EXISTENCE OF STATIONARY SOLUTIONS FOR SOME INTEGRO-DIFFERENTIAL EQUATIONS WITH ANOMALOUS DIFFUSION

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Abstract: The article deals with the existence of solutions of an integro-differential equation arising in population dynamics in the case of anomalous diffusion involving the negative Laplace operator raised to a certain fractional power. The proof of existence of solutions is based on a fixed point technique. Solvability conditions for non-Fredholm elliptic operators in unbounded domains along with the Sobolev inequality for a fractional Laplacian are being used.

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1. Introduction

The present article is devoted to the studies of the existence of stationary solutions of the following integro-differential equation

$$\frac{\partial u}{\partial t} = -D(-\Delta)^s u + \int_{\mathbb{R}^3} K(x-y)g(u(y,t))dy + f(x), \quad \frac{1}{4} < s < \frac{3}{4} \qquad (1.1)$$

appearing in cell population dynamics. The space variable x corresponds to the cell genotype, $u(x,t)$ denotes the cell density as a function of their genotype and time. The right side of this equation describes the evolution of cell density by means of cell proliferation, mutations and cell influx. Here the anomalous diffusion term is correspondent to the change of genotype via small random mutations, and the nonlocal term describes large mutations. In this context $g(u)$ stands for the rate of cell birth which depends on u (density dependent proliferation), and the function $K(x-y)$ gives the proportion of newly born cells which change their genotype from y to x. Let us assume that it is dependent on the distance between the genotypes. Finally, the last term in the right side of this equation designates the influx of cells for different genotypes.

The operator $(-\Delta)^s$, $\frac{1}{4} < s < \frac{3}{4}$ in equation (1.1) represents a particular case of the anomalous diffusion actively studied in relation with various applications in plasma physics and turbulence [13], [14], surface diffusion [15], [16], semiconductors [17] and so on. The physical meaning of the anomalous diffusion is that the random process occurs with longer jumps in comparison with normal diffusion. The moments of jump length distribution is finite in the case of normal diffusion, but this is not the case for superdiffusion. The operator $(-\Delta)^s$, $\frac{1}{4} < s < \frac{3}{4}$ is defined by means of the spectral calculus. A similar problem in the case of the standard Laplace operator in the diffusion term was studied recently in [28].

Let us set $D = 1$ and establish the existence of solutions of the problem

$$-(-\Delta)^s u + \int_{\mathbb{R}^3} K(x-y)g(u(y))dy + f(x) = 0, \quad \frac{1}{4} < s < \frac{3}{4}. \tag{1.2}$$

The particular case of this equation when $s = \frac{1}{2}$ was treated recently in [29]. We consider the case when the linear part of this operator does not satisfy the Fredholm property. Therefore, conventional methods of nonlinear analysis may not be applicable. Let us use solvability conditions for non Fredholm operators along with the method of contraction mappings.

Consider the equation

$$-\Delta u + V(x)u - au = f, \tag{1.3}$$

where $u \in E = H^2(\mathbb{R}^d)$ and $f \in F = L^2(\mathbb{R}^d)$, $d \in \mathbb{N}$, a is a constant and the scalar potential function $V(x)$ is either zero identically or converges to 0 at infinity. For $a \geq 0$, the essential spectrum of the operator $A : E \to F$ corresponding to the left side of equation (1.3) contains the origin. As a consequence, this operator fails to satisfy the Fredholm property. Its image is not closed, for $d > 1$ the dimension of its kernel and the codimension of its image are not finite. The present work is devoted to the studies of some properties of the operators of this kind. Note that elliptic problems with non Fredholm operators were studied actively in recent years. Approaches in weighted Sobolev and Hölder spaces were developed in [2], [3], [4], [5], [6]. The non Fredholm Schrödinger type operators were treated with the methods of the spectral and the scattering theory in [18], [21], [23]. The Laplace operator with drift from the point of view of non Fredholm operators was studied in [22] and linearized Cahn-Hilliard problems in [24] and [26]. Nonlinear non Fredholm elliptic equations were treated in [25] and [27]. Significant applications to the theory of reaction-diffusion equations were developed in [8], [9]. Operators without Fredholm property arise also when studying wave systems with an infinite number of localized traveling waves (see [1]). Particularly, when $a = 0$ the operator A is Fredholm in some properly chosen weighted spaces (see [2], [3], [4], [5], [6]). However, the case of $a \neq 0$ is considerably different and the approach developed in these works cannot be used. Front propagation problems with anomalous diffusion were studied actively in recent years (see e.g. [30], [31]).

We set $K(x) = \varepsilon \mathcal{K}(x)$ with $\varepsilon \geq 0$ and suppose that the assumption below is satisfied.

Assumption 1. *Consider $\frac{1}{4} < s < \frac{3}{4}$. Let $f(x) : \mathbb{R}^3 \to \mathbb{R}$ be nontrivial, such that $f(x) \in L^1(\mathbb{R}^3)$ and $(-\Delta)^{1-s} f(x) \in L^2(\mathbb{R}^3)$. Assume also that $\mathcal{K}(x) : \mathbb{R}^3 \to \mathbb{R}$ and $\mathcal{K}(x) \in L^1(\mathbb{R}^3)$. Additionally, $(-\Delta)^{1-s}\mathcal{K}(x) \in L^2(\mathbb{R}^3)$, such that*

$$Q := \|(-\Delta)^{1-s}\mathcal{K}(x)\|_{L^2(\mathbb{R}^3)} > 0.$$

Let us choose the space dimension $d = 3$, which is related to the solvability conditions for the linear Poisson type problem (3.22) established in Lemma 5. From the point of view of applications, the space dimension is not limited to $d = 3$ since the space variable is correspondent to cell genotype but not to the usual physical space.

We use the Sobolev inequality for the fractional Laplacian (see e.g. Lemma 2.2 of [10], also [11])

$$\|f(x)\|_{L^{\frac{6}{4s-1}}(\mathbb{R}^3)} \leq c_s \|(-\Delta)^{1-s} f(x)\|_{L^2(\mathbb{R}^3)}, \quad \frac{1}{4} < 1 - s < \frac{3}{4} \tag{1.4}$$

along with the assumption above and the standard interpolation argument, which yields

$$f(x) \in L^2(\mathbb{R}^3).$$

Let us use the Sobolev spaces

$$H^{2s}(\mathbb{R}^3) := \{u(x) : \mathbb{R}^3 \to \mathbb{R} \mid u(x) \in L^2(\mathbb{R}^3),\ (-\Delta)^s u \in L^2(\mathbb{R}^3)\}, \quad 0 < s \leq 1$$

equipped with the norm

$$\|u\|^2_{H^{2s}(\mathbb{R}^3)} := \|u\|^2_{L^2(\mathbb{R}^3)} + \|(-\Delta)^s u\|^2_{L^2(\mathbb{R}^3)}. \tag{1.5}$$

The standard Sobolev embedding tells that

$$\|u\|_{L^\infty(\mathbb{R}^3)} \leq c_e \|u\|_{H^2(\mathbb{R}^3)}, \tag{1.6}$$

where $c_e > 0$ is the constant of the embedding. When the nonnegative parameter $\varepsilon = 0$, we obtain the linear Poisson type equation (3.22). By virtue of Lemma 5 below along with Assumption 1 equation (3.22) has a unique solution

$$u_0(x) \in H^{2s}(\mathbb{R}^3), \quad \frac{1}{4} < s < \frac{3}{4},$$

such that no orthogonality conditions are required. By means of Lemma 5, when $\frac{3}{4} \leq s < 1$, a specific orthogonality relation (3.24) is needed to be able to solve problem (3.22) in $H^{2s}(\mathbb{R}^3)$. On the other hand, we need $s > \dfrac{1}{4}$ to be able to use the Sobolev type inequality (1.4). By means of Assumption 1, using that

$$-\Delta u(x) = (-\Delta)^{1-s} f(x) \in L^2(\mathbb{R}^3),$$

we have for the unique solution of the linear equation (3.22) that $u_0(x) \in H^2(\mathbb{R}^3)$. We seek the resulting solution of the nonlinear problem (1.2) as

$$u(x) = u_0(x) + u_p(x). \tag{1.7}$$

Clearly, we arrive at the perturbative equation

$$(-\Delta)^s u_p = \varepsilon \int_{\mathbb{R}^3} \mathcal{K}(x-y) g(u_0(y) + u_p(y)) dy, \quad \frac{1}{4} < s < \frac{3}{4}. \tag{1.8}$$

We introduce a closed ball in the Sobolev space

$$B_\rho := \{u(x) \in H^2(\mathbb{R}^3) \mid \|u\|_{H^2(\mathbb{R}^3)} \leq \rho\}, \quad 0 < \rho \leq 1. \tag{1.9}$$

Let us seek the solution of equation (1.8) as the fixed point of the auxiliary nonlinear problem

$$(-\Delta)^s u = \varepsilon \int_{\mathbb{R}^3} \mathcal{K}(x-y) g(u_0(y) + v(y)) dy, \quad \frac{1}{4} < s < \frac{3}{4} \tag{1.10}$$

in ball (1.9). For a given function $v(y)$ this is an equation with respect to $u(x)$. The left side of (1.10) contains the non Fredholm operator $(-\Delta)^s : H^{2s}(\mathbb{R}^3) \to L^2(\mathbb{R}^3)$. Its essential spectrum fills the nonnegative semi-axis $[0, +\infty)$. Therefore, this operator has no bounded inverse. The similar situation appeared in articles [25] and [27] but as distinct from the present work, the problems studied there required orthogonality relations. The fixed point technique was used in [20] to estimate the perturbation to the standing solitary wave of the Nonlinear Schrödinger (NLS) equation when either the external potential or the nonlinear term in the NLS were perturbed but the Schrödinger operator involved in the nonlinear problem there had the Fredholm property (see Assumption 1 of [20], also [7]). We define the interval on the real line

$$I := [-c_e \|u_0\|_{H^2(\mathbb{R}^3)} - c_e, \; c_e \|u_0\|_{H^2(\mathbb{R}^3)} + c_e]. \tag{1.11}$$

Let us make the following assumption on the nonlinear part of problem (1.2).

Assumption 2. *Let $g(z) : \mathbb{R} \to \mathbb{R}$, such that $g(0) = 0$ and $g'(0) = 0$. It is also assumed that $g(z) \in C_2(\mathbb{R})$, such that*

$$a_2 := sup_{z \in I} |g''(z)| > 0.$$

Clearly, $a_1 := sup_{z \in I}|g'(z)| > 0$ as well, otherwise the function $g(z)$ will be constant on the interval I and then $a_2 = 0$. For instance, $g(z) = z^2$ obviously satisfies the assumption above.

We introduce the operator T_g, such that $u = T_g v$, where u is a solution of equation (1.10). Our main proposition is as follows.

Theorem 3. *Let Assumptions 1 and 2 hold. Then problem (1.10) defines the map* $T_g : B_\rho \to B_\rho$, *which is a strict contraction for all* $0 < \varepsilon < \varepsilon*$ *for some* $\varepsilon* > 0$. *The unique fixed point* $u_p(x)$ *of this map* T_g *is the only solution of equation (1.8) in* B_ρ.

Evidently, the resulting solution of problem (1.2) given by (1.7) will be nontrivial because the source term $f(x)$ is nontrivial and $g(0) = 0$ according to our assumptions. Let us make use of the following trivial lemma.

Lemma 4. *For* $R \in (0, +\infty)$ *consider the function*

$$\varphi(R) := \alpha R^{3-4s} + \frac{\beta}{R^{4s}}, \quad \frac{1}{4} < s < \frac{3}{4}, \quad \alpha, \beta > 0.$$

It achieves the minimal value at $R^* = \left(\frac{4\beta s}{\alpha(3-4s)} \right)^{\frac{1}{3}}$, *which is given by*

$$\varphi(R^*) = 3(3-4s)^{\frac{4s}{3}-1}(4s)^{-\frac{4s}{3}}\alpha^{\frac{4s}{3}}\beta^{1-\frac{4s}{3}}.$$

Let us proceed to the proof of our main result.

2. The existence of the perturbed solution

Proof of Theorem 3. We choose arbitrarily $v(x) \in B_\rho$ and denote the term involved in the integral expression in the right side of problem (1.10) as

$$G(x) := g(u_0(x) + v(x)).$$

Let us apply the standard Fourier transform (3.25) to both sides of equation (1.10), which yields

$$\widehat{u}(p) = \varepsilon (2\pi)^{\frac{3}{2}} \frac{\widehat{\mathcal{K}}(p)\widehat{G}(p)}{|p|^{2s}}.$$

Thus for the norm we have

$$\|u\|^2_{L^2(\mathbb{R}^3)} = (2\pi)^3 \varepsilon^2 \int_{\mathbb{R}^3} \frac{|\widehat{\mathcal{K}}(p)|^2 |\widehat{G}(p)|^2}{|p|^{4s}} dp. \tag{2.12}$$

As distinct from articles [25] and [27] involving the standard Laplacian operator in the diffusion term, here we do not try to control the norm

$$\left\| \frac{\widehat{\mathcal{K}}(p)}{|p|^{2s}} \right\|_{L^\infty(\mathbb{R}^3)}.$$

We estimate the right side of (2.12) using inequality (3.26) with $R > 0$ as

$$(2\pi)^3 \varepsilon^2 \int_{|p|\leq R} \frac{|\widehat{\mathcal{K}}(p)|^2 |\widehat{G}(p)|^2}{|p|^{4s}} dp + (2\pi)^3 \varepsilon^2 \int_{|p|>R} \frac{|\widehat{\mathcal{K}}(p)|^2 |\widehat{G}(p)|^2}{|p|^{4s}} dp \leq$$

$$\leq \varepsilon^2 \|\mathcal{K}\|^2_{L^1(\mathbb{R}^3)} \left\{ \frac{1}{2\pi^2} \|G(x)\|^2_{L^1(\mathbb{R}^3)} \frac{R^{3-4s}}{3-4s} + \frac{1}{R^{4s}} \|G(x)\|^2_{L^2(\mathbb{R}^3)} \right\}. \tag{2.13}$$

Due to the fact that $v(x) \in B_\rho$, we get

$$\|u_0 + v\|_{L^2(\mathbb{R}^3)} \leq \|u_0\|_{H^2(\mathbb{R}^3)} + 1.$$

The Sobolev embedding (1.6) gives us

$$|u_0 + v| \leq c_e \|u_0\|_{H^2(\mathbb{R}^3)} + c_e.$$

By means of the formula $G(x) = \displaystyle\int_0^{u_0+v} g'(z)dz$ with the interval I defined in (1.11), we obtain

$$|G(x)| \leq sup_{z\in I} |g'(z)| |u_0 + v| = a_1 |u_0 + v|.$$

Hence

$$\|G(x)\|_{L^2(\mathbb{R}^3)} \leq a_1 \|u_0 + v\|_{L^2(\mathbb{R}^3)} \leq a_1 (\|u_0\|_{H^2(\mathbb{R}^3)} + 1).$$

Evidently, $G(x) = \displaystyle\int_0^{u_0+v} dy \Big[\int_0^y g''(z)dz \Big]$. This yields

$$|G(x)| \leq \frac{1}{2} sup_{z\in I} |g''(z)| |u_0 + v|^2 = \frac{a_2}{2} |u_0 + v|^2,$$

$$\|G(x)\|_{L^1(\mathbb{R}^3)} \leq \frac{a_2}{2} \|u_0 + v\|^2_{L^2(\mathbb{R}^3)} \leq \frac{a_2}{2} (\|u_0\|_{H^2(\mathbb{R}^3)} + 1)^2. \tag{2.14}$$

Hence we arrive at the estimate from above for the right side of (2.13) as

$$\varepsilon^2 \|\mathcal{K}\|^2_{L^1(\mathbb{R}^3)} (\|u_0\|_{H^2(\mathbb{R}^3)} + 1)^2 \left\{ \frac{a_2^2}{8\pi^2(3-4s)} (\|u_0\|_{H^2(\mathbb{R}^3)} + 1)^2 R^{3-4s} + \frac{a_1^2}{R^{4s}} \right\},$$

where $R \in (0, +\infty)$. Lemma 4 gives us the minimal value of the expression above. Therefore,

$$\|u\|^2_{L^2(\mathbb{R}^3)} \le \varepsilon^2 \|\mathcal{K}\|^2_{L^1(\mathbb{R}^3)} (\|u_0\|_{H^2(\mathbb{R}^3)} + 1)^{2+\frac{8s}{3}} \frac{3 a_2^{\frac{8s}{3}} a_1^{2-\frac{8s}{3}}}{(3-4s) s^{\frac{4s}{3}} \pi^{\frac{8s}{3}} 2^{4s+\frac{8s}{3}}}. \tag{2.15}$$

Clearly, (1.10) yields

$$-\Delta u = \varepsilon (-\Delta)^{1-s} \int_{\mathbb{R}^3} \mathcal{K}(x-y) G(y) dy.$$

By virtue of (3.26) along with (2.14) we arrive at

$$\|\Delta u\|^2_{L^2(\mathbb{R}^3)} \le \varepsilon^2 \|G\|^2_{L^1(\mathbb{R}^3)} Q^2 \le \varepsilon^2 \frac{a_2^2}{4} (\|u_0\|_{H^2(\mathbb{R}^3)} + 1)^4 Q^2. \tag{2.16}$$

Therefore, by means of the definition of the norm (1.5) along with inequalities (2.15) and (2.16) we derive the upper bound for $\|u\|_{H^2(\mathbb{R}^3)}$ as

$$\varepsilon (\|u_0\|_{H^2(\mathbb{R}^3)} + 1)^2 a_2 \times$$

$$\times \left[\|\mathcal{K}\|^2_{L^1(\mathbb{R}^3)} \left(\frac{a_2(\|u_0\|_{H^2(\mathbb{R}^3)} + 1)}{a_1} \right)^{\frac{8s}{3}-2} \frac{3}{(3-4s) s^{\frac{4s}{3}} \pi^{\frac{8s}{3}} 2^{4s+\frac{8s}{3}}} + \frac{Q^2}{4} \right]^{\frac{1}{2}} \le \rho$$

for all $\varepsilon > 0$ sufficiently small. Therefore, $u(x) \in B_\rho$ as well. If for a certain $v(x) \in B_\rho$ there exist two solutions $u_{1,2}(x) \in B_\rho$ of equation (1.10), their difference $w(x) := u_1(x) - u_2(x) \in L^2(\mathbb{R}^3)$ solves

$$(-\Delta)^s w = 0.$$

Because the operator $(-\Delta)^s$ considered in the whole space does not possess nontrivial square integrable zero modes, $w(x) = 0$ a.e. in $\mathbb{R}^3$. Hence, equation (1.10) defines a map $T_g : B_\rho \to B_\rho$ for all $\varepsilon > 0$ small enough.

Our goal is to establish that this map is a strict contraction. Let us choose arbitrarily $v_{1,2}(x) \in B_\rho$. The argument above gives us $u_{1,2} = T_g v_{1,2} \in B_\rho$ as well. By means of (1.10)

$$(-\Delta)^s u_1 = \varepsilon \int_{\mathbb{R}^3} \mathcal{K}(x-y) g(u_0(y) + v_1(y)) dy, \tag{2.17}$$

$$(-\Delta)^s u_2 = \varepsilon \int_{\mathbb{R}^3} \mathcal{K}(x-y) g(u_0(y) + v_2(y)) dy, \tag{2.18}$$

$\frac{1}{4} < s < \frac{3}{4}$. We define

$$G_1(x) := g(u_0(x) + v_1(x)), \quad G_2(x) := g(u_0(x) + v_2(x))$$

and apply the standard Fourier transform (3.25) to both sides of equations (2.17) and (2.18). This yields

$$\widehat{u_1}(p) = \varepsilon(2\pi)^{\frac{3}{2}}\frac{\widehat{\mathcal{K}}(p)\widehat{G_1}(p)}{|p|^{2s}}, \quad \widehat{u_2}(p) = \varepsilon(2\pi)^{\frac{3}{2}}\frac{\widehat{\mathcal{K}}(p)\widehat{G_2}(p)}{|p|^{2s}}.$$

Obviously,

$$\|u_1 - u_2\|^2_{L^2(\mathbb{R}^3)} = \varepsilon^2(2\pi)^3\int_{\mathbb{R}^3}\frac{|\widehat{\mathcal{K}}(p)|^2|\widehat{G_1}(p) - \widehat{G_2}(p)|^2}{|p|^{4s}}dp.$$

Evidently, it can be estimated from above by virtue of (3.26) by

$$\varepsilon^2\|\mathcal{K}\|^2_{L^1(\mathbb{R}^3)}\left\{\frac{1}{2\pi^2}\|G_1(x) - G_2(x)\|^2_{L^1(\mathbb{R}^3)}\frac{R^{3-4s}}{3-4s} + \|G_1(x) - G_2(x)\|^2_{L^2(\mathbb{R}^3)}\frac{1}{R^{4s}}\right\},$$

where $R \in (0, +\infty)$. We use the identity

$$G_1(x) - G_2(x) = \int_{u_0+v_2}^{u_0+v_1} g'(z)dz.$$

Hence

$$|G_1(x) - G_2(x)| \le sup_{z\in I}|g'(z)||v_1 - v_2| = a_1|v_1 - v_2|.$$

Thus

$$\|G_1(x) - G_2(x)\|_{L^2(\mathbb{R}^3)} \le a_1\|v_1 - v_2\|_{L^2(\mathbb{R}^3)} \le a_1\|v_1 - v_2\|_{H^2(\mathbb{R}^3)}.$$

Clearly,

$$G_1(x) - G_2(x) = \int_{u_0+v_2}^{u_0+v_1} dy\Big[\int_0^y g''(z)dz\Big].$$

Let us obtain the estimate from above for $G_1(x) - G_2(x)$ in the absolute value as

$$\frac{1}{2}sup_{z\in I}|g''(z)||(v_1 - v_2)(2u_0 + v_1 + v_2)| = \frac{a_2}{2}|(v_1 - v_2)(2u_0 + v_1 + v_2)|.$$

The Schwarz inequality yields the upper bound for the norm $\|G_1(x) - G_2(x)\|_{L^1(\mathbb{R}^3)}$ as

$$\frac{a_2}{2}\|v_1-v_2\|_{L^2(\mathbb{R}^3)}\|2u_0+v_1+v_2\|_{L^2(\mathbb{R}^3)} \le a_2\|v_1-v_2\|_{H^2(\mathbb{R}^3)}(\|u_0\|_{H^2(\mathbb{R}^3)}+1). \quad (2.19)$$

Thus we arrive at the estimate from above for the norm $\|u_1(x) - u_2(x)\|^2_{L^2(\mathbb{R}^3)}$ given by

$$\varepsilon^2\|\mathcal{K}\|^2_{L^1(\mathbb{R}^3)}\|v_1 - v_2\|^2_{H^2(\mathbb{R}^3)}\Big\{\frac{{a_2}^2}{2\pi^2}(\|u_0\|_{H^2(\mathbb{R}^3)} + 1)^2\frac{R^{3-4s}}{3-4s} + \frac{{a_1}^2}{R^{4s}}\Big\}.$$

Lemma 4 allows us to minimize the expression above over $R \in (0, +\infty)$ to obtain the upper bound for $\|u_1(x) - u_2(x)\|^2_{L^2(\mathbb{R}^3)}$ as

$$\varepsilon^2 \|\mathcal{K}\|^2_{L^1(\mathbb{R}^3)} \|v_1 - v_2\|^2_{H^2(\mathbb{R}^3)} \frac{3a_1^{2-\frac{8s}{3}}}{(3-4s)2^{4s}s^{\frac{4s}{3}}} \left[\frac{a_2(\|u_0\|_{H^2(\mathbb{R}^3)} + 1)}{\pi}\right]^{\frac{8s}{3}}. \tag{2.20}$$

Identities (2.17) and (2.18) give us

$$(-\Delta)(u_1 - u_2) = \varepsilon(-\Delta)^{1-s} \int_{\mathbb{R}^3} \mathcal{K}(x-y)[G_1(y) - G_2(y)]dy.$$

By means of inequalities (3.26) and (2.19) we arrtive at

$$\|\Delta(u_1 - u_2)\|^2_{L^2(\mathbb{R}^3)} \le \varepsilon^2 \|(-\Delta)^{1-s}\mathcal{K}\|^2_{L^2(\mathbb{R}^3)} \|G_1 - G_2\|^2_{L^1(\mathbb{R}^3)} \le$$

$$\le \varepsilon^2 Q^2 a_2^2 \|v_1 - v_2\|^2_{H^2(\mathbb{R}^3)} (\|u_0\|_{H^2(\mathbb{R}^3)} + 1)^2. \tag{2.21}$$

By virtue of (2.20) and (2.21) the norm $\|u_1 - u_2\|_{H^2(\mathbb{R}^3)}$ is estimated from above by the expression $\varepsilon a_2(\|u_0\|_{H^2(\mathbb{R}^3)} + 1)\times$

$$\times \left\{ \frac{3\|\mathcal{K}\|^2_{L^1(\mathbb{R}^3)}}{(3-4s)2^{4s}s^{\frac{4s}{3}}\pi^{\frac{8s}{3}}} \left[\frac{a_2(\|u_0\|_{H^2(\mathbb{R}^3)} + 1)}{a_1}\right]^{\frac{8s}{3}-2} + Q^2 \right\}^{\frac{1}{2}} \|v_1 - v_2\|_{H^2(\mathbb{R}^3)}.$$

This yields that the map $T_g : B_\rho \to B_\rho$ defined by equation (1.10) is a strict contraction for all values of $\varepsilon > 0$ small enough. Its unique fixed point $u_p(x)$ is the only solution of problem (1.8) in the ball B_ρ. The resulting $u(x) \in H^2(\mathbb{R}^3)$ given by (1.7) is a solution of equation (1.2). ■

3. Auxiliary results

Let us obtain the solvability conditions for the following linear Poisson type equation with a square integrable right side

$$(-\Delta)^s u = f(x), \quad x \in \mathbb{R}^3, \quad 0 < s < 1. \tag{3.22}$$

We designate the inner product as

$$(f(x), g(x))_{L^2(\mathbb{R}^3)} := \int_{\mathbb{R}^3} f(x)\bar{g}(x)dx, \tag{3.23}$$

with a slight abuse of notations when the functions involved in (3.23) are not square integrable, like for instance the ones present in orthogonality conditions (3.24) and (3.34) below. Indeed, if $f(x) \in L^1(\mathbb{R}^3)$ and $g(x) \in L^\infty(\mathbb{R}^3)$, then the integral in the right side of (3.23) is well defined. Our technical statement is as follows.

Lemma 5. *Let* $f(x) \in L^2(\mathbb{R}^3)$.

1) When $0 < s < \frac{3}{4}$ *and additionally* $f(x) \in L^1(\mathbb{R}^3)$, *problem (3.22) has a unique solution* $u(x) \in H^{2s}(\mathbb{R}^3)$.

2) When $\frac{3}{4} \le s < 1$ *and in addition* $|x|f(x) \in L^1(\mathbb{R}^3)$, *equation (3.22) admits a unique solution* $u(x) \in H^{2s}(\mathbb{R}^3)$ *if and only if the orthogonality condition*

$$(f(x), 1)_{L^2(\mathbb{R}^3)} = 0 \tag{3.24}$$

holds.

Proof. Let us first note that by virtue of the norm definition (1.5) along with the square integrability of the right side of (3.22), it would be sufficient to establish the solvability of equation (3.22) in $L^2(\mathbb{R}^3)$. The solution $u(x) \in L^2(\mathbb{R}^3)$ will clearly belong to $H^{2s}(\mathbb{R}^3),\ 0 < s < 1$ as well.

We prove the uniqueness of solutions for problem (3.22). Suppose $u_{1,2}(x) \in H^{2s}(\mathbb{R}^3)$ both solve (3.22). Then their difference $w(x) := u_1(x) - u_2(x)$ satisfies the homogeneous equation

$$(-\Delta)^s w = 0.$$

Since the operator $(-\Delta)^s$ in $\mathbb{R}^3$ does not possess nontrivial square integrable zero modes, $w(x)$ vanishes a.e. in the whole space. We will use the standard Fourier transform

$$\widehat{f}(p) := \frac{1}{(2\pi)^{\frac{3}{2}}} \int_{\mathbb{R}^3} f(x) e^{-ipx} dx. \tag{3.25}$$

Obviously, we have the inequality

$$\|\widehat{f}(p)\|_{L^\infty(\mathbb{R}^3)} \le \frac{1}{(2\pi)^{\frac{3}{2}}} \|f(x)\|_{L^1(\mathbb{R}^3)}. \tag{3.26}$$

We apply (3.25) to both sides of problem (3.22). This yields

$$\widehat{u}(p) = \frac{\widehat{f}(p)}{|p|^{2s}} \chi_{\{p \in \mathbb{R}^3 \ | \ |p| \le 1\}} + \frac{\widehat{f}(p)}{|p|^{2s}} \chi_{\{p \in \mathbb{R}^3 \ | \ |p| > 1\}}, \tag{3.27}$$

where χ_A denotes the characteristic function of a set $A \subseteq \mathbb{R}^3$. Clearly for all $0 < s < 1$ the second term in the right side of (3.27) is square integrable due to the estimate

$$\int_{\mathbb{R}^3} \frac{|\widehat{f}(p)|^2}{|p|^{4s}} \chi_{\{p \in \mathbb{R}^3 \ | \ |p| > 1\}} dp \le \|f\|^2_{L^2(\mathbb{R}^3)} < \infty.$$

To prove the square integrability of the first term in the right side of (3.27) when $0 < s < \frac{3}{4}$, we use estimate (3.26), which yields

$$\int_{\mathbb{R}^3} \frac{|\widehat{f}(p)|^2}{|p|^{4s}} \chi_{\{p \in \mathbb{R}^3 \ | \ |p| \le 1\}} dp \le \frac{\|f(x)\|^2_{L^1(\mathbb{R}^3)}}{2\pi^2(3 - 4s)} < \infty,$$

which completes the proof of part 1) of the lemma.

To study the solvability of equation (3.22) when $\frac{3}{4} \leq s < 1$, we use the expansion

$$\widehat{f}(p) = \widehat{f}(0) + \int_0^{|p|} \frac{\partial \widehat{f}(s, \omega)}{\partial s} ds.$$

Here and below ω denotes the angle variables on the sphere. This enables us to write the first term in the right side of (3.27) as

$$\frac{\widehat{f}(0)}{|p|^{2s}} \chi_{\{p \in \mathbb{R}^3 \ | \ |p| \leq 1\}} + \frac{\int_0^{|p|} \frac{\partial \widehat{f}(s,\omega)}{\partial s} ds}{|p|^{2s}} \chi_{\{p \in \mathbb{R}^3 \ | \ |p| \leq 1\}}. \tag{3.28}$$

Definition (3.25) yields

$$\left| \frac{\partial \widehat{f}(|p|, \omega)}{\partial |p|} \right| \leq \frac{1}{(2\pi)^{\frac{3}{2}}} \| |x| f(x) \|_{L^1(\mathbb{R}^3)} < \infty$$

by means of one of our assumptions. Thus,

$$\left| \frac{\int_0^{|p|} \frac{\partial \widehat{f}(s,\omega)}{\partial s} ds}{|p|^{2s}} \chi_{\{p \in \mathbb{R}^3 \ | \ |p| \leq 1\}} \right| \leq \frac{1}{(2\pi)^{\frac{3}{2}}} \| |x| f \|_{L^1(\mathbb{R}^3)} |p|^{1-2s} \chi_{\{p \in \mathbb{R}^3 \ | \ |p| \leq 1\}} \in L^2(\mathbb{R}^3).$$

The remaining term in (3.28) $\dfrac{\widehat{f}(0)}{|p|^{2s}} \chi_{\{p \in \mathbb{R}^3 \ | \ |p| \leq 1\}} \in L^2(\mathbb{R}^3)$ if and only if $\widehat{f}(0) = 0$, which yields orthogonality condidition (3.24) in case 2) of the lemma. ■

Note that for the lower values of the power of the negative Laplacian $0 < s < \frac{3}{4}$ under the assumptions stated above no orthogonality relations are needed to solve the linear Poisson type equation (3.22) in $H^{2s}(\mathbb{R}^3)$.

We prove that one can incorporate a shallow, short-range potential into the linear Poisson type equation considered above and generalize the result of Lemma 5. Let us consider the equation with a square integrable right side

$$(-\Delta + V(x))^s u = f(x), \quad x \in \mathbb{R}^3, \quad 0 < s < 1 \tag{3.29}$$

with the operator $(-\Delta + V(x))^s$ defined by means of the spectral calculus. Under our assumptions the operator $-\Delta + V(x)$ on $L^2(\mathbb{R}^3)$ is nonnegative as discussed below.

Assumption 6. *The potential function $V(x) : \mathbb{R}^3 \to \mathbb{R}$ satisfies the estimate*

$$|V(x)| \leq \frac{C}{1 + |x|^{3.5+\varepsilon}}$$

with some $\varepsilon > 0$ *and* $x \in \mathbb{R}^3$ *a.e. such that*

$$4^{\frac{1}{9}}\frac{9}{8}(4\pi)^{-\frac{2}{3}}\|V\|_{L^\infty(\mathbb{R}^3)}^{\frac{1}{9}}\|V\|_{L^{\frac{4}{3}}(\mathbb{R}^3)}^{\frac{8}{9}} < 1 \quad \textit{and} \quad \sqrt{c_{HLS}}\|V\|_{L^{\frac{3}{2}}(\mathbb{R}^3)} < 4\pi. \tag{3.30}$$

This is analogous to Assumption 1.1 of [21] under which by means of Lemma 2.3 of [21] our Schrödinger operator $-\Delta+V(x)$ is self-adjoint and unitarily equivalent to $-\Delta$ on $L^2(\mathbb{R}^3)$ via the wave operators. Hence the essential spectrum of $(-\Delta+V(x))^s$ on $L^2(\mathbb{R}^3)$ fills the nonnegative semi-axis $[0,+\infty)$. Therefore, such operator does not have a bounded inverse and therefore it fails to satisfy the Fredholm property. Here C denotes a finite, positive constant and c_{HLS} the constant in the Hardy-Littlewood-Sobolev inequality

$$\left|\int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\frac{f_1(x)f_1(y)}{|x-y|^2}dxdy\right| \le c_{HLS}\|f_1\|^2_{L^{\frac{3}{2}}(\mathbb{R}^3)}, \quad f_1 \in L^{\frac{3}{2}}(\mathbb{R}^3)$$

given on p.98 of [12]. The functions of the continuous spectrum of our Schrödinger operator satisfy

$$(-\Delta+V(x))\varphi_k(x) = k^2\varphi_k(x), \quad k \in \mathbb{R}^3,$$

in the integral formulation the Lippmann-Schwinger equation (see e.g. p.98 of [19])

$$\varphi_k(x) = \frac{e^{ikx}}{(2\pi)^{\frac{3}{2}}} - \frac{1}{4\pi}\int_{\mathbb{R}^3}\frac{e^{i|k||x-y|}}{|x-y|}(V\varphi_k)(y)dy \tag{3.31}$$

and the orthogonality relations

$$(\varphi_k(x),\varphi_q(x))_{L^2(\mathbb{R}^3)} = \delta(k-q), \quad k,q \in \mathbb{R}^3.$$

They form a complete system in $L^2(\mathbb{R}^3)$. When the wave vector k vanishes, we deal with the function $\varphi_0(x)$ used in orthogonality relation (3.34) below. We denote by tilde the generalized Fourier transform with respect to these functions, such that

$$\tilde{f}(k) := (f(x),\varphi_k(x))_{L^2(\mathbb{R}^3)}, \quad k \in \mathbb{R}^3. \tag{3.32}$$

The integral operator involved in the right side of equation (3.31) is

$$(Q\varphi)(x) := -\frac{1}{4\pi}\int_{\mathbb{R}^3}\frac{e^{i|k||x-y|}}{|x-y|}(V\varphi)(y)dy, \quad \varphi \in L^\infty(\mathbb{R}^3).$$

We consider $Q : L^\infty(\mathbb{R}^3) \to L^\infty(\mathbb{R}^3)$. By virtue of Lemma 2.1 of [21] under Assumption 6 above on the scalar potential we have $\|Q\|_\infty < 1$. Furthermore, this norm is bounded above by $I(V)$, which denotes the left side of the first inequality in (3.30). Corollary 2.2 of [21] yields the estimate

$$|\tilde{f}(k)| \leq \frac{1}{(2\pi)^{\frac{3}{2}}} \frac{1}{1 - I(V)} \|f\|_{L^1(\mathbb{R}^3)}. \tag{3.33}$$

We have the following proposition.

Lemma 7. *Let the potential $V(x)$ satisfy Assumption 6 and $f(x) \in L^2(\mathbb{R}^3)$.*

a) Let $0 < s < \frac{3}{4}$ and additionally $f(x) \in L^1(\mathbb{R}^3)$. Then equation (3.29) admits a unique solution $u(x) \in L^2(\mathbb{R}^3)$.

b) Let $\frac{3}{4} \leq s < 1$ and in addition $|x|f(x) \in L^1(\mathbb{R}^3)$. Then problem (3.29) has a unique solution $u(x) \in L^2(\mathbb{R}^3)$ if and only if the orthogonality condition

$$(f(x), \varphi_0(x))_{L^2(\mathbb{R}^3)} = 0 \tag{3.34}$$

holds.

Proof. We first suppose that equation (3.29) admits two solutions $u_{1,2}(x) \in L^2(\mathbb{R}^3)$. Then their difference $w(x) := u_1(x) - u_2(x) \in L^2(\mathbb{R}^3)$ satisfies the homogeneous equation

$$(-\Delta + V(x))^s w(x) = 0,$$

which cannot have nontrivial square integrable solutions due to the fact that our self-adjoint operator $-\Delta + V(x)$ is unitarily equivalent to $-\Delta$ on $L^2(\mathbb{R}^3)$. Hence $w(x) = 0$ a.e. in $\mathbb{R}^3$.

We apply the generalized Fourier transform (3.32) to both sides of problem (3.29), which yields

$$\tilde{u}(k) = \frac{\tilde{f}(k)}{|k|^{2s}} \chi_{\{k \in \mathbb{R}^3 \ | \ |k| \leq 1\}} + \frac{\tilde{f}(k)}{|k|^{2s}} \chi_{\{k \in \mathbb{R}^3 \ | \ |k| > 1\}}. \tag{3.35}$$

The second term in the right side of (3.35) is square integrable for all $0 < s < 1$, since

$$\int_{\mathbb{R}^3} \frac{|\tilde{f}(k)|^2}{|k|^{4s}} \chi_{\{k \in \mathbb{R}^3 \ | \ |k| > 1\}} dk \leq \|f\|^2_{L^2(\mathbb{R}^3)} < \infty.$$

We use inequality (3.33) to obtain the upper bound on the square of the L^2 norm of the first term in the right side of (3.35) as

$$\int_{\mathbb{R}^3} \frac{|\tilde{f}(k)|^2}{|k|^{4s}} \chi_{\{k \in \mathbb{R}^3 \ | \ |k| \leq 1\}} dk \leq \frac{1}{2\pi^2(3 - 4s)} \frac{1}{(1 - I(V))^2} \|f(x)\|^2_{L^1(\mathbb{R}^3)} < \infty$$

for $0 < s < \frac{3}{4}$, which completes the proof of part a) of the lemma.

To study the solvability of equation (3.29) when $\frac{3}{4} \leq s < 1$, we use the identity

$$\tilde{f}(k) = \tilde{f}(0) + \int_0^{|k|} \frac{\partial \tilde{f}(s,\omega)}{\partial s} ds,$$

which enables us to express the first term in the right side of (3.35) as

$$\frac{\tilde{f}(0)}{|k|^{2s}} \chi_{\{k \in \mathbb{R}^3 \ | \ |k| \leq 1\}} + \frac{\int_0^{|k|} \frac{\partial \tilde{f}(s,\omega)}{\partial s} ds}{|k|^{2s}} \chi_{\{k \in \mathbb{R}^3 \ | \ |k| \leq 1\}}. \tag{3.36}$$

Note that $\nabla_k \tilde{f}(k) \in L^\infty(\mathbb{R}^3)$ by virtue of Lemma 2.4 of [21]. Hence the second term in (3.36) can be estimated from above in the absolute value by

$$\|\nabla_k \tilde{f}(k)\|_{L^\infty(\mathbb{R}^3)} |k|^{1-2s} \chi_{\{k \in \mathbb{R}^3 \ | \ |k| \leq 1\}} \in L^2(\mathbb{R}^3), \quad \frac{3}{4} \leq s < 1.$$

The remaining term in (3.36) $\dfrac{\tilde{f}(0)}{|k|^{2s}} \chi_{\{k \in \mathbb{R}^3 \ | \ |k| \leq 1\}} \in L^2(\mathbb{R}^3)$ if and only if $\tilde{f}(0)$ vanishes, which gives us orthogonality relation (3.34) in case 2) of the lemma. ■

Note that in case a) of the lemma above, when our Schrödinger operator is raised to a lower power $0 < s < \frac{3}{4}$, under the given assumptions no orthogonality conditions are needed for solving equation (3.29) in $L^2(\mathbb{R}^3)$. The particular case of Lemma 7, when $s = \frac{1}{2}$ was treated recently in [29].

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