or* (A.9) *or* (A.10). *Then, there is $s_0>0, r>0$ such that, for any $U\in C^K_{*\mathbb{R}}(I;\dot{H}^s)\cap B^K_{s_0}(I;r)$, for any $s>0$, the equation* (A.7) *has a unique solution $\Phi^\theta(U)$ satisfying:*

(i) the linear map $\Phi^\theta(U)$ *is invertible and, for some* $C_s>0$,

$$\|\partial_t^k\Phi^\theta(U)[v]\|_{\dot H^{s-k}}+\|\partial_t^k(\Phi^\theta(U))^{-1}[v]\|_{\dot H^{s-k}}\le\|v\|_{k,s}\big(1+C_s\|U\|_{K,s_0}\big)\,,\ \forall 0\le k\le K-K'\,,\tag{A.11}$$

$$C_s^{-1}\|v\|_{\dot H^s}\le\|\Phi^\theta(U)[v]\|_{\dot H^s}\le C_s\|v\|_{\dot H^s}\,,\tag{A.12}$$

for any $v\in C_*^{K-K'}(I;\dot H^s)$ *and uniformly in* $\theta\in[0,1]$*;*
(ii) the map $\Phi^\theta(U)$ *admits an expansion in multilinear maps as* $\Phi^\theta(U)-\mathrm{Id}\in\Sigma\mathcal M_{K,K',1}$, $\theta\in[0,1]$. *More precisely there are* $M_1(U)$ *in* $\widetilde{\mathcal M}_1$, *and* $M_2^{(1)}(U),M_2^{(2)}(U)$ *in* $\widetilde{\mathcal M}_2$ *(independent of* θ*) such that*

$$\Phi^\theta(U)[U]=U+\theta\big(M_1(U)[U]+M_2^{(1)}(U)[U]\big)+\theta^2M_2^{(2)}(U)[U]+M_{\ge3}(\theta;U)[U]\tag{A.13}$$

where $M_{\ge3}(\theta;U)$ *is in* $\mathcal M^m_{K,K',3}$ *with estimates uniform in* $\theta\in[0,1]$.

The same result holds for a matrix valued system $\partial_\theta\mathbf\Phi^\theta(U)=\mathbf B(U)\mathbf\Phi^\theta(U)$, $\mathbf\Phi^0(U)=\mathrm{Id}$, *where* $\mathbf B(U)=Op^{\mathrm{BW}}(B(U;x,\xi))$ *and* $B(U;x,\xi)$ *is a matrix of symbols in* $\Sigma\Gamma^0_{K,K',1}\otimes\mathcal M_2(\mathbb C)$.

Proof. See Lemma 3.22 in [13]. The translation invariance property (2.26) of the flow map $\Phi^\theta(U)$ defined by (A.7) follows by the fact that the homogeneous components of the symbol $f(\theta,U;x,\xi)$ satisfy (2.11). □

The proof of the next lemma follows by standard theory of Banach space ODEs.

Lemma A.3. (Linear flows generated by a smoothing operator) *Assume that* $\mathbf A(U)$ *in* (A.1) *is a smoothing operator in* $\Sigma\mathcal R^{-\rho}_{K,0,1}\otimes\mathcal M_2(\mathbb C)$ *for some* $\rho\ge0$. *Then, there is* $s_0>0,r>0$ *such that, for any* $U\in B^K_s(I;r)$, *for any* $s>s_0$, *the equation* (A.1) *has a unique solution* $\mathbf\Phi^\theta(U)$ *satisfying, for some* $C_s>0$,

$$\|\partial_t^k(\mathbf\Phi^\theta(U))^{\pm1}[v]\|_{\dot H^{s+\rho-k}}\le\|v\|_{k,s}\big(1+C_s\|U\|_{K,s_0}\big)+C_s\|v\|_{k,s_0}\|U\|_{K,s}\,,\tag{A.14}$$

for any $v\in C_*^{K-K'}(I;\dot H^s)$, $0\le k\le K-K'$, *and uniformly in* $\theta\in[0,1]$. *Moreover* $\mathbf\Phi^\theta(U)$ *satisfies a bound like* (A.12) *and* (ii) *of Lemma A.2.*

We now provide the conjugation rules of a paradifferential operator under the flow $\Phi^\theta(U)$ in (A.7). We first give the result in the case when f has the form (A.8), i.e. (A.7) is a transport equation.

Lemma A.4. (Conjugation of a paradifferential operator under transport flow) *Let* $\Phi^\theta(U)$ *be the flow of* (A.7) *given by Lemma A.2 with* $f(\theta,U;x,\xi)$ *as in* (A.8) *and* $U\in C^K_{*\mathbb R}(I;\dot H^{s_0})\cap B^K_{s_0}(I;r)$. *Consider the diffeomorphism of* $\mathbb T$ *given by*

$$\psi_U:x\mapsto x+\beta(U;x)\,.$$

Let $a(U;x,\xi)$ *be a symbol in* $\Sigma\Gamma^m_{K,K',q}$ *for some* $q\in\mathbb N$, $q\le2$, $K'\le K$, $r>0$ *and* $m\in\mathbb R$. *If* s_0 *is large enough and* r *small enough then there is a symbol* $a_\Phi(U;x,\xi)$ *in* $\Sigma\Gamma^m_{K,K',q}$ *such that*

$$\Phi^1(U)Op^{\mathrm{BW}}(a(U;x,\xi))(\Phi^1(U))^{-1}=Op^{\mathrm{BW}}(a_\Phi(U;x,\xi))+R(U)\tag{A.15}$$

where $R(U)$ *is a smoothing remainder in* $\Sigma\mathcal R^{-\rho+m}_{K,K',q+1}$. *Moreover* a_Φ *admits an expansion as*

$$a_\Phi(U;x,\xi)=a_\Phi^{(0)}(U;x,\xi)+a_\Phi^{(1)}(U;x,\xi)\tag{A.16}$$

where

$$a_\Phi^{(0)}(U;x,\xi)=a\big(U;\psi_U(t,x),\xi\partial_y(\psi_U^{-1}(t,y))_{|y=\psi_U(t,x)}\big)\in\Sigma\Gamma^m_{K,K',q}\tag{A.17}$$

and $a_\Phi^{(1)}(U;x,\xi)$ *is a symbol in* $\Sigma\Gamma^{m-2}_{K,K',q+1}$. *In addition, if* $a(U;x,\xi)=g(U;x)\xi$ *then* $a_\Phi^{(1)}=0$.

Furthermore, the symbol $a_\Phi^{(0)}$ *in* (A.17) *admits an expansion in degrees of homogeneity as*

$$a_\Phi^{(0)}=a+\{\beta\xi,a\}+\frac12\Big(\{\beta\xi,\{\beta\xi,a\}\}+\{-\beta\beta_x\xi,a\}\Big)\tag{A.18}$$

up to a symbol in $\Gamma^m_{K,K',3}$.

Proof. Formulas (A.15)-(A.17) are proved in Theorem 3.27 of [13] (with homogeneity degree $N=3$), where it is shown that the symbol $a^{(0)}_\Phi(U;x,\xi) = a_0(\theta,U;x,\xi)_{|\theta=1}$ and $a_0(\theta)$ solves the transport equation

$$\frac{d}{d\theta} a_0(\theta) = \{b(\theta,U;x)\xi, a_0(\theta)\}\,, \quad a_0(0) = a\,. \tag{A.19}$$

The claim that, if $a(U;x,\xi) = g(U;x)\xi$ then $a^{(1)}_\Phi = 0$ follows because in formula (3.5.37) of [13], the symbol $r_{-\rho,3} = 0$. Finally we deduce (A.18) by a Taylor expansion in θ using (A.19) (note that b and β have degree of homogeneity 1 in u). Since the homogeneous components of $\beta(U;x)$ satisfy the invariance condition (2.11), the flow $\Phi^1(U)$ satisfies (2.26), and so the left hand side in (A.15). The proof shows that the symbol a_Φ in (A.16) satisfies the invariance condition (2.11) and therefore the remainder $R(U)$ in (A.15) satisfies (2.26) by difference. □

Lemma A.5. (Conjugation of ∂_t under transport flow) *Let* $\Phi^\theta(U)$ *be the flow of* (A.7) *given by Lemma A.2 with* $f(\theta,U;x,\xi)$ *as in* (A.8). *Then*

$$\big(\partial_t \Phi^1(U)\big)\big(\Phi^1(U)\big)^{-1} = \mathrm{i}Op^{\mathrm{BW}}(g(U;x)\xi) + R(U) \tag{A.20}$$

where $g(U;x)$ *is a function in* $\Sigma\mathcal{F}^{\mathbb{R}}_{K,K'+1,1}$ *and* $R(U)$ *is a smoothing operator in* $\Sigma\mathcal{R}^{-\rho}_{K,K'+1,1}$. *In addition, the function* $g(U;x)$ *admits the expansion in degrees of homogeneity*

$$g(U;x) = \beta_t - \beta_x\beta_t + g_{\geq 3}(U;x)\,, \quad g_{\geq 3}(U;x) \in \mathcal{F}^{\mathbb{R}}_{K,K'+1,3}\,. \tag{A.21}$$

Proof. By the proof of Proposition 3.28 of [13] the operator $P(\theta) := (\partial_t\Phi^\theta(U))(\Phi^\theta(U))^{-1}$ solves

$$\frac{d}{d\theta}P(\theta) = \big[\mathrm{i}Op^{\mathrm{BW}}(b(\theta,U;x)\xi), P(\theta)\big] + \mathrm{i}Op^{\mathrm{BW}}(\partial_t b(\theta,U;x)\xi)\,, \quad P(0) = 0\,. \tag{A.22}$$

The solution of (A.22) is, up to smoothing remainders, given by $P(\theta) = Op^{\mathrm{BW}}(p_0(\theta,x,\xi))$, where the symbol $p_0(\theta,x,\xi)$ solves the forced transport equation

$$\frac{d}{d\theta}p_0(\theta,x,\xi) = \{b(\theta,U;x)\xi, p_0(\theta,x,\xi)\} + \mathrm{i}\partial_t b(\theta,U;x)\xi\,, \quad p_0(0) = 0\,. \tag{A.23}$$

The solution of (A.23) is

$$p_0(\theta,x,\xi) = \mathrm{i}\int_0^\theta \partial_t f(s,U;\phi^{\theta,s}(x,\xi))\,ds \quad \text{where} \quad f(s,U;x,\xi) := b(s,U;x)\xi$$

and $\phi^{\theta,s}(x,\xi)$ is the solution of the characteristic Hamiltonian system

$$\begin{cases} \frac{d}{ds}x(s) = -b(s,x(s)) \\ \frac{d}{ds}\xi(s) = b_x(s,x(s))\xi(s)\,. \end{cases}$$

with initial condition $\phi^{\theta,\theta} = \mathrm{Id}$. Moreover, by Lemma 3.23 in [13], $f(s,U;\phi^{\theta,s}(x,\xi))$ is in $\Sigma\Gamma^1_{K,K',1}$ with estimates uniform in $|\theta|,|s| \leq 1$. Then (A.20) follows with $\mathrm{i}g(U;x)\xi := p_0(1,x,\xi)$. Finally we deduce (A.21) by a Taylor expansion in θ of the symbol $p_0(\theta)$, using (A.23). The function $\beta_t - \beta_x\beta_t$ satisfies the translation invariance property (2.11) as β. As in Lemma A.1 the operator $(\partial_t\Phi^1(U))(\Phi^1(U))^{-1}$ in (A.20) is translation invariant and $R(U)$ satisfies the property (2.26) by difference. □

We now provide the conjugation of a paradifferential operator under the flow $\Phi^\theta(U)$ in (A.7), if f has the form (A.9) or (A.10).

Lemma A.6. (Conjugation of a paradifferential operator) *Let $\Phi^\theta(U)$ be the flow of* (A.7) *given by Lemma A.2 with symbol $f(U;x,\xi)$ in $\Sigma\Gamma^m_{K,K',1}$ with $m\le 1/2$, of the form* (A.9) *or* (A.10). *Let $a(U;x,\xi)$ be a symbol in $\Sigma\Gamma^{m'}_{K,K',q}$ for some $q\in\mathbb{N}$, $q\le 2$, $K'\le K$, $r>0$ and $m'\in\mathbb{R}$. Then*

$$\begin{aligned}&\Phi^1(U)Op^{\mathrm{BW}}(a(U;x,\xi))(\Phi^1(U))^{-1} =\\ &Op^{\mathrm{BW}}\Big(a+\{f,a\}+\frac12\{f,\{f,a\}\}+r_1+r_2+r_3\Big)+R(U)\end{aligned}\tag{A.24}$$

where $r_1\in\Sigma\Gamma^{m+m'-3}_{K,K',q+1}$, $r_2\in\Sigma\Gamma^{2m+m'-4}_{K,K',q+2}$, $r_3\in\Gamma^{3m+m'-3}_{K,K',3}$ and $R(U)\in\Sigma\mathcal{R}^{-\rho}_{K,K',q+1}$.

Proof. The result follows by a Lie expansion. Using (A.3) we have, for $L\ge 3$,

$$\begin{aligned}&\Phi^1(U)Op^{\mathrm{BW}}(a)(\Phi^1(U))^{-1} = Op^{\mathrm{BW}}(a)+\big[Op^{\mathrm{BW}}(\mathrm{i}f),Op^{\mathrm{BW}}(a)\big]+\frac12\mathrm{Ad}^2_{Op^{\mathrm{BW}}(\mathrm{i}f)}[Op^{\mathrm{BW}}(a)]+\\ &\sum_{k=3}^{L}\frac{1}{k!}\mathrm{Ad}^k_{Op^{\mathrm{BW}}(\mathrm{i}f)}[Op^{\mathrm{BW}}(a)]+\frac{1}{L!}\int_0^1(1-\theta)^L\Phi^\theta(U)\big(\mathrm{Ad}^{L+1}_{Op^{\mathrm{BW}}(\mathrm{i}f)}[Op^{\mathrm{BW}}(a)]\big)(\Phi^\theta(U))^{-1}d\theta\,.\end{aligned}$$

By applying Propositions 2.9, 2.10 replacing the smoothing index ρ by some $\tilde\rho$ to be chosen below large enough, we get

$$\mathrm{Ad}_{Op^{\mathrm{BW}}(\mathrm{i}f)}[Op^{\mathrm{BW}}(a)]=\big[Op^{\mathrm{BW}}(\mathrm{i}f),Op^{\mathrm{BW}}(a)\big]=Op^{\mathrm{BW}}\big(\{f,a\}+r_1\big)\,,\quad r_1\in\Sigma\Gamma^{m+m'-3}_{K,K',q+1}\,,$$

up to a smoothing operator in $\Sigma\mathcal{R}^{-\tilde\rho+m+m'}_{K,K',q+1}$. Moreover

$$\mathrm{Ad}^2_{Op^{\mathrm{BW}}(\mathrm{i}f)}[Op^{\mathrm{BW}}(a)]=Op^{\mathrm{BW}}(\{f,\{f,a\}\}+r_2)\,,\quad r_2\in\Sigma\Gamma^{2m+m'-4}_{K,K',q+2}\,,$$

up to a smoothing operator in $\Sigma\mathcal{R}^{-\tilde\rho+2m+m'}_{K,K',q+2}$. By induction, for $k\ge 3$ we have

$$\mathrm{Ad}^k_{Op^{\mathrm{BW}}(\mathrm{i}f)}[Op^{\mathrm{BW}}(a)]=Op^{\mathrm{BW}}(b_k),\quad b_k\in\Sigma\Gamma^{k(m-1)+m'}_{K,K',q+k}\,,$$

up to a smoothing operator in $\Sigma\mathcal{R}^{-\tilde\rho+m'+km}_{K,K',q+k}$. We choose L in such a way that $(L+1)(1-m)-m'\ge\rho$ and $L+1\ge 3$, so that the operator $Op^{\mathrm{BW}}(b_{L+1})$ belongs to $\mathcal{R}^{-\rho}_{K,K',3}$. The integral Taylor remainder in (A.2) belongs to $\mathcal{R}^{-\rho}_{K,K',3}$ as well, see Lemma 5.6 in [13]. Then we choose $\tilde\rho$ large enough so that $\tilde\rho-m'-(L+1)m\ge\rho$ and the remainders are ρ-smoothing. □

Lemma A.7. (Conjugation of ∂_t) *Let $\Phi^\theta(U)$ be the flow of* (A.7) *with symbol $f(U;x,\xi)$ in $\Sigma\Gamma^m_{K,K',1}$ with $m\le 1/2$, of the form* (A.9) *or* (A.10). *Then*

$$\big(\partial_t\Phi^1(U)\big)\big(\Phi^1(U)\big)^{-1}=\mathrm{i}Op^{\mathrm{BW}}\big(\partial_t f+\frac12\{f,\partial_t f\}\big)+Op^{\mathrm{BW}}(r_1+r_2)+R(U)\tag{A.25}$$

where $r_1\in\Sigma\Gamma^{2m-3}_{K,K'+1,2}$, $r_2\in\Gamma^{3m-2}_{K,K'+1,3}$ and $R(U)\in\Sigma\mathcal{R}^{-\rho}_{K,K'+1,2}$.

Proof. By the Lie expansion (A.4) we have

$$\begin{aligned}\big(\partial_t\Phi^1(U)\big)\big(\Phi^1(U)\big)^{-1}&=Op^{\mathrm{BW}}(\mathrm{i}\partial_t f)+\sum_{k=2}^{L}\frac{1}{k!}\mathrm{Ad}^{k-1}_{Op^{\mathrm{BW}}(\mathrm{i}f)}[Op^{\mathrm{BW}}(\mathrm{i}\partial_t f)]\\ &+\frac{1}{L!}\int_0^1(1-\theta)^L\Phi^\theta(U)\Big(\mathrm{Ad}^L_{Op^{\mathrm{BW}}(\mathrm{i}f)}[Op^{\mathrm{BW}}(\mathrm{i}\partial_t f)]\Big)(\Phi^\theta(U))^{-1}d\theta\end{aligned}$$

and the lemma follows noting that $f\#_\rho f_t-f_t\#_\rho f=\frac{1}{\mathrm{i}}\{f,f_t\}$ plus a symbol of order $2m-3$. The translation invariance properties (2.11), (2.26) follow since $f(U;x,\xi)$ and $(\partial_t f)(U;x,\xi)$ satisfy (2.11) as well, and then arguing as in the proof of Lemma A.1. □

A.3. **Lie expansions of vector fields up to quartic degree.** In this subsection the variable U may denote both the couple of complex variables $(u, \overline{u})$ or the real variables (η, ψ).

Lemma A.8. (Inverse of $\mathbf{F}^{\theta}_{\leq 3}(U)$ up to $O(u^4)$) *Consider a map $\theta \mapsto \mathbf{F}^{\theta}_{\leq 3}(U)$, $\theta \in [0,1]$, of the form*

$$\mathbf{F}^{\theta}_{\leq 3}(U) = U + \theta\big(M_1(U)[U] + M_2^{(1)}(U)[U]\big) + \theta^2 M_2^{(2)}(U)[U] \tag{A.26}$$

where $M_1(U)$ is in $\widetilde{\mathcal{M}}_1 \otimes \mathcal{M}_2(\mathbb{C})$ and the maps $M_2^{(1)}(U), M_2^{(2)}(U)$ are in $\widetilde{\mathcal{M}}_2 \otimes \mathcal{M}_2(\mathbb{C})$. Then there is a family of maps $\mathbf{G}^{\theta}_{\leq 3}(V)$ of the form

$$\mathbf{G}^{\theta}_{\leq 3}(V) = V - \theta\big(M_1(V)[V] + M_2^{(1)}(V)[V]\big) + \theta^2 \breve{M}_2^{(2)}(V)[V] \tag{A.27}$$

where $\breve{M}_2^{(2)}(V)$ is in $\widetilde{\mathcal{M}}_2 \otimes \mathcal{M}_2(\mathbb{C})$, such that

$$\mathbf{G}^{\theta}_{\leq 3} \circ \mathbf{F}^{\theta}_{\leq 3}(U) = U + M_{\geq 3}(\theta; U)[U]\,, \quad \mathbf{F}^{\theta}_{\leq 3} \circ \mathbf{G}^{\theta}_{\leq 3}(V) = V + M_{\geq 3}(\theta; U)[U]\,, \tag{A.28}$$

where $M_{\geq 3}(\theta; U)$ is a polynomial in θ and finitely many monomials $M_p(U)[U]$ for maps $M_p(U) \in \widetilde{\mathcal{M}}_p \otimes \mathcal{M}_2(\mathbb{C})$, $p \geq 3$.

Proof. Setting $V = \mathbf{F}^{\theta}_{\leq 3}(U)$ we have, by (A.26),

$$U = V - \theta\big(M_1(U)[U] + M_2^{(1)}(U)[U]\big) - \theta^2 M_2^{(2)}(U)[U]\,. \tag{A.29}$$

Substituting iteratively twice the relation (A.29) into itself, and using the last two remarks under Definition 2.7, we get

$$U = V - \theta M_1(V)[V] + \theta^2 M_1(V)[M_1(V)[V]] + \theta^2 M_1(M_1(V)[V])[V] - \theta^2 M_2^{(2)}(V)[V] - \theta M_2^{(1)}(V)[V]$$

up to a polynomial $M_{\geq 3}(\theta; U)[U]$ in θ and U which has degree of homogeneity at least three in U (recall that $\mathbf{F}^{\theta}_{\leq 3}(U)$ in (A.26) is a polynomial in U). This expansion defines $\mathbf{G}^{\theta}_{\leq 3}$ in (A.27), and proves (A.28). ☐

We regard the map $\theta \mapsto \mathbf{G}^{\theta}_{\leq 3}(V)$ in (A.27) as the formal flow of a non-autonomous vector field $S(\theta; U)$ up a remainder of degree of homogeneity four, see (A.30).

Lemma A.9. *Consider a map $\mathbf{F}^{\theta}_{\leq 3}(U)$ as in* (A.26) *and let $\mathbf{G}^{\theta}_{\leq 3}(V)$ be its approximate inverse as in* (A.27) *up to quartic remainders. Then*

$$\partial_\theta \mathbf{G}^{\theta}_{\leq 3}(V) = S(\theta; \mathbf{G}^{\theta}_{\leq 3}(V)) + M_{\geq 3}(\theta; U)[U]\,, \quad \mathbf{G}^{0}_{\leq 3}(V) = V\,, \tag{A.30}$$

where $S(\theta; U)$ is a vector field of the form

$$S(\theta; U) = S_1(U)[U] + \theta S_2(U)[U] \tag{A.31}$$

where $S_1(U)$ is a map in $\widetilde{\mathcal{M}}_1 \otimes \mathcal{M}_2(\mathbb{C})$ and $S_2(U)$ in $\widetilde{\mathcal{M}}_2 \otimes \mathcal{M}_2(\mathbb{C})$, and $M_{\geq 3}(\theta; U)$ is a polynomial in θ and finitely many monomials $M_p(U)[U]$ for maps $M_p(U) \in \widetilde{\mathcal{M}}_p \otimes \mathcal{M}_2(\mathbb{C})$, $p \geq 3$.

Proof. Differentiating (A.27) we have

$$\partial_\theta \mathbf{G}^{\theta}_{\leq 3}(V) = -M_1(V)[V] - M_2^{(1)}(V)[V] + 2\theta \breve{M}_2^{(2)}(V)[V]\,. \tag{A.32}$$

Then set

$$\breve{S}(\theta; U) := -M_1(\mathbf{F}^{\theta}_{\leq 3}(U))[\mathbf{F}^{\theta}_{\leq 3}(U)] - M_2^{(1)}(\mathbf{F}^{\theta}_{\leq 3}(U))[\mathbf{F}^{\theta}_{\leq 3}(U)] + 2\theta \breve{M}_2^{(2)}(\mathbf{F}^{\theta}_{\leq 3}(U))[\mathbf{F}^{\theta}_{\leq 3}(U)]\,. \tag{A.33}$$

By (A.26) and the last two remarks under Definition 2.7, we deduce that $\breve{S}(\theta; U)$ is equal to a vector field $S(\theta; U)$ as in (A.31) plus a term $M_{\geq 3}(\theta; U)[U]$ which is a polynomial in θ of monomials $M_p(U)[U]$ for maps $M_p(U) \in \widetilde{\mathcal{M}}_p \otimes \mathcal{M}_2(\mathbb{C})$, $p \geq 3$. By (A.33) and the second identity in (A.28) we deduce

$$\breve{S}(\theta; \mathbf{G}^{\theta}_{\leq 3}(V)) = -M_1(V)[V] - M_2^{(1)}(V)[V] + 2\theta \breve{M}_2^{(2)}(V)[V] \tag{A.34}$$

plus another polynomial $M_{\geq 3}(\theta;U)[U]$ of degree at least three. Comparing (A.34) with (A.30) the lemma follows. ☐

Given polynomials vector fields $X(U)$ and $Y(U)$ we define the nonlinear commutator

$$[\![X,Y]\!](U) := d_U Y(U)[X(U)] - d_U X(U)[Y(U)]\,. \tag{A.35}$$

Under the same notation of Lemmata A.8, A.9, we have the following result.

Lemma A.10. (Lie expansion) *Consider a vector field X of the form $X(U) = M(U)U$ for some map $M(U) = M_0 + M_1(U) + M_2(U)$ where M_0 is in $\widetilde{\mathcal{M}}_0 \otimes \mathcal{M}_2(\mathbb{C})$, $M_1(U)$ is in $\widetilde{\mathcal{M}}_1 \otimes \mathcal{M}_2(\mathbb{C})$ and $M_2(U)$ in $\widetilde{\mathcal{M}}_2 \otimes \mathcal{M}_2(\mathbb{C})$. Consider a transformation $\mathbf{F}^\theta_{\leq 3}(U)$ as in* (A.26) *and let $S(\theta;U)$ be the vector field of the form* (A.31) *such that* (A.30) *holds true. Then, if U solves*

$$\partial_t U = X(U)\,, \tag{A.36}$$

the function $V := \mathbf{F}^1_{\leq 3}(U)$ solves

$$\partial_t V = X(V) + [\![S,X]\!]_{|\theta=0}(V) + \frac{1}{2}[\![S,[\![S,X]\!]]\!]_{|\theta=0}(V) + \frac{1}{2}[\![\partial_\theta S_{|\theta=0}, X]\!](V) + \cdots \tag{A.37}$$

up to terms of degree of homogeneity greater or equal to 4.

Proof. In order to find the quadratic and cubic components of the transformed system, it is sufficient to write $V := \mathbf{F}^\theta_{\leq 3}(U)$, $\theta \in [0,1]$, and the first identity in (A.28) as $U = \mathbf{G}^\theta_{\leq 3}(V) + M_{\geq 3}(\theta;U)[U]$. Then, differentiating with ∂_t the first identity in (A.28), and using (A.36), we obtain, up to a quartic term,

$$X(\mathbf{G}^\theta_{\leq 3}(V)) = d\mathbf{G}^\theta_{\leq 3}(V)[V_t] = (\mathrm{Id} - M(\theta;V))[V_t] \tag{A.38}$$

where $M(\theta;V) = \theta(\check{M}_1(V) + \check{M}_2^{(1)}(V)) + \theta^2 \check{M}_2(V)$ for suitable maps $\check{M}_1(V)$ in $\widetilde{\mathcal{M}}_1 \otimes \mathcal{M}_2(\mathbb{C})$ and $\check{M}_2(V), \check{M}_2^{(1)}(V)$ in $\widetilde{\mathcal{M}}_2 \otimes \mathcal{M}_2(\mathbb{C})$, recall (A.27). Applying in (A.38) the "pseudo-inverse"

$$\big(d\mathbf{G}^\theta_{\leq 3}(V)\big)^{-1} := \mathrm{Id} + M(\theta;V) + M^2(\theta;V)\,,$$

and since, by (A.36), we have $\partial_t V = \partial_t U$ plus a quadratic term in U, we deduce that, up to a quartic term,

$$\big(d\mathbf{G}^\theta_{\leq 3}(V)\big)^{-1} X(\mathbf{G}^\theta_{\leq 3}(V)) = V_t\,.$$

The left hand side of this formula can be expanded in Taylor at $\theta = 0$ up to degree 2, obtaining, using (A.30), the usual Lie formula (see e.g. [36])

$$X(V) + \theta[\![S,X]\!]_{|\theta=0}(V) + \frac{\theta^2}{2}\Big([\![S,[\![S,X]\!]]\!]_{|\theta=0}(V) + [\![(\partial_\theta S(\theta))_{|\theta=0}, X]\!](V)\Big) \tag{A.39}$$

up to terms of degree 4. Evaluating (A.39) at $\theta = 1$ we get (A.37). ☐

A.4. **Proof of** (4.10)-(4.11)**.**

Proof. For any function $m_{-1}(U;x)$ in $\Sigma\mathcal{F}_{K,1,1}[r]$, by Lemma A.2 the flow in (4.9) is well defined. We claim that

$$(\mathbf{\Psi}^\theta_{-1}(U))_{|\theta=1} = \exp\{Op^{\mathrm{BW}}(M_{-1})\} = Op^{\mathrm{BW}}(\exp\{M_{-1}\}) + R(U) \tag{A.40}$$

where R is in $\Sigma\mathcal{R}^{-\tilde\rho}_{K,1,1} \otimes \mathcal{M}_2(\mathbb{C})$ for any $\tilde\rho > 0$. Indeed

$$\exp\{Op^{\mathrm{BW}}(M_{-1})\} = \mathrm{Id} + Op^{\mathrm{BW}}(M_{-1}) + \frac{1}{2}Op^{\mathrm{BW}}(M_{-1})Op^{\mathrm{BW}}(M_{-1}) + \sum_{k\geq 3}\frac{1}{k!}\big(Op^{\mathrm{BW}}(M_{-1})\big)^k\,.$$

By Proposition 2.9 (applied with some $\tilde{\rho}$ to be chosen later) we have that $Op^{\rm BW}(M_{-1})Op^{\rm BW}(M_{-1})$ is equal to $Op^{\rm BW}((M_{-1})^2)$ plus a smoothing remainder in $\Sigma\mathcal{R}^{-\tilde{\rho}}_{K,1,2}$. Furthermore, by Proposition 3.6 in [31] we deduce that

$$\sum_{k\geq 3}\frac{1}{k!}\Big(\big(Op^{\rm BW}(M_{-1})\big)^k - Op^{\rm BW}\big((M_{-1})^k\big)\Big)$$

belongs to the class of non-homogeneous smoothing remainders $\mathcal{R}^{-\tilde{\rho}}_{K,1,3}$. This proves (A.40). By an explicit computation (see the proof of Corollary 3.1 in [31]) we have

$$\exp\{M_{-1}\} := \begin{pmatrix} g_1 & g_2 \\ \overline{g_2} & g_1 \end{pmatrix}, \quad g_1 := 1+\Psi_1(|m_{-1}|^2), \qquad g_2 := m_{-1}\Psi_2(|m_{-1}|^2), \tag{A.41}$$

where $\Psi_1(y), \Psi_2(y)$ are the analytic functions

$$\Psi_1(y) := \frac{y}{2} + \sum_{k\geq 2}\frac{y^k}{(2k)!}, \quad \Psi_2(y) := 1 + \sum_{k\geq 1}^{\infty}\frac{y^k}{(2k+1)!}.$$

We now choose $m_{-1}(U;x)$ in such a way that $\exp\{M_{-1}\} := C^{-1}(U;x)$, namely (recalling (4.7), (4.5)) we have to solve the following equations

$$\Psi_1(|m_{-1}|^2) = f-1, \quad m_{-1}\Psi_2(|m_{-1}|^2) = -g.$$

Notice that $\Psi_1(y)$ is locally invertible near 0 and that, since $1+a+\lambda_+ \geq 1/2$, the function f in (4.5) satisfies $f^2-1 = \frac{|a|^2}{2\lambda_+(1+a+\lambda_+)} \geq 0$, thus $f-1\geq 0$. Therefore we have

$$|m_{-1}|^2 = \Psi_1^{-1}(f-1), \quad m_{-1} = -\frac{g}{\Psi_2(|m_{-1}|^2)}. \tag{A.42}$$

Since $f-1$ and g belong to $\Sigma\mathcal{F}_{K,1,1}$ and Ψ_1^{-1}, Ψ_2 are analytic, it follows that the function m_{-1} belongs to $\Sigma\mathcal{F}_{K,1,1}$ as well. Formulas (A.40)-(A.42) prove (4.10).

Let us prove (4.11). The flow $(\mathbf{\Psi}^{\theta}_{-1}(U))_{|\theta=1}$ is invertible and, setting $Q(U) := (\mathbf{\Psi}^{\theta}_{-1}(U))^{-1}_{|\theta=1} - Op^{\rm BW}(C)$, we have

$$\mathrm{Id} = (\mathbf{\Psi}^{\theta}_{-1}(U))^{-1}_{|\theta=1}(\mathbf{\Psi}^{\theta}_{-1}(U))_{|\theta=1} \overset{(4.10)}{=} Op^{\rm BW}(C)Op^{\rm BW}(C^{-1}) + Op^{\rm BW}(C)R(U) + Q(U)(\mathbf{\Psi}^{\theta}_{-1}(U))_{|\theta=1}.$$

Hence, using Propositions 2.9, 2.10, we deduce that $Q(U) = \tilde{R}(U)(\mathbf{\Psi}^{\theta}_{-1}(U))^{-1}_{|\theta=1}$, for some $\tilde{R}$ in the class $\Sigma\mathcal{R}^{-\tilde{\rho}}_{K,1,1}$. We conclude that Q is in $\Sigma\mathcal{R}^{-\rho}_{K,1,1}$, using that $(\mathbf{\Psi}^{\theta}_{-1}(U))^{-1}_{|\theta=1}$ is in $\Sigma\mathcal{M}_{K,1,0}\otimes\mathcal{M}_2(\mathbb{C})$ by item (ii) of Lemma A.2, and choosing $\tilde{\rho}$ large enough. □

References

[1] Alazard T., Baldi P., *Gravity capillary standing water waves.* Arch. Rat. Mech. Anal, 217, 3, 741-830, 2015.

[2] Alazard T., Burq N., Zuily C., *On the water-wave equations with surface tension.* Duke Math. J., 158, 413-499, 2011.

[3] Alazard T., Burq N., Zuily C., *On the Cauchy problem for gravity water waves.* Invent. Math., 198, 71–163, 2014.

[4] Alazard T., Delort J-M., *Sobolev estimates for two dimensional gravity water waves.* Astérisque, 374, viii + 241, 2015.

[5] Alazard T., Delort J-M., *Global solutions and asymptotic behavior for two dimensional gravity water waves.* Ann. Sci. Éc. Norm. Supér., 48, no. 5, 1149-1238, 2015.

[6] Alazard T., Métivier G., *Paralinearization of the Dirichlet to Neumann operator, and regularity of the three dimensional water waves.* Comm. Partial Differential Equations 34, no. 10-12, 1632-1704, 2009.

[7] Ambrose D.M., *Well-posedness of vortex sheets with surface tension.* SIAM J. Math. Anal. 35, no. 1, 211-244, 2003.

[8] Baldi P., Berti M., Montalto R., *KAM for autonomous quasi-linear perturbations of KdV*. Ann. Inst. H. Poincaré Anal. Non Linéaire 33, no. 6, 1589-1638, 2016.

[9] Baldi P., Berti M., Haus E., Montalto R., *Time quasi-periodic gravity water waves in finite depth*. Invent. Math. 214, 2, 739–911, 2018.

[10] Bambusi D., *Birkhoff normal form for some nonlinear PDEs*. Comm. Math. Phys. 234, no. 2, 253-285, 2003.

[11] Bambusi D., Delort J.-M., Grébert B., Szeftel J., *Almost global existence for Hamiltonian semilinear Klein-Gordon equations with small Cauchy data on Zoll manifolds*. Comm. Pure Appl. Math. 60, 2007.

[12] Bambusi D., Grébert B., *Birkhoff normal form for Pdes with tame modulus*. Duke Math. J., 135, no. 3, 507-567, 2006.

[13] Berti M., Delort J.-M., *Almost Global Solutions of Capillary-gravity Water Waves Equations on the Circle*. UMI Lecture Notes 2018 (awarded UMI book prize 2017), ISBN 978-3-319-99486-4.

[14] Berti M., Montalto R., *Quasi-periodic standing wave solutions of gravity-capillary water waves*. To appear in Mem. Amer. Math. Soc., Memo 891, arXiv:1602.02411.

[15] Beyer K., Günther M., *On the Cauchy problem for a capillary drop. I. Irrotational motion*. Math. Methods Appl. Sci. 21, no. 12, 1149-1183, 1998.

[16] Christianson H., Hur V., Staffilani G., *Strichartz estimates for the water-wave problem with surface tension*. Comm. Partial Differential Equations 35, no. 12, 2195-2252, 2010.

[17] Christodoulou D., Lindblad H., *On the motion of the free surface of a liquid*. Comm. Pure Appl. Math. 53, no. 12, 1536-1602, 2000.

[18] Coutand D., Shkoller S., *Well-posedness of the free-surface incompressible Euler equations with or without surface tension*. J. Amer. Math. Soc. 20, no. 3, 829-930, 2007.

[19] Craig W., *An existence theory for water waves and the Boussinesq and Korteweg-de Vries scaling limits*. Comm. Partial Differential Equations, 10, no. 8, 787-1003, 1985.

[20] Craig W., Sulem C., *Mapping properties of normal forms transformations for water waves*. Boll. Unione Mat. Ital. 9, no. 2, 289-318, 2016.

[21] Craig W., Sulem C., *Numerical simulation of gravity waves.* J. Comput. Phys., 108(1):73–83, 1993.

[22] Craig W., Worfolk P., *An integrable normal form for water waves in infinite depth.* Phys. D, 84, no. 3-4, 513-531, 1995.

[23] Delort J.-M., *Long-time Sobolev stability for small solutions of quasi-linear Klein-Gordon equations on the circle*. Trans. Amer. Math. Soc. 361, 4299-4365, 2009.

[24] Delort J.-M., *A quasi-linear Birkhoff normal forms method. Application to the quasi-linear Klein-Gordon equation on S1*. Astérisque No. 341, vi+113 pp., 2012.

[25] Delort J.-M., *Quasi-linear perturbations of Hamiltonian Klein-Gordon equations on spheres*. Mem. Amer. Math. Soc. 234, no. 1103, vi+80 pp, 2015.

[26] Delort J.-M., Szeftel J., *Long-time existence for small data nonlinear Klein-Gordon equations on tori and spheres*. Int. Math. Res. Not. 37, 1897-1966, 2004.

[27] Delort J.-M., Szeftel J., *Long-time existence for semi-linear Klein-Gordon equations with small Cauchy data on Zoll manifolds*. Amer. J. Math. 128, 1187-1218, 2006.

[28] Deng Y., Ionescu A. D., Pausader B., Pusateri F., *Global solutions for the 3D gravity-capillary water waves system*. Acta Math. 219, no. 2, 213-402, 2017.

[29] De Poyferré T., *Blow-up conditions for gravity water-waves*, arXiv:1407.6881.

[30] Dyachenko A.I., Lvov Y.V., Zakharov V.E., *Five-wave interaction on the surface of deep fluid*. Physica D 87, 233-261, 1995.

[31] Feola R., Iandoli F., *Local well-posedness for quasi-linear NLS with large Cauchy data on the circle*. In press on Ann. Inst. H. Poincaré Anal. Non Linéaire, 2018.

[32] Feola R., Iandoli F., *Long time existence for fully nonlinear NLS with small Cauchy data on the circle*, preprint arXiv:1806.03437, 2018.

[33] Germain P., Masmoudi N., Shatah J., *Global solutions for the gravity water waves equation in dimension 3*. Ann. of Math. (2), 175, 691–754, 2012.

[34] Germain P., Masmoudi N., Shatah J., *Global solutions for capillary waves equation in dimension 3*. Comm. Pure Appl. Math., 68, no. 4, 625-687, 2015.

[35] Harrop-Griffiths B., Ifrim M., Tataru D., *Finite depth gravity water waves in holomorphic coordinates*. Ann. PDE 3, no. 1, Art. 4, 102 pp., 2017.

[36] Henrard J., *Equivalence for Lie transforms*. Celestial Mechanics 10, 497–512, 1974.

[37] Hunter J., Ifrim M., Tataru D., *Two dimensional water waves in holomorphic coordinates*. Comm. Math. Phys. 346, 483-552, 2016.

[38] Ifrim M., Tataru D., *Two dimensional water waves in holomorphic coordinates II: global solutions*. Bull. Soc. Math. France 144, 369-394, 2016.

[39] Ifrim M., Tataru D., *The lifespan of small data solutions in two dimensional capillary water waves*. Arch. Ration. Mech. Anal. 225, no. 3, 1279-1346, 2017.
[40] Ionescu A., Pusateri F., *Global solutions for the gravity water waves system in 2d*. Invent. Math., 199, 3, 653-804, 2015.
[41] Ionescu A., Pusateri F., *Global analysis of a model for capillary water waves in 2D*. Comm. Pure and Appl. Math. 69, no. 11, 2015-2071, 2016.
[42] Ionescu A., Pusateri F., *Global regularity for 2d water waves with surface tension*. Preprint arXiv:1408.4428. To appear in Mem. Amer. Math. Soc.
[43] Ionescu A., Pusateri F., *Recent advances on the global regularity for water waves*. Philos. Trans. Roy. Soc. A., vol 376, issue 2111, 2018.
[44] Iooss G., Plotnikov P., Toland J., *Standing waves on an infinitely deep perfect fluid under gravity.* Arch. Ration. Mech. Anal., 177(3):367–478, 2005.
[45] Lannes D., *Well-posedness of the water-waves equations*. J. Amer. Math. Soc., 3, 605–654, 18, 2005.
[46] Lannes D., *The water waves problem: mathematical analysis and asymptotics*. Mathematical Surveys and Monographs, 188, 2013.
[47] Lindblad H., *Well-posedness for the motion of an incompressible liquid with free surface boundary*. Ann. of Math., 162, no. 1, 109-194, 2005.
[48] Moser J., *Convergent series expansions for quasi-periodic motions*. Math. Ann. **169**, 136–176, 1967.
[49] Nalimov V. I., *The Cauchy-Poisson problem*. Dinamika Splosn. Sredy Vyp. 18 Dinamika Zidkost. so Svobod. Granicami, 10-210, 254, 1974.
[50] Plotnikov P., Toland J., *Nash-Moser theory for standing water waves.* Arch. Rat. Mech. Anal., 159, 1–83, 2001.
[51] Shatah J., Zeng C., *Local well-posedness for the fluid interface problem*. Arch. Ration. Mech. Anal. 199, no. 2, 653-705, 2011.
[52] Totz N., Wu S., *A rigorous justification of the modulation approximation to the 2D full water wave problem*. Comm. Math. Phys. 310, no. 3, 817-883, 2012.
[53] Wang X., *Global infinite energy solutions for the 2D gravity water waves system*. Preprint arXiv: 1502.00687.
[54] Wang X., *On 3D water waves system above a flat bottom*. Preprint arXiv:1508.06223.
[55] Wang X., *Global solution for the 3D gravity water waves system above a flat bottom*. Preprint arXiv:1508.06227.
[56] Wu S., *Well-posedness in Sobolev spaces of the full water waves problem in 2-D*. Invent. Math. 130(1), 39-72, 1997.
[57] Wu S., *Well-posedness in Sobolev spaces of the full water waves problem in 3-D*. J. Am. Math. Soc. 12(2), 445-495, 1999.
[58] Wu S., *Almost global wellposedness of the 2-D full water wave problem*. Invent. Math., 1, 177, 45–135, 2009.
[59] Wu S., *Global well-posedness of the 3-D full water wave problem*. Invent. Math., 1, 184, 125–220, 2011.
[60] Yosihara H., *Gravity waves on the free surface of an incompressible perfect fluid of finite depth*. Publ. Res. Inst. Math. Sci. 18, 49-96, 1982.
[61] Zakharov V.E., *Stability of periodic waves of finite amplitude on the surface of a deep fluid*. Zhurnal Prikladnoi Mekhaniki i Teckhnicheskoi Fiziki 9, no.2, 86-94, 1969.
[62] Zakharov V.E., Dyachenko A.I., *Is free-surface hydrodynamics an integrable system?* Physics Letters A 190 144-148, 1994.

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