Acoustic fields in the presence of an impedance cone

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Abstract

We consider the boundary-value problem for the Helmholtz equation connected with an infinite circular cone with an impedance boundary on its face. The scheme of solution includes applying the Kontorovich-Lebedev (KL) transform, derivation of an integral equation satisfied by the KL spectral amplitude and obtaining alternative near and far field representations together with the conditions of validity of these representations for a given set of problem parameters. As such, this article presents a second application to a problem solving strategy that was proposed in [1].

1 Introduction

One of the important problems of acoustic field theory concerns the evaluation of source excited acoustic fields in the presence of an impedance cone. This class of boundary conditions is usually considered to approximate imperfectly reflecting surfaces. In fact, near field evaluation as well as the scattering and diffraction of waves by an impedance cone are very difficult problems to solve in closed form analytically. To this day, there is no explicit closed-form solution for these problems. Recently ([2-3]) a KL-based formulation has been proposed for the rigorous solution of the problem of the diffraction by a cone with mixed boundary conditions of constant impedance type. Additionally [1] has presented a strategy to solve the problem of diffraction by a material cone. The formulation presented here is an attempt to adapt the techniques of [1] to the impedance cone case in order to evaluate the near fields as well as to obtain alternative representations for the far fields.

In section 2 the problem is formulated. Section 3 derives the integral equation satisfied by the KL spectral function. In section 4 the field representations are given. Conclusions are given in section 5. Appendix A presents a scheme to numerically solve the integral equation. In Appendix

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B the singularities of the KL spectral function are identified and quantified. The formal structure of this paper is essentially the same as in [1], and the results which follow represent the impedance cone simplifications of the material cone results in [1].

It should be mentioned that the integral equation given here is a particular case of the one found by Bernard [2], Bernard and Lyalinov [3-4] and recovered by Antipov [5]. Additionally, as mentioned and shown in [2-3], these integral equations are not of the L2 type, so that Fredholm theorems on existence and uniqueness do not apply. However, the existence and uniqueness can be proved by some specific process [2-3]. Also given in [3] that uniqueness could fail if $\operatorname{Re} Z_s = 0$, where Z_s is the surface impedance of the cone.

2 Formulation

We consider the inhomogeneous Helmholtz equation [6]

$$(\nabla^2 + k^2)p(r,\theta) = -\frac{\delta(r-r_0)\ \delta(\theta-\theta_0)}{r_0 \sin\theta_0} \tag{1}$$

in a cone $C = \{0 < r < \infty, 0 \leq \theta < \beta, -\pi \leq \phi \leq \pi\}$ with the impedance boundary condition on its face

$$V_\theta(r,\beta) = Y_s p(r,\beta). \tag{2}$$

$p(r,\theta)$ is the acoustic pressure, $V_\theta(r,\theta)$ is the component of the velocity field, $\mathbf{V}(r,\theta)$, normal to the cone surface with [6]

$$\mathbf{V}(r,\theta) = \frac{-i}{\omega\rho}[\mathbf{r}_0 \frac{\partial}{\partial r} + \boldsymbol{\theta}_0 \frac{1}{r}\frac{\partial}{\partial \theta}]p(r,\theta). \tag{3}$$

$k = \frac{\omega}{c}$ is the wave number, $c = \sqrt{\frac{\lambda}{\rho}}$is the acoustic speed with ρ and λ are respectively the density and incompressibility of the medium in C. ∇^2 stands for the Laplacian in spherical coordinates (r,θ,ϕ). A time factor $\exp\{-i\omega t\}$ is assumed and omitted throughout. Acoustic field excitation is provided by an impressed rotationally invariant unit force density located at (r_0, θ_0). Y_s is the surface admittance of the cone.

Similar to [1], for a while we shall assume that k is purely imaginary with $\arg k = \frac{\pi}{2}$.

The fields are required to decay exponentially to zero as $r \to \infty$. This replaces the Sommerfeld radiation condition for the $\operatorname{Im} k = 0$ case.

An edge condition [6] is imposed on the field behavior near the tip of the cone which requires

$$p = O(r^{\nu - \frac{1}{2}}),\ |\mathbf{V}| = O(r^{\nu - \frac{3}{2}}), r \to 0, \nu > 0. \qquad (4)$$

Similar to [1], we propose to solve the problem by means of a KL transform pair [6] :

$$P(\nu, \theta) = \int_0^{\infty} p(r, \theta) h^{(1)}_{\nu - \frac{1}{2}}(kr) dr, \qquad (5)$$

$$p(r, \theta) = \frac{k}{\pi} \int_{-i\infty}^{i\infty} \nu j_{\nu - \frac{1}{2}}(kr) P(\nu, \theta) d\nu, \qquad (6)$$

where $j_{\nu-\frac{1}{2}}(kr)$ and $h^{(1)}_{\nu-\frac{1}{2}}(kr)$ are spherical Bessel functions given by

$$j_\nu(z) = (\pi/2z)^{\frac{1}{2}} J_{\nu+\frac{1}{2}}(z),\ h^{(1)}_\nu(z) = (\pi/2z)^{\frac{1}{2}} H^{(1)}_{\nu+\frac{1}{2}}(z).$$

$J_\nu(z)$ and $H^{(1)}_\nu(z)$ are the standard Bessel and Hankel functions respectively and ν is purely imaginary.

Since $H^{(1)}_{-\nu}(z) = e^{i\pi\nu} H^{(1)}_\nu(z)$, the definition in Eq. (5) implies that

$$P(-\nu, \theta) = e^{i\pi\nu} P(\nu, \theta). \qquad (7)$$

There are two additional ways of writing the inverse transform of Eq. (5) which will be needed during the course of the analysis:

$$p(r, \theta) = \frac{k}{2\pi} \int_{-i\infty}^{i\infty} \nu h^{(1)}_{\nu - \frac{1}{2}}(kr) P(\nu, \theta) d\nu, \qquad (8)$$

$$p(r,\theta) = \frac{-ik}{\pi}\int_0^{i\infty} \nu e^{i\pi\nu}\sin\pi\nu h^{(1)}_{\nu-\frac{1}{2}}(kr)P(\nu,\theta)d\nu. \tag{9}$$

From Eq. (3), $V_\theta(r,\theta)$ is given by

$$V_\theta(r,\theta) = \frac{-k}{\pi\omega\rho}\frac{1}{r}\frac{\partial}{\partial\theta}\int_0^{i\infty} \nu e^{i\pi\nu}\sin\pi\nu h^{(1)}_{\nu-\frac{1}{2}}(kr)P(\nu,\theta)d\nu. \tag{10}$$

Thus,

$$V_\theta(r,\theta) = \frac{-k}{\pi\omega\rho}\frac{1}{r}\int_0^{i\infty} \nu e^{i\pi\nu}\sin\pi\nu h^{(1)}_{\nu-\frac{1}{2}}(kr)\frac{d}{d\theta}P(\nu,\theta)d\nu. \tag{11}$$

Applying the KL-transform to Eq. (1), we get the ordinary differential equation

$$[\frac{1}{\sin\theta}\frac{d}{d\theta}(\sin\theta\frac{d}{d\theta})+(\nu^2-\frac{1}{4})]P(\nu,\theta) = -\frac{h^{(1)}_{\nu-\frac{1}{2}}(kr_0)\delta(\theta-\theta_0)}{\sin\theta_0},\ 0\le\theta<\beta, \tag{12}$$

whose general solutions are the Legendre functions [6].

We represent the field in C as the sum over an unperturbed field $p^{(0)}, \mathbf{V}^{(0)}$ plus a scattered field $p^{(1)}, \mathbf{V}^{(1)}$ due to the presence of the cone. Hence

$$p(r,\theta) = p^{(0)}(r,\theta) + p^{(1)}(r,\theta) \tag{13}$$

leading to

$$P(\nu,\theta) = P^{(0)}(\nu,\theta) + P^{(1)}(\nu,\theta). \tag{14}$$

2.1 The unperturbed field

Since $P^{(0)}(\nu,\theta)$ satisfies the source conditions of Eq. (12) and must be bounded at $\theta = 0, \pi$, we obtain (see [1, Eq. (22)]),

$$P^{(0)}(\nu,\theta) = \frac{\pi}{2}\frac{h^{(1)}_{\nu-\frac{1}{2}}(kr_0)}{\sin\theta_0 \sin(\nu-\frac{1}{2})\pi} P_{\nu-\frac{1}{2}}(\cos\theta_<) P_{\nu-\frac{1}{2}}(-\cos\theta_>), \tag{15}$$

where $\theta_>|\theta_<$ is the greater|lesser of θ and θ_0.
From Eqs. (11) and (15), $V^{(0)}_\theta(r,\theta)$ is given by

$$V^{(0)}_\theta(r,\theta) = \frac{-k}{\pi\omega\rho}\frac{1}{r}\int_0^{i\infty} \nu e^{i\pi\nu} \sin\pi\nu h^{(1)}_{\nu-\frac{1}{2}}(kr) \frac{d}{d\theta} P^{(0)}(\nu,\theta) d\nu, \tag{16}$$

where, for $\theta > \theta_0$

$$\frac{d}{d\theta}P^{(0)}(\nu,\theta) = \frac{\pi}{2}\frac{h^{(1)}_{\nu-\frac{1}{2}}(kr_0)}{\sin\theta_0 \sin(\nu-\frac{1}{2})\pi} P_{\nu-\frac{1}{2}}(\cos\theta_0) \frac{d}{d\theta} P_{\nu-\frac{1}{2}}(-\cos\theta) \tag{17}$$

and for $\theta < \theta_0$

$$\frac{d}{d\theta}P^{(0)}(\nu,\theta) = \frac{\pi}{2}\frac{h^{(1)}_{\nu-\frac{1}{2}}(kr_0)}{\sin\theta_0 \sin(\nu-\frac{1}{2})\pi} P_{\nu-\frac{1}{2}}(-\cos\theta_0) \frac{d}{d\theta} P_{\nu-\frac{1}{2}}(\cos\theta). \tag{18}$$

2.2 The scattered field

Since $P(\nu,\theta)$ must be bounded at $\theta = 0$ we represent $P^{(1)}(\nu,\theta)$ by

$$P^{(1)}(\nu,\theta) = A(\nu) P_{\nu-\frac{1}{2}}(\cos\theta). \tag{19}$$

$A(\nu)$ is a KL spectral function to be determined from the boundary condition. Eq. (7) together with [6]

$$P_{\nu-\frac{1}{2}}(x) = P_{-\nu-\frac{1}{2}}(x), \tag{20}$$

enforce

$$A(-\nu) = e^{i\pi\nu} A(\nu). \tag{21}$$

Additionally, the convergence of the KL integrals [7] at $\theta = \beta$ implies that, when Im $\nu \to +\infty$, the spectral function, $A(\nu)$, must vanish as

$$A(\nu) = O\left[\exp(-\beta \text{ Im } \nu)\right]. \quad (22)$$

From the above we obtain

$$p^{(1)}(r,\theta) = \frac{-ik}{\pi} \int_0^{i\infty} \nu e^{i\pi\nu} \sin \pi\nu h^{(1)}_{\nu-\frac{1}{2}}(kr) A(\nu) P_{\nu-\frac{1}{2}}(\cos\theta) d\nu, \quad (23)$$

$$V_\theta^{(1)}(r,\theta) = \frac{-k}{\pi\omega\rho}\frac{1}{r} \int_0^{i\infty} \nu e^{i\pi\nu} \sin \pi\nu h^{(1)}_{\nu-\frac{1}{2}}(kr) A(\nu) \frac{d}{d\theta} P_{\nu-\frac{1}{2}}(\cos\theta) d\nu. \quad (24)$$

Eq. (2) now reads

$$V_\theta^{(0)}(r,\beta) + V_\theta^{(1)}(r,\beta) = Y_s[p^{(0)}(r,\beta) + p^{(1)}(r,\beta)]. \quad (25)$$

3 The integral equation derivation

The integral equation on $A(\nu)$ is derived by multiplying Eq. (25) by r $h^{(1)}_{\mu-\frac{1}{2}}(kr)$ and integrating with respect to r from 0 to ∞. Making use of [8, formula 6.576 (4)],

$$\int_0^\infty r^{-\lambda} K_{i\mu}(-ikr) K_{i\nu}(-iNkr) dr = \frac{2^{-2-\lambda}(-ik)^{-i\nu+\lambda-1}(-iNk)^{i\nu}}{\Gamma(1-\lambda)} G_1(\lambda,\mu,\nu) G_2(\lambda,\mu,\nu),$$

where

$$G_1(\lambda,\mu,\nu) = \Gamma(\frac{1-\lambda+i\mu+i\nu}{2})\Gamma(\frac{1-\lambda-i\mu+i\nu}{2})\Gamma(\frac{1-\lambda+i\mu-i\nu}{2})\Gamma(\frac{1-\lambda-i\mu-i\nu}{2}),$$

$$G_2(\lambda,\mu,\nu) = F(\frac{1-\lambda+i\mu+i\nu}{2}, \frac{1-\lambda-i\mu+i\nu}{2}; 1-\lambda; 1-N^2),$$

with $\mathrm{Re}(-ik - iNk) > 0$ and $\mathrm{Re}\,\lambda < 1 - |\,\mathrm{Re}\,\mu| - |\,\mathrm{Re}\,\nu|$.
For $\lambda = 0$, $N = 1$ and using [9, formula 6.1.30]

$$\Gamma(\frac{1}{2} + iy)\Gamma(\frac{1}{2} - iy) = \frac{\pi}{\cosh \pi y},$$

$$\cosh(x - y)\cosh(x + y) = \frac{1}{2}(\cosh 2x + \cosh 2y),$$

we obtain

$$I_2^*(\nu, \mu) = \int_0^\infty r\ h^{(1)}_{i\mu-\frac{1}{2}}(kr)h^{(1)}_{i\nu-\frac{1}{2}}(kr)dr = -i\pi e^{(\nu+\mu)\pi/2}\frac{1}{\cosh \pi\nu + \cosh \pi\mu}. \tag{26.a}$$

From [10] we get

$$D(\nu, \mu) = \int_0^\infty h^{(1)}_{\mu-\frac{1}{2}}(kr)h^{(1)}_{\nu-\frac{1}{2}}(kr)dr = \frac{\pi k}{\mu\ \sin\ \mu\pi}[e^{\mu\pi}\delta(\mathrm{Im}\,\nu - \mathrm{Im}\,\mu) + \delta(\mathrm{Im}\,\nu + \mathrm{Im}\,\mu)]. \tag{26.b}$$

For the particular case which is considered here ($\mathrm{Im}\,\mu > 0$, $\mathrm{Im}\,\nu > 0$), the $\delta(\mathrm{Im}\,\nu + \mathrm{Im}\,\sigma)$ in Eq. (26.b) will be removed. We obtain

$$s(\mu) - \frac{iY_s}{\pi Y}\int_0^{i\infty} K(\nu, \mu)A(\nu)d\nu = f(\mu)A(\mu), \mathrm{Im}\,\mu \geqslant 0, \mathrm{Re}\,\mu = 0, \tag{27}$$

where

$$s(\mu) = s_1(\mu) + s_2(\mu), \tag{28.a}$$

$$s_1(\mu) = -\frac{\pi}{2}\frac{h^{(1)}_{\mu-\frac{1}{2}}(kr_0)}{\sin\theta_0 \sin(\mu - \frac{1}{2})\pi}P_{\mu-\frac{1}{2}}(\cos\theta_0)\frac{d}{d\beta}P_{\mu-\frac{1}{2}}(-\cos\beta), \tag{28.b}$$

$$s_2(\mu) = -\frac{iY_s}{\pi Y}\int_0^{i\infty} e^{i\nu\pi}\nu\sin\nu\pi \hat{I}_2^*(\nu,\mu)P_1^{(0)}(\nu,\beta)d\nu, \tag{28.c}$$

$$P^{(0)}(\nu,\beta) = \frac{\pi}{2}\frac{h^{(1)}_{\nu-\frac{1}{2}}(kr_0)}{\sin\theta_0\sin(\nu-\frac{1}{2})\pi}P_{\nu-\frac{1}{2}}(\cos\theta_0)P_{\nu-\frac{1}{2}}(-\cos\beta), \tag{28.d}$$

where $\frac{d}{d\beta}P_{\mu-\frac{1}{2}}(-\cos\beta)$ stands for $\frac{d}{d\theta}P_{\mu-\frac{1}{2}}(-\cos\theta)|_{\theta=\beta}$, which notation will be used throughout.

Additionally,

$$K(\nu,\mu) = \nu e^{i\nu\pi}\sin\nu\pi \hat{I}_2^*(\nu,\mu)P_{\nu-\frac{1}{2}}(\cos\beta), \tag{29.a}$$

$$f(\mu) = \frac{d}{d\beta}P_{\mu-\frac{1}{2}}(\cos\beta), \tag{29.b}$$

$$\hat{I}_2^*(\nu,\mu) = -i\pi e^{-i(\nu+\mu)\pi/2}\frac{1}{\cos\pi\nu+\cos\pi\mu}, \tag{29.c}$$

$$Y = \frac{1}{\rho c}, \tag{29.d}$$

where Y is the acoustic admittance.

In order to adapt the integral equation for the numerical scheme of Appendix A, we re-write Eq. (27) as

$$\tilde{s}(\mu) - \frac{iY_s}{\pi Y}\frac{1}{f(\mu)}\int_0^{i\infty} K(\nu,\mu)\tilde{A}(\nu)d\nu = \tilde{A}(\mu), \mathrm{Im}\,\mu \geqslant 0, \mathrm{Re}\,\mu = 0, \tag{30.a}$$

where

$$\tilde{A}(\mu) = A(\mu) + \{P^{(0)}(\mu,\beta)/P_{\mu-\frac{1}{2}}(\cos\beta)\}, \tag{30.b}$$

$$\tilde{s}(\mu) = -\frac{h^{(1)}_{\mu-\frac{1}{2}}(kr_0)}{\sin\theta_0 \sin\beta} \frac{P_{\mu-\frac{1}{2}}(\cos\theta_0)}{P_{\mu-\frac{1}{2}}(\cos\beta)\frac{d}{d\beta}P_{\mu-\frac{1}{2}}(\cos\beta)}. \tag{30.c}$$

In Appendix A an approximate solution for the integral equation in Eq. (30) using a collocation method is given. The scheme is inspired by the one used by Antipov [5]. In Appendix B the spectrum of $A(\mu)$ is analytically continued from the imaginary axis into the right half of the complex μ-plane.

4 Field representations

With the results of Appendices A and B, we proceed to derive near and far field representations.

4.1 The near field

As was shown in Appendix B, $A(\mu)$ is a meromorphic function whose only singularities in the complex μ-plane are poles. The conditions under which some of these poles are of higher order have also been established. The second order poles that may be present indicate that the acoustic pressure (velocity) field near the tip of the cone may contain the logarithm of the distance in addition to its power, namely $r^{\mu-\frac{1}{2}}\log r$ ($r^{\mu-\frac{3}{2}}\log r$).

For $r < r_0$, the near field is calculated from Eq. (13). To that end we re-write Eq. (27) as

$$A(\mu) = \frac{s_1(\mu)}{f(\mu)} - \frac{iY_s}{\pi Y}\frac{1}{f(\mu)}\int_0^{i\infty} K(\nu,\mu)\tilde{A}(\nu)d\nu, \tag{31}$$

where $\tilde{A}(\nu)$ is given in Eq. (30.b).

Utilizing the KL representation in Eq. (6),

$$p^{(1)}(r,\theta) = \frac{k}{\pi}\int_{-i\infty}^{i\infty} \nu j_{\nu-\frac{1}{2}}(kr)A(\nu)P_{\nu-\frac{1}{2}}(\cos\theta)d\nu, \tag{32}$$

and substituting for $A(\nu)$, from Eq.(31), in Eq. (32) we obtain

$$p^{(1)}(r,\theta) = p_1(r,\theta) + p_2(r,\theta), \tag{33}$$

where

$$p_1(r,\theta) = \{p_1(r,\theta)\}_1 + \{p_1(r,\theta)\}_2 + \{p_1(r,\theta)\}_3, \tag{34.a}$$

with

$$\{p_1(r,\theta)\}_1 = -2ik \sum_{\nu_p}{}' \nu_p \mathrm{Res}[\frac{s_1(\nu)}{f(\nu)}]_{\nu_p} j_{\nu_p-\frac{1}{2}}(kr) P_{\nu_p-\frac{1}{2}}(\cos\theta), \tag{34.b1}$$

where the prime on the sum over ν_p denotes omission of the $\nu_p = \frac{1}{2}$ term and

$$\mathrm{Res}[\frac{s_1(\nu)}{f(\nu)}]_{\nu_p} = \frac{-\pi h^{(1)}_{\nu-\frac{1}{2}}(kr_0) P_{\nu-\frac{1}{2}}(\cos\theta_0) \frac{d}{d\beta} P_{\nu-\frac{1}{2}}(-\cos\beta)}{2\sin\theta_0 \sin[(\nu-\frac{1}{2})\pi] \frac{d}{d\nu}\frac{d}{d\beta} P_{\nu-\frac{1}{2}}(\cos\beta)}|_{\nu_p}. \tag{34.b2}$$

From the identity [8, formula 8.733(1)]

$$\frac{d}{d\theta} P_{\nu-\frac{1}{2}}(\pm\cos\theta) = \mp(\nu-\frac{1}{2})(\nu+\frac{1}{2}) P^{-1}_{\nu-\frac{1}{2}}(\pm\cos\theta), \tag{35.a}$$

where $P^{-1}_{\nu-\frac{1}{2}}(\pm\cos\theta)$ are Associated Legendre functions, from [8, formula 8.721(4)]

$$P_{\nu}(\cos\phi) \sim \sqrt{\frac{2}{\nu\pi\ sin\ \phi}} \cos[(\nu+\frac{1}{2})\phi - \frac{\pi}{4}], \nu \to \infty, \tag{35.b}$$

$$\nu P_{\nu}^{-1}(\cos\phi) \sim \sqrt{\frac{2}{\nu\pi\ sin\ \phi}} \cos[(\nu+\frac{1}{2})\phi - \frac{\pi}{2}], \nu \to \infty, \tag{35.c}$$

from [6]

$$J_\nu(z) \sim \frac{1}{\sqrt{2\pi\nu}}(\frac{2\nu}{ez})^{-\nu},\ \nu \to \infty, \tag{36.a}$$

$$H_\nu^{(1)}(z) \sim \frac{1}{\sqrt{2\pi\nu}}(\frac{2\nu}{ez})^{\nu},\ \nu \to \infty, \tag{36.b}$$

the behavior of the summand of the series in Eq. (34.b1) is given by

$$\frac{1}{\nu_p}(\frac{r}{r_0})^{\nu_p},\ \nu_p \to \infty. \tag{37}$$

$$\{p_1(r,\theta)\}_2 = -2ik\sum_{s=1}^{\infty}(s+\frac{1}{2})\mathrm{Res}[\frac{s_1(\nu)}{f(\nu)}]_{\nu_s} j_s(kr)P_s(\cos\theta), \tag{38.a1}$$

where

$$\mathrm{Res}[\frac{s_1(\nu)}{f(\nu)}]_{\nu_s} = \frac{(-1)^{s+1}h_s^{(1)}(kr_0)P_s(\cos\theta_0)\frac{d}{d\beta}P_s(-\cos\beta)}{2\sin\theta_0\frac{d}{d\beta}P_s(\cos\beta)}, s = 1,2,3,...,\infty. \tag{38.a2}$$

The summand behaves as

$$\frac{1}{s}(\frac{r}{r_0})^{s},\ s \to \infty. \tag{38.a3}$$

The contribution from the pole at $\nu = \frac{1}{2}$ is given by

$$\{p_1(r,\theta)\}_3 = \frac{-ikh_0^{(1)}(kr_0)\cot^2(\frac{\beta}{2})}{2\sin\theta_0} j_0(kr). \tag{39}$$

Additionally,

$$p_2(r,\theta) = -\frac{ikY_s}{\pi^2 Y}\int_{-i\infty}^{i\infty} \nu j_{\nu-\frac{1}{2}}(kr)\hat{I}(\nu)P_{\nu-\frac{1}{2}}(\cos\theta)d\nu, \tag{40.a}$$

where

$$\hat{I}(\nu) = \frac{1}{f(\nu)} \int_0^{i\infty} K(\nu', \nu) \tilde{A}(\nu') d\nu'. \tag{40.b}$$

From Eq. (35.a) and [8, formula 8.721(1)]

$$P_{\nu-\frac{1}{2}}(\cos\theta) \sim e^{|\nu|\theta}/|\nu|^{\frac{1}{2}}, \nu \to \pm i\infty, \tag{41.a}$$

$$P^{-1}_{\nu-\frac{1}{2}}(\cos\theta) \sim e^{|\nu|\theta}/|\nu|^{\frac{3}{2}}, \nu \to \pm i\infty, \tag{41.b}$$

and from[6]

$$J_\nu(z) \sim e^{|\nu|\frac{\pi}{2} + i\nu \arg z}/|\nu|^{1/2}, \nu \to \pm i\infty, \tag{42}$$

we infer that the integrand in Eq. (40.a) decays exponentially as

$$e^{-|\nu|(\beta-\theta)+i\nu\phi}/|\nu|^{1/2}, \nu \to \pm i\infty, \, \phi = \arg k, \tag{43}$$

and the integral exists when

$$\theta < \beta - \phi. \tag{44}$$

From Eq. (44) we infer that for the lossless case, $\phi = 0$, the representation given in Eq. (40.a) exists for all the observation angles, whereas for the lossy case it has a gap, as given by Eq. (44), in the angular domain where it cannot be used.

The above field representation has the advantage of numerical efficiency and the disadvantages of hiding part of the strength of the field singularities near the tip of the cone associated with $\nu_p < \frac{3}{2}$ and all the strength of the field singularities associated with $\nu_{p1} < \frac{3}{2}$ as well as, for the lossy case, the angular gap where the representation cannot be used. However, if the main interest is to quantify the singular field behavior near the tip of the cone, it is possible to recover the full strength of the field singularities associated with $(\nu_p, \nu_{p1}) < \frac{3}{2}$ by isolating

$$p^{(s)}(r,\theta) = \{p^{(s)}(r,\theta)\}_1, \tag{45.a}$$

where

$$\{p^{(s)}(r,\theta)\}_1 = -2ik \sum^{\nu_p<\frac{1}{2}} \nu_p j_{\nu_p-\frac{1}{2}}(kr)\mathrm{Res}[A(\nu_p)]P_{\nu_p-\frac{1}{2}}(\cos\theta) \tag{45.b}$$

and

$$\mathbf{V}^{(s)}(r,\theta) = \frac{-i}{\omega\rho}[\mathbf{r}_0\frac{\partial}{\partial r} + \boldsymbol{\theta}_0\frac{1}{r}\frac{\partial}{\partial\theta}](\{p^{(s)}(r,\theta)\}_1 + \{p^{(s)}(r,\theta)\}_2), \tag{45.c}$$

where $\{p^{(s)}(r,\theta)\}_1$ is given in Eq. (45.b) with the sum running to $\nu_p < \frac{3}{2}$ and

$$\{p^{(s)}(r,\theta)\}_2 = -2ik \sum^{\nu_{p1}<\frac{3}{2}} \nu_{p1} j_{\nu_{p1}-\frac{1}{2}}(kr)\mathrm{Res}[A(\nu_{p1})]P_{\nu_{p1}-\frac{1}{2}}(\cos\theta), \tag{45.d}$$

where $p^{(s)}(r,\theta)$ ($\mathbf{V}^{(s)}(r,\theta)$) is the singular pressure (velocity) field near the tip of the cone. One then needs to compute the residues of $A(\nu)$ from Eqs. (B.9)-(B.10) only for $(\nu_{p0}, \nu_{p1}) < \frac{3}{2}$. It should be noted that the contribution from the pole at $\nu = \frac{1}{2}$ is not included since the corresponding field is non-singular.

An alternative field representation is available by closing contours of Eq. (32) in the right-hand side of the complex ν-plane and collecting residue contributions from the four pole sets of $A(\nu)$. This field representation has the advantage of being valid for all observation angles (lossless and lossy cases) and the disadvantage of very cumbersome expressions for the residues of the poles. It should also be emphasized that since it is possible for the poles ν_{pl} to accumulate at infinity, forming a dense set on the real line, in addition to the possibility of higher order poles, either of these possibilities could avoid the convergence of the residue sum. The truncated residue sum should, therefore, be understood as giving some asymptotic approximation to the field in terms of first identified poles.

The contribution of $p^{(0)}(r,\theta)$ results from its residue sum over the poles at ν_s, $s=0,1,2,....$

The velocity field is also computed through the same procedures except for $\theta=\beta$, where it is better to compute it through the boundary condition since there some of the poles of $A(\nu)$ cancel with $\frac{d}{d\beta}P_{\nu-\frac{1}{2}}(\cos\beta)$.

4.2 The far field

For $r>r_0$, the field representation is given by the scattered field $p^{(1)}(r,\theta)$ and adding the contribution of $p^{(0)}(r,\theta)$. Making use of the KL representation in Eqs. (8) and (9) and substituting for $A(\nu)$ from Eq. (31), we obtain

$$p^{(1)}(r,\theta)=\hat{p}_1(r,\theta)+\hat{p}_2(r,\theta). \tag{46.a}$$

$$\hat{p}_1(r,\theta)=\frac{-k}{\pi^2}\frac{Y_s}{Y}\int_0^{i\infty}\nu\sin\nu\pi e^{i\nu\pi}h^{(1)}_{\nu-\frac{1}{2}}(kr)\hat{I}(\nu)P_{\nu-\frac{1}{2}}(\cos\theta)d\nu, \tag{46.b}$$

where $\hat{I}(\nu)$ as given in Eq. (40.b) and

$$\hat{p}_2(r,\theta)=\frac{k}{\pi}\int_{-i\infty}^{i\infty}\nu h^{(1)}_{\nu-\frac{1}{2}}(kr)Q(\nu)P_{\nu-\frac{1}{2}}(\cos\theta)d\nu, \tag{46.c}$$

where

$$Q(\nu)=-\frac{\pi}{2f(\nu)}\frac{j_{\nu-\frac{1}{2}}(kr_0)}{\sin\theta_0\sin(\nu-\frac{1}{2})\pi}P_{\nu-\frac{1}{2}}(\cos\theta_0)\frac{d}{d\beta}P_{\nu-\frac{1}{2}}(-\cos\beta). \tag{46.d}$$

Equation (46.c) is obtained after making use of

$$h^{(1)}_{\nu-\frac{1}{2}}(k_1r_0)=2j_{\nu-\frac{1}{2}}(k_1r_0)-h^{(2)}_{\nu-\frac{1}{2}}(k_1r_0),$$

Eq. (20) and that from $H^{(1),(2)}_{-\nu}(z)=e^{\pm i\pi\nu}H^{(1),(2)}_{\nu}(z)$, $[h^{(1)}_{\nu-\frac{1}{2}}(kr)h^{(2)}_{\nu-\frac{1}{2}}(kr_0)]$ is an even function of ν.

From [6]

$$H^{(1)}_{\nu-\frac{1}{2}}(z) \sim e^{|\nu|(\frac{\pi}{2}-\arg z)}/|\nu|^{1/2}, \nu \to i\infty, \tag{47}$$

Eqs. (35.a), (41.a) and (41.b), the integrand of Eq. (46.b) decays exponentially and the integral exists when

$$\theta < \beta + \phi \tag{48}$$

which is trivially satisfied by all observation angles when $\phi > 0$.

Similar to section 4.1, the representation in Eq. (46.c) is further reduced to sums over the residues of the poles at ν_p, ν_s and $\nu = \frac{1}{2}$. The expressions of these series are similar to those given in Eqs. (34.b1), (34.b2), (38.a1), (38.a2) and (39) with r and r_0 interchanged. The behavior of the summands is given by Eqs. (37) and (38.a3) with r and r_0 interchanged.

Another field representation which invokes the reciprocity principle is also available. Thus to calculate the fields when $r > r_0$ we employ the sum on the residues of all the poles of $A(\nu)$ with the source located at (r, θ) and the observer located at (r_0, θ_0), and the truncated sum is understood as an asymptotic approximation related to first identified poles. It is worth mentioning that this alternative cannot be used to derive far field results for plane wave illumination, since the convergence is an asymptotic result for source near the tip. This field representation has the disadvantage of very cumbersome expressions for the residues of the poles. It is mentioned here only for the completeness of the discussion.

Far field ($kr >> 1$) expressions are obtained by utilizing the large argument asymptotic approximation for the Hankel function [6]

$$H^{(1)}_{\nu}(z) \sim \sqrt{\frac{2}{\pi z}} e^{i(z-\nu\pi/2-\pi/4)}. \tag{49}$$

Substituting from Eq. (49) in Eqs. (46.b) and (46.c), we get

$$\hat{p}_1(r,\theta) = \frac{-1}{\pi^2 r}\frac{Y_s}{Y} e^{i(kr-\pi/4)} \int_0^{i\infty} \nu \sin\nu\pi e^{i\nu\pi/2} \hat{I}(\nu) P_{\nu-\frac{1}{2}}(\cos\theta) d\nu. \tag{50}$$

The integrand of Eq. (50) decays exponentially and the integral exists when

$$\theta < \beta. \tag{51}$$

$$\hat{p}_2(r,\theta) = \frac{1}{\pi r} e^{i(kr-\pi/4)} \int_{-i\infty}^{i\infty} \nu e^{-i\nu\pi/2} Q(\nu) P_{\nu-\frac{1}{2}}(\cos\theta) d\nu. \tag{52}$$

The representation in Eq. (52) is further reduced to sums over the residues of the poles at ν_p, ν_s and $\nu = \frac{1}{2}$. From Eqs. (35.a), (35.b), (35.c) and (36.a), the summands from Eq. (52) behave as

$$\frac{1}{|\bar{\nu}|^{1/2}} (\frac{e\ |k| r_0}{2\bar{\nu}})^{\bar{\nu}}, \bar{\nu} \to \infty. \tag{53}$$

where $\bar{\nu}$ stands for ν_p and ν_s.

5 Conclusion

The integral equation satisfied by the KL spectral function, $A(\nu)$, has been given in Eq. (27). A collocation scheme for the numerical evaluation of the KL spectral function in Eq. (30.a) was constructed in Appendix A. It was established in Eqs. (A.13a)-(A.13b) that the scheme has a source-boundary separation requirement for its validity. In Appendix B, analytic continuation was utilized to establish the meromorphic nature of $A(\nu)$ and to identify its pole singularities. Near field alternative representations, together with the advantages and disadvantages of each, were given in Sec. 4.1, Eqs. (34.a), (40.a). Situations under which the residue sum over the poles of $A(\nu)$ turns into an asymptotic approximation were discussed in Sec. 4.1. The condition under which Eq. (40.a) is not valid, for a given set of problem parameters, has been established in Sec. 4.1, Eq. (44) revealing an angular gap in which Eq. (40.a) cannot be used. Far fields were given in Sec. 4.2, Eqs. (46.a) and (50), (52) . The conditions of their validity have been established in Sec. 4.2, Eqs. (48) and (51) respectively. The application presented here together with that in [1] establish the KL formulation as a viable solution strategy for diffraction problems involving cones and wedges with impedance-type and field continuity-type boundary conditions on their faces. The ambitious goal of deriving analytical, exact or approximate, solutions to the integral equations describing the KL spectral amplitudes, if obtained, would widely expand the power of the scheme. This issue is now under study.

Appendix A. Numerical scheme to solve the integral equation in Eq. (30.a)

The scheme used here is inspired by the one used by Antipov [5]. The structure of this Appendix is the same as for the material cone in [1, Appendix A], and the results which follow are simplifications pertaining to the impedance cone.

Let $\{\nu_{m-1}\}$, $m = 1, 2, ..., M+1$, be a set of points defined on the imaginary axis of the complex ν-plane such that

$$\nu_{m-1} = i\ \delta(m-1)^{\epsilon},\ \delta > 0,\ \epsilon > 0. \tag{A.1a}$$

Let also

$$\mu_m = (\nu_{m-1} + \nu_m)/2,\ \ m = 1, 2, ..., M. \tag{A.1b}$$

We approximate the integral equation in Eq. (30.a) with the linear system of algebraic equations

$$\tilde{A}(\mu_n) = \tilde{s}(\mu_n) - \frac{iY_s}{\pi Y f(\mu_n)} \sum_{m=1}^{M} \int_{\nu_{m-1}}^{\nu_m} K_1(\nu, \mu_n)\tilde{A}(\nu) \frac{\sin\ (\pi\nu)}{\cos\ (\pi\nu) + \cos\ (\pi\mu_n)} d\nu, \tag{A.2a}$$

with $n = 1, 2, ..., M$ and

$$K_1(\nu, \mu_n) = -i\pi\nu e^{i(\nu-\mu_n)\pi/2} P_{\nu-\frac{1}{2}}(\cos\beta). \tag{A.2b}$$

Eq. (A.2a) is further approximated as

$$\tilde{A}(\mu_n) = \tilde{s}(\mu_n) - \frac{iY_s}{\pi Y f(\mu_n)} \sum_{m=1}^{M} K_1(\hat{\nu}_m, \mu_n)\tilde{A}(\mu_m) I_{nm}, n = 1, 2, ..., M, \tag{A.3a}$$

where

$$\hat{\nu}_m = (\nu_{m-1} + \nu_m)/2, \tag{A.3b}$$

$$I_{nm} = \int_{\nu_{m-1}}^{\nu_m} \frac{\sin\ (\pi\nu)}{\cos\ (\pi\nu) + \cos\ (\pi\mu_n)} d\nu. \tag{A.3c}$$

The integral in Eq. (A.3c) is given by

$$I_{nm} = \frac{-1}{\pi} \ln\left(\frac{\cos\ (\pi\nu_m) + \cos\ (\pi\mu_n)}{\cos\ (\pi\nu_{m-1}) + \cos\ (\pi\mu_n)}\right). \tag{A.3d}$$

Thus we can re-write the linear system as

$$\tilde{\mathbf{A}} = \tilde{\mathbf{s}} - \mathbf{C}^* \tilde{\mathbf{A}}, \tag{A.4a}$$

with $\tilde{\mathbf{A}}$ and $\tilde{\mathbf{s}}$ as vectors and $\mathbf{C}^*$ as the matrix

$$C^*_{nm} = \frac{iY_s}{\pi Y f(\mu_n)} K_1(\hat{\nu}_m, \mu_n) I_{nm}. \tag{A.4b}$$

Next we estimate the behavior of $\tilde{s}(\mu_n)$ as $n \to \infty$ ($\mu_n \to i\infty$). From Eqs. (35.a), (41.a), (41.b) and (47), we infer that

$$\tilde{s}(\mu_n) = O(e^{-|\mu_n|(2\beta+\phi-\frac{\pi}{2}-\theta_0)}/|\mu_n|), n \to \infty. \tag{A.5a}$$

Therefore, $\tilde{s}(\mu_n)$ decays exponentially as $n \to \infty$ ($\mu_n \to i\infty$) when

$$\frac{\pi}{2} + \theta_0 < 2\beta + \phi. \tag{A.5b}$$

Additionally, for m fixed and $n \to \infty$, writing I_{nm} as

$$I_{nm} = \frac{-1}{\pi}[\ln(1 + \frac{\cos\ (\pi\nu_m)}{\cos\ (\pi\mu_n)}) - [\ln(1 + \frac{\cos\ (\pi\nu_{m-1})}{\cos\ (\pi\mu_n)})], \tag{A.6a}$$

followed by the series representation

$$\ln(1+x) = x - \frac{x^2}{2} + O(x^3), x < 1, \tag{A.6b}$$

we obtain

$$I_{nm} = O(e^{-|\mu_n|\pi}), n \to \infty, m \text{ is fixed.} \tag{A.6c}$$

From Eqs. (35.a), (41.b) and (A.6c) , we estimate the behavior of C^*_{nm} as $n \to \infty$ ($\mu_n \to i\infty$), m is fixed, as

$$C^*_{nm} = O(e^{-|\mu_n|(\beta+\frac{\pi}{2})}/|\mu_n|^{1/2}), n \to \infty, m \text{ is fixed.} \tag{A.7}$$

Hence, C^*_{nm} decays exponentially as $n \to \infty$, m is fixed.
Additionally, for n fixed and $m \to \infty$,

$$I_{nm} \sim \frac{-1}{\pi} \ln \frac{\cos\ (\pi\nu_m)}{\cos\ (\pi\nu_{m-1})} \to -(\nu_m - \nu_{m-1}) \tag{A.8a}$$

leads, on account of Eq. (A.1a), to

$$I_{nm} = O(|\nu_m|/m), m \to \infty, n \text{ is fixed.} \tag{A.8b}$$

From Eqs. (41.a) and (A.8b)

$$C^*_{nm} = O(|\nu_m|^{3/2} \frac{e^{-|\nu_m|(\frac{\pi}{2}-\beta)}}{m}), m \to \infty, n \text{ is fixed,} \tag{A.9}$$

where we have made use of the fact that $\hat{\nu}_m$ and ν_m are practically equal for $m \to \infty$.

Thus C^*_{nm} decays exponentially as $m \to \infty$, n is fixed when

$$\beta < \frac{\pi}{2}. \tag{A.10}$$

Assuming that the inequalities in Eqs. (A.5b) and (A.10) are satisfied and that an inverse exists for the matrix $\{\delta_{nm} + C^*_{nm}\}$ $(n, m = 1, 2, ..., M)$, then the approximate solution $\tilde{A}^{(M)}$ converges to the exact one $\tilde{A}^*$ and the rate of convergence is exponential (see [5]).

The two inequalities in Eqs. (A.5b) and (A.10) could be changed by a slight modification to the numerical scheme through a normalization process. This is outlined as follows:

a) Normalize $\tilde{A}(\mu)$ in Eq. (30.a) as

$$\tilde{A}(\mu) = \hat{A}(\mu) e^{-i\mu(\frac{\pi}{2}-\beta-\phi)}, \tag{A.11}$$

leading to modifying Eq. (30.a) to

$$\hat{A}(\mu) = \hat{s}(\mu) - \hat{C}(\mu) \int_0^{i\infty} \hat{K}_1(\nu,\mu)\hat{A}(\nu) \frac{\sin\ (\pi\nu)}{\cos\ (\pi\nu) + \cos\ (\pi\mu)} d\nu, \mathrm{Im}\,\mu \geqslant 0, \mathrm{Re}\,\mu = 0, \tag{A.12a}$$

where

$$\hat{s}(\mu) = \tilde{s}(\mu) e^{i\mu(\frac{\pi}{2}-\beta-\phi)}, \tag{A.12b}$$

$$\hat{C}(\mu) = \frac{iY_s}{\pi Y f(\mu)} e^{i\mu(\frac{\pi}{2}-\beta-\phi)}, \tag{A.12c}$$

$$\hat{K}_1(\nu,\mu) = K_1(\nu,\mu) e^{-i\nu(\frac{\pi}{2}-\beta-\phi)}. \tag{A.12d}$$

b) Use the numerical scheme to find $\hat{A}(\mu)$.

Assuming that the involved matrix has an inverse and that the inequalities

$$\beta > \theta_0, \tag{A.13a}$$

$$\phi > 0 \tag{A.13b}$$

are satisfied, then the approximate solution $\hat{A}^{(M)}(\mu)$ converges to the exact one $\hat{A}^{*}$.

c) De-normalize $\hat{A}(\mu)$.

Hence the two inequalities in Eqs. (A.5b) and (A.10) are changed to Eqs. (A.13a)-(A.13b).

Appendix B. Poles and Residues of $A(\mu)$ in Eq. (27)

Here again, the stepwise results are the impedance cone reductions of the material cone results in [1, App. B].

Let us cast Eq. (27) as

$$A(\mu) = \frac{1}{f(\mu)}\{s_1(\mu)+\frac{Y_s}{2\pi Y}[s_2(\mu)+\int_{-i\infty}^{i\infty} K(\nu,\mu)A(\nu)d\nu]\}, \mathrm{Re}\,\mu = 0, \mathrm{Im}\,\mu \in (-\infty,\infty), \tag{B.1}$$

where

$$s_2(\mu) = \int_{-i\infty}^{i\infty} \nu \hat{I}_2^{*}(\nu,\mu)p^{(0)}(\nu,\beta)d\nu, \tag{B.2a}$$

$$K(\nu,\mu) = \nu \hat{I}_2^{*}(\nu,\mu)P_{\nu-\frac{1}{2}}(\cos\beta), \tag{B.2b}$$

with $\hat{I}_2^{*}(\nu,\mu)$ as given in Eq. (29.c).

Eq. (B.1) defines $A(\mu)$ on the imaginary axis of the complex μ-plane and is also valid for the strip $S_0\{\mathrm{Re}\,\mu \in (0,1), \mathrm{Im}\,\mu \in (-\infty,\infty)\}$ and reveals that the singularities of $A(\mu)$ in S_0 are:

(A) One set of poles located at μ_p with

$$\frac{d}{d\beta}P_{\mu_p-\frac{1}{2}}(\cos\beta) = 0. \tag{B.3a}$$

(C) A pole located at

$$\mu = \frac{1}{2}. \tag{B.3b}$$

To continue $A(\mu)$ into the strip $S_1\{\operatorname{Re}\mu \in (1,3), \operatorname{Im}\mu \in (-\infty,\infty)\}$, we collect two residue contributions from the poles of $\hat{I}_2^*(\nu,\mu)$ located at $\mu = \pm\nu + 1$. Poles of $\hat{I}_2^*(\nu,\mu)$ at $\mu = \pm\nu - 1$ are located in the left-hand side of the complex μ-plane and, therefore, are not captured by the process of analytic continuation to $\operatorname{Re}\mu > 0$.

Utilizing Eqs. (20) and (21), $A(\mu)$ in the S_1 strip is given by

$$A(\mu) = \frac{1}{f(\mu)}\{s_1(\mu)+\frac{Y_s}{2\pi Y}[s_2(\mu)+\cot\mu\pi(s_2^{+1}(\mu)+K^{+1}(\mu)A(\mu-1))+\int_{-i\infty}^{i\infty} K(\nu,\mu)A(\nu)d\nu]\}, \tag{B.4a}$$

where

$$s_2^{+1}(\mu) = -4(\mu-1)p^{(0)}(\mu-1,\beta), \tag{B.4b}$$

$$K^{+1}(\mu) = 4\pi i(\mu-1)P_{(\mu-1)-\frac{1}{2}}(\cos\beta). \tag{B.4c}$$

Thus for $\mu \in S_1$, $A(\mu)$ has the singularities:

(A) A set of poles at μ_p for which

$$\frac{d}{d\beta}P_{\mu_p-\frac{1}{2}}(\cos\beta) = 0. \tag{B.5a}$$

(B) A set of poles satisfying

$$\frac{d}{d\beta}P_{(\mu-1)-\frac{1}{2}}(\cos\beta) = 0. \tag{B.5b}$$

(C) Two poles located at

$$\mu = \frac{3}{2}, \frac{5}{2}. \tag{B.5c}$$

(D) A pole located at

$$\mu = 2. \tag{B.5d}$$

We generalize to the q^{th} strip $S_q\{\mathrm{Re}\,\mu \in (2q-1, 2q+1), \mathrm{Im}\,\mu \in (-\infty, \infty)\}$, $q = 2, 3, ...,$

$$\begin{aligned} A(\mu) &= \frac{1}{f(\mu)}\{s_1(\mu) + \frac{Y_s}{2\pi Y}[s_2(\mu) + \cot\mu\pi \sum_{j=1}^{j=q}(s_2^{+j}(\mu) + K^{+j}(\mu)A(\mu - 2j + 1)) \\ &\quad + \int_{-i\infty}^{i\infty} K(\nu, \mu)A(\nu)d\nu]\}, \end{aligned} \tag{B.6a}$$

where

$$s_2^{+j}(\mu) = 4(-1)^j(\mu - 2j + 1)p^{(0)}(\mu - 2j + 1, \beta), \tag{B.6b}$$

$$K^{+j}(\mu) = 4\pi i(-1)^{j+1}(\mu - 2j + 1)P_{(\mu-2j+1)-\frac{1}{2}}(\cos\beta). \tag{B.6c}$$

We infer from Eq. (B.6a) that for $\mu \in S_q$, $q > 1$, $A(\mu)$ has the singularities:

(A) A set of poles at μ_p for which

$$\frac{d}{d\beta}P_{\mu_p-\frac{1}{2}}(\cos\beta) = 0. \tag{B.7a}$$

(B) q sets of poles satisfying

$$\frac{d}{d\beta}P_{(\mu-2j+1)-\frac{1}{2}}(\cos\beta) = 0,\ j = 1, 2, ..q. \tag{B.7b}$$

(C) Two poles at

$$\mu = 2q \pm \frac{1}{2}. \tag{B.7c}$$

(D) Two poles at

$$\mu = 2q - 1, \mu = 2q. \tag{B.7d}$$

In order to assess the order of the poles in Eq. (B.7d), we need to investigate the products $\{\cot \mu\pi A(\mu - 2j + 1)\}$ in Eq. (B.6a). For $q = 2$, at $\mu = 3$, both $\cot \mu\pi$ and $A(\mu - 1)$ have a simple pole. Hence the pole of $A(\mu)$ at $\mu = 3$ is a second order pole. At $\mu = 4$, $\cot \mu\pi$ has a simple pole while $A(\mu - 1)$ has a second order pole. Hence the pole of $A(\mu)$ at $\mu = 4$ is a third order pole. Moving on to $q = 3, 4, ...$, we infer that the pole at $\mu = 2q - 1$ is of order $2q - 2$ and the pole at $\mu = 2q$ is of order $2q - 1$, $q \geq 2$.

Throughout, we will adopt the terminology:
(i) μ_p for poles of type (A),

$$\frac{d}{d\beta} P_{\mu_p - \frac{1}{2}}(\cos \beta) = 0, p = 1, 2, 3, ..., \infty. \tag{B.8a}$$

These are simple poles.
(ii) μ_{pl} for poles of type (B),

$$\mu_{pl} = \mu_p + 2l - 1, l = 1, 2, .., \infty; p = 1, 2, 3, .., \infty, \tag{B.8b1}$$

with μ_p satisfying Eq. (B.8a).
When μ_{pl} happens to be equal to $\mu_{p'}$,

$$\mu_{p'} = \mu_p + 2l - 1, \tag{B.8b2}$$

then $A(\mu)$ has a second order pole at μ_{pl}.
(iii) μ_s for poles of type (C),

$$\mu_s = s + \frac{1}{2}, s = 0, 1, 2, ..., \infty. \tag{B.8c}$$

These are simple poles.
(iv) μ_m for poles of type (D),

$$\mu_m = m,\ m = 2, ..., \infty. \tag{B.8d}$$

These poles are of order $m-1$.

Residues Computation

a) **Poles of the type** μ_p in the S_q strip

The residue is given by

$$\mathrm{Res}[A(\mu_p)] = \frac{1}{f'(\mu_p)}\{\Lambda(\mu)+\frac{Y_s}{2\pi Y}\cot\mu\pi\sum_{j=1}^{j=q}(s_2^{+j}(\mu)+K^{+j}(\mu)A(\mu-2j+1))\}_{\mu_p}, \tag{B.9a}$$

where

$$\Lambda(\mu) = s_1(\mu) + \frac{Y_s}{2\pi Y}[s_2(\mu) + \int_{-i\infty}^{i\infty} K(\nu,\mu)A(\nu)d\nu], \tag{B.9b}$$

$$f^{'}(\mu_p) = \frac{df(\mu)}{d\mu}|_{\mu_p}. \tag{B.9c}$$

b) **Poles of the type** μ_{pl}, $l = 1, 2, ..., \infty$

The residues of these poles are computed once the μ_p residues are computed and are given by

$$\mathrm{Res}[A(\mu_{pl)})] = \{\frac{\cot\mu\pi}{f(\mu)}\frac{Y_s}{2\pi Y}K^{+l}(\mu)\}_{\mu_{pl}}\mathrm{Res}[A(\mu_p)], l = 1, 2, 3, ..., \infty. \tag{B.10}$$

The residues when one or some poles of μ_{pl} are of second order are not detailed here but are straightforward and require the utilization of the second order residue formula instead of the first order formula used in this analysis.

The residues of the poles μ_s and μ_m are straightforward, though the ones for μ_m require the use of the formulae for higher order poles.

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