Aharonov and Bohm *vs.* Welsh eigenvalues

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Abstract. We consider a class of two-dimensional Schrödinger operator with a singular interaction of the δ type and a fixed strength β supported by an infinite family of concentric, equidistantly spaced circles, and discuss what happens below the essential spectrum when the system is amended by an Aharonov-Bohm flux $\alpha \in [0, \frac{1}{2}]$ in the center. It is shown that if $\beta \neq 0$, there is a critical value $\alpha_{\text{crit}} \in (0, \frac{1}{2})$ such that the discrete spectrum has an accumulation point when $\alpha < \alpha_{\text{crit}}$, while for $\alpha \geq \alpha_{\text{crit}}$ the number of eigenvalues is at most finite, in particular, the discrete spectrum is empty for any fixed $\alpha \in (0, \frac{1}{2})$ and $|\beta|$ small enough.

Mathematics Subject Classification (2010). 81Q10, 35J10.

Keywords. Singular Schrödinger operator, radial symmetry, discrete spectrum, Aharonov-Bohm flux.

1 Introduction

Schrödinger operators with radially periodic potentials attracted attention because they exhibit interesting spectral properties. It was noted early [10] that the essential spectrum threshold of such an operator coincides with that of the one-dimensional Schrödinger operator describing the radial motion. More surprising appeared to be the structure of the essential spectrum which may consist of interlacing intervals of dense point and absolutely continuous nature as was first illustrated using potentials of cosine shape [11].

While this behavior can be observed in any dimension ≥ 2, the two-dimensional case is of a particular interest because here these operators can also have a discrete spectrum below the threshold of the essential one. This fact was first observed in [5] and the national pride inspired the authors to refer to this spectrum as to *Welsh eigenvalues*; it was soon established that that their number is infinite if the radially symmetric potential is nonzero and belongs to L^1_{loc} [16]. Moreover, the effect persists if such a regular potential is replaced by a periodic array of δ interactions or more general singular interactions [7, 8].

The question addressed in this paper is how are the Welsh eigenvalues influenced by a local magnetic field preserving the rotational symmetry. For simplicity we will choose the simplest setting, the two-dimensional system with δ potential of a fixed strength β supported on a concentric family of circles $\{\mathcal{C}_{r_n}\}_{n\in\mathbb{N}}$ or radii $r_n = d(n+\frac{1}{2})$, $n = 0, 1, \dots,$ with $d > 0$. Without the presence of the magnetic field the corresponding Hamiltonian can be symbolically written as

$$H_\beta = -\Delta + \beta \sum_n \delta(x - \mathcal{C}_{r_n})\,, \quad \beta \in \mathbb{R}\,,$$

which can be given meaning as a self-adjoint operator in $L^2(\mathbb{R}^2)$ as we will recall below. As we have said, the discrete spectrum of H_β is infinite [8], which is a direct consequence of the fact that the effective potential in the s-wave component contains the term $-\frac{1}{4r^2}$ producing an infinite number of eigenvalues below $E_\beta := \inf \sigma_{\mathrm{ess}}(H_\beta)$.

The magnetic interaction we add is also chosen in the simplest possible way, namely as an Aharonov-Bohm flux α at the origin of the coordinates, measured is suitable units, that gives rise to the magnetic field vanishing outside this point. The corresponding Hamiltonian will be denoted $H_{\alpha,\beta}$ and as we will argue, it is sufficient to consider flux values up to half of the quantum, $\alpha \in (0, \frac{1}{2})$. Since singular interactions are involved, it is maybe useful to stress that we consider an Aharonov-Bohm flux alone, without an additional point interactions at origin *à la* [1, 6]. It is known that local magnetic fields generally, and Aharonov-Bohm fluxes in particular, can reduce the discrete spectrum, if combined with an effective potential that behaves like r^{-2}, on the borderline between short and long range, the effect can be dramatic [14].

We are going to show that in the present model the Aharonov-Bohm field also influences the discrete spectrum but the dependence on the flux value is more complicated. Specifically, we claim that

- there is an $\alpha_{\mathrm{crit}}(\beta) = \alpha_{\mathrm{crit}} \in (0, \frac{1}{2})$ such that for $\alpha \in (0, \alpha_{\mathrm{crit}})$ the discrete spectrum of $H_{\alpha,\beta}$ is infinite accumulating at the threshold E_0, while for $\alpha \in [\alpha_{\mathrm{crit}}, \frac{1}{2})$ there is at most finite number of eigenvalues below E_0,
- the critical value $\alpha_{\mathrm{crit}}(\beta)$ admits the following asymptotics,

$$\alpha_{\mathrm{crit}}(\beta) \to \frac{1}{2}- \quad \text{for} \quad \beta \to \pm\infty$$

  and

$$\alpha_{\mathrm{crit}}(\beta) \to 0+ \quad \text{for} \quad \beta \to 0\,,$$

- for any fixed $\alpha \in (0, \frac{1}{2})$ there exists $\beta_0 > 0$ such that for any $|\beta| \le \beta_0$ we have $\sigma_{\mathrm{d}}(H_{\alpha,\beta}) = \emptyset$, and moreover, $\sigma_{\mathrm{d}}(H_{\frac{1}{2},\beta}) = \emptyset$ holds for any $\beta \in \mathbb{R}$.

These properties will be demonstrated in Sections 3 and 4; before coming to that, in the next section we introduce properly the Hamiltonian and derive its elementary properties.

2 Preliminaries

We consider a magnetic flux ϕ perpendicular to the plane to which the particle is confined and placed at the origin of the coordinates corresponding to the vector potential

$$A(x,y) = \frac{\phi}{2\pi}\left(-\frac{y}{r^2}, \frac{x}{r^2}\right)\,.$$

In the rational units we use the flux quantum is 2π, thus it is natural to introduce $\alpha := \frac{\phi}{2\pi}$. Given this A we define the 'free' Aharonov-Bohm Hamiltonian

$$H_\alpha := (-i\nabla - A)^2\,, \quad D(H_\alpha) = \{f \in L^2(\mathbb{R}^2) :\, (-i\nabla - A)^2 f \in L^2\}\,,$$

where the domain is sometimes dubbed magnetic Sobolev space. Since the integer part of a given α can be removed by a simple gauge transformation, it is sufficient to consider $\alpha \in (0,1)$ only.

The radial symmetry allows us to describe H_α in terms of the partial wave decomposition. To this aim we introduce unitary operator $U : L^2(\mathbb{R}_+, r\mathrm{d}r) \to L^2(\mathbb{R}_+)$ acting as $Uf(r) = r^{1/2}f(r)$. This naturally leads to

$$L^2(\mathbb{R}^2) = \bigoplus_{l\in\mathbb{Z}} U^{-1}L^2(\mathbb{R}_+) \otimes S_l \,,$$

where S_l is the l-th eigenspace of Laplace operator on the unit circle, and the corresponding decomposition of the Hamiltonian

$$H_\alpha = \bigoplus_{l} U^{-1}H_{\alpha,l}U \otimes I_l \,,$$

where I_l is the identity operator on S_l and the radial part is

$$H_{\alpha,l} := -\frac{d^2}{d^2r} + \frac{1}{r^2}c_{\alpha,l}\,, \quad c_{\alpha,l} := -\frac{1}{4} + (l+\alpha)^2\,, \tag{2.1}$$

$$\begin{aligned} D(H_{\alpha,l}) := \{ f \in L^2(\mathbb{R}_+) \,:\, -f'' + \frac{c_{\alpha,l}}{r^2}f \in L^2(\mathbb{R}_+)\,, \\ \lim_{r\to 0^+} r^{\alpha-1/2}f(r) = 0\,,\; l = 0\,, \\ \lim_{r\to 0^+} r^{1-\alpha-1/2}f(r) = 0\,,\; l = -1\}. \end{aligned} \tag{2.2}$$

We recall that this operator describes a 'pure' Aharonov-Bohm field without an additional singular interaction at the origin [1, 6]. This corresponds to the choice of $H_{\alpha,l}$, $l = 0, -1$, as appropriate self-adjoint extensions of the operator $-\frac{d^2}{d^2r} + \frac{1}{r^2}c_{\alpha,l}$ restricted to $C_0^\infty(\mathbb{R}_+)$. For all the other values of l the centrifugal term ensures the essential self-adjointness, here we choose the conditions which exclude the more singular of the two solutions at the origin, $r^{1/2}K_\alpha(\kappa r)$ and $r^{1/2}K_{1-\alpha}(\kappa r)$, respectively.

In the next step we consider the δ *interaction supported by concentric circles*; we amend the system governed by H_α by a singular radially periodic potential supported by concentric circles $\mathcal{C}_{r_n}$ of the radii $r_n = d(n+\frac{1}{2})$, $d > 0$, the strength of which is characterized by a nonzero coupling constant $\beta \in \mathbb{R}$. Since the radial symmetry is preserved, the resulting Hamiltonian can be again expressed in terms of its partial-wave components,

$$H_{\alpha;\beta} = \bigoplus_{l} U^{-1}H_{\alpha;\beta,l}U \otimes I_l \,, \tag{2.3}$$

where

$$D(H_{\alpha;\beta,l}) := \{f, \in W^{2,2}(\mathbb{R}^2 \setminus \cup_{n\in\mathbb{N}} \mathcal{C}_{r_n}) :\ f \text{ satisfies (2.2)} \tag{2.4}$$

$$\text{and } \partial_r f(r_n^+) - \partial_r f(r_n^-) = \beta f(r_n)\,,\ n \in \mathbb{N}\}\,; \tag{2.5}$$

it is easy to check that operator $H_{\alpha;\beta}$ is self-adjoint.

As in [7] it is useful to introduce a one-dimensional comparison operator which is the usual *Kronig-Penney Hamiltonian* with equidistantly spaced δ interactions supported by the set $\{x_n := d(n+\frac{1}{2})\ :\ n \in \mathbb{Z}\}$. We denote it h_β, it acts as $\mathrm{h}_\beta f = -f''$ on the domain

$$D(\mathrm{h}_\beta) = \{f \in W^{2,2}(\mathbb{R} \setminus \cup_{n\in\mathbb{Z}}\{x_n\})\ :\ f'(x_n^+) - f'(x_n^-) = \beta f(x_n)\,,\ n \in \mathbb{Z}\}\,.$$

Let E_0 stand for the spectral threshold of h_β,

$$E_0 := \inf \sigma(\mathrm{h}_\beta)\,; \tag{2.6}$$

mimicking the argument used in [8] we can check easily that this quantity determines the essential spectrum of $H_{\alpha;\beta}$, namely

$$\sigma_{\mathrm{ess}}(H_{\alpha;\beta}) = [E_0, \infty)\,. \tag{2.7}$$

Although it is not important for the present work, let us add that the reasoning made in [8] remains valid if the centrifugal coefficients in (2.1) replace their nonmagnetic values $c_{0,l}$, and consequently, the essential spectrum is not affected by the Aharonov-Bohm flux consisting of the absolutely continuous bands that coincide with the spectral bands of h_β and the dense point part filling the spectral gaps of h_β.

Our interest here concerns the spectrum of $H_{\alpha;\beta}$ in the interval $(-\infty, E_0)$ which is discrete according to (2.7). Let us first collect its elementary properties.

Proposition 2.1 *Suppose that $\beta \neq 0$, then*

(i) $\sharp\sigma_{\mathrm{disc}}(H_{0;\beta}) = \infty$

(ii) $\sigma_{\mathrm{disc}}(H_{\frac{1}{2};\beta}) = \emptyset$

(iii) $\sigma_{\mathrm{disc}}(H_{\alpha;\beta}) = \sigma_{\mathrm{disc}}(H_{1-\alpha;\beta})$

(iv) eigenvalues of $H_{\alpha;\beta}$ are nondecreasing in $[0, \frac{1}{2}]$, $\lambda_j(\alpha') \geq \lambda_j(\alpha)$ if $\alpha' \geq \alpha$

Proof. Claim (i) follows from [8, Thm 5.1]. Partial wave operators in the decomposition (2.3) can contribute to $\sigma_{\rm disc}(H_{\alpha;\beta})$ only if $c_{\alpha,l} < 0$. Indeed, if $c_{\alpha,l} = 0$ the spectrum of $H_{\alpha;\beta,l}$ coincides, up to multiplicity, with that of the operator h_β amended according to (2.2) with Dirichlet condition at $x = 0$, hence (2.6) in combination with a bracketing argument [15, Sec. XIII.15] shows that the discrete spectrum is empty and yields assertion (ii). Furthermore, in view of the min-max principle [15, Sec. XIII.1] this verifies the above claim and shows that the discrete spectrum comes from $H_{\alpha;\beta,0}$ if $\alpha \in [0, \frac{1}{2})$ and from $H_{\alpha;\beta,-1}$ if $\alpha \in (\frac{1}{2}, 1)$. The third claim follows from the identity $c_{\alpha,0} = c_{1-\alpha,-1}$ valid for $\alpha \in (0, 1)$, and the last one we get employing the min-max principle again. ∎

It is therefore clear, as indicated in the introduction, that to describe the discrete spectrum it is sufficient to limit our attention to the values $\alpha \in (0, \frac{1}{2})$ and to consider the operator $H_{\alpha;\beta,0}$.

3 Properties of the discrete spectrum

The previous discussion shows that the discrete spectrum comes for $\alpha \in (0, \frac{1}{2})$ from the partial wave operator $H_{\alpha;\beta,0}$ and the decisive quantity is the coefficient $c_{\alpha,0} = \alpha^2 - \frac{1}{4}$. Let y be the solution of

$$H_{\alpha;\beta,0} g = E_0 g\,, \tag{3.1}$$

where E_0 is the threshold value (2.6). We are going to employ the oscillation theory; following its general strategy we introduce the Prüfer variables (ρ, θ) as follows

$$\begin{pmatrix} y \\ y' \end{pmatrix} = \rho \begin{pmatrix} \cos\theta \\ \sin\theta \end{pmatrix}.$$

As it is usually the case with singular potentials [8], we can rephrase the discrete spectrum analysis as investigation of the asymptotic behavior of the function $r \mapsto \theta(r)$; for the reader convenience the needed facts from the oscillation theory are collected in Sec. 5 below.

To formulate the first main result we denote by u the d-periodic real-valued solution of the one-dimensional comparison problem,

$$\mathrm{h}_\beta u = E_0 u\,. \tag{3.2}$$

Then we can make the following claim:

Theorem 3.1 *Suppose that $\alpha \in (0, \frac{1}{2})$ and put*

$$c_{\text{crit}} := -\frac{1}{4} \left(\frac{1}{d} \int_0^d \frac{1}{u^2}\, \mathrm{d}x \right)^{-1} \left(\frac{1}{d} \int_0^d u^2\, \mathrm{d}x \right)^{-1}$$

then E_0 is an accumulation point of $\sigma_{\text{disc}}(H_{\alpha;\beta,0})$ provided $\frac{c_{\alpha,0}}{c_{\text{crit}}} > 1$, while for $\frac{c_{\alpha,0}}{c_{\text{crit}}} \leq 1$ the operator has at most finite number of eigenvalues below E_0 with the multiplicity taken into account.

Proof. The asymptotic properties of the function θ can be found in a way similar to that used in [17]. Let u, v be linearly independent real-valued solutions of equation (3.2), where u is the mentioned d-periodic function involved in the definition of c_{crit}, chosen in such a way that the Wronskian $W[u, v] = 1$. Furthermore, we introduce the generalized Prüfer variables

$$\begin{pmatrix} y \\ y' \end{pmatrix} = \begin{pmatrix} u & v \\ u' & v' \end{pmatrix} a \begin{pmatrix} \sin\gamma \\ -\cos\gamma \end{pmatrix}, \tag{3.3}$$

where a is a smooth positive function and γ is continuous in view of [8, Lemma 3.4]. On the other hand, according to [17, Prop. 1] the functions $\gamma(\cdot)$ and $\theta(\cdot)$ have the same asymptotics up to the constant. Consequently, it is sufficient to investigate the asymptotics of $\gamma(\cdot)$ which we will do using the expression

$$\gamma' = \frac{c_{\alpha,0}}{r^2} (u \sin\gamma - v\cos\gamma)^2 = c_{\alpha,0}\, u^2 \cos^2\gamma \left(\frac{1}{r} \tan\gamma - \frac{v}{ru} \right)^2,$$

which can be obtained from (3.3) by a direct computation using (3.1) and the Wronskian properties of the functions u, v. In the next step we employ the Kepler transformation

$$\tan\phi = \frac{1}{r} \tan\gamma - \frac{1}{r}\frac{v}{u},$$

which yields

$$\phi' = \frac{1}{r} \left(-\sin\phi\cos\phi + \mathcal{B}(r)\sin^2\phi + \mathcal{A}(r)\cos^2\phi \right), \tag{3.4}$$

where $\mathcal{A}$ and $\mathcal{B}$ are the d-periodic functions defined by

$$\mathcal{B}(r) := c_{\alpha,0} u(r)^2 \quad \text{and} \quad \mathcal{A}(r) := -\frac{1}{u(r)^2}. \tag{3.5}$$

The Kepler transformation preserves the asymptotics, i.e. $\gamma(r) = \phi(r) + \mathcal{O}(1)$ holds as $r \to \infty$, thus we may inspect the asymptotics of $\phi(\cdot)$. This can be done in the same way as for regular period potentials. Specifically, we define

$$\overline{\phi}(r) := \frac{1}{d} \int_r^{r+d} \phi(\xi)\, \mathrm{d}\xi\,, \quad r > R_0\,, \tag{3.6}$$

for some $R_0 > 0$. Proposition 2 of [17] allows us to conclude that $\overline{\phi}(r) = \phi(r) + o(1)$ and

$$\overline{\phi}'(r) = \frac{1}{r}\left(-\sin\overline{\phi}\cos\overline{\phi} + B\sin^2\overline{\phi} + A\cos^2\overline{\phi}\right) + \mathcal{O}(r^{-2})\,, \tag{3.7}$$

where

$$A := \frac{1}{d}\int_0^d \mathcal{A}(r)\,\mathrm{d}r \quad \text{and} \quad B := \frac{1}{d}\int_0^d \mathcal{B}(r)\,\mathrm{d}r\,.$$

Now we apply Proposition 3 of [17] which states that $\overline{\phi}$ is bounded provided $4AB < 1$ and unbounded if $4AB > 1$. Combining this fact with the observation that

$$4AB = -4c_{\alpha,0}\left(\frac{1}{d}\int_0^d \frac{1}{u^2}\,\mathrm{d}r\right)\left(\frac{1}{d}\int_0^d u^2\,\mathrm{d}r\right) = \frac{c_{\alpha,0}}{c_{\mathrm{crit}}}$$

we come to the claim of theorem for any $c_{\alpha,0}$ apart from the case $c_{\alpha,0} = c_{\mathrm{crit}}$. To complete the proof we note that

$$\lim_{r\to 0} r(\log r)^2 \left(\overline{\phi}'(r) - \frac{1}{r}\left(-\sin\overline{\phi}\cos\overline{\phi} + B\sin^2\overline{\phi} + A\cos^2\overline{\phi}\right)\right) = 0\,,$$

cf. (3.7). Applying now Proposition 4 of [17] we conclude that, if $4AB = 1$ then $\overline{\phi}$ is globally bounded. This equivalently means that for $c_{\alpha,0} = c_{\mathrm{crit}}$ at most finite number of discrete spectrum below E_0 can exist. ∎

This allows us to prove the following claim.

Theorem 3.2 *There exists an $\alpha_{\mathrm{crit}}(\beta) = \alpha_{\mathrm{crit}} \in (0, \frac{1}{2})$ such that for $\alpha \in (0, \alpha_{\mathrm{crit}})$ the operator $H_{\alpha,\beta}$ has infinitely many eigenvalues accumulating at the threshold E_0, the multiplicity taken into account, while for $\alpha \in [\alpha_{\mathrm{crit}}, \frac{1}{2})$ the cardinality of discrete spectrum is finite.*

Proof. The function $\alpha \mapsto c_{\alpha,0} = \alpha^2 - \frac{1}{4}$ is increasing in $(0, \frac{1}{2})$. Thus it suffices to show that $c_{\rm crit} \in (-\frac{1}{4}, 0)$ which is an easy consequence of Schwartz inequality,

$$c_{\rm crit} := -\frac{1}{4}\left(\frac{1}{d}\int_0^d \frac{1}{u^2}\,{\rm d}x\right)^{-1}\left(\frac{1}{d}\int_0^d u^2\,{\rm d}x\right)^{-1} > -\frac{1}{4}\left(\frac{1}{d}\int_0^d {\rm d}x\right)^{-2} = -\frac{1}{4}\,;$$

note that the inequality is sharp because the function u is nonconstant. The claim then follows from Theorem 3.1 if we set $\alpha_{\rm crit} := \sqrt{c_{\rm crit} + \frac{1}{4}}$. ∎

Moreover, in our present case the critical value can be computed explicitly because we know the function u which is equal to

$$u(x) = \begin{cases} {\rm e}^{-\kappa_0(x-d/2)} + {\rm e}^{\kappa_0 d}{\rm e}^{\kappa_0(x-d/2)} & \text{for } 0 < x < \frac{d}{2}\,, \\ {\rm e}^{\kappa_0 d}{\rm e}^{-\kappa_0(x-d/2)} + {\rm e}^{\kappa_0(x-d/2)} & \text{for } \frac{d}{2} < x < d \end{cases} \tag{3.8}$$

cf. [2, Sec. III.2.3], where $i\kappa_0 = k_0$, $k_0^2 = E_0$. Note that the function is obviously real-valued if $\beta < 0$ so that $E_0 < 0$ and $\kappa_0 > 0$, in the opposite case with $\beta > 0$ we have $E_0 > 0$ and κ_0 is purely imaginary, nevertheless u is a multiple of a real-valued function again. A straightforward calculations then yields

$$D_1 := \frac{1}{d}\int_0^d u^2\,{\rm d}x = \frac{2}{d}\,{\rm e}^{\kappa_0 d}\left(\frac{1}{2\kappa_0}\left({\rm e}^{\kappa_0 d} - {\rm e}^{-\kappa_0 d}\right) + d\right)$$

and

$$D_2 := \frac{1}{d}\int_0^d \frac{1}{u^2}\,{\rm d}x = \frac{1}{d\kappa_0}\,{\rm e}^{-\kappa_0 d}\left(\frac{1}{2} - \frac{1}{1+{\rm e}^{\kappa_0 d}}\right)\,.$$

Using this notation we have

$$c_{\rm crit} = -\frac{1}{4}\frac{1}{D_1 D_2}\,.$$

These expressions allow us, in particular, to find the behavior of the critical flux values in the asymptotic regimes. In the *weak coupling constant case*, $\beta \to 0$, we have $\kappa_0 \to 0$ and the quantities D_1 and D_2 have the following limits

$$D_1 \to 4\,, \quad D_2 \to \frac{1}{4} \quad \text{as} \quad \beta \to 0\,.$$

This implies $c_{\rm crit} \to -\frac{1}{4}$, and therefore

$$\alpha_{\rm crit}(\beta) \to 0+ \quad \text{as} \quad \beta \to 0 \tag{3.9}$$

which is certainly not surprising in view of the fact that the discrete spectrum is empty for $\beta = 0$.

In the *strong coupling constant case* one has to take the sign of β into account as the spectral condition takes a different form,

$$\coth\left(\frac{1}{2}\kappa d\right) = \frac{2\kappa}{|\beta|} \qquad \text{and} \qquad \cot\left(\frac{1}{2}kd\right) = \frac{2k}{\beta}$$

for $\mp\beta \to \infty$, and $E_0(\beta)$ tends to $-\infty$ and $\left(\frac{\pi}{d}\right)^2$, respectively. In both cases, however, $c_{\rm crit}$ tends to zero, exponentially fast for the attractive δ interactions when

$$c_{\rm crit} \approx -\frac{d^2}{8}\, \mathrm{e}^{-|\beta| d/2}\,.$$

Furthermore, this yields

$$\alpha_{\rm crit}(\beta) \to \frac{1}{2}- \quad \text{as} \quad \beta \to \pm\infty\,. \tag{3.10}$$

4 Nonexistence of the discrete spectrum for weak δ interactions

The above results tell us nothing about the spectrum of $H_{\alpha;\beta}$ for $\alpha \in [\alpha_{\rm crit}, \frac{1}{2})$, in particular, we do not know whether the operator may have some eigenvalues. Our aim now is to show that for a fixed α, with the exception of the nonmagnetic and half-of-the-quantum cases, we have

$$\sigma_{\rm disc}(H_{\alpha;\beta}) = \emptyset\,.$$

provided the involved δ interaction is sufficiently weak. Using a modified version of the Hardy inequality, we are going to prove the following claim:

Theorem 4.1 *Given $\alpha \in (0, \frac{1}{2})$ there exists a $\beta_0 > 0$ such that for any $|\beta| < \beta_0$ the operator $H_{\alpha;\beta}$ has no discrete spectrum.*

Proof. To show that the discrete spectrum is void it suffices to investigate the 'lowest' partial-wave component $H_{\alpha;\beta,0}$. Consider the quadratic form associated with the 'shifted' operator $H_{\alpha;\beta,0} - E_0$,

$$q_{\alpha;\beta,0}[f] := \tag{4.1}$$
$$\int_0^\infty |f(r)'|^2 \,\mathrm{d}r + c_{\alpha,0} \int_0^\infty \frac{1}{r^2} |f(r)|^2 \,\mathrm{d}r + \beta \sum_n \int_{\mathcal{C}_{r_n}} |f(r)|^2 \,\mathrm{d}\mu_{\mathcal{C}_{r_n}} - E_0 \|f\|^2 \,,$$

where $\mu_{\mathcal{C}_{r_n}}$ defines the arc length measure on $\mathcal{C}_{r_n}$ and $c_{\alpha,0} \in (-1/4, 0)$, moreover, $f \in D(H_{\alpha;\beta,0})$, i.e. that it satisfies the boundary conditions given by (2.4) and (2.5). Without loss of generality we may assume that f is a real function. As in the previous discussion u stands for the periodic function defining the 'lowest'generalized eigenfunction of $H_{\alpha;\beta,0}$. We may assume that u is positive, then from the explicit expression (3.8) we see that for a fixed $\beta_1 > 0$ there exists a $C_{\min} > 0$ such that $u \geq C_{\min}$ holds for any $|\beta| \leq \beta_1$. Furthermore, we put $\chi = \frac{f}{u}$; one can easily check that $\chi \in H_0^{2,2}(\mathbb{R}_+)$. Integrating by parts and using the boundary conditions (2.4) and (2.5) we get

$$q_{\alpha;\beta,0}[u\chi] = -\int_0^\infty u\chi(u\chi)'' \,\mathrm{d}r + c_{\alpha,0} \int_0^\infty u^2 \frac{\chi^2}{r^2} \,\mathrm{d}r - E_0 \|u\chi\|^2 \,.$$

After expanding the second derivative and using the equation that u as a generalized eigenfunction satisfies we get

$$\begin{aligned} q_{\alpha;\beta,0}[u\chi] &= \int_0^\infty u^2 \left(-\chi\chi'' + \frac{c_{\alpha,0}}{r^2} \chi^2 \right) \mathrm{d}r - \int_0^\infty (u^2)' \chi\chi' \,\mathrm{d}r \\ &= \int_0^\infty u^2 (\chi')^2 \,\mathrm{d}r + c_{\alpha,0} \int_0^\infty u^2 \frac{\chi^2}{r^2} \,\mathrm{d}r \,, \end{aligned} \tag{4.2}$$

where in the second step we performed integration by parts in the last expression of the first line with the boundary term vanishing due to (2.2). The following lemma will be useful in the further discussion.

Lemma 4.2 *We have*

$$q_{\alpha;\beta,0}[u\chi] > \alpha^2 \int_0^\infty u^2 \frac{\chi^2}{r^2} \,\mathrm{d}r - \frac{1}{2} \int_0^\infty (u^2)' \frac{\chi^2}{r} \,\mathrm{d}r \,. \tag{4.3}$$

Proof. To prove the claim we start from the expression

$$\int_0^\infty u^2((r^{-1/2}\chi)')^2\, r\mathrm{d}r = \int_0^\infty u^2 \left(-\frac{1}{2}r^{-3/2}\chi + r^{-1/2}\chi'\right)^2 r\mathrm{d}r$$
$$= \int_0^\infty u^2 \left(\frac{1}{4r^2}\chi^2 - \frac{1}{r}\chi\chi' + (\chi')^2\right) \mathrm{d}r\,. \qquad (4.4)$$

On the other hand, the second term in (4.4) can be rewritten as

$$-\int_0^\infty u^2 \frac{\chi\chi'}{r}\, \mathrm{d}r = -\frac{1}{2}\int_0^\infty u^2 \frac{(\chi^2)'}{r} \mathrm{d}r = \frac{1}{2}\int_0^\infty \left(\frac{(u^2)'}{r} - \frac{u^2}{r^2}\right) \chi^2\, \mathrm{d}r\,,$$

where we have again employed integration by parts in combination with (2.2); inserting this to (4.4) we get

$$\int_0^\infty u^2((r^{-1/2}\chi)')^2\, r\mathrm{d}r = \int_0^\infty u^2 \left((\chi')^2 - \frac{\chi^2}{4r^2}\right) \mathrm{d}r + \frac{1}{2}\int_0^\infty \frac{(u^2)'}{r}\chi^2\, \mathrm{d}r\,.$$

Since $\int_0^\infty u^2((r^{-1/2}\chi)')^2 r\mathrm{d}r > 0$, taking into account expression (4.2) and using $c_{\alpha,0} = \alpha^2 - \frac{1}{4}$ we obtain the claim of lemma. ∎

With a further purpose in mind we introduce a symbol for the second term at the right-hand side of (4.3),

$$\tilde{q}[\chi] := -\frac{1}{2}\int_0^\infty (u^2)'\, \frac{\chi^2}{r}\, \mathrm{d}r\,.$$

Our next aim it to show that $\tilde{q}[\cdot]$ is small with respect to $q_{\alpha;\beta,0}[\cdot]$. This the contents of the following lemma.

Lemma 4.3 *We have*

$$|\tilde{q}[\chi]| \le \eta(\beta)\int_0^\infty (\chi')^2\, \mathrm{d}r\,, \qquad (4.5)$$

where the function $\eta(\cdot)$ *behave asymptotically as*

$$\eta(\beta) = \mathcal{O}(\kappa_0(\beta))$$

for β *small. Here* $\kappa_0 = \kappa_0(\beta)$ *is the quantity introduced in* (3.8) *and the expression on the right-hand side does not depend on* χ*.*

Proof. Note first that an integration by parts in combination with conditions (2.2) yields

$$\tilde{q}[\chi] = \frac{1}{2}\int_0^\infty u^2 \left(\frac{\chi^2}{r}\right)' \mathrm{d}r = \frac{1}{2}\int_0^\infty \left(u^2(r) - u^2(0)\right) \left(\frac{\chi^2}{r}\right)' \mathrm{d}r =$$

$$= \frac{1}{2}\int_0^\infty \left(u^2(r) - u^2(0)\right) \left(\frac{2\chi\chi'}{r} - \frac{\chi^2}{r^2}\right) \mathrm{d}r\,.$$

On the other hand, from the explicit expression (3.8) we get easily

$$\left|u^2(r) - u^2(0)\right| = \mathcal{O}(\kappa_0(\beta))$$

as $\beta \to 0$ where the right-hand side does not depend on r since the function u is periodic, and naturally neither on χ. Consequently,

$$|\tilde{q}[\chi]| \le \eta_1(\beta) \left(\int_0^\infty \frac{|2\chi\chi'|}{r}\, \mathrm{d}r + \int_0^\infty \frac{(\chi)^2}{r^2}\, \mathrm{d}r\right), \tag{4.6}$$

where $\eta_1(\beta)$ behaves asymptotically as $\eta_1(\beta) = \mathcal{O}(\kappa_0(\beta))$. Our next aim is to estimate the first integral on the right-hand side of (4.6),

$$\tilde{q}_1[\chi] := 2\int_0^\infty \frac{|\chi\chi'|}{r}\, \mathrm{d}r\,.$$

Applying the Schwartz inequality together with the classical Hardy inequality,

$$\int_0^\infty (\chi')^2\, \mathrm{d}r > \frac{1}{4}\int_0^\infty \frac{\chi^2}{r^2}\, \mathrm{d}r\,,$$

one obtains

$$\tilde{q}_1[\chi] \le 2\left(\int_0^\infty \frac{(\chi)^2}{r^2}\, \mathrm{d}r\right)^{1/2} \left(\int_0^\infty (\chi')^2\, \mathrm{d}r\right)^{1/2}$$

$$\le \int_0^\infty \frac{(\chi)^2}{r^2}\, \mathrm{d}r + \int_0^\infty (\chi')^2\, \mathrm{d}r < 5\int_0^\infty (\chi')^2\, \mathrm{d}r\,.$$

Applying the Hardy inequality again to (4.6) and combining this with the above result we get

$$|\tilde{q}[\chi]| \le \eta(\beta) \int_0^\infty (\chi')^2\, \mathrm{d}r\,,$$

where $\eta(\beta) := 6\eta_1(\beta)$. This completes the proof of lemma. ∎

Proof of Theorem 4.1, continued: As we noted above, for any β satisfying $|\beta| \le \beta_1$ we have $\min_{r\ge 0} u(r) \ge C_{\min}$. Then the above lemma tells us that

$$|\tilde{q}[\chi]| \le \eta(\beta)\frac{1}{C_{\min}}\int_0^\infty u^2(\chi')^2\,\mathrm{d}r\,, \tag{4.7}$$

Since by Lemma 4.2 we have

$$q_{\alpha;\beta,0}[u\chi] > \alpha^2\int_0^\infty u^2\frac{\chi^2}{r^2}\mathrm{d}r + \tilde{q}[\chi]\,, \tag{4.8}$$

relation (4.7) yields

$$|\tilde{q}[\chi]| \le \tilde{\eta}(\beta)\int_0^\infty u^2(\chi')^2\mathrm{d}r \quad \text{with} \quad \tilde{\eta}(\beta) = \mathcal{O}(\kappa_0(\beta))\,. \tag{4.9}$$

Combining relations (4.9) and (4.8) we get

$$(1+\tilde{\eta}(\beta))\left(\int_0^\infty u^2(\chi')^2\,\mathrm{d}r + c_{\alpha,0}\int_0^\infty u^2\frac{\chi^2}{r^2}\,\mathrm{d}r\right) > \left(\alpha^2 + c_{\alpha,0}\tilde{\eta}(\beta)\right)\int_0^\infty u^2\frac{\chi^2}{r^2}\,\mathrm{d}r$$

which implies

$$q_{\alpha;\beta,0}[u\chi] > \frac{\alpha^2 + c_{\alpha,0}\tilde{\eta}(\beta)}{1+\tilde{\eta}(\beta)}\int_0^\infty u^2\frac{\chi^2}{r^2}\,\mathrm{d}r\,.$$

By Lemma 4.3 there is a $\beta_0 \in (0,\beta_1)$ such that for any β satsisfying $|\beta| \le \beta_0$ the pre-integral factor in the last formula is positive which means that we have

$$(H_{\alpha;\beta,0}f, f) - E_0\|f\|^2 > 0$$

for any real function $f \in D(H_{\alpha;\beta,0})$. The same holds *mutatis mutandis* for the full Hamiltonian $H_{\alpha;\beta}$ which completes the proof. ∎

5 Oscillation theory tools

To make the paper self-contained, we collect in this section the needed results of oscillation theory for singular potentials derived in [8]. Note that they extend the theory of Wronskian zeros for regular potentials developed in [9], related results can also be found in [18].

Consider points interaction localized at $x_n \in (l_-, l_+)$, where $n \in M \in \mathbb{N}$. Moreover, assume that $q \in L^1_{\rm loc}(l_-, l_+)$ and combine the singular and regular potential in the operator on $L^2(l_-, l_+)$ acting as

$$Tu(x) = -u''(x) + q(x)u(x)\,,$$

with the domain

$$\begin{aligned} D(T) :=&\{f, f, \in AC_{\rm loc}(l_-, l_+) \setminus (\cup_{n\in M}\{x_n\}) \; : \\ & Tu \in L^2(l_-, l_+)\,,\; \partial_r f(x_n^+) - \partial_r f(x_n^-) = \beta f(x_n)\,,\; n \in M\}\,. \end{aligned}$$

In general, the operator T is symmetric and we denote by H its self-adjoint extension satisfying either one of the following conditions

- T is limit point in at least one endpoint $l_\pm$
- H is defined by separated boundary conditions at the endpoints

Suppose that there exist $\psi_\pm$ that satisfy the boundary conditions defining H at $l_\pm$ and $T\psi_\pm = E\psi_\pm$. Furthermore, let $W_0(u_1, u_2)$ stand for the number of zeros of the Wronskian $W(u_1, u_2) = u_1 u_2' - u_1' u_2$ in (l_-, l_+) and denote $N_0(E_1, E_2) := \dim \mathrm{Ran} P_{(E_1,E_2)}$, where $E_1 < E_2$ and $P_{(E_1,E_2)}$ is the corresponding spectral measure of H. Then we have [8]

$$W_0(\psi_-(E_1), \psi_+(E_2)) = N_0(E_1, E_2)\,. \tag{5.1}$$

In particular, the above equivalence allows us to estimate the cardinality of the discrete spectrum below the essential spectrum threshold E_0. Indeed, suppose $E < E_0$. Then, in the same way as for regular potentials, there exist $u = \psi_\pm(E)$ with the corresponding Prüfer angle θ bounded for E large negative. Expressing the Wronskian in the terms of the Prüfer variables $W[\psi_-(E), \psi_+(E_0)] = \rho(x)\rho_0(x)\sin(\theta_0(x) - \theta(x))$ we come to the conclusion that *the number of discrete spectrum points of H below E_0 is finite* iff $\theta_0(\cdot)$ *is bounded.*

6 Concluding remarks

The main aim of this letter is to show that the influence of a local magnetic field on the Welsh eigenvalues depends nontrivially on the magnetic flux. In order to make the exposition simple we focused on the simple setting with radial δ potentials and an Aharonov-Bohm field, however, we are convinced that the conclusions extend to other potentials magnetic field profiles, as long as the radial symmetry and periodicity are preserved. This could be a subject of further investigation, as well as the remaining spectral properties of the present simple model such the eigenvalue accumulation for $\alpha \in (0, \alpha_{\text{crit}})$ or (non)existence of eigenvalues for $\alpha \in [\alpha_{\text{crit}}, \frac{1}{2})$ and an arbitrary $\beta \neq 0$. It would be also interesting to revisit from the present point of view situations in which the radially periodic interaction is of a purely magnetic type with zero total flux [12].

Acknowledgements

The work was supported by the project 17-01706S of the the Czech Science Foundation (GAČR) and the project DEC-2013/11/B/ST1/03067 of the Polish National Science Centre (NCN).

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