

June 11, 2015

## A CONTINUOUS FAMILY OF EQUILIBRIA IN FERROMAGNETIC MEDIA ARE GROUND STATES

XIFENG SU AND RAFAEL DE LA LLAVE

**ABSTRACT.** We show that a foliation of equilibria (a continuous family of equilibria whose graph covers all the configuration space) in ferromagnetic models are ground states.

The result we prove is very general, and it applies to models with long range interactions and many body. As an application, we consider several models of networks of interacting particles including models of Frenkel-Kontorova type on  $\mathbb{Z}^d$  and one-dimensional quasi-periodic media.

The result above is an analogue of several results in the calculus variations (fields of extremals) and in PDE's. Since the models we consider are discrete and long range, new proofs need to be given. We also note that the main hypothesis of our result (the existence of foliations of equilibria) is the conclusion (using KAM theory) of several recent papers. Hence, we obtain that the KAM solutions recently established are minimizers when the interaction is ferromagnetic (and transitive).

**Keywords:** Ground states, quasi-periodic solutions, Hilbert integrals, mimimizers, Frenkel-Kontorova models

**MSC:**[2010] 82B20, 37J50, 49J21, 82D30

### 1. INTRODUCTION

Many physical problems lead to variational problems for functions described in discrete sets.

A model to keep in mind as motivation is the Frenkel-Kontorova model [FK39] which considers configurations  $u = \{u_i\}_{i \in \mathbb{Z}}$  and tries to find those that minimize the energy given by the formal sum

$$(1) \quad \mathcal{S}(u) = \sum_{i \in \mathbb{Z}} \left[ \frac{1}{2}(u_i - u_{i+1})^2 - V(u_i) \right].$$

There are several physical interpretations of the FK model [BK04, Sel92], the original one is the interaction of planar dislocations in a 3-D crystal, but

---

X. S supported by National Natural Science Foundation of China (Grant No. 11301513) and “the Fundamental Research Funds for the Central Universities”.

R.L. has been partially supported by NSF grant DMS-1500943. R.L. also acknowledges the support of the Tang Aoqing visiting professorship in Jilin University.

it has appeared as a model of other situations. We can think of  $u_i$  as describing the position of the  $i^{\text{th}}$  atom deposited over a  $1 - D$  medium. The first part of the sum describes the interaction between the nearest particles. The function  $V$  models the interaction of the atoms with the medium, which is assumed to be periodic or quasi-periodic in models of crystals and quasi-crystals (in this paper, periodicity or quasiperiodicity is not assumed). Note that, with many of these interpretations, it is natural to consider also more general models which involve longer range interactions, multi site interactions, or higher dimensional crystals. Hence, in this paper we will include these generalizations.

In the case that  $V$  is a periodic function, the problem of showing existence of plane-like minimizers (i.e. minimizers that differ from a linear function by a bounded function) of (1) with a well defined frequency independently by Mather [Mat82] and Aubry [ALD83], which is now referred as Aubry-Mather theory. Several authors ( see [Bla89, Bla90, KdILR97, CdIL98, dILV07b, dILV07a, dILV10] and references therein) generalized the setting of Aubry-Mather theory to higher dimensional crystals, more general media and for many-body interactions. Related models appear in PDE's [Mos89, RS11], minimal surfaces [CdIL01, Val04, Tor04], fractional laplacian operators [dILV09, Dáv13].

In the case that  $V$  in (1) is a quasi-periodic function, the problem to establish all the results of Aubry-Mather theory for periodic systems is still open. Notably, the existence of plane-like minimizers is still not settled. In [LS03] there are examples of quasi-periodic potentials for which no plane-like minimizer exists. On the other hand, when the potential  $V$  is small enough, the papers [SdIL12b, SdIL12a, SZdIL15, ZSdIL15] use a rather unusual KAM theory to construct families of equilibria which are plane-like. The results of this paper show that the families constructed by KAM method are minimizers when the problem is ferromagnetic. Hence a very interesting problem is to study the transition – now known to exist – from models with plane-like minimizers to models without them. The papers [SdIL12b, SdIL12a] lead to efficient numerical algorithms which were implemented in [BdIL13] and lead to several conjectures about the transition between plane-like and non plane-like minimizers. Notably [BdIL13] discovered numerically scaling relations similar to those in phase transitions in the breakdown of analyticity of plane-like solutions in quasi-periodic media. The papers [SZdIL15, ZSdIL15] also present efficient algorithms for the computation of other solutions, but they have not been implemented yet.

The goal of this paper is to show that for ferromagnetic models when there are continuous families of equilibria whose graphs cover the whole phase space they are actually ground states (also called class-A minimizers).

In particular, the solutions produced by KAM theory in [SdlL12b, SdlL12a, SZdlL15, ZSdlL15] are ground states.

Results establishing that families of equilibria are minimizers are very common place in the standard calculus of variations. They are proved by either the methods of *fields of extremals* or *Hilbert integral* [Car99]. In our case, since we are considering discrete space and long range models, these methods do not seem to apply directly and we have to use a different method.

We note that, as it is customary, the non-degeneracy equations of KAM theory are weaker than those in the variational theory. Roughly speaking, the KAM theory just requires that certain operators are invertible. The variational theory requires that these quantities are positive definite. On the other hand, the KAM theory is more sensitive to quantitative features. For example, in (1) and periodic  $V$ , the KAM only applies for  $V$  which are small in a smooth norm, whereas the variational methods apply for any differentiable  $V$ .

In Section 2.1, we present the results in a very general set up, patterned after the general set up of statistical mechanics [Rue69] allowing multi-body and long range interactions. In Section 3 we present again the results for some concrete models, which have appeared in the literature. Even if this could have been avoided logically since the models in Section 3 are particular cases of those in Section 2.1 we hope that this will add to the readability of the paper and as motivation for those interested in the concrete models and in numerical implementations. Also, the methods of proof used in Section 3 are different from the methods used in the proof of the general theorem and closer to the arguments in the classical calculus of variations.

## 2. FORMULATION OF THE MAIN RESULT

**2.1. A very general set-up.** We consider a very general setup motivated by the formulation in [Rue69] of statistical mechanics. Later, in Section 3, we will present more details for less general set ups, which may be more familiar.

*2.1.1. General assumptions on the systems and its configurations.* We consider a discrete countable set  $\Lambda$ . Its elements will be called sites. The set  $\Lambda$  may be imagined as a network of particles. Many cases in statistical mechanics consider that  $\Lambda$  is an integer lattice, corresponding physically to a crystal.

We assume that the state of each site is given by a real number. Hence, the state of the system is given by a function  $u : \Lambda \rightarrow \mathbb{R}$  which assigns to each site  $i \in \Lambda$  the value  $u_i$ . For our purposes, it is crucial that the order parameter at each site is a one-dimensional number. We do not know

how to deal with two-dimensional phase spaces. Indeed in [Bla90] presents counterexamples to several crucial statement in our setting when the order parameter are 2-dimensional. The papers [Mat91, Mañ97] contains rather satisfactory analogues of several other results of Aubry-Mather theory for higher dimensional order parameters but they do not consider higher dimensional independent variables.

We associate to the finite subsets  $B$  of  $\Lambda$  an energy function  $H_B : \mathbb{R}^B \rightarrow \mathbb{R}$ , which models the (possibly many-body and long range) interaction. In Physical terms, the interaction may be even among the different sites or among the sites and a substratum. The total energy associated to a configuration  $u$  is given by the formal sum:

$$(2) \quad \mathcal{S}(u) = \sum_{\substack{B \subseteq \Lambda \\ \#B < \infty}} H_B(u) \quad \forall u \in \mathbb{R}^\Lambda$$

where  $H_B(u)$  depends only on  $u|_B$ .

**Remark 1.** In this paper, we will not assume any periodicity properties of the set  $\Lambda$  and of the interaction, since this will not play any role in our arguments. On the other hand, we note that the main hypothesis of this paper (the existence of a foliation of equilibria) is the conclusion of several other papers which use periodicity. In [CdL98, dLV10], there is a very general setup for quasi-periodicity involving the action of a group  $G$  on  $\Lambda$  and on the interaction.  $G$  is assumed to satisfy some mild growth properties.

In Section 3 we will present the results for some finite range models which are concrete examples of the set up and for which our main hypotheses are verified.

2.1.2. *Critical points and ground states.* The following definitions are very standard in the calculus of variations.

**Definition 2.** We say that a configuration  $u$  is an equilibrium for an energy (2) when

$$(3) \quad \frac{\partial}{\partial u_i} \mathcal{S}(u) \equiv \sum_{B \ni i} \partial_{u_i} H_B(u) = 0 \quad \forall i \in \Lambda.$$

For simplicity, we denote  $\frac{\partial}{\partial u_i} \mathcal{S}(u)$  by  $E_i(u)$  and  $E(u) = \{E_i(u)\}_{i \in \Lambda}$ .

We note that, even if the sum defining  $\mathcal{S}$  is formal, the equilibrium equations (3) are meant to be well defined equations. This can happen for example if  $H_B \equiv 0$  whenever  $\text{diam}(B) \geq L$ . (These are called finite range interactions and Frenkel-Kontorova models are an example.) In Section 2.1.7, we will formulate a condition, more general than finite range which is enough for our purposes.

We are interested in the existence of the following special class of equilibria.

**Definition 3** (Ground states). *We say that a configuration  $u$  is a ground state ( or a class-A minimizer in the terminology of Morse [Mor24]) if for any configuration  $\varphi$  whose support is a finite subset of  $\Lambda$  we have*

$$(4) \quad \mathcal{S}(u) - \mathcal{S}(u + \varphi) \leq 0.$$

Note that (4) should be understood cancelling all the terms that are identical. That is

$$(5) \quad \sum_{\substack{\#B < \infty \\ B \cap \text{supp}(\varphi) \neq \emptyset}} H_B(u) - H_B(u + \varphi) \leq 0.$$

We note that the conditions (5) make sense when the interactions are finite range since the sum in (5) involves only finitely many terms. In Section 2.1.7, we will make assumptions more general than finite range that ensure that the sum in (5) make sense. It is clear that the main idea is that we will assume the terms in the sum (5) as well as their derivatives decay fast enough for all  $u$  in a class of functions. We will postpone the precise formulation till we have specified which classes of functions we will consider.

Since expressions similar to (5) will appear often in our calculations, we will introduce the notation

$$(6) \quad \Gamma(\varphi; u, \tilde{B}) \equiv \sum_{\substack{\#B < \infty \\ B \cap \tilde{B} \neq \emptyset}} H_B(u) - H_B(u + \varphi).$$

We remark that if  $\text{supp}(\varphi) \subset \tilde{B}$ , we have

$$\Gamma(\varphi; u, \tilde{B}) = \Gamma(\varphi; u, \text{supp}(\varphi)).$$

The reason is that, the sums defining the two  $\Gamma$  differ only in sets  $B$  which do not intersect the support of  $\varphi$ . Hence, the corresponding term in the sum is zero.

It is easy to check that a ground state is an equilibrium.

**2.1.3. Foliation by equilibria.** We say that a collection of configurations  $\{u^\beta\}_{\beta \in \mathbb{R}}$  is a foliation when:

- (A1)  $E(u^\beta) = 0$ , i.e.  $E_i(u^\beta) = 0$  for any  $\beta \in \mathbb{R}, i \in \Lambda$ ;
- (A2)  $u^\beta$  is increasing with respect to  $\beta$ , i.e. if  $\beta_1 \leq \beta_2$ ,  $u_i^{\beta_1} \leq u_i^{\beta_2}$  for any  $i \in \Lambda$ ;
- (A3)  $u_i^\beta \rightarrow \pm\infty$  as  $\beta$  goes to  $\pm\infty$  for any  $i \in \Lambda$ ;
- (A4)  $\{u_i^\beta \mid \beta \in \mathbb{R}\} = \mathbb{R}$ .

The most crucial assumption for us is (A4). This means that, as we move the parameters  $\beta$ , the graphs of the functions  $u^\beta$  sweep out all the space  $\Lambda \times \mathbb{R}$ .

We say a foliation is strict if

$$(A2)' \quad \beta_1 < \beta_2 \implies u_i^{\beta_1} < u_i^{\beta_2} \text{ for all } i \in \Lambda.$$

Having a family of critical points satisfying (A1)-(A4) is extremely analogous to the assumption on the *fields of extremals* in the calculus of variations [Car99]<sup>1</sup>.

The usual Aubry-Mather theory for a fixed frequency  $\omega$  produces a family satisfying (A1)-(A3) – but in general not (A4). On the other hand, for Diophantine  $\omega$  and (and some models) we can use KAM theory to produce families satisfying (A1)-(A4). The calculus of variations methods do not assume that the system is close to integrable, but they require positive definite assumptions on the interaction. On the other hand, KAM methods do not require that the system is convex (positive definite Jacobian) but they require that the system admits an approximate solution to the invariance equation (in particular, this is satisfied for systems close to integrable).

In the applications to Aubry-Mather theory which we will discuss later in Section 3, the set  $\Lambda$  will be  $\mathbb{Z}^d$  and the functions  $u^\beta$  will be roughly linear.

We point out, however that in the case of no interactions, in dimensions bigger or equal than 2 one could have also harmonic polynomials, which are minimizers. It is marginally pointed out in [Mos86, Mos89] that developing a variational theory starting from the harmonic polynomials of higher degree would be very interesting.

Note that the subsequent properties we will assume depend on the class of functions  $u^\beta$ .

#### 2.1.4. Ferromagnetic properties.

**Definition 4** (Ferromagnetic condition). *We say that the  $C^2$  interaction potential  $H$  satisfies the ferromagnetic condition if*

$$(7) \quad \frac{\partial^2 H_B}{\partial u_p \partial u_q}(u) \leq 0 \quad \forall p, q \in \Lambda, p \neq q,$$

where  $u$  is any configuration on  $\Lambda$  and  $B$  is any finite subset of  $\Lambda$ .

**Definition 5** (Ferromagnetic Transitive). *We say that a ferromagnetic interaction in  $\Lambda$  is transitive for a class of configurations  $u^\beta$  when, given any  $p, q \in \Lambda$  there exist an integer  $k \geq 1$ , a sequence  $p_0, \dots, p_k$  in  $\Lambda$  with*

<sup>1</sup>Note that the fields of extremals in [Car99] are formulated for functions of a one dimensional variable taking values into any dimensional space. Here we are in the opposite situation: we are considering functions of many variables, but taking values in a one dimensional space.

$p_0 = p, p_k = q$  and sets  $B_i$  containing a pair  $p_i, p_{i+1}$  for  $i = 0, \dots, k-1$  such that, for any  $\varphi$  with compact support,

$$\partial_{p_i} \partial_{p_{i+1}} H_{B_i}(u^\beta + \varphi) < 0.$$

In the main cases of interest, such as the Frenkel-Kontorova models, we will see that the  $\partial_{p_i} \partial_{p_{i+1}} H_{B_i}$  are independent of the configuration, so that this assumption will be very easy to verify in several models of practical importance.

The assumption in Definition 5 appeared in [dILV07a] where it was shown that it implies that the gradient flow of the formal energy  $\mathcal{S}$  in (2) satisfies a strong comparison principle. In the PDE case, a comparison principle for the gradient flow would give a very quick proof of our results, but the long range of the interactions requires an extra argument. See Remark 9.

**Remark 6.** The ferromagnetism assumptions, when  $\Lambda = \mathbb{Z}$ , and the interactions are nearest neighbor, become the twist conditions in Aubry-Mather theory. We also note that they can be thought of as analogues of ellipticity conditions for continuous variational problems. See [CdIL98, dILV07b] for some more explanations of these analogies.

2.1.5. *Graph theoretic language to describe the Ferromagnetic assumptions.* We can reformulate some of the assumptions of Section 2.1.4 in the language of graph theory. The introduction of a new language is purely cosmetic, but allows us to express future arguments concisely and it may be illuminating.

The key observation is that we can interpret Definition 5 as the existence of a graph structure on  $\Lambda$ .

Whenever there exists  $B$  such that for all  $u^\beta$

$$(8) \quad \partial_p \partial_q H_B(u^\beta + \varphi) < 0$$

then the sites  $p, q$  are linked.

The physical meaning of (8) is that the configuration at  $p$  affects the forces experienced at the site  $q$  (and viceversa, in agreement with the action-reaction principle). The Definition 5 can be interpreted as saying that any site can influence any other site, if not directly, through influencing intermediate sites that in turn influence some others.

It is natural to endow  $\Lambda$  with a graph structure by considering the points of  $\Lambda$  as vertices and drawing an edge among two linked sites in the sense of (8).

The assumption in Definition 5 can be interpreted as saying that, starting from any site, we can reach any other jumping only through linked sites or that the graph is connected.

The graph structure allows us to introduce two notions that are standard in graph theory: distance and connectedness.

Given a path  $\gamma$  in the graph, we define the  $|\gamma|$  the length of a path  $\gamma$  as the number of edges it contains.

We define the *distance between two sites*  $i, j \in \Lambda$  as

$$(9) \quad d(i, j) = \inf\{ |\gamma| \mid \gamma \text{ joins } i, j \}.$$

This  $d$  satisfies the usual assumptions of distance.

We also define the *distance of a point*  $i$  to a set  $S \subset \Lambda$  as

$$(10) \quad d(i, S) = \inf\{ d(i, j) \mid j \in S \}.$$

(Since the  $d(i, j)$  takes values in integers, it is clear that the infimum in (10) is a minimum.)

We can also define that a set  $S$  is *connected* when any pair of points can be joined by paths all whose edges have end points in the set  $S$ .

It will be important for us later that, given a finite set  $B$ , we can obtain another finite set  $\text{Con}(B)$  which is connected and which contains  $B$ .

If Definition 5 holds, given any pair  $i, j \in \Lambda$  we can find a path joining  $i$  to  $j$ . We denote this path as  $\gamma_{i,j}$ . Given a path  $\gamma_{i,j}$  we denote  $v(\gamma_{i,j})$  the *vertices* of the path. Hence, given a set  $B$ , we define

$$(11) \quad \text{Con}(B) = \cup_{i,j \in B} v(\gamma_{i,j}).$$

Clearly,  $B \subset \text{Con}(B)$  and  $\text{Con}(B)$  is connected because we note that given any pair of points  $a \in \gamma_{i,j}$ ,  $b \in \gamma_{i,j}$  we can find a path joining them by starting in  $a$ , following  $\gamma_{i,j}$  till  $j$ , then  $\gamma_{j,i}$  and then  $\gamma_{i,j}$  till we arrive to  $b$ .

The following elementary remark will play an important role for us later, so we formulate it now. It is mainly an exercise in the notation.

**Proposition 7.** *Assume that the interaction satisfies Definition 5.*

*Given any finite set  $S$ , the set*

$$S_1 = \{ i \mid d(i, S) \leq 1 \} = \{ i \mid j \in S, d(i, j) \leq 1 \}$$

*contains at least a point which is not in  $S$ .*

The totally trivial proof of Proposition 7 is the observation that, if there was no path that stepped out of  $S$ , it would be impossible for any point in  $S$  to be joined to other points far away.  $\square$

**2.1.6. Coerciveness Assumption.** Given a family  $u^\beta$  as before, we will assume that for any compactly supported  $\varphi$  and any  $i \in \text{supp}(\varphi)$  we have

$$(12) \quad \lim_{|t| \rightarrow \infty} \sum_{B \cap \text{supp}(\varphi) \neq \emptyset} \left[ H_B(u^\beta + \varphi + \delta_i t) - H_B(u^\beta + \varphi) \right] = +\infty,$$

where  $\delta_i$  denotes the Kronecker function which takes the value 1 at  $i$  and 0 at any other point.



Note that (12) says that if we make a test function grow at just one point, then the relative energy grows.

2.1.7. *A regularity assumption.* We will be performing some calculations with the equilibrium equations. In order to justify them, we will need some assumptions on the convergence of the  $E_i$  and their derivatives.

The following assumption is sufficient for the methods used in this paper. We note that the finite range of the interaction easily implies our assumption. Many models of interest (e.g. the Frenkel-Kontorova models) are finite range, but there are also models of physical interest which are not. See [SdlL12b] for a discussion of when hierarchical models satisfy the assumptions.

Given a class  $u^\beta$  satisfying (A1)-(A4) we say that the interaction  $\{H_B\}$  is  $u^\beta$  summable when: for  $\varphi$  satisfying either

- a)  $\varphi$  with compact support
- b)  $\varphi = (u^{\tilde{\beta}} - u^{\hat{\beta}})$  for any  $\tilde{\beta}, \hat{\beta} \in \mathbb{R}$ ,

we have for all  $\beta \in \mathbb{R}$ ,

$$(13) \quad \begin{aligned} \lim_{L \rightarrow \infty} \sum_{\substack{\text{diam}(B) \geq L \\ B \cap \text{supp}(\varphi) \neq \emptyset}} |\partial_{u_i} H_B(u^\beta + t\varphi)| &= 0 \\ \lim_{L \rightarrow \infty} \sum_{\substack{\text{diam}(B) \geq L \\ B \cap \text{supp}(\varphi) \neq \emptyset}} |\partial_{u_i} \partial_{u_j} H_B(u^\beta + t\varphi)| &= 0 \end{aligned}$$

and the limit in (13) is uniform in  $t \in [0, 1]$

We note that in the case  $u^{\tilde{\beta}} - u^{\hat{\beta}} \in \ell^\infty$  (as it happens in KAM theory) and in all families of plane-like equilibria of fixed slope  $b$  is implied by a). We also note that if we assume (A2)' instead of (A2), the case b) can be dispensed with.

The following result is a very simple corollary of the coercivity and regularity assumption.

**Proposition 8.** *Let  $u^\beta$  be a family of configurations and  $\{H_B\}_{\substack{B \subset \Lambda \\ \#B < \infty}}$  be a family of interactions that satisfy the coerciveness and the regularity assumptions with respect to them.*

*Fix any function  $u^\beta$  in the family and a finite set  $\tilde{B}$ .*

*Then, there is a function  $\varphi^*$  such that*

- $\text{supp}(\varphi^*) \subset \tilde{B}$

- 

$$\Gamma(\varphi^*; u^\beta, \tilde{B}) = \inf \{ \Gamma(\varphi; u^\beta, \tilde{B}) \mid \text{supp}(\varphi) \subset \tilde{B} \}$$

*where we use the notation introduced in (6).*

- 

$$(14) \quad E_i(u^\beta + \varphi^*) = 0 \quad \forall i \in \tilde{B}.$$

The proof of Proposition 8 is very easy. We note that we are considering a function of finitely many real variables (the values of  $\varphi^*$  at the sites of  $\tilde{B}$ ). By the assumption of regularity this function is differentiable and tends to infinity as any of its arguments goes to infinity. Hence, this function reaches its minimum and the minimum has zero derivative.

Of course, the support of the minimizing function could be smaller than  $\tilde{B}$  is some of the values of the minimizing function happens to be zero.

**Remark 9.** Note that in Proposition 8 we do not obtain that  $u + \varphi^*$  is an equilibrium. In (14), we only obtain that the equilibrium equations hold in the finite set  $\tilde{B}$ .

Even if  $u^\beta$  satisfies the equilibrium equations in  $\Lambda$ , when the interaction is long range, modifying the configuration in  $\tilde{B}$  can affect the equilibrium equations everywhere.

This is an important difference with the PDE models in the classical calculus of variations and this is the reason why our arguments need to be different.

#### 2.1.8. Statement of the main general result.

**Theorem 10.** *Let  $H$  be a  $C^2$  ferromagnetic interaction potential. Assume that there exists a collection of configurations  $\{u^\beta\}_{\beta \in \mathbb{R}}$  such that (A1)-(A4) hold. Moreover, assume that, with respect to  $u^\beta$  the interaction satisfies the ferromagnetic transitivity, coercivity and regularity assumptions above.*

*Then, all the equilibria  $u^\beta$  are ground states.*

Suppose by contradiction that there exist a number  $\beta_0$  and a configuration  $\varphi$  whose support is nonempty and finite such that

$$(15) \quad \mathcal{S}(u^{\beta_0} + \varphi) - \mathcal{S}(u^{\beta_0}) < 0.$$

That is, using the notation (6)

$$(16) \quad \Gamma(\varphi; u^{\beta_0}, \text{supp}(\varphi)) < 0.$$

Denote by  $\tilde{B} = \text{Con}(\text{supp}(\varphi))$  the connected subset constructed in (11),  $\text{supp}(\varphi) \subset \tilde{B}$ .

Using Proposition 8 there is a configuration  $\varphi^*$  with support in  $\tilde{B}$  such that

$$(17) \quad \Gamma(\varphi^*; u^{\beta_0}, \tilde{B}) = \min_{\text{supp}(\varphi_1) \subseteq \tilde{B}} \Gamma(\varphi_1; u^{\beta_0}, \tilde{B}).$$

We note that, since we can take  $\varphi$  as a test function  $\varphi_1$  we have

$$(18) \quad \Gamma(\varphi^*; u^{\beta_0}, \tilde{B}) = \Gamma(\varphi; u^{\beta_0}, \tilde{B}) = \Gamma(\varphi; u^{\beta_0}, \text{supp}(\varphi)) < 0.$$

Hence, if the function  $u^{\beta_0}$  was not a ground state, we could find a non-trivial  $\varphi^*$ . We will show that this is impossible and, therefore that  $u^{\beta_0}$  is a ground state.

We denote

$$(19) \quad \begin{aligned} \beta_+ &= \inf\{\beta \in \mathbb{R} \mid u^\beta > u^{\beta_0} + \varphi^*\}, \\ \beta_- &= \sup\{\beta \in \mathbb{R} \mid u^\beta < u^{\beta_0} + \varphi^*\}, \end{aligned}$$

where the partial ordering  $u < v$  is defined by  $u_i < v_i$  for any  $i \in \Lambda$ . Analogous definitions hold for “>”, “ $\geq$ ” and “ $\leq$ ”. Consequently, we have that assumption (A2) can be formulated just as  $u^{\beta_+} \geq u^{\beta_0} \geq u^{\beta_-}$ .

By the choice of  $\varphi^*$  and  $\beta_+$ , we have

$$(20) \quad \begin{aligned} E_i(u^{\beta_0} + \varphi^*) &= 0, \quad i \in \tilde{B} \\ E(u^{\beta_+}) &= 0. \end{aligned}$$

Moreover, we have  $u^{\beta_0} + \varphi^* \leq u^{\beta_+}$ .

The following is an elementary calculation using the fundamental theorem of calculus which holds for any configuration  $u^*$  and any  $\eta$  so that the regularity assumptions hold.

$$(21) \quad \begin{aligned} &E_{i^*}(u^* + \eta) - E_{i^*}(u^*) \\ &= \int_0^1 dt \left[ \sum_{j \in \Lambda} \frac{\partial^2 H_B}{\partial u_{i^*} \partial u_j}(u^* + t\eta) \eta_j \right] \\ &= \eta_{i^*} \int_0^1 dt \frac{\partial^2 H_B}{\partial u_{i^*} \partial u_{i^*}}(u^* + t\eta) \\ &\quad + \sum_{\substack{j \in \Lambda \\ j \neq i^*}} \eta_j \int_0^1 \frac{\partial^2 H_B}{\partial u_{i^*} \partial u_j}(u^* + t\eta). \end{aligned}$$

The identity (21) leads immediately to the following proposition.

**Proposition 11.** *Assume that, in the conditions of (21) we have*

$$\begin{aligned} E_{i^*}(u^*) &= E_{i^*}(u^* + \eta) \\ \eta &\geq 0 \text{ ( or } \eta \leq 0 \text{ )}. \end{aligned}$$

*Then, we have that  $\eta_j = 0$  for all  $j$  such that  $d(i^*, j) = 1$  where  $d$  is the graph distance introduced in (9).*

The proof of Proposition 11 is just observing that since  $\eta_{i^*} = 0$ , and all the other terms in (21) have the same sign, we should have that all of the terms in the sum in (21) should be zero. Hence, either  $\eta_j = 0$  or  $\int_0^1 \frac{\partial^2 H_B}{\partial u_{i^*} \partial u_j}(u^* + t\eta)$ , but for the points  $j$  at distance 1, this integral is not zero.  $\square$

Applying repeatedly Proposition 11 we have the following result for functions which satisfy the equilibrium equation on a set.

**Proposition 12.** *Assume that, in the conditions of (21) we have*

•

$$\begin{aligned} 0 &= E_{i^*}(u^*) = E_{i^*}(u^* + \eta) \quad \forall i^* \in \tilde{S} \\ \eta &\geq 0 \text{ ( or } \eta \leq 0 \text{ )}. \end{aligned}$$

- *The set  $\tilde{S}$  is connected in the sense introduced in Section 2.1.5.*

*Then, we have that  $\eta_j = 0$  for all  $j$  such that  $d(i^*, \tilde{S}) \leq 1$  where  $d$  is the graph distance introduced in (9).*

The proof of Proposition 12 is to proceed by induction starting on the point  $i^*$ . Applying Proposition 11 we obtain that for all the points  $j$  such that  $d(i^*, j) = 1$ , we should have  $\eta_j = 0$ . Now, for such points  $j$  that belong to  $S$ , the argument can restart. Therefore, proceeding by induction, we can always prolong the paths that land in  $S$ . Because  $S$  is connected, we can cover all the set  $S$  and obtain that  $\eta_j = 0$  for all the  $j$  in  $S$ . Note also that in the last step, we can get also that  $\eta$  vanishes in the set of points that are at distance 1 from  $S$ . It will be important for future purposes that, as observed in Proposition 8, we have that the set where we can obtain that  $\eta = 0$  is strictly larger than the set  $S$ .

□

**Remark 13.** With the analogies in Remark 6, we note Proposition 12 is reminiscent to the proof of the comparison principle for elliptic equations. Of course, the proof in the discrete case is different. The subtlety that we obtain the comparison in a larger set than the set where the equation holds does not have any analogue in the elliptic equations case.

Now, we come back to the proof of Theorem 10.

Since we have that  $u^{\beta_+} \geq u_0^\beta + \varphi^*$  and that the  $\beta_+$  is the smallest possible, we have alternatives:

- A) There is a point where  $\varphi^*$  is strictly positive;
- B)  $\varphi_i^* < 0$  for all  $i \in \tilde{S}$ ;
- C)  $\varphi^* \equiv 0$ .

Theorem 10 will be established when we show that all these alternatives are impossible. Hence, we conclude that  $\varphi^*$  in (14) could not exist and, hence no  $\varphi$  satisfying (30) could exist.

The case C) can be excluded because we argued in (18) that  $\varphi^*$  should be non trivial.

In case A), there exists  $i^* \in \tilde{S}$  such that

$$u_{i^*}^{\beta_+} = u_{i^*}^{\beta_0} + \varphi_{i^*}^*.$$

In this case, recalling that  $u^{\beta_+}$  satisfies the equilibrium equations in the whole  $\Lambda$  and that  $u^{\beta_0} + \varphi^*$  satisfies them in  $\tilde{S}$ , we can apply Proposition 12

with  $\eta = \varphi^*$  and obtain that

$$u_j^{\beta_+} = u_j^{\beta_0} + \varphi_j^*, \quad d(j, \tilde{S}) \leq 1.$$

The important point of the above observation is that there is such a point  $j$  outside of  $\tilde{S}$ . That is, a point  $j$  outside of the support of  $\varphi^*$ . Hence, there is point  $j^*$  such that

$$(22) \quad u_j^{\beta_+} = u_j^{\beta_0}.$$

When we apply Proposition 12 with  $\eta = u^{\beta_+} - u^{\beta_0}$  we obtain that  $u^{\beta_+} = u^{\beta_-}$ . This is a contradiction with  $\varphi^*$  being strictly positive.

Note that if we assume (A2)' from (22) we could obtain the conclusion without applying Proposition 8.

Excluding Case B) is very similar to excluding case A), but actually easier. Since there is point  $j$  where  $\varphi_j^*$  is strictly negative, we can have that there is a  $j^*$  where  $u^{\beta_-}$  touches from below the  $u^{\beta_0} + \varphi^*$ . Applying again Proposition 12, we derive that  $u^{\beta_-} = u^{\beta_0}$  which is a contradiction with the assumption that  $\varphi^*$  was strictly negative.

### 3. SOME CONCRETE EXAMPLES OF THE MODELS CONSIDERED

In this section, we will show how very different models fit simultaneously in the framework developed. The fact that we can obtain results for different models at the same times is due to the generality of the methods we present here. In some cases, we will also present different proofs.

**3.1. General one-dimensional periodic models.** The papers [dIL08] considers one dimensional models given by energies of the form:

$$(23) \quad \mathcal{L}(u) = \sum_k \sum_L H_{L,k}(u_k, u_{k+1}, \dots, u_{k+L})$$

where  $u : \mathbb{Z} \rightarrow \mathbb{R}$  and  $H_{L,k}$

Note that the models in (23) enjoy a translation invariance, which is not present in our general set up, but which is physically justified.

The corresponding equilibrium equation for the models (23) are:

$$(24) \quad \mathcal{E}_i(u) = \sum_k \sum_L \sum_j \partial_{j+1} H_{L,k}(u_{i-j}, \dots, u_i, \dots, u_{L-k+i-j}).$$

The paper [dIL08] includes coercivity and regularity similar to ours, it includes an extra periodicity property

$$H_L(u_k, u_{k+1}, \dots, u_{k+L}) = H_L(u_k + 1, u_{k+1} + 1, \dots, u_{k+L} + 1)$$

as well as higher regularity assumption. On the other hand, the paper [dIL08] does not use the full strength of the ferromagnetic property and

indeed they allows some antiferromagnetic terms. Note that, since the systems in [dIL08] are translation invariant, the ferromagnetic transitive is implied by the ferromagnetic property of nearest neighbors (there are other assumptions such as the strict ferromagnetic for other sets of interactions).

The papers [dIL08] consider only equilibrium configurations given by a hull function

$$u_k = \omega k + h(\omega k)$$

where  $h$  is a periodic function called the “*hull function*”. The function is such that  $t + u(t)$  is an increasing function.

It is easy to see that – it is shown with many details in [dIL08] that if  $h$  is the hull function for a critical point so is  $u^\beta$  given by

$$h^\beta(\theta) = \beta + h(\theta + \beta).$$

We observe that, when  $h$  is a smooth function and  $|h|_{L^\infty} < 1$ , the configurations obtained for all these hull functions produce a foliation in our sense.

Hence, applying Theorem 10, we obtain the following result:

**Theorem 14.** *Assume the setup of [dIL08] assume furthermore, that, for some hull function, the system satisfies the ferromagnetic property*

$$\partial_i \partial_j H_L \leq 0$$

and that

$$\partial_1 \partial_2 H_1(x, y) \leq -\eta < 0$$

*Then, the quasi-periodic solutions produced in [dIL08] are ground states.*

**3.2. Application to the Frenkel-Kontorova models on quasi-periodic media.** These class of models was considered in [SdIL12b] with nearest neighbor interactions. In [SdIL12a] for more general interactions, many body interactions. The papers [SdIL12b, SdIL12a] consider quasiperiodic solutions which are non-resonant (inded Diophantine) with the frequency of the medium. The papers [SZdIL15, ZSdIL15] study quasi-periodic solutions which are resonant with the frequency of the medium in models in which the interactions are only nearest neighbor. Using the results of this paper, we can conclude that the solutions are ground states provided that we assume transitive ferromagnetic conditions.

In this section, we will consider only the problem in [SdIL12b], which will allow us to give a more direct proof of the results. We note that in the models based on the Frenkel-Kontorova models with next neighbor interaction, the transitive ferromagnetic hypothesis is automatic.

We consider the following formal energy

$$(25) \quad \mathcal{S}(\{u\}_{i \in \mathbb{Z}}) = \sum_{n \in \mathbb{Z}} \frac{1}{2} (u_n - u_{n+1})^2 - V(u_n \alpha),$$

where  $V : \mathbb{T}^d \rightarrow \mathbb{R}$  and  $\alpha \in \mathbb{R}^d$  satisfy  $k \cdot \alpha \neq 0$  when  $k \in \mathbb{Z}^d - \{0\}$  where  $d \geq 2$ .

For simplicity, we denote  $H(x, y) = \frac{1}{2}(x - y)^2 - V(x\alpha)$ . Consequently,  $\partial_{xy}H(x, y) = \partial_{yx}H(x, y) = -1$ .

Under the assumption of [SdlL12b, ZSdlL15], using KAM method, we prove the existence of quasi-periodic solutions of the equilibrium equation

$$(26) \quad u_{n+1} + u_{n-1} - 2u_n + \partial_\alpha V(u_n \alpha) = 0,$$

where  $\partial_\alpha V \equiv (\alpha \cdot \nabla)V$ .

Indeed, the solutions of (26) we found are given by a hull function

$$u_n = n\omega + h(n\omega\alpha)$$

for some given  $\omega \in \mathbb{R}$ . Therefore, the equilibrium equation we solve in terms of  $h$  is

$$(27) \quad h(\sigma + \omega\alpha) + h(\sigma - \omega\alpha) - 2h(\sigma) + \partial_\alpha V(\sigma + \alpha \cdot h(\sigma)) = 0$$

The papers [SdlL12b, ZSdlL15] have very different non-resonance assumptions from the assumptions in [SdlL12b, SdlL12a] and require very different methods. Nevertheless, from the point of view of the arguments of this paper, to show that the quasi-periodic solutions produced in both papers are ground states, we can use the same argument.

It is easy to see that if  $h(\sigma)$  is a solution (27), for any  $\beta \in \mathbb{R}$ ,  $h(\sigma + \beta\alpha) + \beta$  is a solution. We denote  $h_\beta(\sigma) = h(\sigma + \beta\alpha) + \beta$ .

Hence, let us denote  $u_n^\beta = n\omega + h_\beta(n\omega\alpha)$  which is a continuum of equilibria of (26) with respect to the parameter  $\beta \in \mathbb{R}$ . It is easy to see that, for every fixed  $n \in \mathbb{Z}$ ,  $u_n^\beta$  is monotone with respect to  $\beta$ , i.e.,

$$(28) \quad \frac{\partial u_n^\beta}{\partial \beta} = 1 + \partial_\alpha h(n\omega\alpha + \beta\alpha) \neq 0.$$

Without loss of generality, we assume  $u_n^\beta$  is monotone increasing with respect to  $\beta$ .

The following result is a particular case of Theorem 10, but in this section, we will present a different proof.

**Theorem 15.** *For every  $\beta \in \mathbb{R}$ , the configurations  $u^\beta \equiv \{u_i^\beta\}_{i \in \mathbb{Z}}$  are ground states of (25).*

*Proof.* Suppose by contradiction that there exists  $\beta_0$  such that  $\{u_i^{\beta_0}\}_{i \in \mathbb{Z}}$  is not a ground state of (25). That is, there exists two integers  $m < n$  and a configuration  $\{v_i\}_{i \in \mathbb{Z}}$  satisfying  $v_i = u_i^{\beta_0}$  for any  $i \leq m$  or  $i > n$  such that

$$(29) \quad \mathcal{S}_m^n(\{v_i\}_{i \in \mathbb{Z}}) \equiv \sum_{i=m}^n \frac{1}{2}(v_i - v_{i+1})^2 - V(v_i \alpha) < \sum_{i=m}^n \frac{1}{2}(u_i^{\beta_0} - u_{i+1}^{\beta_0})^2 - V(u_i^{\beta_0} \alpha).$$

Since  $\min_{v \in \mathbb{R}^Z} \mathcal{S}_m^n(v)$  is a minimizing problem of finite variables and  $\mathcal{S}_m^n(v)$  is bounded from below, there exists a minimizing segment  $\{w_i\}_{i=m}^{n+1}$  of  $\mathcal{S}_m^n$  with the boundary condition  $w_i = u_i^{\beta_0}$  for  $i = m$  or  $i = n + 1$ . One can suppose, without loss of generality, that there exists  $m < i_0 \leq n$  such that  $w_{i_0} > u_{i_0}^{\beta_0}$ . Since  $u^\beta$  is a foliation, there exist  $\beta_1 > \beta_0$  and  $m < i_1 \leq n$  such that

$$(30) \quad w_{i_1} = u_{i_1}^{\beta_1}, \text{ and } w_i \leq u_i^{\beta_1}, \quad \forall m \leq i \leq n + 1.$$

Indeed, one can choose  $m < i_2 \leq n$  such that  $w_{i_2} = u_{i_2}^{\beta_1}$  and  $w_{i_2-1} < u_{i_2-1}^{\beta_1}$ .

We use an adaptation of the standard technique of the Hilbert integral in calculus of variations (see also [CdLL98]). For every  $m \leq i \leq n$  we calculate

$$(31) \quad \begin{aligned} 0 &= \partial_x H(w_i, w_{i+1}) + \partial_y H(w_{i-1}, w_i) + \partial_x H(u_i^\beta, u_{i+1}^\beta) + \partial_y H(u_{i-1}^\beta, u_i^\beta) \\ &= \int_0^1 \frac{d}{dt} \left[ \partial_x H(tw_i + (1-t)u_i^\beta, tw_{i+1} + (1-t)u_{i+1}^\beta) \right. \\ &\quad \left. + \partial_y H(tw_{i-1} + (1-t)u_{i-1}^\beta, tw_i + (1-t)u_i^\beta) \right] dt \\ &= \int_0^1 \left[ (\partial_{xx} H)(w_i - u_i^\beta) + (\partial_{xy} H)(w_{i+1} - u_{i+1}^\beta) \right. \\ &\quad \left. + (\partial_{yx} H)(w_{i-1} - u_{i-1}^\beta) + (\partial_{yy} H)(w_i - u_i^\beta) \right] dt. \end{aligned}$$

Let  $i = i_2$ ,  $\beta = \beta_1$  in the above calculation, we obtain

$$0 = w_{i_2+1} - u_{i_2+1}^{\beta_1} + w_{i_2-1} - u_{i_2-1}^{\beta_1}.$$

Hence, due to the choice of  $i_2$ , we have  $w_{i_2+1} - u_{i_2+1}^{\beta_1} = u_{i_2-1}^{\beta_1} - w_{i_2-1} > 0$ , which contradicts (30).  $\square$

#### ACKNOWLEDGEMENTS

The authors thank the hospitality of the JLU-GT joint institute for theoretical Sciences, which made possible the collaboration. R. L. also thanks Beijing Normal University for hospitality.

#### REFERENCES

- [ALD83] S. Aubry and P. Y. Le Daeron. The discrete Frenkel-Kontorova model and its extensions. I. Exact results for the ground-states. *Phys. D*, 8(3):381–422, 1983.
- [BdlL13] Timothy Blass and Rafael de la Llave. The analyticity breakdown for Frenkel-Kontorova models in quasi-periodic media: numerical explorations. *J. Stat. Phys.*, 150(6):1183–1200, 2013.
- [BK04] O. M. Braun and Y. S. Kivshar. *The Frenkel-Kontorova model*. Texts and Monographs in Physics. Springer-Verlag, Berlin, 2004.
- [Bla89] M. L. Blank. Metric properties of minimal solutions of discrete periodical variational problems. *Nonlinearity*, 2(1):1–22, 1989.



- [Bla90] M. L. Blank. Chaos and order in the multidimensional Frenkel-Kontorova model. *Teoret. Mat. Fiz.*, 85(3):349–367, 1990.
- [Car99] C. Carathéodory. *Calculus of variations and partial differential equations of the first order*. Translated from the German by Robert B. Dean, Julius J. Brandstatter, translating editor. AMS Chelsea Publishing, 1999.
- [CdLL98] A. Candel and R. de la Llave. On the Aubry-Mather theory in statistical mechanics. *Comm. Math. Phys.*, 192(3):649–669, 1998.
- [CdLL01] Luis A. Caffarelli and Rafael de la Llave. Planelike minimizers in periodic media. *Comm. Pure Appl. Math.*, 54(12):1403–1441, 2001.
- [Dáv13] Gonzalo Dávila. Plane-like minimizers for an area-Dirichlet integral. *Arch. Ration. Mech. Anal.*, 207(3):753–774, 2013.
- [dLL08] Rafael de la Llave. KAM theory for equilibrium states in 1-D statistical mechanics models. *Ann. Henri Poincaré*, 9(5):835–880, 2008.
- [dLLV07a] Rafael de la Llave and Enrico Valdinoci. Critical points inside the gaps of ground state laminations for some models in statistical mechanics. *J. Stat. Phys.*, 129(1):81–119, 2007.
- [dLLV07b] Rafael de la Llave and Enrico Valdinoci. Ground states and critical points for generalized Frenkel-Kontorova models in  $\mathbb{Z}^d$ . *Nonlinearity*, 20(10):2409–2424, 2007.
- [dLLV09] Rafael de la Llave and Enrico Valdinoci. A generalization of Aubry-Mather theory to partial differential equations and pseudo-differential equations. *Ann. Inst. H. Poincaré Anal. Non Linéaire*, 26(4):1309–1344, 2009.
- [dLLV10] Rafael de la Llave and Enrico Valdinoci. Ground states and critical points for Aubry-Mather theory in statistical mechanics. *J. Nonlinear Sci.*, 20(2):153–218, 2010.
- [FK39] J. Frenkel and T. Kontorova. On the theory of plastic deformation and twinning. *Acad. Sci. URSS, J. Physics*, 1:137–149, 1939.
- [KdLLR97] Hans Koch, Rafael de la Llave, and Charles Radin. Aubry-Mather theory for functions on lattices. *Discrete Contin. Dynam. Systems*, 3(1):135–151, 1997.
- [LS03] Pierre-Louis Lions and Panagiotis E. Souganidis. Correctors for the homogenization of Hamilton-Jacobi equations in the stationary ergodic setting. *Comm. Pure Appl. Math.*, 56(10):1501–1524, 2003.
- [Mañ97] Ricardo Mañé. Lagrangian flows: the dynamics of globally minimizing orbits. *Bol. Soc. Brasil. Mat. (N.S.)*, 28(2):141–153, 1997.
- [Mat82] John N. Mather. Existence of quasiperiodic orbits for twist homeomorphisms of the annulus. *Topology*, 21(4):457–467, 1982.
- [Mat91] John N. Mather. Action minimizing invariant measures for positive definite Lagrangian systems. *Math. Z.*, 207(2):169–207, 1991.
- [Mor24] Harold Marston Morse. A fundamental class of geodesics on any closed surface of genus greater than one. *Trans. Amer. Math. Soc.*, 26(1):25–60, 1924.
- [Mos86] Jürgen Moser. Minimal solutions of variational problems on a torus. *Ann. Inst. H. Poincaré Anal. Non Linéaire*, 3(3):229–272, 1986.
- [Mos89] Jürgen Moser. Minimal foliations on a torus. In *Topics in calculus of variations (Montecatini Terme, 1987)*, volume 1365 of *Lecture Notes in Math.*, pages 62–99. Springer, Berlin, 1989.
- [RS11] Paul H. Rabinowitz and Edward W. Stredulinsky. *Extensions of Moser-Bangert theory*. Progress in Nonlinear Differential Equations and their Applications, 81. Birkhäuser/Springer, New York, 2011. Locally minimal solutions.

- [Rue69] David Ruelle. *Statistical mechanics: Rigorous results*. W. A. Benjamin, Inc., New York-Amsterdam, 1969.
- [SdlL12a] Xifeng Su and Rafael de la Llave. KAM theory for quasi-periodic equilibria in 1D quasi-periodic media: II. Long-range interactions. *J. Phys. A*, 45(45):455203, 24, 2012.
- [SdlL12b] Xifeng Su and Rafael de la Llave. KAM Theory for Quasi-periodic Equilibria in One-Dimensional Quasi-periodic Media. *SIAM J. Math. Anal.*, 44(6):3901–3927, 2012.
- [Sel92] W Selke. Spatially modulated structures in systems with competing interactions. In *Phase transitions and critical phenomena, Volume 15*, pages 1–72. Academic Press, 1992.
- [SZdlL15] Xifeng Su, Lei Zhang, and Rafael de la Llave. Resonant equilibrium configurations in quasi-periodic media i: perturbative expansions. 2015. Preprint available at <http://arxiv.org/abs/1503.03304>.
- [Tor04] Monica Torres. Plane-like minimal surfaces in periodic media with exclusions. *SIAM J. Math. Anal.*, 36(2):523–551, 2004.
- [Val04] Enrico Valdinoci. Plane-like minimizers in periodic media: jet flows and Ginzburg-Landau-type functionals. *J. Reine Angew. Math.*, 574:147–185, 2004.
- [ZSdlL15] Lei Zhang, Xifeng Su, and Rafael de la Llave. Equilibrium quasi-periodic configurations in quasi-periodic media with resonant frequencies II: KAM theory. 2015. Preprint available at <http://arxiv.org/abs/1503.03311>.

SCHOOL OF MATHEMATICAL SCIENCES, BEIJING NORMAL UNIVERSITY, No. 19, XINJIEKOUWAI ST., HAI DIAN DISTRICT, BEIJING 100875, P. R. CHINA  
*E-mail address:* xfsu@bnu.edu.cn

SCHOOL OF MATHEMATICS, GEORGIA INSTITUTE OF TECHNOLOGY, 686 CHERRY ST., ATLANTA GA 30332, USA

JLU-GT JOINT INSTITUTE FOR THEORETICAL SCIENCE, JILIN UNIVERSITY, CHANGCHUN, 130012, CHINA  
*E-mail address:* rafael.delallave@math.gatech.edu