Acoustic diffraction by an impedance wedge

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Abstract. We consider the boundary-value problem for the Helmholtz equation connected with an infinite wedge with an impedance boundary on its face. The scheme of solution includes applying the Kontorovich-Lebedev (KL) transform, derivation of an integral equation satisfied by the KL spectral amplitude and obtaining near and far field representations together with the conditions of validity of these representations.

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1 Introduction

One of the important problems of acoustic field theory concerns the evaluation of source excited acoustic fields in the presence of an impedance wedge. This class of boundary conditions is usually considered to approximate imperfectly reflecting surfaces. In fact, near field evaluation as well as the scattering and diffraction of waves by an impedance wedge are very difficult problems to solve in closed form analytically. To this day, there is no explicit closed-form solution for these problems. The problem was first solved by Malyuzhinets [1] in the form of a Sommerfeld integral with a new special function, namely the Malyuzhinets function. The formulation presented here is an attempt to adapt the technique of the KL-transform to the problem thus allowing for alternative representations for the near and far fields.

In section 2 the problem is formulated. Section 3 derives the integral equation satisfied by the KL spectral function. In section 4 the field representations are given. Conclusions are given in section 5. Appendix A presents a scheme to numerically solve the integral equation. In Appendix B the singularities of the KL spectral function are identified and quantified.

It should be mentioned that, in analogy with the impedance cone problem

2 Formulation

We consider the problem of scattering of harmonic acoustic waves by an infinite impedance wedge imbedded in an infinite medium. In what follows (r, ϕ, z) denotes the usual cylindrical coordinates such that on the surface of the wedge one has $\phi = \pm\beta$ with $0 < \beta < \pi$. $C = \{0 < r < \infty, -\beta < \phi < \beta, -\infty < z < \infty\}$ defines the region external to the wedge. A time factor $\exp\{-i\omega t\}$ is assumed and omitted throughout. k, ρ and λ are respectively the wave number, density and incompressibility of the medium in C. Since the problem under consideration is one of scattering and diffraction the wave number k is real (complex) for the lossless (lossy) case. However, for a while we shall assume that k are such that

$$\phi = \arg k = \frac{\pi}{2}. \tag{1}$$

The condition in Eq. (1) has been shown by Osipov [4] to guarantee the convergence of the subsequent KL integral representations, which in turn is required for the possibility to use the boundary conditions conveniently, leading to the derivation of an integral equation on the KL spectrum. Once an integral equation is derived and a numerical scheme is proposed to solve it, we examine the restrictions under which k could be extended to real or complex values (to tackle the original scattering and diffraction problem) while maintaining the validity of the numerical scheme.

The acoustic field is describable by the pressure p and velocity $\mathbf{V}$ obeying the Euler field equations [5]

$$\frac{-i\omega}{\lambda} p + \nabla.\mathbf{V} = -s, \tag{2a}$$

$$\nabla p - i\omega\rho\mathbf{V} = -\mathbf{f}. \tag{2b}$$

The excitation terms s and $\mathbf{f}$ represent the scalar particle source and the impressed vector force densities, respectively; ∇ is the spatial gradient operator.

With ρ and λ assumed constant and from Eqs. (2a) and (2b), the acoustic pressure in C, $p(r,\phi)$, satisfies the inhomogeneous Helmholtz equation

$$(\nabla^2 + k^2)p(r,\phi) = -\frac{\delta(r-r_0)\,\delta(\phi-\phi_0)}{r_0}, \tag{3}$$

with the impedance boundary condition on its face

$$Z_s V_\phi(r,\pm\beta) = p(r,\pm\beta). \tag{4}$$

$k = \frac{\omega}{c}$, $c = \sqrt{\frac{\lambda}{\rho}}$is the acoustic speed. ∇^2 stands for the Laplacian.$V_\phi(r,\phi)$ is the component of the velocity field, $\mathbf{V}(r,\phi)$, normal to the wedge surface with

$$\mathbf{V}(r,\phi) = \frac{-i}{\omega\rho}[\mathbf{r}_0\frac{\partial}{\partial r} + \boldsymbol{\phi}_0\frac{1}{r}\frac{\partial}{\partial\phi}]p(r,\phi). \tag{5}$$

The fields are required to decay exponentially to zero as $r \to \infty$. This replaces the Sommerfeld radiation condition for the $\mathrm{Im}\,k = 0$ case.

Following the condition of Meixner [6], we assume that the energy stored in any finite neighborhood of the edge of the wedge must be finite, that is,

$$\int_S (\rho|\mathbf{V}|^2 + \frac{1}{\rho c^2}p^2)dS \to 0 \tag{6}$$

as the surface S contracts to the neighborhood of the edge. Owing to this condition the field behavior near the edge of the wedge is

$$p = O(r^\nu),\ |\mathbf{V}| = O(r^{\nu-1}), r \to 0, \nu > 0. \tag{7}$$

By using the symmetry of the problem structure with respect to the planes $\phi = 0, \pm\pi$, we split the problem into two independent sub-problems. The boundary conditions on the symmetry planes correspond to either a hard wall (normal velocity is zero on the wall) or a soft wall (acoustic pressure is zero on the wall). Without loss of generality, we confine our attention to the case of a soft wall only and the angular domain $0 \le \phi \le \pi$.

$$p(r,\phi) = \frac{1}{2}\int_{-i\infty}^{i\infty} \nu J_\nu(kr)P(\nu,\phi)d\nu, \tag{9}$$

where $J_\nu(z)$ and $H_\nu^{(1)}(z)$ are the standard Bessel and Hankel functions respectively and ν is purely imaginary.

Since [5]

$$H_{-\nu}^{(1)}(z) = e^{i\pi\nu}H_\nu^{(1)}(z) \tag{10a}$$

the definition in Eq. (10a) implies that

$$P(-\nu,\phi) = e^{i\pi\nu}P(\nu,\phi). \tag{10b}$$

There are two additional ways of writing the inverse transform of Eq. (9):

a) Making use of

$$J_\nu(z) = \frac{1}{2}[H_\nu^{(1)}(z) + H_\nu^{(2)}(z)], \tag{11a}$$

$$H_{-\nu}^{(2)}(z) = e^{-i\pi\nu}H_\nu^{(2)}(z), \tag{11b}$$

we obtain

$$p(r,\phi) = \frac{1}{4}\int_{-i\infty}^{i\infty} \nu H_\nu^{(1)}(kr)P(\nu,\phi)d\nu. \tag{12}$$

b) Substituting with Eqs. (10) in Eq. (12), we obtain

$$p(r,\phi) = \frac{-i}{2}\int_{0}^{i\infty} \nu e^{i\pi\nu}\sin\pi\nu H_\nu^{(1)}(kr)P(\nu,\phi)d\nu. \tag{13}$$

The velocity field is derived from Eq. (5). Relevant to the problem under consideration is the velocity component normal to the wedge surface, namely $V_\phi(r,\phi)$

Applying the KL-transform to Eq. (3) , we get the ordinary differential equation

$$[\frac{d^2}{d\phi^2}+\nu^2]P(\nu,\phi)=-H_\nu^{(1)}(kr_0)\delta(\phi-\phi_0),\ 0\leq\phi<\beta, \tag{16}$$

From Eqs. (5) and (13), the impedance boundary condition now reads

$$\frac{Z_s}{\omega\rho}\frac{1}{r}\int_0^{i\infty}\nu e^{i\pi\nu}\sin\pi\nu H_\nu^{(1)}(kr)\frac{d}{d\beta}P(\nu,\beta)d\nu=i\int_0^{i\infty}\nu e^{i\pi\nu}\sin\pi\nu H_\nu^{(1)}(kr)P(\nu,\beta)d\nu, \tag{17}$$

where $\frac{d}{d\beta}P(\nu,\beta)$ stands for $\frac{d}{d\phi}P(\nu,\phi)|_{\phi=\beta}$, which notation will be used throughout.

We represent the field in C as the sum over an unperturbed field $p^{(0)}$, $\mathbf{V}^{(0)}$ plus a scattered field $p^{(1)}$, $\mathbf{V}^{(1)}$ due to the presence of the wedge. Hence

$$p(r,\phi)=p^{(0)}(r,\phi)+p^{(1)}(r,\phi) \tag{18}$$

leading to

$$P(\nu,\phi)=P^{(0)}(\nu,\phi)+P^{(1)}(\nu,\phi). \tag{19}$$

2.1 The unperturbed field

$P^{(0)}(\nu,\phi)$ satisfies the source conditions of Eq. (16), namely

a) $P^{(0)}(\nu,\phi)$ is continuous across $\phi=\phi_0$

$$P^{(0)}(\nu,\phi_0-0)=P^{(0)}(\nu,\phi_0+0), \tag{20a}$$

b) $\frac{d}{d\phi}P^{(0)}(\nu,\phi)$ is discontinuous across $\phi=\phi_0$

$$\frac{d}{d\phi}P^{(0)}(\nu,\phi)|_{\phi_0+0}-\frac{d}{d\phi}P^{(0)}(\nu,\phi)|_{\phi_0-0}=-H_\nu^{(1)}(kr_0). \tag{20b}$$

From the two conditions in Eqs. (20) and the soft boundary requirement

$V_\phi^{(0)}(r,\phi)$ is thus given by

$$V_\phi^{(0)}(r,\phi) = \frac{-1}{2\omega\rho_1}\frac{1}{r}\int_0^{i\infty} \nu e^{i\pi\nu} \sin\pi\nu H_\nu^{(1)}(kr)\frac{d}{d\phi}P^{(0)}(\nu,\phi)d\nu, \tag{22a}$$

where, for $\phi > \phi_0$

$$\frac{d}{d\phi}P^{(0)}(\nu,\phi) = -\frac{H_\nu^{(1)}(kr_0)\cos(\nu[\pi-\phi])\sin(\nu\phi_0)}{\sin(\nu\pi)} \tag{22b}$$

and for $\phi < \phi_0$

$$\frac{d}{d\phi}P^{(0)}(\nu,\phi) = \frac{H_\nu^{(1)}(k_1 r_0)\sin(\nu[\pi-\phi_0])\cos(\nu\phi)}{\sin(\nu\pi)}. \tag{22c}$$

2.2 The scattered field

Since $P(\nu,\phi)$ must satisfy the soft boundary condition at $\phi = 0$, we represent the scattered field by

$$P^{(1)}(\nu,\phi) = A(\nu)\sin(\nu\phi). \tag{23}$$

$A(\nu)$ is a KL spectrum to be determined from the impedance boundary condition. Eq. (10b) enforces

$$A(-\nu) = -e^{i\pi\nu}A(\nu), \tag{24}$$

Additionally, the convergence of the KL integrals [4] at $\phi = \beta$ implies that, when $\mathrm{Im}\ \nu \to +\infty$, the spectral function must vanish as

$$A(\nu) = O\left[\exp(-\beta\ \mathrm{Im}\ \nu)\right],$$

From the above we obtain

leading to

$$\frac{Z_s}{\omega\rho r}\int_0^{i\infty}\nu e^{i\pi\nu}\sin\pi\nu H_\nu^{(1)}(kr)[F_1(\nu)+\nu A(\nu)\cos(\nu\beta)]d\nu =$$
$$i\int_0^{i\infty}\nu e^{i\pi\nu}\sin\pi\nu H_\nu^{(1)}(kr)[F_2(\nu)+A(\nu)\sin(\nu\beta)]d\nu, \tag{26a}$$

where

$$F_1(\nu) = -\frac{H_\nu^{(1)}(kr_0)}{\sin(\nu\pi)}\sin(\nu\phi_0)\cos(\nu[\pi-\beta]), \tag{26b}$$

$$F_2(\nu) = \frac{H_\nu^{(1)}(kr_0)}{\nu\sin(\nu\pi)}\sin(\nu[\pi-\beta])\sin(\nu\phi_0). \tag{26c}$$

3 The integral equation derivation

The integral equation on $A(\nu)$ is derived by multiplying Eq. (26a) by $H_\mu^{(1)}(kr)$ and integrating with respect to r from 0 to ∞. Making use of [8, formula 6.576 (4)],

$$\int_0^\infty r^{-\lambda}K_{i\mu}(-ikr)K_{i\nu}(-iNkr)dr = \frac{2^{-2-\lambda}(-ik)^{-i\nu+\lambda-1}(-iNk)^{i\nu}}{\Gamma(1-\lambda)}G_1(\lambda,\mu,\nu)G_2(\lambda,\mu,\nu),$$

where

$$G_1(\lambda,\mu,\nu) = \Gamma(\frac{1-\lambda+i\mu+i\nu}{2})\Gamma(\frac{1-\lambda-i\mu+i\nu}{2})\Gamma(\frac{1-\lambda+i\mu-i\nu}{2})\Gamma(\frac{1-\lambda-i\mu-i\nu}{2}),$$

$$G_2(\lambda,\mu,\nu) = F(\frac{1-\lambda+i\mu+i\nu}{2},\frac{1-\lambda-i\mu+i\nu}{2};1-\lambda;1-N^2),$$

$$\cosh(x-y)\cosh(x+y)=\frac{1}{2}(\cosh 2x+\cosh 2y),$$

we obtain

$$\int_0^\infty H_\mu^{(1)}(kr)H_\nu^{(1)}(kr)dr=\frac{-2i}{k}e^{-i(\nu+\mu)\pi/2}\frac{1}{\cos\pi\nu+\cos\pi\mu}. \tag{27a}$$

From [10] we get

$$\int_0^\infty H_\mu^{(1)}(kr)H_\nu^{(1)}(kr)\frac{dr}{r}=\frac{2}{\mu\ \sin\mu\pi}e^{-i(\mu+\nu)\pi/2}[\delta(\operatorname{Im}\nu-\operatorname{Im}\mu)+\delta(\operatorname{Im}\nu+\operatorname{Im}\mu)]. \tag{27b}$$

For the particular case which is considered here ($\operatorname{Im}\mu>0$, $\operatorname{Im}\nu>0$), the $\delta(\operatorname{Im}\nu+\operatorname{Im}\sigma)$ in Eq. (27b) will be removed.

We obtain the integral equation on $A(\mu)$

$$s(\mu)+\frac{Z}{Z_s}\int_0^{i\infty}K(\nu,\mu)\sin\nu\beta A(\nu)d\nu=\mu\cos\mu\beta A(\mu),\operatorname{Im}\mu\geqslant 0,\operatorname{Re}\mu=0, \tag{28}$$

where

$$s(\mu)=s_1(\mu)+s_2(\mu), \tag{29a}$$

$$s_1(\mu)=-F_1(\mu), \tag{29b}$$

$$s_2(\mu)=\frac{Z}{Z_s}\int_0^{i\infty}K(\nu,\mu)F_2(\nu)d\nu, \tag{29c}$$

Additionally,

In order to adapt the integral equation for the numerical scheme of Appendix A, we re-write Eq. (28) as

$$\tilde{s}(\mu) + \frac{Z}{Z_s}\sin\mu\beta \int_0^{i\infty} K(\nu,\mu)\tilde{A}(\nu)d\nu = \mu\cos\mu\beta\tilde{A}(\mu), \operatorname{Im}\mu \geqslant 0, \operatorname{Re}\mu = 0, \tag{31a}$$

where

$$\tilde{A}(\mu) = F_2(\mu) + \sin\mu\beta A(\mu), \tag{31b}$$

$$\tilde{s}(\mu) = H_\mu^{(1)}(kr_0)\sin\mu\phi_0. \tag{31c}$$

In Appendix A an approximate solution for the integral equation in Eq. (31a) using a collocation method is given. The scheme is inspired by the one used by Antipov [11]. In Appendix B the spectrum of $A(\mu)$ is analytically continued from the imaginary axis into the right half of the complex μ-plane.

4 Field representations

With the results of Appendices A and B, we proceed to derive near and far field representations.

4.1 The near field

As was shown in Appendix B, $A(\mu)$ is a meromorphic function whose only singularities in the complex μ-plane are poles. The conditions under which some of these poles are of second order have also been established. The second order poles that may be present indicate that the acoustic pressure (velocity) field near the edge of the wedge may contain the logarithm of the distance in addition to its power, namely $r^\mu \log r$ ($r^{\mu-1}\log r$).

For $r < r_0$, making use of the KL representation in Eq. (9) we obtain

$$p^{(1)}(r,\phi) = p_1^{(1)}(r,\phi) + p_2^{(1)}(r,\phi) + p_3^{(1)}(r,\phi), \tag{33}$$

where

$$p_1^{(1)}(r,\phi) = -\pi i \sum_{\nu_{pl}} \nu_{pl} J_{\nu_{pl}}(kr) \mathrm{Res}[A(\nu_{pl})] \sin(\nu_{pl}\phi), \tag{34}$$

where the residues $\mathrm{Res}[A_2(\nu_{pl})]$ are given in Eq. (B.9c) in Appendix B. Eq. (34) is valid for first order poles only. The series summand is dominated by

$$J_{\nu_{pl}}(kr) H_{\nu_{pl}}^{(1)}(kr_0), \nu_{pl} \to \infty. \tag{35}$$

From [5]

$$J_\nu(z) \sim \frac{1}{\sqrt{2\pi\nu}} (\frac{2\nu}{ez})^{-\nu}, \ \nu \to \infty, \tag{36a}$$

$$H_\nu^{(1)}(z) \sim \frac{1}{\sqrt{2\pi\nu}} (\frac{2\nu}{ez})^{\nu}, \ \nu \to \infty, \tag{36b}$$

Eq. (35), characterizing the dominant behavior of the series summand, reduces to

$$\frac{1}{\nu_{pl}} (\frac{r}{r_0})^{\nu_{pl}}, \nu_{pl} \to \infty. \tag{37}$$

Modifications are required for second order poles if they exist.

Since it is possible for the poles ν_{pl} to accumulate at infinity, forming a dense set on the real line, in addition to the possibility of higher order poles, either of these possibilities could avoid the convergence of the residue sum. The truncated residue sum should, therefore, be understood as giving some asymptotic approximation to the field in terms of first identified poles.

$$p_2^{(1)}(r,\phi) = -\pi i \sum^{\infty} \nu_p J_{\nu_p}(kr) \mathrm{Res}[A_1(\nu_p)] \sin(\nu_p\phi). \tag{38}$$

and the series converges exponentially.

$$p_3^{(1)}(r,\phi) = -\pi i \sum_{s=1}^{\infty} s J_s(kr) \mathrm{Res}[A_1(s)] \sin(s\phi). \tag{40}$$

The residues are given in Eq. (B.9b) and the dominant behavior of the series summand is

$$\frac{1}{s}(\frac{r}{r_0})^s, s \to \infty, \tag{41}$$

and the series converges exponentially.

Finally we add the contribution from $p^{(0)}(r,\phi)$ in order to get the total near field

$$p^{(0)}(r,\phi) = -i \sum_{s=1}^{\infty} (-1)^s H_s^{(1)}(kr_0) J_s(kr) \sin(s[\pi - \phi_>]) \sin(s\phi_<). \tag{42}$$

The series summand behaves as given in Eq. (41) and the series converges exponentially.

The above field representation is valid for all observation angles in C and, due to the exponential convergence, only few poles are needed to get an accurate representation for the near field.

Meanwhile, if the main interest is to compute the singular behavior of the field near the edge of the wedge (iff $\beta > \pi/2$), then one could use

$$p^{(s)}(r,\phi) = -\pi i \nu_p J_{\nu_p}(kr) \mathrm{Res}[A(\nu_p)] \sin(\nu_p[\pi - \phi]), \nu_p = \frac{\pi}{2\beta}, \tag{43}$$

whose velocity field, as per Eq. (5) gives the singular field near the edge of the wedge.

4.2 The far field

cumbersome for the impedance wedge problem. We propose an alternative to calculate the far field. That is, to invoke the reciprocity principle.

The field expressions are similar to those given above with (r, ϕ) and (r_0, ϕ_0) interchanged. The convergence properties of the series representations is as discussed before in Sec. 4.1.

Remark. The above residue sums cannot be used to derive far field results for plane wave illumination, since the convergence is an asymptotic result for source near the edge of the wedge.

4.3 The plane wave illumination case

The far field due to a normally (with respect to z) incident plan wave is recovered by:

a) Replacing $H_\nu^{(1)}(kr_0)$ by $\exp(-i\pi\nu/2)$,

b) Substituting in Eq. (9) with [8]

$$J_\nu(kr) = \frac{e^{-i\pi\nu/2}}{2\pi} \int_\gamma e^{ikr\cos\alpha + i\alpha\nu} d\alpha, \tag{44}$$

where γ is the Sommerfeld integration path going from $-\frac{\pi}{2}+i\infty$ to $\frac{3\pi}{2}+i\infty$ in the complex α plane.

We obtain the far field representation

$$p(r, \phi) = \int_\gamma e^{ikr\cos\alpha} f(\alpha) d\alpha, \tag{45}$$

where

$$f(\alpha) = f^{(0)}(\alpha) + f_1^{(1)}(\alpha) + f_2^{(1)}(\alpha) + f_3^{(1)}(\alpha), \tag{46}$$

with

$$f^{(0)}(\alpha) = \frac{-i}{2} \sum_{s=1}^{\infty} e^{i\alpha s} e^{-i\pi s/2} \sin(s\phi_<) \sin(s[\pi - \phi_>]), \tag{47a}$$

$$f_3^{(1)}(\alpha) = \frac{-i}{2}\sum_{s=1}^{\infty} e^{i\alpha s} e^{-i\pi s/2} s\mathrm{Res}[A(s)]\sin(s\phi). \tag{47d}$$

The series in Eqs. (47a), (47c) and (47.d) converge exponentially and the truncated residue sum in Eq. (47b) understood as giving some asymptotic approximation in terms of first identified poles.

5 Conclusion

The integral equation satisfied by the KL spectral function, $A(\nu)$, has been given in Eq. (26). A collocation scheme for the numerical evaluation of the KL spectral function in Eq. (31) was constructed in Appendix A. In Appendix B, analytic continuation was utilized to establish the meromorphic nature of $A(\nu)$ and to identify its pole singularities. Representations for the near and far fields, in terms of the singularities of $A(\nu)$, were given in Sec. 4. Situations under which these representations turns into an asymptotic approximation were discussed. The approach of this paper is applicable for 2- and 3-D problems of thermal conductivity, electromagnetics and elastodynamics in a wedge and a cone configuration with boundary conditions, both of the continuity and impedance types, on the radial direction.

Appendix A. Numerical scheme to solve the integral equation in Eq. (31)

The scheme used here is inspired by the one used by Antipov [11].

Let $\{\nu_{m-1}\}$, $m = 1, 2, ..., M+1$, be a set of points defined on the imaginary axis of the complex ν-plane such that

$$\nu_{m-1} = i\ \delta(m-1)^{\epsilon},\ \delta > 0,\ \epsilon > 0. \tag{A.1a}$$

Let also

$$\mu_m = (\nu_{m-1} + \nu_m)/2,\ \ m = 1, 2, ..., M. \tag{A.1b}$$

$$\tilde{A}(\mu_n) = s^*(\mu_n) + \frac{Z \sin \mu_n \beta}{Z_s \mu_n \cos \mu_n \beta} \sum_{m=1}^{M} \int_{\nu_{m-1}}^{\nu_m} K_1(\nu, \mu_n) \tilde{A}(\nu) \frac{\sin\ (\pi\nu)}{\cos\ (\pi\nu) + \cos\ (\pi\mu_n)} d\nu, \tag{A.2a}$$

with $n = 1, 2, ..., M$ and

$$K_1(\nu, \mu_n) = \nu e^{i(\nu - \mu_n)\pi/2}, \tag{A.2b}$$

$$s^*(\mu_n) = \frac{\tilde{s}(\mu_n)}{\mu_n \cos \mu_n \beta}. \tag{A.2c}$$

Eq. (A.2a) is further approximated as

$$\tilde{A}(\mu_n) = s^*(\mu_n) + \frac{Z \sin \mu_n \beta}{Z_s \mu_n \cos \mu_n \beta} \sum_{m=1}^{M} K_1(\hat{\nu}_m, \mu_n) \tilde{A}(\mu_m) I_{nm}, n = 1, 2, ..., M, \tag{A.3a}$$

where

$$\hat{\nu}_m = (\nu_{m-1} + \nu_m)/2, \tag{A.3b}$$

$$I_{nm} = \int_{\nu_{m-1}}^{\nu_m} \frac{\sin\ (\pi\nu)}{\cos\ (\pi\nu) + \cos\ (\pi\mu_n)} d\nu. \tag{A.3c}$$

The integral in Eq. (A.3c) is given by

$$I_{nm} = \frac{-1}{\pi} \ln\left(\frac{\cos\ (\pi\nu_m) + \cos\ (\pi\mu_n)}{\cos\ (\pi\nu_{m-1}) + \cos\ (\pi\mu_n)}\right). \tag{A.3d}$$

Thus we re-write the linear system as

$$\tilde{\mathbf{A}} = \mathbf{s}^* + \mathbf{C}^* \tilde{\mathbf{A}}, \tag{A.4a}$$

with $\tilde{\mathbf{A}}$ and $\mathbf{s}^*$ vectors and $\mathbf{C}^*$ the matrix

$$H_\nu^{(1)}(z) \sim e^{|\nu|(\frac{\pi}{2}-\arg z)}/|\nu|^{1/2}, \nu \to i\infty,$$

we infer that

$$s^*(\mu_n) = O(e^{-|\mu_n|(\beta+\phi-\frac{\pi}{2}-\theta_0)}/|\mu_n|^{3/2}), n \to \infty. \tag{A.5a}$$

Therefore, $s^*(\mu_n)$ decays exponentially as $n \to \infty$ ($\mu_n \to i\infty$) when

$$\frac{\pi}{2} + \theta_0 < \beta + \phi. \tag{A.5b}$$

Additionally, for m fixed and $n \to \infty$, writing I_{nm} as

$$I_{nm} = \frac{-1}{\pi}[\ln(1 + \frac{\cos\ (\pi\nu_m)}{\cos\ (\pi\mu_n)}) - [\ln(1 + \frac{\cos\ (\pi\nu_{m-1})}{\cos\ (\pi\mu_n)})], \tag{A.6a}$$

followed by the series representation

$$\ln(1+x) = x - \frac{x^2}{2} + O(x^3), x < 1, \tag{A.6b}$$

we obtain

$$I_{nm} = O(e^{-|\mu_n|\pi}), n \to \infty, m \text{ is fixed}. \tag{A.6c}$$

Hence, we estimate the behavior of C^*_{nm} as $n \to \infty$ ($\mu_n \to i\infty$), m is fixed, as

$$C^*_{nm} = O(e^{-|\mu_n|\frac{\pi}{2}}/|\mu_n|), n \to \infty, m \text{ is fixed}, \tag{A.7}$$

and C^*_{nm} decays exponentially as $n \to \infty$, m is fixed.

Additionally, for n fixed and $m \to \infty$,

$$I_{nm} \sim \frac{-1}{\pi} \ln \frac{\cos\ (\pi\nu_m)}{\cos\ (\pi\nu_{m-1})} \to -(\nu_m - \nu_{m-1}) \tag{A.8a}$$

leads, on account of Eq. (A.1a), to

where we have made use of the fact that $\hat{\nu}_m$ and ν_m are practically equal for $m \to \infty$.

Thus C^*_{nm} decays exponentially as $m \to \infty$, n is fixed.

Assuming that the inequality in Eq. (A.5b) is satisfied and that an inverse exists for the matrix $\{\delta_{nm} - C^*_{nm}\}$ $(n, m = 1, 2, ..., M)$, then the approximate solution $\tilde{A}^{(M)}$ converges to the exact one $\tilde{A}^*$ and the rate of convergence is exponential (see [11]).

The inequality in Eq. (A.5b) could be changed by a slight modification to the numerical scheme through a normalization process. This is outlined as follows:

a) Normalize $\tilde{A}(\mu)$ in Eq. (31a) as

$$\tilde{A}(\mu) = \hat{A}(\mu)e^{-i\mu(\frac{\pi}{2}+\phi_0-\beta)}, \tag{A.10}$$

leading to modifying Eq. (31a) to

$$\hat{A}(\mu) = \hat{s}(\mu)+\hat{C}(\mu)\int_0^{i\infty} \hat{K}_1(\nu,\mu)\hat{A}(\nu)\frac{\sin\ (\pi\nu)}{\cos\ (\pi\nu)+\cos\ (\pi\mu)}d\nu, \mathrm{Im}\,\mu \geqslant 0, \mathrm{Re}\,\mu = 0, \tag{A.11a}$$

where

$$\hat{s}(\mu) = \tilde{s}(\mu)e^{i\mu(\frac{\pi}{2}+\phi_0-\beta)}, \tag{A.11b}$$

$$\hat{C}(\mu) = \frac{Z\sin\mu\beta}{Z_s\mu\cos\mu\beta}e^{i\mu(\frac{\pi}{2}+\phi_0-\beta)}, \tag{A.11c}$$

$$\hat{K}_1(\nu,\mu) = K_1(\nu,\mu)e^{-i\nu(\frac{\pi}{2}+\phi_0-\beta)}. \tag{A.11d}$$

b) Use the numerical scheme to find $\hat{A}(\mu)$.

Assuming that the involved matrix has an inverse and that the inequalities

$$\beta > \theta_0, \tag{A.12a}$$

Hence the two inequality in Eq. (A.5b) is changed to Eqs. (A.12a)-(A.12b).

Appendix B. Poles and Residues of $A(\mu)$ in Eq. (26)

Let us cast Eq. (26) as

$$A(\mu)=\frac{1}{\mu\cos\mu\beta}\{-F_1(\mu)+\frac{iZe^{-i\mu\pi/2}}{2Z_s}\int_{-i\infty}^{i\infty}K(\nu,\mu)[F_2(\nu)+A(\nu)\sin\nu\beta d\nu]\},\mathrm{Re}\,\mu=0,\mathrm{Im}\,\mu\in(-\infty,\infty), \tag{B.1}$$

where

$$K(\nu,\mu)=\frac{\nu e^{-i\nu\pi/2}}{\cos\nu\pi+\cos\mu\pi}. \tag{B.2}$$

Eq. (B.1) defines $A(\mu)$ on the imaginary axis of the complex μ-plane and is also valid for the strip $S_0:\{\mathrm{Re}\,\mu\in(-1,1),\mathrm{Im}\,\mu\in(-\infty,\infty)\}$ and reveals that the singularities of $A(\mu)$ in S_0 are:

(A) Iff $\beta>\frac{\pi}{2}$: a pole located at μ_p,

$$\mu_p=\frac{\pi}{2\beta}. \tag{B.3a}$$

(C) A pole located at

$$\mu=1. \tag{B.3b}$$

To continue $A(\mu)$ into the strip $S_1:\{\mathrm{Re}\,\mu\in(1,3),\mathrm{Im}\,\mu\in(-\infty,\infty)\}$, we collect two residue contributions from the poles of $K(\nu,\mu)$ located at $\mu=\pm\nu+1$. Poles of $K(\nu,\mu)$ at $\mu=\pm\nu-1$ are located in the left-hand side of the complex μ-plane and, therefore, are not captured by the process of analytic continuation to $\mathrm{Re}\,\mu>0$.

Utilizing Eq. (24), $A(\mu)$ in the S_1 strip is given by

$$A(\mu)=\frac{1}{\mu\cos\mu\beta}\{-F_1(\mu)+\frac{iZe^{-i\mu\pi/2}}{2Z_s}(F_2^{+1}(\mu)+K^{+1}(\mu)A(\mu-1)+\int_{-i\infty}^{i\infty}K(\nu,\mu)[F_2(\nu)+A(\nu)\sin\nu\beta]d\nu)\},$$

$$K^{+1}(\mu) = 4\hat{\mu}\sin\hat{\mu}\beta e^{i\hat{\mu}\pi/2}. \tag{B.4c}$$

Thus for $\mu \in S_1$, $A(\mu)$ has the singularities:
(A) Poles at μ_p for which

$$\cos\mu_p\beta = 0,\ 1 < \mu_p < 3. \tag{B.5a}$$

(B) Poles satisfying

$$\cos(\mu_{p1} - 1)\beta = 0,\ 1 < \mu_{p1} < 3. \tag{B.5b}$$

(C) Two poles located at

$$\mu = 2, 3. \tag{B.5c}$$

We generalize to the q^{th} strip S_q : $\{\mathrm{Re}\,\mu \in (2q-1, 2q+1), \mathrm{Im}\,\mu \in (-\infty, \infty)\}$, $q = 2, 3, ...,$

$$A(\mu) = \frac{1}{\mu\cos\mu\beta}\{-F_1(\mu)+\frac{iZe^{-i\mu\pi/2}}{2Z_s}(\sum_{j=1}^{j=q} F_2^{+j}(\mu)+K^{+j}(\mu)A(\mu-2j+1)+\int_{-i\infty}^{i\infty} K(\nu,\mu)[F_2(\nu)+A(\nu)\sin\nu\beta]d\nu)\}, \tag{B.6a}$$

where

$$F_2^{+j}(\mu) = \frac{-4\sin\hat{\mu}\phi_0\sin\hat{\mu}(\pi-\beta)H_{\hat{\mu}}^{(1)}(kr_0)e^{i\hat{\mu}\pi/2}}{\sin\mu\pi},\ \hat{\mu} = \mu - 2j + 1, \tag{B.6b}$$

$$K^{+1}(\mu) = 4\hat{\mu}\sin\hat{\mu}\beta e^{i\hat{\mu}\pi/2}. \tag{B.6c}$$

We infer from Eq. (B.6a) that for $\mu \in S_q$, $q \geq 1$, $A(\mu)$ has the singularities:
(A) Poles at μ_p for which

(C) Two poles at

$$\mu = 2q - 1, \mu = 2q. \tag{B.7c}$$

Throughout we will adopt the terminology:

(i) μ_p for poles of type (A),

$$\cos \mu_p \beta = 0, p = 1, 2, 3, ..., \infty. \tag{B.8a}$$

(ii) μ_{pl} for poles of type (B),

$$\mu_{pl} = \mu_p + 2l - 1, l = 1, 2, .., \infty; p = 1, 2, 3, .., \infty, \tag{B.8b}$$

with μ_p satisfying Eq. (B.8a).

(iii) μ_s for poles of type (C),

$$\mu_s = s, s = 1, 2, ..., \infty. \tag{B.8c}$$

Remarks.

1. The number of poles of types (A) and (B) in each strip depends on the value of β as per Eqs. (B.7a)-(B.7b).

2. Second order poles will exist when β is a rational multiple of π under either of the following conditions:

i) $\beta = \frac{\pi}{n}$ with n even integer, $n \geq 4$. All the μ_p poles will coalesce with some of the μ_s poles rendering second order poles.

ii) $\beta = \frac{m}{n}\pi$, m odd and n even. Some of the μ_p poles will coalesce with some of the μ_s poles rendering second order poles.

iii) $\beta = \frac{m}{n}\pi$, n odd. Some of the μ_{pl} poles will coalesce with some of the μ_p poles rendering second order poles.

Residues Computation

a) **Poles of the type** μ_p in the S_q strip

The residue is given by

$$\mathrm{Res}[A(\mu_p)] = \frac{-1}{\ldots}\{\ldots E(\mu_p) + \frac{iZe^{-i\mu_p\pi/2}}{\ldots}(\sum^{j=q} F^{\pm j}(\mu_p) \ldots K^{\pm j}(\mu_p) A(\mu_p - 2j + 1) \ldots$$

$$\mathrm{Res}[A(\mu_s)] = \frac{(-1)^s \sin \pi\mu_s}{\pi\mu_s \cos \mu_s\beta}\{-F_1(\mu_s) + \frac{iZe^{-i\mu_s\pi/2}}{2Z_s}\sum_{j=1}^{j=q} F_2^{+j}(\mu_s)\}. \quad \text{(B.9b)}$$

c) **Poles of the type** μ_{pl}

The residues of these poles are computed once the μ_p residues are computed and are given by

$$\mathrm{Res}[A(\mu_{pl)})] = \{\frac{1}{\mu \cos \mu\beta}\frac{iZe^{-i\mu\pi/2}}{2Z_s}K^{+j}(\mu)\}_{\mu_{pl}}A(\mu_p),\ l = 1, 2, 3, ..., \infty. \quad \text{(B.9c)}$$

The residues of second order poles, if they exist, are not detailed here but are straightforward and require the utilization of the second order residue formula instead of the first order formula used in this analysis.

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