On the existence of stationary solutions for some systems of non-Fredholm integro-differential equations with superdiffusion

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Abstract. We establish the existence of stationary solutions for certain systems of reaction-diffusion equations with superdiffusion. The corresponding elliptic problem involves the operators with or without Fredholm property. The fixed point technique in appropriate H^2 spaces of vector functions is employed.

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1 Introduction

Let us recall that a linear operator L acting from a Banach space E into another Banach space F has the Fredholm property when its image is closed, the dimension of its kernel and the codimension of its image are finite. As a consequence, the equation $Lu = f$ is solvable if and only if $\phi_k(f) = 0$ for a finite number of functionals ϕ_k from the dual space F^*. These properties of Fredholm operators are broadly used in various methods of linear and nonlinear analysis.

Elliptic equations studied in bounded domains with a sufficiently smooth boundary satisfy the Fredholm property when the ellipticity condition, proper ellipticity and Lopatinskii conditions are satisfied (see e.g. [1], [13], [20]), which is the main result of the theory of linear elliptic equations. When working in unbounded domains, these conditions may not be sufficient and the Fredholm property may not be fulfilled. For example, for the Laplace operator, $Lu = \Delta u$ considered in $\mathbb{R}^d$ Fredholm property does not hold when the problem is studied either in Hölder spaces, such that $L : \mathbb{C}^{2+\alpha}(\mathbb{R}^d) \to \mathbb{C}^{\alpha}(\mathbb{R}^d)$ or in Sobolev spaces, $L : H^2(\mathbb{R}^d) \to L^2(\mathbb{R}^d)$.

For linear elliptic equations considered in unbounded domains the Fredholm property is satisfied if and only if, in addition to the conditions stated above, the limiting operators are invertible (see [21]). In some trivial cases, the limiting operators can be constructed explicitly. For example, when

$$Lu = a(x)u'' + b(x)u' + c(x)u, \quad x \in \mathbb{R},$$

with the coefficients of the operator having limits at infinity,

$$a_\pm = \lim_{x \to \pm\infty} a(x), \quad b_\pm = \lim_{x \to \pm\infty} b(x), \quad c_\pm = \lim_{x \to \pm\infty} c(x),$$

the limiting operators are given by

$$L_\pm u = a_\pm u'' + b_\pm u' + c_\pm u.$$

Because the coefficients here are constants, the essential spectrum of the operator, which is the set of complex numbers λ for which the operator $L - \lambda$ does not have the Fredholm property, can be found explicitly via the standard Fourier transform, such that

$$\lambda_\pm(\xi) = -a_\pm \xi^2 + b_\pm i\xi + c_\pm, \quad \xi \in \mathbb{R}.$$

The limiting operators are invertible if and only if the origin does not belong to the essential spectrum.

For general elliptic equations the analogous assertions hold. The Fredholm property is satisfied when the essential spectrum does not contain the origin or when the limiting operators are invertible. These conditions may not be written explicitly.

For non-Fredholm operators we may not apply the standard solvability conditions and in a general case solvability conditions are unknown. However, solvability conditions were obtained recently for some classes of operators. For example, consider the following problem

$$Lu \equiv \Delta u + au = f \tag{1.1}$$

in $\mathbb{R}^d$, $d \in \mathbb{N}$ with a positive constant a. Here the operator L and its limiting operators coincide. The corresponding homogeneous problem has a nontrivial bounded solution, such that the Fredholm property is not satisfied. Because the differential operator contained in (1.1) has constant coefficients, we are able to obtain the solution explicitly by applying the standard Fourier transform. In [31] we derived the following solvability relations. Let $f(x) \in L^2(\mathbb{R}^d)$ and $xf(x) \in L^1(\mathbb{R}^d)$. Then equation (1.1) has a unique solution in $H^2(\mathbb{R}^d)$ if and only if

$$\left(f(x), \frac{e^{ipx}}{(2\pi)^{\frac{d}{2}}} \right)_{L^2(\mathbb{R}^d)} = 0, \quad p \in S^d_{\sqrt{a}} \quad a.e.$$

Here and below S^d_r denotes the sphere in $\mathbb{R}^d$ of radius r centered at the origin. Thus, although the Fredholm property is not satisfied for this operator, we can formulate solvability relations similarly. Note that this similarity is only formal because the range of the operator is not closed. In the situation when the operator contains a scalar potential, such that

$$Lu \equiv \Delta u + b(x)u = f,$$

we are not able to use the standard Fourier transform directly. However, solvability relations in three dimensions can be obtained by virtue of the spectral and the scattering theory of Schrödinger type operators (see [23]). Analogously to the constant coefficient case, solvability relations are expressed in terms of orthogonality to solutions of the adjoint homogeneous equation. We derive solvability conditions for several other examples of non Fredholm linear elliptic operators (see [21] – [29], [31]).

Solvability relations are crucial in the analysis of nonlinear elliptic problems. In the presence of non-Fredholm operators, in spite of some progress in studies of linear equations, nonlinear non-Fredholm operators were analyzed only in few examples (see [7]– [9], [30], [31]). Evidently, this situation can be explained by the fact that the majority of methods of linear and nonlinear analysis rely on the Fredholm property. In the present article we study some systems of N nonlinear integro-differential reaction-diffusion type equations, for which the Fredholm property may not be satisfied:

$$\frac{\partial u_k}{\partial t} = -\sqrt{-\Delta}u_k + \int_\Omega G_k(x-y)F_k(u_1(y,t), u_2(y,t), ..., u_N(y,t), y)dy + a_k u_k, \ 1 \le k \le N. \tag{1.2}$$

Here $\{a_k\}_{k=1}^N$ are nonnegative constants, $\Omega \subseteq \mathbb{R}^d$, $\quad d = 1, 2, 3$ are the more physically relevant dimensions. The operator $\sqrt{-\Delta}$ is defined via the spectral calculus. System (1.2) describes a particular case of superdiffusion actively treated in the context of various applications in plasma physics and turbulence (see e.g. [6], [19]), surface diffusion (see e.g. [14], [16]), semiconductors (see e.g. [18]) and so on. The superdiffusion can be understood as a random process of particle motion characterized by the probability density distribution of jump length. The moments of this density distribution are finite for normal diffusion, but this is not the case for superdiffusion. Asymptotic behavior at infinity of the probability density function determines the value of the power of the negative Laplacian (see e.g. [15]).

In population dynamics the integro-differential equations are used to describe biological systems with intra-specific competition and nonlocal consumption of resources (see e.g. [2], [4], [10]). The stability issues for the travelling fronts of reaction- diffusion type problems with the essential spectrum of the linearized operator crossing the imaginary axis were also treated in [3] and [11]. Note that the single equation of (1.2) type has been studied in [32]. Reaction-diffusion type problems in which in the diffusion term the Laplacian is replaced by the nonlocal operator with an integral kernel were treated in [17].

The nonlinear terms of system (1.2) will fulfill the following regularity requirements.

Assumption 1. *Functions $F_k(u,x) : \mathbb{R}^N \times \Omega \to \mathbb{R}, \ 1 \le k \le N$ are such that*

$$\sqrt{\sum_{k=1}^N F_k^2(u,x)} \le Q|u|_{\mathbb{R}^N} + h(x) \quad for \quad u \in \mathbb{R}^N, \ x \in \Omega, \tag{1.3}$$

with a constant $Q > 0$ *and* $h(x) : \Omega \to \mathbb{R}^+, \quad h(x) \in L^2(\Omega)$*. Furthermore, they are Lipschitz continuous functions, such that*

$$\sqrt{\sum_{k=1}^{N}(F_k(u^{(1)}, x) - F_k(u^{(2)}, x))^2} \leq l|u^{(1)} - u^{(2)}|_{\mathbb{R}^N} \quad for \quad any \quad u^{(1),(2)} \in \mathbb{R}^N, \quad x \in \Omega, \tag{1.4}$$

where a constant $l > 0$*.*

Here and below we use the notations for a vector $u := (u_1, u_2, ..., u_N) \in \mathbb{R}^N$ and its norm $|u|_{\mathbb{R}^N} := \sqrt{\sum_{k=1}^{N} u_k^2}$. Evidently, the stationary solutions of problem (1.2), if any exist, will satisfy the system of nonlocal elliptic equations

$$-\sqrt{-\Delta}u_k + \int_{\Omega} G_k(x-y)F_k(u_1(y), u_2(y), ..., u_N(y), y)dy + a_k u_k = 0, \;\; a_k \geq 0, \;\; 1 \leq k \leq N.$$

For the technical purposes we consider the auxiliary semi-linear system

$$\sqrt{-\Delta}u_k - a_k u_k = \int_{\Omega} G_k(x-y)F_k(v_1(y), v_2(y), ..., v_N(y), y)dy, \;\; 1 \leq k \leq N. \tag{1.5}$$

We denote $(f_1(x), f_2(x))_{L^2(\Omega)} := \int_{\Omega} f_1(x)\bar{f}_2(x)dx$, with a slight abuse of notations in the case when these functions do not belong to $L^2(\Omega)$, like for example those used in the orthogonality relations of the assumption below. Indeed, if $f_1(x) \in L^1(\Omega)$ and $f_2(x)$ is bounded there, then the integral over Ω mentioned above is well defined. We begin the article with the treatment of the whole space case, such that $\Omega = \mathbb{R}^d$ and the corresponding Sobolev space is equipped with the norm

$$\|u\|^2_{H^2(\mathbb{R}^d, \; \mathbb{R}^N)} := \sum_{k=1}^{N} \|u_k\|^2_{H^2(\mathbb{R}^d)} = \sum_{k=1}^{N} \{\|u_k\|^2_{L^2(\mathbb{R}^d)} + \|\Delta u_k\|^2_{L^2(\mathbb{R}^d)}\},$$

where $u(x) : \mathbb{R}^d \to \mathbb{R}^N$. The primary obstacle in solving problem (1.5) is that operators $\sqrt{-\Delta} - a_k : H^2(\mathbb{R}^d) \to L^2(\mathbb{R}^d), \;\; a_k \geq 0$ fail to satisfy the Fredholm property. The similar situations in linear equations, which can be self- adjoint or non self-adjoint involving non Fredholm second, fourth and sixth order differential operators or even systems of equations including non Fredholm operators have been treated actively in recent years (see [23]-[29]). We are able to prove that system of equations (1.5) defines a map $T_a : H^2(\mathbb{R}^d, \; \mathbb{R}^N) \to H^2(\mathbb{R}^d, \; \mathbb{R}^N), \;\; a_k \geq 0, \;\; 1 \leq k \leq N$, which is a strict contraction under stated technical conditions. We make the following assumption on the integral kernels contained in the nonlocal parts of system (1.5).

Assumption 2. *Let* $G_k(x) : \mathbb{R}^d \to \mathbb{R}, \;\; G_k(x) \in W^{1,1}(\mathbb{R}^d), \;\; 1 \leq k \leq N, \;\; 1 \leq d \leq 3$ *and* $1 \leq m \leq N-1, \;\; m \in \mathbb{N}$ *with* $N \geq 2$*.*

I) Let $a_k > 0,\ 1 \leq k \leq m$, assume that $xG_k(x) \in L^1(\mathbb{R}^d)$ and

$$\left(G_k(x), \frac{e^{\pm i a_k x}}{\sqrt{2\pi}}\right)_{L^2(\mathbb{R})} = 0 \text{ when } d = 1. \tag{1.6}$$

$$\left(G_k(x), \frac{e^{ipx}}{(2\pi)^{\frac{d}{2}}}\right)_{L^2(\mathbb{R}^d)} = 0 \text{ for } p \in S^d_{a_k} \text{ a.e. when } d = 2, 3. \tag{1.7}$$

II) Let $a_k = 0,\ m+1 \leq k \leq N$, assume that $xG_k(x) \in L^1(\mathbb{R}^d)$ and

$$(G_k(x), 1)_{L^2(\mathbb{R}^d)} = 0. \tag{1.8}$$

Let us use the hat symbol here and below to designate the standard Fourier transform, such that

$$\widehat{G_k}(p) := \frac{1}{(2\pi)^{\frac{d}{2}}} \int_{\mathbb{R}^d} G_k(x) e^{-ipx} dx,\ p \in \mathbb{R}^d. \tag{1.9}$$

Thus

$$\|\widehat{G_k}(p)\|_{L^\infty(\mathbb{R}^d)} \leq \frac{1}{(2\pi)^{\frac{d}{2}}} \|G_k\|_{L^1(\mathbb{R}^d)}.$$

Let us introduce the following auxiliary quantities

$$M_k := \max\left\{\left\|\frac{\widehat{G_k}(p)}{|p| - a_k}\right\|_{L^\infty(\mathbb{R}^d)},\ \left\|\frac{p^2\widehat{G_k}(p)}{|p| - a_k}\right\|_{L^\infty(\mathbb{R}^d)}\right\},\ 1 \leq k \leq m. \tag{1.10}$$

$$M_k := \max\left\{\left\|\frac{\widehat{G_k}(p)}{p}\right\|_{L^\infty(\mathbb{R}^d)},\ \left\|p\widehat{G_k}(p)\right\|_{L^\infty(\mathbb{R}^d)}\right\},\ m+1 \leq k \leq N. \tag{1.11}$$

Note that expressions (1.10) and (1.11) are finite by virtue of Lemma A1 in one dimension and Lemma A2 for $d = 2, 3$ of the Appendix of [32] under our Assumption 2. Therefore, we define

$$M := \max M_k,\ 1 \leq k \leq N \tag{1.12}$$

with M_k given by (1.10) and (1.11). We have the following proposition.

Theorem 3. *Let $\Omega = \mathbb{R}^d,\ d = 1, 2, 3$, Assumptions 1 and 2 hold and $\sqrt{2}(2\pi)^{\frac{d}{2}} M l < 1$. Then the map $T_a v = u$ on $H^2(\mathbb{R}^d,\ \mathbb{R}^N)$ defined by the system of equations (1.5) possesses a unique fixed point $v_a(x) : \mathbb{R}^d \to \mathbb{R}^N$, which is the only stationary solution of problem (1.2) in $H^2(\mathbb{R}^d,\ \mathbb{R}^N)$. This fixed point $v_a(x)$ is nontrivial provided the intersection of supports of the Fourier transforms of functions $supp\widehat{F_k(0, x)}(p) \cap supp\widehat{G_k}(p)$ is a set of nonzero Lebesgue measure in $\mathbb{R}^d$ for some $1 \leq k \leq N$.*

Then we turn our attention to the studies of the analogous problem on the interval $\Omega = I := [0,\ 2\pi]$ with periodic boundary conditions for the solution vector function and its first derivative. We assume the following about the integral kernels present in the nonlocal parts of system (1.5) in such case.

Assumption 4. *Let $G_k(x) : I \to \mathbb{R},\ G_k(x) \in W^{1,1}(I)$ with $G_k(0) = G_k(2\pi),\ 1 \le k \le N$, where $N \ge 3$ and $1 \le m < q \le N-1,\ m, q \in \mathbb{N}$.*

I) Let $a_k > 0$ and $a_k \ne n,\ n \in \mathbb{N}$ for $1 \le k \le m$.

II) Let $a_k = n_k,\ n_k \in \mathbb{N}$ and

$$\left(G_k(x), \frac{e^{\pm i n_k x}}{\sqrt{2\pi}}\right)_{L^2(I)} = 0 \ for\ m+1 \le k \le q. \tag{1.13}$$

III) Let $a_k = 0$ and

$$(G_k(x), 1)_{L^2(I)} = 0 \ for\ q+1 \le k \le N. \tag{1.14}$$

Let $F_k(u, 0) = F_k(u, 2\pi)$ for $u \in \mathbb{R}^N$ and $k = 1, ..., N$.

We introduce the Fourier transform for periodic functions on the $[0, 2\pi]$ interval as

$$G_{k,\ n} := \int_0^{2\pi} G_k(x) \frac{e^{-inx}}{\sqrt{2\pi}} dx,\ n \in \mathbb{Z} \tag{1.15}$$

and define the following expressions

$$P_k := \max\left\{ \left\| \frac{G_{k,\ n}}{|n| - a_k} \right\|_{l^\infty}, \left\| \frac{n^2 G_{k,\ n}}{|n| - a_k} \right\|_{l^\infty} \right\},\ 1 \le k \le m. \tag{1.16}$$

$$P_k := \max\left\{ \left\| \frac{G_{k,\ n}}{|n| - n_k} \right\|_{l^\infty}, \left\| \frac{n^2 G_{k,\ n}}{|n| - n_k} \right\|_{l^\infty} \right\},\ m+1 \le k \le q. \tag{1.17}$$

$$P_k := \max\left\{ \left\| \frac{G_{k,\ n}}{n} \right\|_{l^\infty}, \left\| n G_{k,\ n} \right\|_{l^\infty} \right\},\ q+1 \le k \le N. \tag{1.18}$$

By virtue of Lemma A3 of the Appendix of [32] under Assumption 4 the quantities given by (1.16), (1.17) and (1.18) are finite, which allows us to define

$$P := \max P_k,\ 1 \le k \le N$$

with P_k stated in formulas (1.16), (1.17) and (1.18). To study the existence of stationary solutions for our system we use the corresponding functional space

$$H^2(I) = \{v(x) : I \to \mathbb{R} \mid v(x), v''(x) \in L^2(I), \quad v(0) = v(2\pi), \quad v'(0) = v'(2\pi)\},$$

aiming at $u_k(x) \in H^2(I),\ 1 \le k \le m$. Then we introduce the following auxiliary constrained subspaces

$$H_k^2(I) := \left\{ v \in H^2(I) \mid \left(v(x), \frac{e^{\pm i n_k x}}{\sqrt{2\pi}}\right)_{L^2(I)} = 0 \right\},\ n_k \in \mathbb{N},\ m+1 \le k \le q,$$

with the goal of having $u_k(x) \in H^2_k(I), \ m+1 \le k \le q$. And, finally

$$H^2_0(I) = \{v \in H^2(I) \mid (v(x), 1)_{L^2(I)} = 0\}, \ q+1 \le k \le N.$$

Our goal is to have $u_k(x) \in H^2_0(I), \ q+1 \le k \le N$. The constrained subspaces defined above are Hilbert spaces as well (see e.g. Chapter 2.1 of [12]). The resulting space used for establishing the existence of solutions $u(x) : I \to \mathbb{R}^N$ of problem (1.5) will be the direct sum of the spaces mentioned above, namely

$$H^2_c(I, \ \mathbb{R}^N) := \oplus_{k=1}^m H^2(I) \oplus_{k=m+1}^q H^2_k(I) \oplus_{k=q+1}^N H^2_0(I),$$

such that the corresponding Sobolev norm is given by

$$\|u\|^2_{H^2_c(I, \ \mathbb{R}^N)} := \sum_{k=1}^N \{\|u_k\|^2_{L^2(I)} + \|u_k''\|^2_{L^2(I)}\},$$

where $u(x) : I \to \mathbb{R}^N$. Let us prove that the system of equations (1.5) in such case defines a map on the space mentioned above, which will be a strict contraction under given conditions.

Theorem 5. *Let $\Omega = I$, Assumptions 1 and 4 hold and $2\sqrt{\pi}Pl < 1$. Then the map $\tau_a v = u$ on $H^2_c(I, \mathbb{R}^N)$ defined by the system of equations (1.5) has a unique fixed point $v_a(x) : I \to \mathbb{R}^N$, the only stationary solution of system (1.2) in $H^2_c(I, \mathbb{R}^N)$. This fixed point $v_a(x)$ is nontrivial provided the Fourier coefficients $G_{k, \ n}F_k(0, x)_n \neq 0$ for some $k = 1, ..., N$ and some $n \in \mathbb{Z}$.*

Note that the constrained subspaces $H^2_k(I)$ and $H^2_0(I)$ involved in the direct sum of spaces $H^2_c(I, \mathbb{R}^N)$ are such that the operators

$$\sqrt{-\frac{d^2}{dx^2} - n_k} : H^2_k(I) \to L^2(I) \ \textit{and} \ \sqrt{-\frac{d^2}{dx^2}} : H^2_0(I) \to L^2(I)$$

having the Fredholm property, possess trivial kernels.

Finally, we turn our attention to the studies of our problem in the layer domain, which is the product of the two spaces, such that one is the I interval with periodic boundary conditions as in the previous part of the work and another is the whole space of dimension either one or two, namely $\Omega = I \times \mathbb{R}^d = [0, 2\pi] \times \mathbb{R}^d, \ d = 1, 2$ and $x = (x_1, x_\perp)$, where $x_1 \in I$ and $x_\perp \in \mathbb{R}^d$. The cumulative Laplacian in this context will be given by $\Delta := \frac{\partial^2}{\partial x_1^2} + \Delta_\perp$, where $\Delta_\perp := \sum_{s=1}^d \frac{\partial^2}{\partial x^2_{\perp, \ s}}$. The corresponding Sobolev space for our problem will be $H^2(\Omega, \ \mathbb{R}^N)$ of vector functions $u(x) : \Omega \to \mathbb{R}^N$, such that for $k = 1, ..., N$

$$u_k(x), \Delta u_k(x) \in L^2(\Omega), \ u_k(0, x_\perp) = u_k(2\pi, x_\perp), \ \frac{\partial u_k}{\partial x_1}(0, x_\perp) = \frac{\partial u_k}{\partial x_1}(2\pi, x_\perp),$$

where $x_\perp \in \mathbb{R}^d$ *a.e.* It is equipped with the norm

$$\|u\|^2_{H^2(\Omega,\ \mathbb{R}^N)} = \sum_{k=1}^{N} \{\|u_k\|^2_{L^2(\Omega)} + \|\Delta u_k\|^2_{L^2(\Omega)}\}.$$

Analogously to the whole space case treated in Theorem 3, the operators $\sqrt{-\Delta} - a_k : H^2(\Omega) \to L^2(\Omega)$ for $a_k \geq 0$ do not possess the Fredholm property. Let us show that system (1.5) in such case defines a map $t_a : H^2(\Omega,\ \mathbb{R}^N) \to H^2(\Omega,\ \mathbb{R}^N)$, which is a strict contraction under the corresponding technical conditions stated below.

Assumption 6. *Let $G_k(x) : \Omega \to \mathbb{R},\ G_k(x) \in W^{1,1}(\Omega),\ G_k(0, x_\perp) = G_k(2\pi, x_\perp)$ and $F_k(u, 0, x_\perp) = F_k(u, 2\pi, x_\perp)$ for $x_\perp \in \mathbb{R}^d$ a.e., $u \in \mathbb{R}^N,\ d = 1, 2$ and $k = 1, ..., N$. Let $N \geq 3$ and $1 \leq m < q \leq N - 1$ with $m, q \in \mathbb{N}$.*

I) Assume for $1 \leq k \leq m$ that we have $n_k < a_k < n_k + 1,\ n_k \in \mathbb{Z}^+ = \mathbb{N} \cup \{0\},\ x_\perp G_k(x) \in L^1(\Omega)$ and

$$\left(G_k(x_1, x_\perp), \frac{e^{inx_1}}{\sqrt{2\pi}} \frac{e^{\pm i\sqrt{a_k^2 - n^2}x_\perp}}{\sqrt{2\pi}}\right)_{L^2(\Omega)} = 0,\ |n| \leq n_k \ for\ d = 1, \tag{1.19}$$

$$\left(G_k(x_1, x_\perp), \frac{e^{inx_1}}{\sqrt{2\pi}} \frac{e^{ipx_\perp}}{2\pi}\right)_{L^2(\Omega)} = 0,\ p \in S^2_{\sqrt{a_k^2 - n^2}}\ a.e.,\ |n| \leq n_k \ for\ d = 2. \tag{1.20}$$

II) Assume for $m + 1 \leq k \leq q$ that we have $a_k = n_k,\ n_k \in \mathbb{N},\ x_\perp^2 G_k(x) \in L^1(\Omega)$ and

$$\left(G_k(x_1, x_\perp), \frac{e^{inx_1}}{\sqrt{2\pi}} \frac{e^{\pm i\sqrt{n_k^2 - n^2}x_\perp}}{\sqrt{2\pi}}\right)_{L^2(\Omega)} = 0,\ |n| \leq n_k - 1 \ for\ d = 1, \tag{1.21}$$

$$\left(G_k(x_1, x_\perp), \frac{e^{inx_1}}{\sqrt{2\pi}} \frac{e^{ipx_\perp}}{2\pi}\right)_{L^2(\Omega)} = 0,\ p \in S^2_{\sqrt{n_k^2 - n^2}}\ a.e.,\ |n| \leq n_k - 1 \ for\ d = 2, \tag{1.22}$$

$$\left(G_k(x_1, x_\perp), \frac{e^{\pm in_k x_1}}{\sqrt{2\pi}}\right)_{L^2(\Omega)} = 0,\ \left(G_k(x_1, x_\perp), \frac{e^{\pm in_k x_1}}{\sqrt{2\pi}} x_{\perp,\ s}\right)_{L^2(\Omega)} = 0,\ 1 \leq s \leq d. \tag{1.23}$$

III) Assume for $q + 1 \leq k \leq N$ that we have $a_k = 0,\ x_\perp G_k(x) \in L^1(\Omega)$ and

$$(G_k(x), 1)_{L^2(\Omega)} = 0. \tag{1.24}$$

Let us use the Fourier transform for functions on such a product of spaces, such that

$$\widehat{G}_{k,\ n}(p) := \frac{1}{(2\pi)^{\frac{d+1}{2}}} \int_{\mathbb{R}^d} dx_\perp e^{-ipx_\perp} \int_0^{2\pi} G_k(x_1, x_\perp) e^{-inx_1} dx_1,\ p \in \mathbb{R}^d,\ n \in \mathbb{Z},\ k = 1, ..., N. \tag{1.25}$$

Hence

$$\|\widehat{G}_{k,\ n}(p)\|_{L^\infty_{n,p}} := \sup_{\{p \in \mathbb{R}^d,\ n \in \mathbb{Z}\}} |\widehat{G}_{k,\ n}(p)| \leq \frac{1}{(2\pi)^{\frac{d+1}{2}}} \|G_k\|_{L^1(\Omega)}.$$

We define the following quantities

$$R_k := \max\left\{\left\|\frac{\widehat{G}_{k,\ n}(p)}{\sqrt{p^2+n^2}-a_k}\right\|_{L^\infty_{n,p}},\ \left\|\frac{(p^2+n^2)\widehat{G}_{k,\ n}(p)}{\sqrt{p^2+n^2}-a_k}\right\|_{L^\infty_{n,p}}\right\},\ k=1,...,m. \tag{1.26}$$

$$R_k := \max\left\{\left\|\frac{\widehat{G}_{k,\ n}(p)}{\sqrt{p^2+n^2}-n_k}\right\|_{L^\infty_{n,p}},\ \left\|\frac{(p^2+n^2)\widehat{G}_{k,\ n}(p)}{\sqrt{p^2+n^2}-n_k}\right\|_{L^\infty_{n,p}}\right\},\ k=m+1,...,q. \tag{1.27}$$

$$R_k := \max\left\{\left\|\frac{\widehat{G}_{k,\ n}(p)}{\sqrt{p^2+n^2}}\right\|_{L^\infty_{n,p}},\ \left\|\sqrt{p^2+n^2}\widehat{G}_{k,\ n}(p)\right\|_{L^\infty_{n,p}}\right\},\ k=q+1,...,N. \tag{1.28}$$

Assumption 6 along with Lemmas A4, A5 and A6 of the Appendix of [32] yield that the expressions given by (1.26), (1.27) and (1.28) are finite. This enables us to define

$$R := \max R_k,\ k=1,...,N$$

with R_k given in (1.26), (1.27) and (1.28). The final proposition of our article is as follows.

Theorem 7. *Let $\Omega = I\times\mathbb{R}^d,\ d=1,2$, Assumptions 1 and 6 hold and $\sqrt{2}(2\pi)^{\frac{d+1}{2}}Rl<1$. Then the map $t_a v = u$ on $H^2(\Omega,\ \mathbb{R}^N)$, which is defined by the system of equations (1.5) admits a unique fixed point $v_a(x):\Omega\to\mathbb{R}^N$, which is the only stationary solution of problem (1.2) in $H^2(\Omega,\ \mathbb{R}^N)$. This fixed point $v_a(x)$ is nontrivial provided that the intersection of supports of the Fourier images of functions $supp\widehat{F_k(0,x)}_n(p)\cap supp\widehat{G}_{k,\ n}(p)$ is a set of nonzero Lebesgue measure in $\mathbb{R}^d$ for some $k=1,...,N$ and some $n\in\mathbb{Z}$.*

Note that the maps discussed in the theorems above are applied to real valued vector functions by means of the assumptions on $F_k(u,x)$ and $G_k(x),\ k=1,...,N$ present in the nonlocal terms of problem (1.5).

2 The System in the Whole Space

Proof of Theorem 3. First let us suppose that when $\Omega=\mathbb{R}^d,\ d=1,2,3$ there exists $v(x)\in H^2(\mathbb{R}^d,\ \mathbb{R}^N)$ such that system (1.5) admits two solutions $u^{(1),(2)}(x)\in H^2(\mathbb{R}^d,\ \mathbb{R}^N)$. Thus the difference vector function $w(x):=u^{(1)}(x)-u^{(2)}(x)\in H^2(\mathbb{R}^d,\ \mathbb{R}^N)$ is a solution of the homogeneous system of equations

$$\sqrt{-\Delta}w_k = a_k w_k,\ 1\le k\le N.$$

Because the $\sqrt{-\Delta}$ operator does not have any nontrivial eigenfunctions belonging to $L^2(\mathbb{R}^d)$, we obtain $w_k(x)=0$ a.e. in $\mathbb{R}^d$ for $k=1,...,N$.

Let us choose arbitrarily a vector function $v(x) \in H^2(\mathbb{R}^d,\ \mathbb{R}^N)$ and apply the standard Fourier transform (1.9) to both sides of problem (1.5). This implies

$$\widehat{u_k}(p) = (2\pi)^{\frac{d}{2}} \frac{\widehat{G_k}(p)\widehat{f_k}(p)}{|p| - a_k},\ k = 1, ..., N. \tag{2.1}$$

Here $\widehat{f_k}(p)$ stands for the Fourier image of $F_k(v(x), x)$. We obtain the elementary estimates using expressions (1.10) and (1.11)

$$|\widehat{u_k}(p)| \leq (2\pi)^{\frac{d}{2}} M_k |\widehat{f_k}(p)| \quad \textit{and} \quad |p^2 \widehat{u_k}(p)| \leq (2\pi)^{\frac{d}{2}} M_k |\widehat{f_k}(p)|,\ k = 1, ..., N.$$

This gives us the upper bound for the norm

$$\|u\|^2_{H^2(\mathbb{R}^d,\ \mathbb{R}^N)} \leq 2(2\pi)^d \sum_{k=1}^N M_k^2 \|F_k(v(x), x)\|^2_{L^2(\mathbb{R}^d)} < \infty$$

by virtue of inequality (1.3) of Assumption 1. Therefore, for any $v(x) \in H^2(\mathbb{R}^d,\ \mathbb{R}^N)$ there exists a unique vector function $u(x) \in H^2(\mathbb{R}^d,\ \mathbb{R}^N)$, which satisfies system (1.5) and its Fourier image is given by (2.1). Hence the map $T_a : H^2(\mathbb{R}^d,\ \mathbb{R}^N) \to H^2(\mathbb{R}^d,\ \mathbb{R}^N)$ is well defined.

This enables us to choose arbitrary $v^{(1),(2)}(x) \in H^2(\mathbb{R}^d,\ \mathbb{R}^N)$ and obtain their images under the map $u^{(1),(2)} := T_a v^{(1),(2)} \in H^2(\mathbb{R}^d,\ \mathbb{R}^N)$ and derive easily the bounds for $k = 1, ..., N$

$$\left|\widehat{u_k^{(1)}}(p) - \widehat{u_k^{(2)}}(p)\right| \leq (2\pi)^{\frac{d}{2}} M \left|\widehat{f_k^{(1)}}(p) - \widehat{f_k^{(2)}}(p)\right|,$$

$$\left|p^2\widehat{u_k^{(1)}}(p) - p^2\widehat{u_k^{(2)}}(p)\right| \leq (2\pi)^{\frac{d}{2}} M \left|\widehat{f_k^{(1)}}(p) - \widehat{f_k^{(2)}}(p)\right|.$$

In this context $\widehat{f_k^{(1),(2)}}(p)$ stand for the Fourier transforms of $F_k(v^{(1),(2)}(x), x)$. This yields the bound on the corresponding norm of the difference of vector functions

$$\|u^{(1)} - u^{(2)}\|^2_{H^2(\mathbb{R}^d,\ \mathbb{R}^N)} \leq 2(2\pi)^d M^2 \sum_{k=1}^N \|F_k(v^{(1)}(x), x) - F_k(v^{(2)}(x), x)\|^2_{L^2(\mathbb{R}^d)}.$$

By virtue of the Sobolev embedding theorem for $k = 1, ..., N$ we have $v_k^{(1),(2)}(x) \in H^2(\mathbb{R}^d) \subset L^\infty(\mathbb{R}^d),\ 1 \leq d \leq 3$. Inequality (1.4) trivially gives us

$$\|T_a v^{(1)} - T_a v^{(2)}\|_{H^2(\mathbb{R}^d,\ \mathbb{R}^N)} \leq \sqrt{2}(2\pi)^{\frac{d}{2}} Ml \|v^{(1)} - v^{(2)}\|_{H^2(\mathbb{R}^d,\ \mathbb{R}^N)}.$$

The constant in the right side of this bound is less than one by means of the assumption of the theorem. Therefore, the Fixed Point Theorem implies the existence of a unique vector function $v_a(x) \in H^2(\mathbb{R}^d, \mathbb{R}^N)$, such that $T_a v_a = v_a$. This is the only stationary solution of system (1.2) in $H^2(\mathbb{R}^d,\ \mathbb{R}^N)$. Finally, let us assume that $v_a(x) = 0$ a.e. in $\mathbb{R}^d$. This will yield the contradiction to the condition that for some $k = 1, ..., N$ the Fourier images of $G_k(x)$ and $F_k(0, x)$ do not vanish simultaneously on some set of nonzero Lebesgue measure in $\mathbb{R}^d$. ■

3 The System on the $[0, 2\pi]$ Interval

Proof of Theorem 5. We first suppose that for some $v(x) \in H_c^2(I, \mathbb{R}^N)$ there exist two solutions $u^{(1),(2)}(x) \in H_c^2(I, \mathbb{R}^N)$ of system (1.5) with $\Omega = I$. Then the difference vector function $w(x) := u^{(1)}(x) - u^{(2)}(x) \in H_c^2(I, \mathbb{R}^N)$ will be a solution of the system of equations

$$\sqrt{-\frac{d^2}{dx^2}} w_k = a_k w_k, \ \ k = 1, ..., N.$$

Due to Assumption 4, we have $a_k \neq n, \ n \in \mathbb{Z}^+ = \mathbb{N} \cup \{0\}$ when $k = 1, ..., m$ and as a consequence, they are not the eigenvalues of the operator $\sqrt{-\frac{d^2}{dx^2}}$ on $L^2(I)$ with periodic boundary conditions. Hence, $w_k(x)$ vanishes a.e. in I when $k = 1, ..., m$. For $k = m+1, ..., q$ the values of a_k are identical to the nonzero eigenvalues of the square root of the negative second derivative operator with periodic boundary conditions on the $[0, 2\pi]$ interval but w_k belong to the constraned subspaces $H_k^2(I)$. Thus, $w_k = 0$ a.e. in I for $k = m+1, ..., q$ since they are orthogonal to the eigenfunctions $\frac{e^{\pm i n_k x}}{\sqrt{2\pi}}$. By virtue of Assumption 4 the constants a_k are zeros for $k = q+1, ..., N$. But w_k belong to the constrained subspace $H_0^2(I)$ of functions orthogonal to the zero mode of $\sqrt{-\frac{d^2}{dx^2}}$ on $L^2(I)$ with periodic boundary conditions. Thus, $w_k(x)$ vanishes a.e. in I when $k = q+1, ..., N$ as well.

We assume that $v(x) \in H_c^2(I, \mathbb{R}^N)$ is arbitrary. Let us apply the Fourier transform (1.15) to both sides of the system of equations (1.5) considered on the interval $[0, 2\pi]$ and obtain

$$u_{k,\ n} = \sqrt{2\pi} \frac{G_{k,\ n} f_{k,\ n}}{|n| - a_k}, \quad n \in \mathbb{Z}, \tag{3.1}$$

where $f_{k,\ n} := F_k(v(x), x)_n$. Apparently, the Fourier coefficients of the second derivatives are given by

$$(-u_k'')_n = \sqrt{2\pi} \frac{n^2 G_{k,\ n} f_{k,\ n}}{|n| - a_k}, \quad n \in \mathbb{Z}.$$

We trivially obtain the estimate from above

$$\|u\|^2_{H_c^2(I,\ \mathbb{R}^N)} = \sum_{k=1}^{N} \left\{ \sum_{n=-\infty}^{\infty} |u_{k,\ n}|^2 + \sum_{n=-\infty}^{\infty} |n^2 u_{k,\ n}|^2 \right\} \leq 4\pi \sum_{k=1}^{N} P_k^2 \|F_k(v(x), x)\|^2_{L^2(I)} < \infty,$$

which comes from inequality (1.3) of Assumption 1. Thus, for an arbitrarily chosen vector function $v(x) \in H_c^2(I,\ \mathbb{R}^N)$ there exists a unique $u(x) \in H_c^2(I,\ \mathbb{R}^N)$, which satisfies the system of equations (1.5) and its Fourier coefficients are given by formula (3.1), such that the map $\tau_a : H_c^2(I,\ \mathbb{R}^N) \to H_c^2(I,\ \mathbb{R}^N)$ is well defined. Note that orthogonality relations (1.13) and (1.14) along with (3.1) yield that for $k = m+1, ..., q$ components $u_k(x)$ are orthogonal to Fourier harmonics $\dfrac{e^{\pm i n_k x}}{\sqrt{2\pi}}$ in $L^2(I)$ and for $k = q+1, ..., N$ functions $u_k(x)$ are orthogonal to 1 in $L^2(I)$, since the corresponding Fourier coeffients can be made equal to zero.

Then we choose arbitrary vector functions $v^{(1),(2)}(x) \in H_c^2(I,\ \mathbb{R}^N)$, such that their images under the map defined above are $u^{(1),(2)} := \tau_a v^{(1),(2)} \in H_c^2(I,\ \mathbb{R}^N)$ and arrive easily at the estimate

$$\|u^{(1)} - u^{(2)}\|^2_{H_c^2(I,\ \mathbb{R}^N)} = \sum_{k=1}^{N} \left\{ \sum_{n=-\infty}^{\infty} |u^{(1)}{}_{k,\ n} - u^{(2)}{}_{k,\ n}|^2 + \sum_{n=-\infty}^{\infty} |n^2(u^{(1)}{}_{k,\ n} - u^{(2)}{}_{k,\ n})|^2 \right\} \leq$$

$$\leq 4\pi \sum_{k=1}^{N} P_k^2 \|F_k(v^{(1)}(x), x) - F_k(v^{(2)}(x), x)\|^2_{L^2(I)}.$$

Evidently, by means of the Sobolev embedding theorem $v_k^{(1),(2)}(x) \in H^2(I) \subset L^\infty(I)$ for $k = 1, ..., N$. Using (1.4) we easily obtain

$$\|\tau_a v^{(1)} - \tau_a v^{(2)}\|_{H_c^2(I,\ \mathbb{R}^N)} \leq 2\sqrt{\pi} P l \|v^{(1)} - v^{(2)}\|_{H_c^2(I,\ \mathbb{R}^N)}.$$

The constant in the right side of this bound is less than one by virtue of the assumption of the theorem. Therefore, the Fixed Point Theorem implies the existence and uniqueness of a vector function $v_a(x) \in H_c^2(I,\ \mathbb{R}^N)$, which satisfies $\tau_a v_a = v_a$. This is the only stationary solution of the system of equations (1.2) in $H_c^2(I,\ \mathbb{R}^N)$. Finally, we suppose that $v_a(x)$ vanishes a.e. in the interval I. This will imply the contradiction to our assumption that the Fourier coefficients $G_{k,\ n} F_k(0, x)_n \neq 0$ for some $k = 1, ..., N$ and some $n \in \mathbb{Z}$. ■

4 The System in the Layer Domain

Proof of Theorem 7. First of all we suppose that there exists $v(x) \in H^2(\Omega,\ \mathbb{R}^N)$ generating $u^{(1),(2)}(x) \in H^2(\Omega,\ \mathbb{R}^N)$, which satisfy system (1.5). Then the difference of such vector functions $w(x) := u^{(1)}(x) - u^{(2)}(x) \in H^2(\Omega,\ \mathbb{R}^N)$ will be a solution to the homogeneous system of equations

$$\sqrt{-\Delta} w_k = a_k w_k,\ \ k = 1, ..., N.$$

We apply the partial Fourier transform with respect to the first variable to this system and obtain

$$\sqrt{-\Delta_\perp + n^2} w_{k,\ n}(x_\perp) = a_k w_{k,\ n}(x_\perp),\ \ k = 1, ..., N,\ \ n \in \mathbb{Z}$$

with $w_{k,\ n}(x_\perp) := \dfrac{1}{\sqrt{2\pi}} \displaystyle\int_0^{2\pi} w_k(x_1, x_\perp) e^{-i n x_1} dx_1$. Clearly,

$$\|w_k\|^2_{L^2(\Omega)} = \sum_{n=-\infty}^{\infty} \|w_{k,\ n}\|^2_{L^2(\mathbb{R}^d)}.$$

Therefore, $w_{k,\ n}(x_\perp) \in L^2(\mathbb{R}^d),\ k=1,...,N,\ n\in\mathbb{Z}$. But the operator $\sqrt{-\Delta_\perp + n^2}$ considered on $L^2(\mathbb{R}^d)$ does not have any nontrivial eigenfunctions. This implies that $w(x)=0$ a.e. in Ω.

Let us choose an arbitrary vector function $v(x) \in H^2(\Omega,\ \mathbb{R}^N)$ and apply the Fourier transform (1.25) to both sides of problem (1.5). This yields

$$\widehat{u}_{k,\ n}(p) = (2\pi)^{\frac{d+1}{2}} \frac{\widehat{G}_{k,\ n}(p)\widehat{f}_{k,\ n}(p)}{\sqrt{p^2+n^2}-a_k},\ k=1,...,N, \quad n\in\mathbb{Z}, \quad p\in\mathbb{R}^d, \quad d=1,2, \tag{4.1}$$

where $\widehat{f}_{k,\ n}(p)$ stands for the Fourier image of $F_k(v(x),x)$. Apparently, for the above mentioned values of $k,\ n$ and p we have the bounds in terms of the quantities given by (1.26), (1.27) and (1.28) as

$$|\widehat{u}_{k,\ n}(p)| \le (2\pi)^{\frac{d+1}{2}} R_k |\widehat{f}_{k,\ n}(p)| \quad \textit{and} \quad |(p^2+n^2)\widehat{u}_{k,\ n}(p)| \le (2\pi)^{\frac{d+1}{2}} R_k |\widehat{f}_{k,\ n}(p)|.$$

By virtue of (1.3) of Assumption 1 we arrive at

$$\|u\|^2_{H^2(\Omega,\ \mathbb{R}^N)} = \sum_{k=1}^N \left\{ \sum_{n=-\infty}^{\infty} \int_{\mathbb{R}^d} |\widehat{u}_{k,\ n}(p)|^2 dp + \sum_{n=-\infty}^{\infty} \int_{\mathbb{R}^d} |(p^2+n^2)\widehat{u}_{k,\ n}(p)|^2 dp \right\} \le$$

$$\le 2(2\pi)^{d+1} \sum_{k=1}^N {R_k}^2 \|F_k(v(x),x)\|^2_{L^2(\Omega)} < \infty.$$

Hence, for any vector function $v(x) \in H^2(\Omega,\ \mathbb{R}^N)$ there exists a unique $u(x) \in H^2(\Omega,\ \mathbb{R}^N)$ which solves the system of equations (1.5) and its Fourier image is given by formula (4.1). Therefore, the map $t_a : H^2(\Omega,\ \mathbb{R}^N) \to H^2(\Omega,\ \mathbb{R}^N)$ is well defined.

We choose two arbitrary vector functions $v^{(1),(2)} \in H^2(\Omega,\ \mathbb{R}^N)$ such that their images under the map discussed above are $u^{(1),(2)} := t_a v^{(1),(2)} \in H^2(\Omega,\ \mathbb{R}^N)$. Hence

$$\|u^{(1)} - u^{(2)}\|^2_{H^2(\Omega,\ \mathbb{R}^N)} = \sum_{k=1}^N \sum_{n=-\infty}^{\infty} \int_{\mathbb{R}^d} dp \Bigg\{ |\widehat{u^{(1)}}_{k,\ n}(p) - \widehat{u^{(2)}}_{k,\ n}(p)|^2 +$$

$$+ |(p^2+n^2)(\widehat{u^{(1)}}_{k,\ n}(p) - \widehat{u^{(2)}}_{k,\ n}(p))|^2 \Bigg\} \le 2(2\pi)^{d+1} R^2 \sum_{k=1}^N \|F_k(v^{(1)}(x),x) - F_k(v^{(2)}(x),x)\|^2_{L^2(\Omega)}.$$

Obviously, by virtue of the Sobolev embedding theorem $v_k^{(1),(2)}(x) \in H^2(\Omega) \subset L^\infty(\Omega)$ for $k=1,...,N$. By means of (1.4) we easily derive the estimate

$$\|t_a v^{(1)} - t_a v^{(2)}\|_{H^2(\Omega,\ \mathbb{R}^N)} \le \sqrt{2}(2\pi)^{\frac{d+1}{2}} Rl \|v^{(1)} - v^{(2)}\|_{H^2(\Omega,\ \mathbb{R}^N)}$$

with the constant in its right side less than one due to our assumption. Therefore, the Fixed Point Theorem gives us the existence and uniqueness of a vector function $v_a(x) \in H^2(\Omega,\ \mathbb{R}^N)$, for which $t_a v_a = v_a$ holds. This is the only stationary solution of problem (1.2) in $H^2(\Omega,\ \mathbb{R}^N)$. Finally, we suppose that the vector function $v_a(x) = 0$ a.e. in Ω. This will contradict to the assumption of the theorem that there exists $k=1,...,N$ and $n\in\mathbb{Z}$, such that $supp\widehat{F_k(0,x)}_n(p) \cap supp\widehat{G}_{k,\ n}(p)$ is a set of nonzero Lebesgue measure in $\mathbb{R}^d$. ∎

5 Discussion

We will conclude the article with a brief discussion of biological interpretations of the results obtained above. All tissues and organs in a biological organism are characterized by cell distribution with respect to their genotype. Without mutations all cells would have an identical genotype. Because of mutations, the genotype changes and represents a certain distribution around its principal value. Stationary solutions of such system give stationary cell distribution with respect to the genotype. Existence of such stationary distributions is a significant property of biological organisms allowing their existence as steady state systems.

Existence of stationary solutions is established in the spaces of integrable functions decaying at infinity, with periodic boundary conditions on an interval and in a mixed situation in a layer.

Biologically this implies that the cell distribution with respect to the genotype decays as the distance from the principal genotype increases. The results of the article establish that conditions should be imposed on cell proliferation, mutations and influx/efflux to obtain such distributions.

In the context of population dynamics, such result is applicable also to biological species which individuals are distributed around a certain average genotype. In such case, existence of stationary solutions is related to the existence of biological species (see [5]).

References

[1] M.S Agranovich: *Elliptic boundary problems,* Encyclopaedia Math. Sci., vol 79, Partial Differential Equations, IX, Springer, Berlin, 1997, pp. 1–144

[2] N. Apreutesei, N. Bessonov, V. Volpert, V. Vougalter: *Spatial Structures and Generalized Travelling Waves for an Integro- Differential Equation,* Discrete Contin. Dyn. Syst. Ser. B **13, 3** (2010), 537–557

[3] M. Beck, A. Ghazaryan, B. Sandstede: *Nonlinear convective stability of travelling fronts near Turing and Hopf instabilities,* J. Differential Equations **246** (2009), 4371–4390

[4] H. Berestycki, G. Nadin, B. Perthame, L. Ryzhik: *The non-local Fisher-KPP equation: travelling waves and steady states,* Nonlinearity **22, 12** (2009), 2813–2844

[5] N. Bessonov, N. Reinberg, V. Volpert: *Mathematics of Darwins Diagram,* Math. Model. Nat. Phenom., **9, 3** (2014), 5–25

[6] B. Carreras, V. Lynch, G. Zaslavsky: *Anomalous diffusion and exit time distribution of particle tracers in plasma turbulence model,* Phys. Plasmas **8** (2001), 5096–5103

[7] A. Ducrot, M. Marion and V. Volpert: *Systemes de réaction-diffusion sans propriété de Fredholm,* CRAS, **340, 9** (2005), 659–664

[8] A. Ducrot, M. Marion and V. Volpert: *Reaction-diffusion problems with non Fredholm operators,* Advances Diff. Equations , **13, 11-12** (2008), 1151–1192

[9] A. Ducrot, M. Marion and V. Volpert. Reaction-diffusion waves (with the Lewis number different from 1). Publibook, Paris, 2009, 113 pp.

[10] S. Genieys, V. Volpert, P. Auger: *Pattern and waves for a model in population dynamics with nonlocal consumption of resources,* Math. Model. Nat. Phenom. **1, 1** (2006), 63–80

[11] A. Ghazaryan, B. Sandstede: *Nonlinear convective instability of Turing-unstable fronts near onset: a case study,* SIAM J. Appl. Dyn. Syst. **6, 2** (2007), 319–347

[12] P.D. Hislop, I.M. Sigal: *Introduction to spectral theory. With applications to Schrödinger operators.* Springer, 1996, 337 pp.

[13] J.L. Lions, E. Magenes: *Problemes aux limites non homogenes et applications.* Vol. 1. Dunod, Paris, 1968, 372 pp.

[14] P. Manandhar, J. Jang, G.C. Schatz, M.A. Ratner, S. Hong: *Anomalous surface diffusion in nanoscale direct deposition processes,* Phys. Rev. Lett., **90** (2003), 4043–4052

[15] R. Metzler, J. Klafter: *The random walk's guide to anomalous diffusion: a fractional dynamics approach,* Phys. Rep., **339** (2000), 1–77

[16] J. Sancho, A. Lacasta, K. Lindenberg, I. Sokolov, A. Romero: *Diffusion on a solid surface: Anomalous is normal,* Phys. Rev. Lett., **92** (2004), 250601

[17] W. Shen, A. Zhang: *Spreading speeds for monostable equations with nonlocal dispersal in space periodic habitats,* J. Differential Equations, **249, 4** (2010), 747–795

[18] H. Scher, E. Montroll: *Anomalous transit-time dispersion in amorphous solids,* Phys. Rev. B, **12** (1975), 2455–2477

[19] T. Solomon, E. Weeks, H. Swinney: *Observation of anomalous diffusion and Levy flights in a two-dimensional rotating flow,* Phys. Rev. Lett., **71** (1993), 3975–3978

[20] L.R. Volevich: *Solubility of boundary value problems for general elliptic systems,* Mat. Sb., **68, 110** (1965), 373–416; English translation: Amer. Math. Soc. Transl., **67, 2** (1968), 182–225.

[21] V. Volpert: *Elliptic partial differential equations.* Volume I. Fredholm theory of elliptic problems in unbounded domains. Birkhäuser, 2011, 639 pp.

[22] V. Volpert, B. Kazmierczak, M. Massot, Z.Peradzynski: *Solvability conditions for elliptic problems with non-Fredholm operators,* Appl. Math., **29, 2** (2002), 219–238

[23] V. Vougalter, V. Volpert: *Solvability conditions for some non Fredholm operators,* Proc. Edinb. Math. Soc. (2), **54, 1** (2011), 249–271

[24] V. Vougalter, V. Volpert: *On the solvability conditions for some non Fredholm operators,* Int. J. Pure Appl. Math., **60, 2** (2010), 169–191

[25] V. Vougalter, V. Volpert: *On the solvability conditions for the diffusion equation with convection terms,* Commun. Pure Appl. Anal., **11, 1** (2012), 365–373

[26] V. Vougalter, V. Volpert: *Solvability relations for some non Fredholm operators,* Int. Electron. J. Pure Appl.Math., **2, 1** (2010), 75–83

[27] V. Volpert, V. Vougalter: *On the solvability conditions for a linearized Cahn-Hilliard equation,* Rend. Istit. Mat. Univ. Trieste, **43** (2011), 1–9

[28] V. Vougalter, V. Volpert: *Solvability conditions for some systems with non Fredholm operators,* Int. Electron. J. Pure Appl.Math., **2, 3** (2010), 183–187

[29] V. Vougalter, V. Volpert: *Solvability conditions for a linearized Cahn-Hilliard equation of sixth order,* Math. Model. Nat. Phenom., **7, 2** (2012), 146–154

[30] V. Vougalter, V. Volpert: *On the existence of stationary solutions for some non-Fredholm integro-differential equations,* Doc. Math., **16** (2011), 561–580

[31] V. Vougalter, V. Volpert: *Solvability conditions for some linear and nonlinear non-Fredholm elliptic problems,* Anal. Math. Phys., **2, 4** (2012), 473–496

[32] V. Vougalter, V. Volpert: *Existence of stationary solutions for some non-Fredholm integro-differential equations with superdiffusion.* Preprint 2016.