Dynamics of two-sided markets

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Abstract

This paper considers a dynamical system, which models a two-sided market. I evaluate the volume of users from each side of the market as a function of time. I formulate and prove theorems describing the long-term behavior and tendency of the market. Even though I consider generic attachment functions, I obtain a concrete result, formulated in terms of properties of attachment functions. I conclude the paper with examples (defined by the most popular in economics attachment functions), which illustrate behavior of the two-sided markets. I also simulate some two-sided market scenarios, showing how adjustments of attachment functions can influence the users' dynamics. Since platforms revenue depends on the number of users, this dynamical systems helps to analyze revenue over time. The dynamical system's approach to the study of two-sided market problem allows natural generalization to multi-sided market, where one can utilize the same technique and obtain similar results in higher dimensions.

Keywords: equilibrium; basin of attraction; two-sided market.

1. Introduction

In many industries, users visit, join, or adopt a *platform* or *intermediary* (such as a hardware device, content distribution service, payment system, or health insurance network) in order to access that platform's goods and services. *Two-sided* *market* arises when two different types of users (health insurance providers & subscribers, video-game users & game developers, auction buyers & sellers, dating web-site's female & male users, real estate buyers & sellers) may realize gains by interacting with one another through one or more such platforms or intermediary.

The benefit of each new user depends on the number of existing users on the platform. A *network effect* is a reflection of such dependency. There would be no benefit of having a telephone in absence of a network of people owning telephones. In case of two-sided market, one can talk about a *cross-side network effect*. It is a dependency of the one type users on the users of different type that platform or intermediary managed to attract. For example, auction buyers have no interest in a platform, which has no auction sellers. A real-valued function expressing interest of one type users in the opposite type of users comes into the dynamical system as a *attachment function*. (Generally speaking, an attachment function is a representation of user's preferences over some set of goods and services.)

In some two-sided markets, users are more or less reluctant to join a platform if the current number of users of their type is very high. This is called *same-side network effect*. In this paper, I assume that the same-side network effect is negative, and proportional to the number of users of this type. For example, buyers prefer platforms with a few buyers to keep prices low. However, some two-sided markets may have a positive same-side network effect (like in the case of multi-player online games, where players benefit from the presence of other players). I do not consider this type of market in this paper.

The purpose of this paper is to study the dynamics of percent of users from each side of the market on a platform. This dynamics is defined by generic attachment functions V and W, and by negative same-side network effect. Accordingly, I consider the system

$$\begin{cases} b' = V(g) - \varepsilon b \\ g' = W(b) - \varepsilon g, \end{cases} \quad b \in [0,1],\ g \in [0,1],\ \varepsilon \geq 1. \tag{1}$$

In section 2 (theorem 2.3), I describe all stationary points of this system (sinks, saddles and stable saddle-nodes), and corresponding properties of attachment functions that are responsible for the types of stationary points. In remark 2.5, I describe how these points can be ”ordered”. In section 3, I consider specific types of attachment functions (step-function, smoothed step-function, linear and convex), and describe their equilibria. In section 4, I define Nash equilibrium for the system 1, and specify Nash equilibria for the examples of section 3. In conclusion (section 5), I simulate some two-sided market scenarios and show how certain adjustments of attachment curves help to obtain higher volume of users.

A large number of publications in economics is devoted to the study of two-sided markets and pricing. These publications usually utilize a game theoretic approach. (For an overview see, for example, [7].) The dynamical system approach allows one to consider multi-sided market, because it is easy to extend the logic of this paper to multidimensional dynamical system. It is also easy to characterize all fixed points of the dynamical system (i.e., equilibria of the market) with arbitrary attachment functions. One can also characterize the market with positive same-side network effect – utilizing dynamical systems technique.

2. Dynamics

Let two-sided market be defined on $\mathscr{B} \times \mathscr{G} = [0,1] \times [0,1] = D$ by the following system of differential equations:

$$\begin{cases} b' = V(g) - \varepsilon b \\ g' = W(b) - \varepsilon g, \end{cases} \quad b \in \mathscr{B},\ g \in \mathscr{G},\ \varepsilon \geq 1. \tag{2}$$

Here $V : \mathscr{G} \mapsto \mathscr{B}$ and $W : \mathscr{B} \mapsto \mathscr{G}$ are non-decreasing functions. I will assume that $V(0) = W(0) = 0$ and $V(1) = W(1) = 1$.

This system describes percents of two different types of users (b and g) interacting through a platform or a intermediary, and representing two different sides of the market. V and W are attachment functions, describing the degree of interest of b and g users in each other. b and g can represent, for example, the percent of sellers and buyers that chose to interact through Amazon.com at each moment of time (time $\in [0, \infty)$).

Theorem 2.1. *Assume that V and W are continuously differentiable functions. Then, limit point of any trajectory of the system 2 with initial conditions $(b_0, g_0) \in D$ lies in D. The limit points can be either stationary points, or non-closed pathes.*

Proof. The vector field of equation 2 on the boundary of D is pointing inside of D. Indeed, if $b = 0$, then $b' \geq 0$; if $b = 1$, then $b' \leq 0$; if $g = 0$, then $g' \geq 0$; if $g = 1$, then $g' \leq 0$.

Since the flow cannot leave D, ω-limit set of the equation 2 lies entirely in D. (A point p is an ω-limit point if there are points $\phi_{t_1}(b_0, g_0), \phi_{t_2}(b_0, g_0), \ldots$ on the trajectory of (b_0, g_0) such that $\phi_{t_i}(b_0, g_0) \to p$, as $t_i \to \infty$.) It is also clear that ω-limit set is non-empty, because it contains at least 2 fixed points: $(0,0)$ and $(1,1)$.

For planar flows, all the possible ω-limit sets fall into three classes: fixed points, closed orbits and the union of fixed points and trajectories connecting them [1]. This statement is sometimes referred to as the generalized Poincaré-Bendixson theorem, even though it is not clear whether Bendixson stated this conclusion.

On the other hand, Bendixson's criterion [3] guaranties that there are no closed orbits generated by our system 2. (A closed orbit is either a periodic orbit, or a closed path that consists of fixed points and trajectories connecting them, or a single fixed point with an attached loop.)

Theorem 2.2 (Bendixson's criterion [3])**.** *If on a simply connected region $D \subset \mathbb{R}^2$ the expression $\frac{\partial f}{\partial x} + \frac{\partial h}{\partial y}$ is not identically zero and does not change sign, then the system $x' = f(x,y), y' = h(x,y)$ has no closed orbits lying entirely in D.*

Application of 2.2 completes the proof. Indeed, the sum

$$\frac{\partial\left(-\varepsilon b+V(g)\right)}{\partial b}+\frac{\partial\left(-\varepsilon g+W(b)\right)}{\partial g}=-2\varepsilon$$

is negative. Then, for our system 2, ω-limit set can be a fixed point or a non-closed path of trajectories connecting fixed points. □

Theorem 2.3. *The ω-limit set of system 2 may have the following 3 types of fixed points: sinks, saddles, and stable saddle-nodes. All of the fixed points lie on the line $\{(b,g) \in D : \varepsilon b = V(g), \varepsilon g = W(b)\}$. The types of the fixed points are defined by the coefficients of Taylor series expansion of the attachment functions V and W. Suppose (b_0, g_0) is an isolated fixed point. Then,*

1. *(b_0, g_0) is a sink, if and only if $V'(g_0) \cdot W'(b_0) < \varepsilon^2$;*
2. *(b_0, g_0) is a saddle, if and only if $V'(g_0) \cdot W'(b_0) > \varepsilon^2$;*

3. (b_0, g_0) *is a stable saddle-nodes, if and only if* $V'(g_0) \cdot W'(b_0) = \varepsilon^2$. *Moreover, if*

$$V''(g_0) \cdot W'(b_0) + \frac{1}{\varepsilon}(V')^2(g_0) \cdot W''(b_0) \neq 0,$$

then one of the branches of central manifold converges to (b_0, g_0)*, but the other one diverges from this fixed point. I will call it stable saddle-node type I. If*

$$V''(g_0) \cdot W'(b_0) + \frac{1}{\varepsilon}(V')^2(g_0) \cdot W''(b_0) = 0,$$

the central manifold's behavior can be investigated via calculations of the higher order derivatives. When both branches of the central manifold converge to the fixed point (b_0, g_0)*, I will call it type II stable saddle-node.*

Proof. Let me define a function $G(g) : \mathscr{G} \mapsto \mathscr{G}$ by

$$G = W\left(V(g)/\varepsilon\right) - \varepsilon g.$$

It is clear that $(V(g_0)/\varepsilon, g_0)$ is a fixed point of the differential equations 2 if and only if $G(g_0) = 0$. Since attachment functions are non-decreasing, there are only real eigenvalues of the linearized system 2, and in each pair of eigenvalues of this two-dimensional system, one of the eigenvalues is negative. Thus, fixed points are either sink nodes, or saddles, or stable saddle-nodes. It is also clear that a fixed point $(V(g_0), g_0)$ is a sink if and only if $G'(g_0) < 0$; $(V(g_0), g_0)$ is a saddle point if and only if $G'(g_0) > 0$; and $(V(g_0), g_0)$ has one local stable manifold and one local central manifold (i.e., g_0 is a stable saddle-node) if and only if $G'(g_0) = 0$.

There are two types of stable saddle-node fixed points that may occur in the system 2. Suppose (b_0, g_0) is an isolated fixed point. Then, either both branches of the central manifold curve converge to the point (b_0, g_0), or one of the branches converges to (b_0, g_0), but the other one diverges from this fixed point. The latter dynamics takes place if

$$V''(g_0)\cdot W'(b_0)+\frac{1}{\varepsilon}(V')^2(g_0)\cdot W''(b_0)\neq 0.$$

Below I present the proof of this fact.

The system 2 has stable saddle-node if and only if $\varepsilon b_0 = V(g_0),\ \varepsilon g_0 = W(b_0)$ and one of the eigenvalues is 0, i.e.,

$$W'(b_0)=\frac{\varepsilon^2}{V'(g_0)}.$$

Then, I can construct linear transformation of the system, such that the new coordinates are the eigenvectors of the system. This transformation, call it T, has the form:

$$T\begin{pmatrix} x \\ y \end{pmatrix}=\begin{pmatrix} V'(g_0) & V'(g_0) \\ \varepsilon & -\varepsilon \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}+\begin{pmatrix} b_0 \\ g_0 \end{pmatrix}. \tag{3}$$

Then,

$$\begin{pmatrix} b \\ g \end{pmatrix}=T\begin{pmatrix} x \\ y \end{pmatrix}, \tag{4}$$

and in the new coordinates the system 2 takes the form:

$$\begin{aligned}\begin{pmatrix} x' \\ y' \end{pmatrix}&=\begin{pmatrix} 0 & 0 \\ 0 & -2\varepsilon \end{pmatrix}\cdot\begin{pmatrix} x \\ y \end{pmatrix}\\ &+\begin{pmatrix} \frac{1}{4}V''(g_0)W'(b_0)(x-y)^2+\frac{1}{4\varepsilon}(V')^2(g_0)W''(b_0)(x+y)^2 \\ \frac{1}{4}V''(g_0)W'(b_0)(x-y)^2-\frac{1}{4\varepsilon}(V')^2(g_0)W''(b_0)(x+y)^2 \end{pmatrix}\\ &+\begin{pmatrix} O_1(x-y)^3+O_2(x+y)^3 \\ O_1(x-y)^3+O_2(x+y)^3 \end{pmatrix}.\end{aligned} \tag{5}$$

Since the central manifold is tangent to $y=0$ space, it can be represented as a (local) graph

$$W^c = \{(x,y) : y = H(x)\}, \ \ H(0) = H'(0) = 0,$$

where $H : U \mapsto \mathbb{R}$ is defined on some neighborhood $U \in \mathbb{R}$ of the origin. I now consider the projection of the vector field on $y = H(x)$ onto $y = 0$ space:

$$\begin{aligned} x' =& \frac{1}{4}V''(g_0)W'(b_0)(x-H(x))^2 + \frac{1}{4\varepsilon}(V'(g_0))^2 W''(b_0)(x+H(x))^2 \\ &+ O_1(x-H(x))^3 + O_2(x+H(x))^3 \end{aligned} \tag{6}$$

By theorem of Henry and Carr ([2] and [5]), the local asymptotic stability of the flow 5 is defined by the local asymptotic stability of the flow 6. (This result also follows from the global linearization theory of Pugh and Shub ([6]).)

Theorem 2.4 (Carr, Henry)**.** *If the origin $x = 0$ of equation 6 is locally asymptotically stable (resp. unstable), then the origin of equation 5 is also locally asymptotically stable (resp. unstable).*

Now, the goal is to calculate or approximate $H(x)$. Consider the expression for y:

$$y' = DH(x)\cdot x'$$

Substituting into this expression equation 5, I obtain

$$\begin{aligned} 0 = H'(x)\cdot\Big[&\frac{1}{4}V''(g_0)W'(b_0)\,(x-H(x))^2 + \frac{1}{4\varepsilon}(V')^2(g_0)W''(b_0)\,(x+H(x))^2 \\ &+O_1\,(x-H(x))^3 + O_2\,(x+H(x))^3\Big] + 2\varepsilon H(x) \\ &-\frac{1}{4}V''(g_0)W'(b_0)\,(x-H(x))^2 + \frac{1}{4\varepsilon}(V')^2(g_0)W''(b_0)\,(x+H(x))^2 \\ &+O_1(x-H(x))^3 + O_2\,(x+H(x))^3 . \end{aligned} \tag{7}$$

Differentiating this expression twice and setting $x = y = 0$, I obtain the following estimate for H:

$$4\varepsilon H''(0) = V''(g_0) \cdot W'(b_0) - \frac{1}{\varepsilon} W''(b_0) \cdot (V')^2(g_0). \tag{8}$$

Then equation 6 takes the form:

$$x' = \frac{1}{4} \cdot \left(V''(g_0) \cdot W'(b_0) + \frac{1}{\varepsilon} W''(b_0) \cdot (V')^2(g_0) \right) \cdot x^2 + O(x^3) = \alpha \cdot x^2 + O(x^3), \tag{9}$$

where

$$\alpha = \frac{1}{4} \cdot \left(V''(g_0) \cdot W'(b_0) + \frac{1}{\varepsilon} W''(b_0) \cdot (V')^2(g_0) \right) \neq 0.$$

Since the degree of the polynomial equation for x' is even, one branch of the central manifold converges to the fixed point, but the other branch diverges. □

Remark 2.5. Let me consider the flow, discussed in 2.3 and two fixed points, defined by the flow. There are only four topologically distinct cases of the fixed points ”order”, that system 2 can generate. By ”order” I mean the order of stationary points, connected with each other via invariant manifolds.) This order is defined by the indices of the fixed points and index of a closed curve containing these fixed points. Below (figures 1, 2, 3, 4) are all possible configurations of the flow neighboring the two consecutive fixed points.

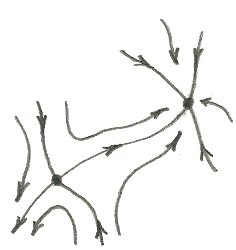

Figure 1: Dynamics in the neighborhood of the saddle fixed point followed by stable saddle-node Type II , or followed by sink.

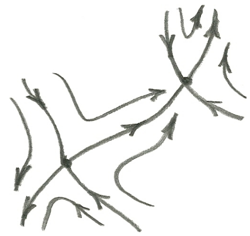

Figure 2: Dynamics in the neighborhood of the saddle fixed point followed by stable saddle-node Type I.

Figure 3: Dynamics in the neighborhood of the stable saddle-node Type I followed by stable saddle-node Type I.

Figure 4: Dynamics in the neighborhood of the sink, or stable saddle-node Type II followed by stable saddle-node Type I.

3. Examples

In this section, I discuss various examples of V and W attachment functions, frequently considered in economics literature. These examples demonstrate that fixed points can be isolated from each other (examples 3.1, 3.2, 3.4), or can form a 1-dimensional manifold (example 3.3). For simplicity, I assume that $\varepsilon = 1$. This assumption does not change the essential ideas of the examples. Two of these examples (3.1 and 3.3) have exact solutions, but I utilize the technique of fixed points characterization instead of solving the equations. Also the ideas of linearization of dynamical system (see, for example [4]) help to understand the dynamics of these two-sided markets.

Example 3.1 ("Staircase" attachment functions)**.** Let functions V and W be defined in the following way:

$$V(g) = \begin{cases} 0, & g \in [0, .25) \\ 1/2, & g \in [.25, .75) \\ 1, & g \in [.75, 1] \end{cases} \tag{10}$$

$$W(b) = \begin{cases} 0, & b \in [0, .25) \\ 1/2, & b \in [.25, .75) \\ 1, & b \in [.75, 1] \end{cases} \tag{11}$$

Solution to equation 2 is defined only on the regions of continuity of its right side, i.e., on the union of 9 rectangles without their common boundaries. The area $[0, .25)^2 \cup (.25, .75)^2 \cup (.75, 1]^2$ has three fixed points: $(0,0)$, $(1/2, 1/2)$ and $(1,1)$. The other 6 rectangles do not have fixed points of the flow defined on each of these rectangles. The fixed points of each of these 6 vector fields are located outside of the regions where the flow is defined. Therefore the flow ends on the boundaries of regions of continuity without reaching its fixed point. (For example, the linear flow on the rectangles $(.25, .75) \times (0, .25)$ is defined by $g = kb + .5$, $k < 0$. It cannot reach its fixed point $(0, .5)$ located outside of the rectangle

$(.25, .75) \times (0, .25)$.)

By 2.3 each of the three fixed points of the area $[0, .25)^2 \cup (.25, .75)^2 \cup (.75, 1]^2$ is a sink. Moreover, the equation is piece-wise linear, and the dynamics is simply defined as in figure 5. Any solution with initial condition in the lower-left square tends to the fixed point $(0,0)$. Any solution with initial conditions in the middle square tends to the fixed point $(.5, .5)$. Any solution with initial conditions in the upper-right square tends to the fixed point $(1, 1)$. All other initial conditions are transferred by the flow to the rectangles' boundaries. See figure 5.

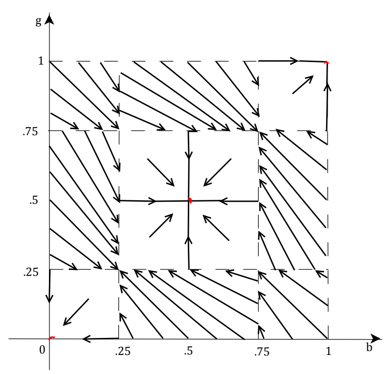

Figure 5: Piece-wise linear flow has three fixed points: (0, 0), (.5, .5) and (1, 1). They are of type sink.

Example 3.2 ("Smooth Staircase" attachment functions**).** Because of the discontinuities in the right side of the differential equation 2, in example 3.1 the flow is not defined on the entire unit square and is not continuous. I can now smooth the singularities of the previous example, and thus extend the solution to the entire unit square. Let functions V and W be defined in the following way:

$$V(g)=\begin{cases}0, & g\in[0,.25-\varepsilon]\\ c_1(g), & g\in(.25-\varepsilon,.25+\varepsilon)\\ 1/2, & g\in[\varepsilon+.25,.75-\varepsilon]\\ c_2(g), & g\in(.75-\varepsilon,.75+\varepsilon)\\ 1, & g\in[\varepsilon+.75,1]\end{cases} \tag{12}$$

$$W(b)=\begin{cases}0, & b\in[0,.25-\varepsilon]\\ c_3(b), & b\in(.25-\varepsilon,.25+\varepsilon)\\ 1/2, & b\in[\varepsilon+.25,.75-\varepsilon]\\ c_4(b), & b\in(.75-\varepsilon,.75+\varepsilon)\\ 1, & b\in[\varepsilon+.75,1]\end{cases} \tag{13}$$

(Here, c_1, c_2, c_3 and c_4 are increasing functions, smoothly connecting discontinuities of the step function.) Then, by 2.3, there are three fixed points of type sink, located in the same positions as in the previous example: (0,0,), (.5,.5) and (1,1). There are also at least two more fixed points. For definiteness, let me assume that c_1 intersects c_3 in a unique point in $(.25-\varepsilon,\ .25+\varepsilon)$, and c_2 intersects c_4 in a unique point in $(.75-\varepsilon,\ .75+\varepsilon)$. Then, it is clear that there are exactly two more fixed points, and one of them belongs to $(0,.25-\varepsilon)\times(0,.25-\varepsilon)$ and another one belongs to $(.75+\varepsilon,1)\times(.75+\varepsilon,1)$. Also, by 2.5 these two fixed points are of saddle types. If the total number of fixed points is more than 5, the order of the various types of nodes must be the same as described in 2.5.

Thus, the flow of equation 2, with V and W defined by 12 and 13, has the same three sinks: $(0,0)$, $(.5,.5)$ and $(1,1)$. However, the flow is defined on the entire unit square D. There are two separatrices, defined by stable manifolds of each of the saddle fixed points. The unit square is partitioned into three basins of attraction, defined by the boundary of the unit square and by the separatrix. Any solution originated in D, but not on a separatrix, converges to the fixed point, located in its basin of attraction. If solution starts on a separatrix, it converges to the corresponding saddle point. See figure 6.

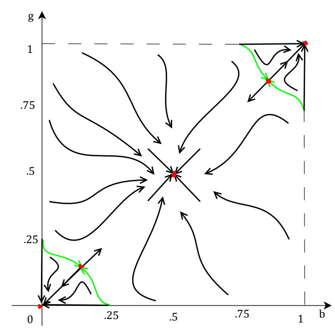

Figure 6: If a solution starts on a separatrix, it converges to the saddle point. Otherwise, any solution with initial conditions in D converges to one of the three contracting fixed points: $(0,0),(.5,.5)$ and $(1,1)$.

Example 3.3 (Linear attachment functions)**.** It is also natural to consider linear functions V and W. Let $V(g)=g$ and $W(b)=b$. Then, the differential equation 2 has a whole segment of fixed points, defined by equation $b=g$ on the unit square D. By 2.3, at each point of the diagonal, there is stable saddle-node. Any solution converges to the nearest point (b_0,b_0) on the diagonal of the unit square, i.e., it converges along stable invariant manifold $W^S=\left\{(b,g)\in D: g=-b+b_0\sqrt{2}\right\}$. See figure 7.

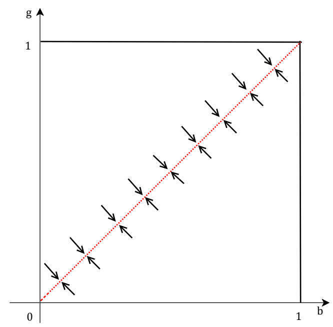

Figure 7: Any solution linearly converges to the nearest point on the diagonal of the unit square.

Example 3.4 (Convex attachment functions)**.** Another typical pair of attachment functions can be defined by $V(g) = \sqrt{g}$ and $W(b) = \sqrt{b}$. The graphs of these curves bound convex region in the plane. The convexity forces the increase of both state variables, b and g, with respect to time. The differential equation 2 has two fixed points: $(0,0)$ and $(1,1)$.

By 2.3, the point $(1,1)$ is a sink and the origin is not continuously differentiable. Thus, any solution with initial conditions in $\{D \setminus (0,0)\}$ converges to the point $(1,1)$. See figure 8.

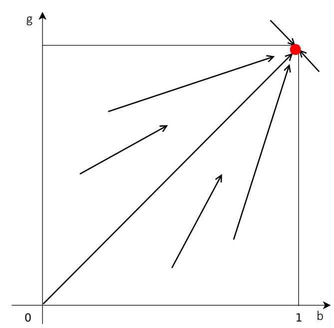

Figure 8: Any solution started in $\{D \setminus (0,0)\}$ converges the fixed point of type sink, located at the point $(1,1)$.

Remark 3.5 (α-degree attachment functions)**.** If attachment functions are of the form $V(g) = g^{\alpha}$, $W(b) = b^{\alpha}$, and $\alpha < 1$, then the dynamics is similar to the last example. The figure 8 applies to all such cases.

4. Nash equilibrium

Let me define a game, characterized by the dynamical system 2, as the following pair of strategy and payoff: strategy is the initial condition $(b_0, g_0) \in \mathscr{D}$; and the payoff is the limit point $(g_\infty(b_0, g_0), b_\infty(b_0, g_0)) \in \mathscr{D}$ of the flow initiated at (b_0, g_0). In other words, if the strategy is such that there are (b_0, g_0) users interacting through a platform now, then the payoff for b-users is g-users in the future $(g_\infty(b_0, g_0))$; and payoff for g-users is b-users in the future $(b_\infty(b_0, g_0))$.

Then, a set of points $(b_0^*, g_0^*) \in \mathscr{D}$ form a Nash equilibrium region $\mathscr{N}$, if for any $(b_0^*, g_0) \notin \mathscr{N}$ and $(b_0, g_0^*) \notin \mathscr{N}$,

$$g_\infty(b_0^*, g_0^*) \geq g_\infty(b_0, g_0^*)$$

and

$$b_\infty(b_0^*, g_0^*) \geq b_\infty(b_0^*, g_0).$$

Now I consider basins of attractions of the examples discussed in section 3.

It is clear that in each of the four examples, the gray area (see pictures below) represents the set of Nash equilibria points. It corresponds to the best strategy, which will give the highest payoff (g-users will receive the largest number of b-users, and b-users will receive the largest number of g-users) in the future.

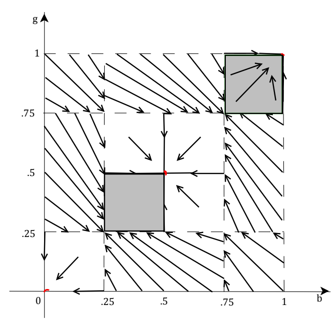

Figure 9: If attachment functions are step-functions, the best results are achieved when there are more than 75% of users of both types. The corresponding gray area is the Nash equilibrium region $\mathcal{N}$.

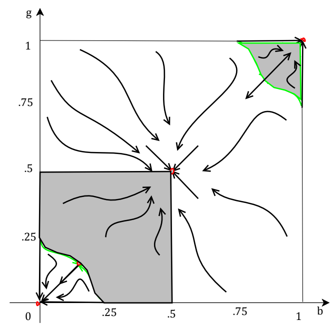

Figure 10: If attachment functions are smoothed step-functions, the best results are achieved when percentage of users is in the upper-right regions, defined by separatrix. The corresponding gray area is the Nash equilibrium region $\mathscr{N}$.

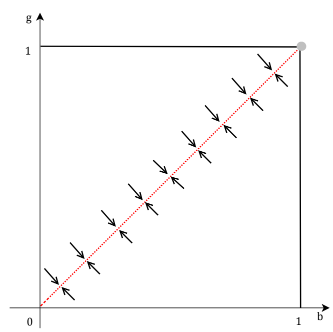

Figure 11: Linear attachment functions do not support simultaneous growth of both types of users. The Nash equilibrium is a single point $(1,1)$.

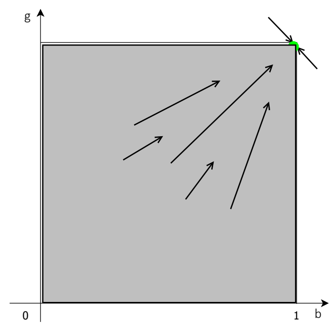

Figure 12: If attachment functions are convex, any strategy eventually will attract all users. The unit square without the origin ($\{D \setminus (0,0)\}$) is the Nash equilibrium region.

5. Numerical results

Let me discuss some applications of the Nash equilibrium in the game defined by the two-sided market dynamics. I will use the examples of section 3 to illustrate some ideas of attachment curves' tuning.

If a platform has no users from either side of the market, there is no dynamics on such platform. As soon as there is any (arbitrarily small) amount of users from any side, we can talk about the dynamics of the volume of users on the platform.

If the attachment curves are convex, the flow of users from both sides will increase. The convexity of the region between V and W curves will bring more customers to the platform. Thus, if we start with a very small number of customers, convex attachment should be created to expand the volume of users. In fact, if we preserve convexity and the curves V, W do not intersect (i.e., do not produce a fixed point), the volume of users can be brought to any desired level.

Of course, keeping the convexity is expensive. But improved volume of users on a platform allows to switch to a weaker attachment curves. For instance, one can tweak the attachment functions to stay constant for some time. It is clear that any constant attachment will eventually produce a fixed point. So we need to tweak the attachment again for switching to a convex one, unless we want to stay at the level, that corresponds to the fixed point. (Note that it may take very long time to reach this fixed point. Theoretically, the exponential decay is infinitely long.) Thus, alternating convex and constant shapes we can create a staircase-like attachment. However, until we are satisfied with the volume of users, we should not allow intersection of V and W curves, which would produce a fixed point. In other words, we cannot allow constant shape for too long time and we should have proper shapes of convex attachments, like in the following example.

In this example, I will assume that V and W are of the same shape in respective coordinates (V and W will be inverse of each other). For the numerical construction of V in Mathematica, I use *Sqrt* function, which produces convex attachment starting from $(0,0)$. I switch from convex to constant at $V(1/9) = 1/3$; then back to convex at $(1/3-\varepsilon, 1/3)$ with $\varepsilon = .07$ (setting $\varepsilon > 0$ allows to avoid intersection with W at $(1/3, 1/3)$). I switch to constant again at $(1/2, \sqrt{1/2-(1/3-\varepsilon)}+1/3)$; back to convex at $(\sqrt{1/2-1/3}+1/3-\varepsilon, \sqrt{1/2-(1/3-\varepsilon)})$; and constant again as soon as the convex attachment reaches the ceiling of $b = 1$. Clearly, this curve lies above the diagonal and has no intersections (except $(0,0)$ and $(1,1)$) with the similarly constructed W. Thus, given any initial presence of users (including arbitrarily small, but different from 0), the flow of users tends to the fixed point $(1,1)$, i.e., this is Nash equilibrium region. In the simulated example, the initial volume of users is $b = 0.05$ and $g = 0$. See figure 13.

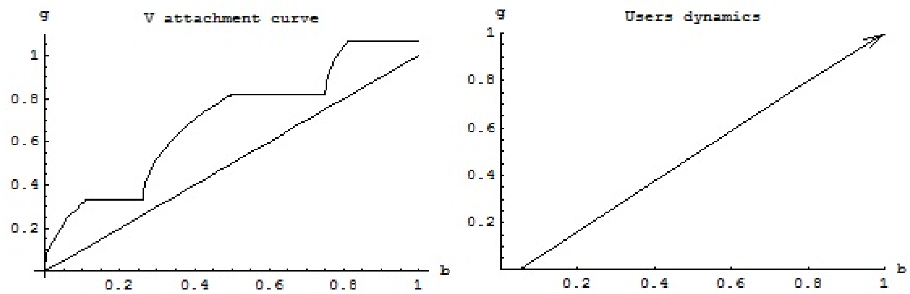

Figure 13: If V and W do not intersect (except for $(0,0)$ and $(1,1)$), the volume of users tends to $(1,1)$.

If we let $\varepsilon = 0$, V and W will intersect at the points, where constant attachments switch to convex. Then, the same initial volume of users can grow only up to the first fixed point, which is located at $(1/3, 1/3)$. See figure 14.

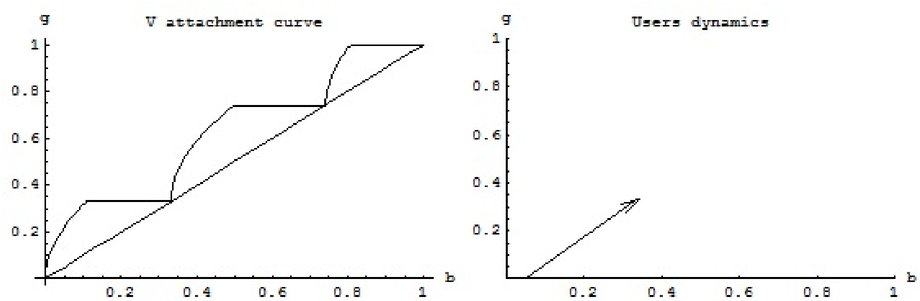

Figure 14: Fixed point (1/3,1/3) limits the growth of users.

With the same attachment curves ($\varepsilon = 0$), if the initial volume of users is "above" the fixed point $(1/3, 1/3)$ (for example, $b_0 = 0.6$ and $g_0 = 0$), the flow of users will tend to the next fixed point. See figure 15.

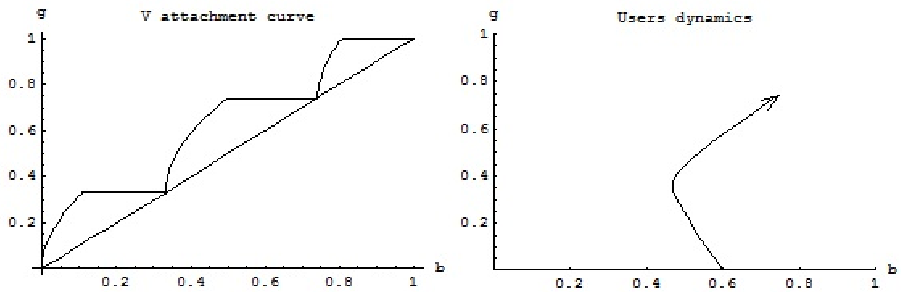

Figure 15: If initial conditions are higher, the next fixed point can be the limit for the growth of users.

If we have constant attachment for a very long time, this can be modeled with $\varepsilon < 0$. Then, only even numbered fixed points (counting $(0,0)$ as the first fixed point) will be attractors. This situation is similar to the example 3.2, but with even attracting fixed points (because of the convexity of the first bounded region). Odd numbered fixed points will belong to separatrix. If attachment curves produced separatrix (saddle) fixed point, the next (closer to $(1,1)$) fixed point will be an attractor, and will help to increase the volume of users, because it belongs to the Nash equilibrium region. Thus, there is no need to tweak the attachment curves so that they have square root grows for a long time after their intersection with W. They can be switched to a constant to produce another attracting fixed point as soon as the desired level of the fixed point (corresponding to the volume of users) is achieved. In this case, the dynamics is sensitive to initial conditions. Depending on whether the initial state belongs to the Nash equilibrium region or not, the volume of users may increase to the fourth fixed point or decrease to the second fixed point. In this example (see figure 15.), for the initial condition $b_0 = .46$, $g_0 = .46$ volume of users increases, while for the initial condition $b_0 = .45$, $g_0 = .45$ it decreases.

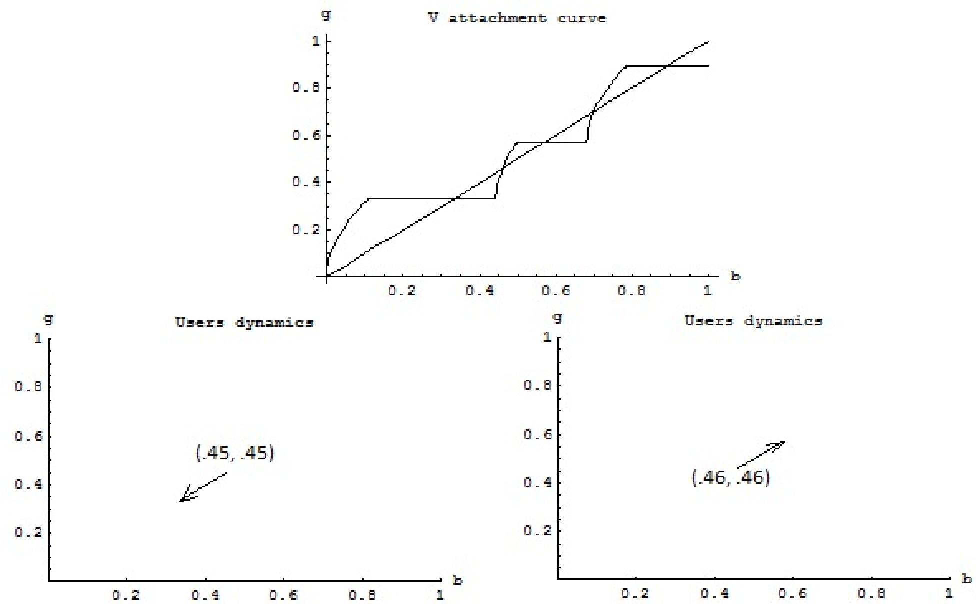

Figure 16: If initial conditions are higher, the next fixed point can be the limit for the growth of users.

6. Conclusions

Understanding the differences between attracting, saddle and repelling fixed points of this model helps to predict the dynamics of users. Basin of attraction, associated with the Nash equilibrium of the game defined by the two-sided market model allows to find appropriate moments for adjustments of attachment curves and thus, to influence dynamics of the volume of users on the platform.

Also, since platform's revenue depends on the volume of users, the dynamical system approach allows to analyze platform's revenue over time.

[1] A.A. Andronov, S.E. Khaiken, E.A. Vitt, Theory of Oscillators, Pergamon Press, Oxford, 1966.

[2] J. Carr, Applications of Centre Manifold Theory, Springer-Verlag, New-York, Heidelberg, Berlin, 1981.

[3] J. Guckenheimer, Ph. Holmes, Nonlinear Oscillations, Dynamical Systems, and Bifurcations of Vector Fields, Springer-Verlag, 1983.

[4] M. Guysinsky, B. Hasselblatt, V. Rayskin, Differentiability of the Hartman-Grobman linearization, Discrete Contin. Dyn. Syst. 9 (2003) 979–984.

[5] D. Henry, Geometric Theory of Semilinear Parabolic Equations, Springer-Verlag, Lecture Notes in Mathematics 840, New-York, Heidelberg, Berlin, 1981.

[6] C.C. Pugh, M. Shub, Linearization of normally hyperbolic diffeomorphisms and flows, Invent. Math. 10 (1970) 187–198.

[7] J.-C. Rochet, J. Tirole, Platform competition in two-sided markets, Journal of the European Economic Association 1(4) (2003) 990–1029.