Rigidity for a class of generalized interval exchange transformations

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Abstract

For almost all irrational $\rho \in (0, 1)$, any two cyclic generalized interval exchange transformations with breaks on the same orbit, with the same rotation number ρ, and the same size of the corresponding breaks, are C^1-smoothly conjugate to each other. In particular, for almost all irrational $\rho \in (0, 1)$, generalized interval exchange transformations of two intervals, with the same rotation number ρ and the same size of the corresponding breaks, are C^1-smoothly conjugate to each other. These results generalize the results of Marmi, Moussa and Yoccoz [8].

1 Introduction and statement of the results

Interval exchange transformations play an important role in dynamics. These are piecewise affine maps $T_{\mathbf{a},\sigma}$, with slope 1, of an interval $I = [0, 1]$, determined by a vector $\mathbf{a} \in (0, 1)^{k-1}$, with components satisfying $0 < a_1 < \cdots < a_{k-1} < 1$, that cuts the interval into k subintervals $I_i = [a_{i-1}, a_i]$, $i = 1, \ldots, k$, where $a_0 = 0$ and $a_k = 1$, and a permutation $\sigma \in S_k$ that permutes them. In a recent paper [8], Marmi, Moussa and Yoccoz introduced generalized interval exchange transformations, obtained by replacing the affine restrictions of $T_{\mathbf{a},\sigma}$ to each I_i with smooth diffeomorphisms. This paper concerns rigidity of generalized interval exchange transformations.

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Marmi, Moussa and Yoccoz showed that sufficiently smooth generalized interval exchange transformations of a certain combinatorial type, which are deformations of standard interval exchange transformations and tangent to them at the points of discontinuities, are smoothly linearizable [8]. Cunha and Smania [1] showed that break-equivalent cyclic generalized interval exchange transformations, with bounded-type rotation numbers and zero mean nonlinearity $\mathcal{N} = \int_I \frac{T''_{\mathbf{a},\sigma}(x)}{2T'_{\mathbf{a},\sigma}(x)}\, dx$, are C^1-smoothly conjugate to each other. We say that two such maps $T_{\mathbf{a},\sigma}$ and $\widetilde{T}_{\widetilde{\mathbf{a}},\sigma}$ are break-equivalent if the break points of one map $\widetilde{a}_i$ are mapped into the break points of the other a_i, by a topological conjugacy φ, satisfying $\widetilde{T}_{\widetilde{\mathbf{a}},\sigma} = \varphi^{-1} \circ T_{\mathbf{a},\sigma} \circ \varphi$, i.e. $a_i = \varphi(\widetilde{a}_i)$, and the corresponding sizes of breaks, $c_i = \sqrt{(T_{\mathbf{a},\sigma})'_-(a_i)/(T_{\mathbf{a},\sigma})'_+(a_i)}$ and $\widetilde{c}_i = \sqrt{(\widetilde{T}_{\widetilde{\mathbf{a}},\sigma})'_-(\widetilde{a}_i)/(\widetilde{T}_{\widetilde{\mathbf{a}},\sigma})'_+(\widetilde{a}_i)}$, are the same, for each $i = 1, \dots, k$. If we identify the end points of the interval I, a generalized interval exchange transformation for which σ is a cyclic permutation of $(1, \dots, k)$ is a circle diffeomorphism with k break points. Correspondingly, $(T_{\mathbf{a},\sigma})'_+(a_k)$ above should be understood as $(T_{\mathbf{a},\sigma})'_+(0)$. Notice that the mean nonlinearity $\mathcal{N} = \log \prod_{i=1}^k c_i$. We call a map a circle diffeomorphism with breaks if it is a piecewise smooth diffeomorphism of a circle with the derivative bounded away from 0. In the following, we will consider cyclic generalized interval exchange transformations as acting on the circle $\mathbb{S}^1 = \mathbb{R}/\mathbb{Z}$. The following claim, that follows directly from our recent results on rigidity of circle maps with a single break point [5, 6], is the first result on rigidity of generalized interval exchange transformations in the general case of non-vanishing mean nonlinearity.

Theorem 1.1 *Let $T_{\mathbf{a},\sigma}$ and $\widetilde{T}_{\widetilde{\mathbf{a}},\sigma}$ be two $C^{2+\alpha}$-smooth, with $\alpha \in (0,1)$, cyclic generalized interval exchange transformations, of $k \geq 2$ intervals, with all break-points on the same orbit and the same nonlinearity $\mathcal{N}$. There is a set $\mathcal{S} \subset (0,1)\backslash\mathbb{Q}$ of Lebesgue measure 1 such that the following holds. If $T_{\mathbf{a},\sigma}$ and $\widetilde{T}_{\widetilde{\mathbf{a}},\sigma}$ have the same rotation number $\rho \in \mathcal{S}$ and if they are break-equivalent, then they are C^1-smoothly conjugate to each other, i.e. there is a C^1-smooth diffeomorphism $\varphi : \mathbb{S}^1 \to \mathbb{S}^1$ such that*

$$\widetilde{T}_{\widetilde{\mathbf{a}},\sigma} = \varphi^{-1} \circ T_{\mathbf{a},\sigma} \circ \varphi. \tag{1.1}$$

Remark 1 The set of rotation numbers $\mathcal{S} = \mathcal{S}(\alpha, \mathcal{N})$ depends on the smoothness exponent $\alpha \in (0,1)$ and the nolinearity of the maps $\mathcal{N}$. In the special case $\mathcal{N} = 0$, it follows essentially from Herman's theory and the analysis below that this set can be taken to be the set of Diophantine numbers of class $\mathcal{D}(\delta)$, for any $\delta \in (0, \alpha)$, for which rigidity holds for circle diffeomorphisms. A number ρ is said to be Diophantine of class $\mathcal{D}(\delta)$ if there exist $\mathcal{C} > 0$ and $\delta \geq 0$ such that $|\rho - p/q| > \mathcal{C}/q^{2+\delta}$, for any $p \in \mathbb{Z}$ and $q \in \mathbb{N}$. In the more general case, $\mathcal{N} \neq 0$, it follows from our results [5, 6], that the set $\mathcal{S}$ can be taken to be the set $\mathcal{S}_{\text{rig}}$, for which C^1-rigidity holds for circle maps with a break. We reserve the term circle map with a break for a circle diffeomorphism with a break of size different than 1. The set $\mathcal{S}_{\text{rig}}$ includes all irrational $\rho \in (0,1)$ for which partial quotients k_n, in the

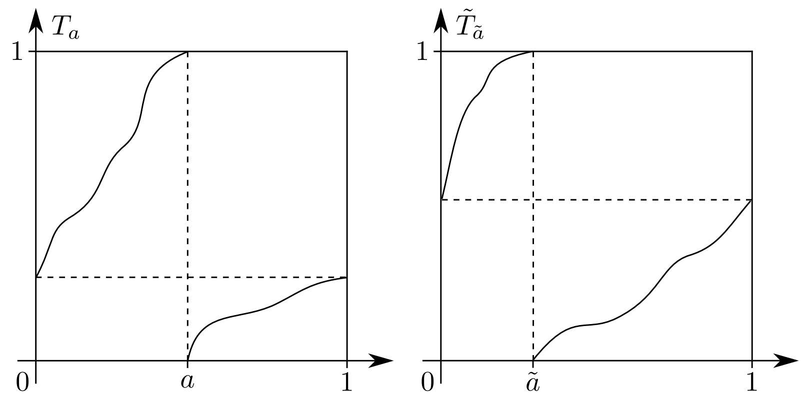

Figure 1: Two break-equivalent generalized interval exchange transformations of two intervals.

continued fraction expansion of the rotation number $\rho = [k_1, k_2, \dots]$, satisfy the following. For some $C_1 > 0$ and $\lambda_1 \in (0,1)$, $k_n \leq C_1 \lambda_1^{-n}$, for all n odd if $\mathcal{N} < 0$ or all n even, if $\mathcal{N} > 0$. We note that the intersection of all these sets is still of full measure and that the set $\mathcal{S}$ can also be chosen independently of α or $\mathcal{N}$.

Remark 2 Strictly speaking, Herman's theory [2, 3, 7, 9] is not sufficient to provide the desired estimates in the case $\mathcal{N} = 0$. Namely, after the conjugation of $T_{\mathbf{a},\sigma}$, the diffeomorphism that we obtain can have discontinuities of the second derivative at points that correspond to the original break points. Nevertheless, outside of these points, the map is still $C^{2+\alpha}$-smooth. It is not difficult to show that this discontinuity will not influence the validity of the results.

Remark 3 The result of Theorem 1.1 cannot be extended to all irrational rotation numbers $\rho \in (0,1)$, as follows from [4].

It is easy to see that a generalized interval exchange transformation T_a, with $a \in (0,1)$, of $k = 2$ intervals, for which $\sigma \in S_2$ is a transposition, is a circle map with two break points 0 and a. Since these two points are on the same orbit of the map, i.e. $T_a(a) = 0$, we immediately have the following.

Let c_a be the size of the break of T_a at point a, i.e. $c_a = \sqrt{\frac{(T_a)'_-(a)}{(T_a)'_+(a)}}$.

Corollary 1.2 *Let T_a and $\widetilde{T}_{\widetilde{a}}$ be two $C^{2+\alpha}$-smooth generalized interval exchange transformations of $k = 2$ intervals. If $\rho = \widetilde{\rho} \in \mathcal{S}$, $c_a = \widetilde{c}_{\widetilde{a}}$ and $\mathcal{N} = \widetilde{\mathcal{N}}$, then T_a and $\widetilde{T}_{\widetilde{a}}$ are C^1-smoothly conjugate to each other.*

Lemma 1.3 *Any $C^{2+\alpha}$-smooth cyclic generalized interval exchange transformation $T_{\mathbf{a},\sigma}$ with breaks of size c_i, $i = 1, \ldots, k$, on the same orbit can be conjugated to a circle diffeomorphism with a single break point $\mathcal{T}$, with a break of size $c = \prod_{i=1}^{k} c_i$, via a piecewise smooth homeomorphism $\mathcal{H} : \mathbb{S}^1 \to \mathbb{S}^1$, with breaks on the same orbit. Moreover, the sizes of breaks of $\mathcal{H}$ are completely determined by the sizes and combinatorics of the breaks of $T_{\mathbf{a},\sigma}$.*

Proof. Let $h : \mathbb{S}^1 \to \mathbb{S}^1$ be a piecewise smooth diffeomorphism of the circle and let $F = h^{-1} \circ T_{\mathbf{a},\sigma} \circ h$. For every $x \in \mathbb{S}^1$, the one-sided derivatives

$$\begin{aligned} F'_\pm(x) &= (h^{-1})'_\pm(T_{\mathbf{a},\sigma} \circ h(x))(T_{\mathbf{a},\sigma})'_\pm(h(x))h'_\pm(x) \\ &= \frac{1}{h'_\pm(F(x))}(T_{\mathbf{a},\sigma})'_\pm(h(x))h'_\pm(x). \end{aligned} \tag{1.2}$$

Let $b_i = h^{-1}(a_i)$, for $i = 1, \ldots, k$. Let h be a diffeomorphism with a single break point and break of size c_i^{-1} at b_i, i.e. $c_i^{-1} = \sqrt{(h'_-(b_i))/(h'_+(b_i))}$.

Let $x \in \mathbb{S}^1$. If $x \neq b_i$ and $x \neq F^{-1}(b_i)$, then

$$\sqrt{\frac{F'_-(x)}{F'_+(x)}} = \sqrt{\frac{(T_{\mathbf{a},\sigma})'_-(h(x))}{(T_{\mathbf{a},\sigma})'_+(h(x))}}, \tag{1.3}$$

and, thus, the size of the break of F at x is the same as the size of the break of $T_{\mathbf{a},\sigma}$ at $h(x)$.

It follows from (1.2) for $x = b_i$ that

$$\sqrt{\frac{F'_-(b_i)}{F'_+(b_i)}} = \sqrt{\frac{(T_{\mathbf{a},\sigma})'_-(a_i)h'_-(b_i)}{(T_{\mathbf{a},\sigma})'_+(a_i)h'_+(b_i)}} = c_i c_i^{-1} = 1, \tag{1.4}$$

and, hence, F has no break at b_i. For $x = F^{-1}(b_i)$, (1.2) gives

$$\sqrt{\frac{F'_-(F^{-1}(b_i))}{F'_+(F^{-1}(b_i))}} = \sqrt{\frac{(T_{\mathbf{a},\sigma})'_-(T_{\mathbf{a},\sigma}^{-1}(a_i))h'_+(b_i)}{(T_{\mathbf{a},\sigma})'_+(T_{\mathbf{a},\sigma}^{-1}(a_i))h'_-(b_i)}} = c_i \sqrt{\frac{(T_{\mathbf{a},\sigma})'_-(T_{\mathbf{a},\sigma}^{-1}(a_i))}{(T_{\mathbf{a},\sigma})'_+(T_{\mathbf{a},\sigma}^{-1}(a_i))}}, \tag{1.5}$$

and, thus, the size of the break of F at $F^{-1}(b_i)$ is c_i times larger than the size of the break of $T_{\mathbf{a},\sigma}$ at $T_{\mathbf{a},\sigma}^{-1}(a_i)$. Therefore, the effect of the transformation of $T_{\mathbf{a},\sigma}$ with a conjugacy h with a single break point at b_i is a multiplicative "shift" of the size of the break c_i to its preimage under the map. If all the break points of $T_{\mathbf{a},\sigma}$ are on the same orbit, the map $\mathcal{H}$ can, therefore, be constructed as a composition of a finite number of maps of the same type as h, shifting all the breaks of $T_{\mathbf{a},\sigma}$ to a single break point. All the break points of $\mathcal{H}$ are on the same orbit of $\mathcal{T} = \mathcal{H}^{-1} \circ T_{\mathbf{a},\sigma} \circ \mathcal{H}$, and their sizes are completely determined by the combinatorics and the sizes of breaks of $T_{\mathbf{a},\sigma}$. The breaks of $T_{\mathbf{a},\sigma}$ belong to a single orbit. Let us order them according to the time along this orbit, i.e. $a_{\pi(1)},\dots,a_{\pi(k)}$, where π is a permutation in S_k. Hence, we have $T^{\ell_n}_{\mathbf{a},\sigma}a_{\pi(n)}=a_{\pi(n+1)}$, for every $n=1,\dots,k-1$. It is easy to see that, for every such n and $j=1,\dots,\ell_n$, $\mathcal{H}$ has a break of size $\prod_{m=n+1}^k c^{-1}_{\pi(m)}$ at $\mathcal{H}^{-1}\circ T^j_{\mathbf{a},\sigma}(a_{\pi(n)})=\mathcal{T}^j\circ\mathcal{H}^{-1}(a_{\pi(n)})$. The claim follows. **QED**

Proof of Theorem 1.1. It follows from Lemma 1.3 that there is a pair of circle diffeomorphisms with a break $\mathcal{T}$ and $\widetilde{\mathcal{T}}$, with a break of size $\prod_{i=1}^k c_i$, and a pair of break-equivalent piecewise-smooth circle diffeomorphisms $\mathcal{H}$ and $\widetilde{\mathcal{H}}$, such that $\mathcal{T}=\mathcal{H}^{-1}\circ T_{\mathbf{a},\sigma}\circ\mathcal{H}$ and $\widetilde{\mathcal{T}}=\widetilde{\mathcal{H}}^{-1}\circ\widetilde{T}_{\widetilde{\mathbf{a}},\sigma}\circ\widetilde{\mathcal{H}}$. If $\prod_{i=1}^k c_i=1$, $\mathcal{T}$ and $\widetilde{\mathcal{T}}$ are circle diffeomorphisms; if $\prod_{i=1}^k c_i\neq 1$, $\mathcal{T}$ and $\widetilde{\mathcal{T}}$ are circle maps with a break. For $\rho\in\mathcal{S}$, it follows from Herman's theory [2,3,7,9] (see also remark 2), if $\prod_{i=1}^k c_i=1$, and from our results on rigidity of circle maps with a single break point [5,6], if $\prod_{i=1}^k c_i\neq 1$, that $\mathcal{T}$ and $\widetilde{\mathcal{T}}$ are C^1-smoothly conjugate to each other, i.e. there is a C^1-smooth diffeomorphism ϕ such that $\widetilde{\mathcal{T}}=\phi^{-1}\circ\mathcal{T}\circ\phi$. It follows that $\widetilde{T}_{\widetilde{\mathbf{a}},\sigma}=\varphi^{-1}\circ T_{\mathbf{a},\sigma}\circ\varphi$, where $\varphi=\mathcal{H}\circ\phi\circ\widetilde{\mathcal{H}}^{-1}$. Since, for every $x\in\mathbb{S}^1$, the one-sided derivatives,

$$\begin{aligned}\varphi'_\pm(x)&=\mathcal{H}'_\pm(\phi\circ\widetilde{\mathcal{H}}^{-1}(x))\phi'(\widetilde{\mathcal{H}}^{-1}(x))(\widetilde{\mathcal{H}}^{-1})'_\pm(x)\\&=\mathcal{H}'_\pm(\phi\circ\widetilde{\mathcal{H}}^{-1}(x))\phi'(\widetilde{\mathcal{H}}^{-1}(x))\frac{1}{\widetilde{\mathcal{H}}'_\pm(\widetilde{\mathcal{H}}^{-1}(x))}.\end{aligned}\tag{1.6}$$

Since ϕ is C^1-smooth, it maps the break point of $\widetilde{\mathcal{T}}$ into the break point of $\mathcal{T}$. Since, by Lemma 1.3, the sizes of breaks of $\mathcal{H}$ at $\phi(\widetilde{y})$ and $\widetilde{\mathcal{H}}$ at $\widetilde{y}$ are the same, it follows that $\varphi'_-(x)=\varphi'_+(x)$. The claim follows. **QED**

References

[1] K. Cunha and D. Smania, Rigidity for piecewise smooth homeomorphisms on the circle, *Adv. Math.* **250** (2014) 193–226.

[2] M. R. Herman, Sur la conjugasion differentiable des difféomorphismes du cercle a de rotations, *Publ. Math. Inst. Hautes Etudes Sci.* **49** (1979), 5–234.

[3] Y. Katznelson, D. Orstein, The differentiability of conjugation of certain diffeomorphisms of the circle, *Ergod. Th. & Dynam. Sys.* **9** (1989), 643–680.

[4] K. Khanin, S. Kocić, Absence of robust rigidity for circle diffeomorphisms with breaks, *Ann. Inst. H. Poincaré Anal. Non Linéaire*, **30** 3 (2013), 385–399.

[5] K. Khanin, S. Kocić, Renormalization conjecture and rigidity theory for circle diffeomorphisms with breaks, *Geom. Funct. Anal.* **24** 6 (2014), 2002–2028.

[6] K. Khanin, S. Kocić, E. Mazzeo, C^1-rigidity of circle diffeomorphisms with breaks for almost all rotation numbers, *Preprint* mp-arc 11-102.

[7] K. Khanin, A. Teplinsky, Herman's theory revisited, *Invent. Math.* **178** (2009), 333–344.

[8] S. Marmi, P. Moussa, J.-C. Yoccoz, Linearization of generalized interval exchange maps, *Ann. Math* **176** 3 (2012) 1583–1646.

[9] J.-C. Yoccoz, Conjugaison differentiable des difféomorphismes du cercle donc le nombre de rotation vérifie une condition Diophantienne, *Ann. Sci. Ec. Norm. Sup.* **17** (1984), 333–361.