

A one-dimensional symmetry result for solutions of an integral equation of convolution type*

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Abstract

We consider an integral equation in the plane, in which the leading operator is of convolution type, and we prove that monotone (or stable) solutions are necessarily one-dimensional.

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Introduction

In this paper, we consider solutions of an integral equation driven by the following nonlocal, linear operator of convolution type:

$$\mathcal{L}u(x) := \int_{\mathbb{R}^n} (u(x) - u(y)) k(x - y) dy. \quad (1)$$

Here we suppose¹ that k is an even, measurable kernel with normalization

$$\int_{\mathbb{R}^n} k(\zeta) d\zeta = 1$$

and such that

$$m_o \chi_{B_{r_o}}(\zeta) \leq k(\zeta) \leq M_o \chi_{B_{R_o}}(\zeta) \quad (2)$$

for any $\zeta \in \mathbb{R}^n$, for some fixed $M_o \geq m_o > 0$ and $R_o \geq r_o > 0$.

We consider here solutions u of the semilinear equation

$$\mathcal{L}u(x) = f(u(x)). \quad (3)$$

In the past few years, there has been an intense activity in this type of equations, both for its mathematical interest and for its relation with biological models, see, among the others [17, 18, 20, 21]. In this case, the solution u is thought as the density of a biological species and the nonlinearity f is often a logistic map, which prescribes the birth and death rate of the population. In this framework, the nonlocal diffusion modeled by \mathcal{L} is motivated by the long-range interactions between the individuals of the species.

The goal of this paper is to study the symmetry properties of solutions of (3) in the light of a famous conjecture of De Giorgi arising in elliptic partial differential equations, see [12]. The original problem consisted in the following question:

¹For the sake of completeness, we point out that assumptions more general than (2) may be taken into account with the same methods as the ones used in this paper. For instance, one could follow assumptions (H1)–(H4) in [11] with $g \geq \alpha$ for some $\alpha > 0$. We focus on the simpler case of assumption (2) for simplicity.

Conjecture 1. *Let u be a bounded solution of*

$$-\Delta u = u - u^3$$

in the whole of \mathbb{R}^n , with

$$\partial_{x_n} u(x) > 0 \text{ for any } x \in \mathbb{R}^n.$$

Then, u is necessarily one-dimensional, i.e. there exist $u_\star : \mathbb{R} \rightarrow \mathbb{R}$ and $\omega \in \mathbb{R}^n$ such that $u(x) = u_\star(\omega \cdot x)$, for any $x \in \mathbb{R}^n$, at least when $n \leq 8$.

The literature has presented several variations of Conjecture 1: in particular, a weak form of it has been investigated when the additional assumption

$$\lim_{x_n \rightarrow \pm\infty} u(x_1, \dots, x_n) = \pm 1 \tag{4}$$

is added to the hypotheses.

When the limit in (4) is uniform in the variables $(x_1, \dots, x_{n-1}) \in \mathbb{R}^{n-1}$, the version of Conjecture 1 obtained in this way is due to Gibbons and is related to problems in cosmology.

In spite of the intense activity of the problem, Conjecture 1 is still open in its generality. Up to now, Conjecture 1 is known to have a positive answer in dimension 2 and 3 (see [19, 2] and also [5, 1]) and a negative answer in dimension 9 and higher (see [14]).

Also, the weak form of Conjecture 1 under the limit assumption in (4) was proved (up to the optimal dimension 8) in [22], and the version of Conjecture 1 under a uniform limit assumption in (4) holds true in any dimension (see [15, 3, 6]).

Since it is almost impossible to keep track in this short introduction of all the research developed on this important topic, we refer to [16] for further details and motivations.

Goal of this paper is to investigate whether results in the spirit of Conjecture 1 hold true when the Laplace operator is replaced by the nonlocal, integral operator in (1). We remark that symmetry results in nonlocal settings have been obtained in [7, 23, 13, 10, 8, 9], but all these works dealt with fractional operators with a regularizing effect. Namely, the integral kernel considered there is not integrable, therefore the solutions of the associated equation enjoy additional regularity and rigidity properties. Also, some of the problems considered in the previous works rely on an extension property of the operator that bring the problem into a local (though higher dimensional and either singular or degenerate) problem.

In this sense, as far as we know, this paper is the first one to take into account integrable kernels, for which the above regularization techniques do not hold and for which equivalent local problems are not available.

In this note, we prove the following one-dimensional result in dimension 2:

Theorem 2. *Let u be a solution of (3) in the whole of \mathbb{R}^2 , with $\|u\|_{C^1(\mathbb{R}^2)} < +\infty$ and $f \in C^1(\mathbb{R})$. Assume that*

$$\partial_{x_2} u(x) > 0 \text{ for any } x \in \mathbb{R}^2. \tag{5}$$

Then, u is necessarily one-dimensional.

The proof of Theorem 2 relies on a technique introduced by [5] and refined in [2], which reduced the symmetry property to a Liouville type property for an associated equation (of course, differently from the classical case, we will have to deal with equations, and in fact inequalities, of integral type, in which the appropriate simplifications are more involved).

For the existence of one-dimensional solutions of (3) under quite general conditions, see Theorem 3.1(b) in [4].

The rest of the paper is devoted to the proof of Theorem 2.

Proof of Theorem 2. We observe that

$$\begin{aligned}
2 \int_{\mathbb{R}^2} \mathcal{L}f(x) g(x) dx &= 2 \int_{\mathbb{R}^2} \left[\int_{\mathbb{R}^2} (f(x) - f(y)) k(x - y) dy \right] g(x) dx \\
&= \int_{\mathbb{R}^2} \left[\int_{\mathbb{R}^2} (f(x) - f(y)) k(x - y) dy \right] g(x) dx \\
&\quad + \int_{\mathbb{R}^2} \left[\int_{\mathbb{R}^2} (f(y) - f(x)) k(x - y) dx \right] g(y) dy \\
&= \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} (f(x) - f(y)) (g(x) - g(y)) k(x - y) dx dy.
\end{aligned} \tag{6}$$

Now we let $u_i := \partial_{x_i} u$, for $i \in \{1, 2\}$. In light of (5), we can define

$$v(x) := \frac{u_1(x)}{u_2(x)}. \tag{7}$$

Also, fixed $R > 1$ (to be taken as large as we wish in the sequel), we consider a cut-off function $\tau := \tau_R \in C_0^\infty(B_{2R})$, such that $\tau = 1$ in B_R and $|\nabla \tau| \leq CR^{-1}$, for some $C > 0$.

By (3), we have that

$$\begin{aligned}
f'(u(x)) u_i(x) &= \partial_{x_i} (f(u(x))) \\
&= \partial_{x_i} (\mathcal{L}u(x)) = \partial_{x_i} \left(\int_{\mathbb{R}^2} (u(x) - u(x - \zeta)) k(\zeta) d\zeta \right) \\
&= \int_{\mathbb{R}^2} (u_i(x) - u_i(x - \zeta)) k(\zeta) d\zeta = \int_{\mathbb{R}^2} (u_i(x) - u_i(y)) k(x - y) dy \\
&= \mathcal{L}u_i(x).
\end{aligned} \tag{8}$$

Accordingly,

$$\begin{aligned}
f'(u) u_1 u_2 &= \mathcal{L}u_1 u_2 \\
\text{and} \quad f'(u) u_1 u_2 &= \mathcal{L}u_2 u_1.
\end{aligned}$$

By subtracting these two identities and using (7), we obtain

$$0 = \mathcal{L}u_1 u_2 - \mathcal{L}u_2 u_1 = \mathcal{L}(vu_2) u_2 - \mathcal{L}u_2 (vu_2).$$

Now, we multiply by $2\tau^2v$ and we integrate. Hence, recalling (6), we conclude that

$$\begin{aligned}
0 &= 2 \int_{\mathbb{R}^2} \mathcal{L}(vu_2)(x) (\tau^2vu_2)(x) dx - 2 \int_{\mathbb{R}^2} \mathcal{L}u_2(x) (\tau^2v^2u_2)(x) dx \\
&= \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} (vu_2(x) - vu_2(y)) (\tau^2vu_2(x) - \tau^2vu_2(y)) k(x-y) dx dy \\
&\quad - \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} (u_2(x) - u_2(y)) (\tau^2v^2u_2(x) - \tau^2v^2u_2(y)) k(x-y) dx dy \\
&=: I_1 - I_2.
\end{aligned} \tag{9}$$

By writing

$$vu_2(x) - vu_2(y) = (u_2(x) - u_2(y)) v(x) + (v(x) - v(y)) u_2(y),$$

we see that

$$\begin{aligned}
I_1 &= \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} (u_2(x) - u_2(y)) (\tau^2vu_2(x) - \tau^2vu_2(y)) v(x) k(x-y) dx dy \\
&\quad + \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} (v(x) - v(y)) (\tau^2vu_2(x) - \tau^2vu_2(y)) u_2(y) k(x-y) dx dy.
\end{aligned} \tag{10}$$

In the same way, if we write

$$\tau^2v^2u_2(x) - \tau^2v^2u_2(y) = (\tau^2vu_2(x) - \tau^2vu_2(y)) v(x) + (v(x) - v(y)) \tau^2vu_2(y),$$

we get that

$$\begin{aligned}
I_2 &= \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} (u_2(x) - u_2(y)) (\tau^2vu_2(x) - \tau^2vu_2(y)) v(x) k(x-y) dx dy \\
&\quad + \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} (u_2(x) - u_2(y)) (v(x) - v(y)) \tau^2vu_2(y) k(x-y) dx dy.
\end{aligned} \tag{11}$$

By (10) and (11), after a simplification we obtain that

$$\begin{aligned}
I_1 - I_2 &= \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} (v(x) - v(y)) (\tau^2vu_2(x) - \tau^2vu_2(y)) u_2(y) k(x-y) dx dy \\
&\quad - \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} (u_2(x) - u_2(y)) (v(x) - v(y)) \tau^2vu_2(y) k(x-y) dx dy.
\end{aligned}$$

Now we notice that

$$\begin{aligned}
&\tau^2vu_2(x) - \tau^2vu_2(y) \\
&= (v(x) - v(y)) \tau^2(x) u_2(x) + (\tau^2(x) - \tau^2(y)) u_2(x) v(y) + (u_2(x) - u_2(y)) \tau^2(y) v(y),
\end{aligned}$$

and so

$$\begin{aligned}
I_1 - I_2 &= \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} (v(x) - v(y))^2 \tau^2(x) u_2(x) u_2(y) k(x-y) dx dy \\
&\quad + \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} (v(x) - v(y)) (\tau^2(x) - \tau^2(y)) v(y) u_2(x) u_2(y) k(x-y) dx dy.
\end{aligned}$$

Thus, using this and (9), and recalling (2) and the support properties of τ , we deduce that

$$\begin{aligned}
J_1 &:= \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} (v(x) - v(y))^2 \tau^2(x) u_2(x) u_2(y) k(x - y) dx dy \\
&\leq \iint_{\mathcal{R}_R} |v(x) - v(y)| |\tau(x) - \tau(y)| |\tau(x) + \tau(y)| |v(y)| u_2(x) u_2(y) k(x - y) dx dy \quad (12) \\
&=: J_2,
\end{aligned}$$

where

$$\begin{aligned}
\mathcal{R}_R &:= \{(x, y) \in \mathbb{R}^2 \times \mathbb{R}^2 \text{ s.t. } |x - y| \leq R_0\} \cap \mathcal{S}_R \\
\text{and } \mathcal{S}_R &:= \left((B_{2R} \times B_{2R}) \setminus (B_R \times B_R) \right) \cup \left(B_{2R} \times (\mathbb{R}^2 \setminus B_{2R}) \right) \cup \left((\mathbb{R}^2 \setminus B_{2R}) \times B_{2R} \right).
\end{aligned}$$

We use the symmetry in the (x, y) variables and the substitution $\zeta := x - y$ to see that

$$\begin{aligned}
|\mathcal{R}_R| &\leq |\{|x - y| \leq R_0\} \cap \{|x|, |y| \leq 2R\}| + 2 |\{|x - y| \leq R_0\} \cap \{|x| \leq 2R \leq |y|\}| \\
&\leq 3 \int_{B_{R_0}} \left[\int_{B_{2R}} dx \right] d\zeta \quad (13) \\
&\leq CR^2,
\end{aligned}$$

for some $C > 0$, possibly depending on R_0 .

Moreover, making use of the Hölder Inequality, we see that

$$\begin{aligned}
J_2^2 &\leq \iint_{\mathcal{R}_R} (v(x) - v(y))^2 (\tau(x) + \tau(y))^2 u_2(x) u_2(y) k(x - y) dx dy \\
&\quad \cdot \iint_{\mathcal{R}_R} (\tau(x) - \tau(y))^2 v^2(y) u_2(x) u_2(y) k(x - y) dx dy. \quad (14)
\end{aligned}$$

Now we claim that

$$u_2(x) \leq C u_2(y) \quad (15)$$

for any $(x, y) \in \mathcal{R}_R$, for a suitable $C > 0$, possibly depending on R_0 . For this, fix x and let $\Omega := B_{R_0}(x)$. Then we use the Harnack Inequality for integral equations (recall (2), (5) and (8), and see Corollary 1.7 in [11]), to obtain that

$$u_2(x) \leq \sup_{\Omega} u_2 \leq C \inf_{\Omega} u_2 \leq C u_2(y),$$

which establishes (15).

From (7) and (15), we obtain that

$$(\tau(x) - \tau(y))^2 v^2(y) u_2(x) u_2(y) \leq CR^{-2} v^2(y) u_2^2(y) = CR^{-2} u_1^2(y) \leq CR^{-2},$$

for some $C > 0$. Hence, by (13),

$$\iint_{\mathcal{R}_R} (\tau(x) - \tau(y))^2 v^2(y) u_2(x) u_2(y) k(x - y) dx dy \leq C,$$

for some $C > 0$. Therefore, recalling (14),

$$J_2^2 \leq C \iint_{\mathcal{R}_R} (v(x) - v(y))^2 (\tau(x) + \tau(y))^2 u_2(x) u_2(y) k(x - y) dx dy. \quad (16)$$

Hence, since

$$(\tau(x) + \tau(y))^2 = \tau^2(x) + \tau^2(y) + 2\tau(x)\tau(y) \leq 3\tau^2(x) + 3\tau^2(y),$$

we can use the symmetric role played by x and y in (16) and obtain that

$$J_2^2 \leq C \iint_{\mathcal{R}_R} (v(x) - v(y))^2 \tau^2(x) u_2(x) u_2(y) k(x - y) dx dy,$$

up to renaming $C > 0$. So, we insert this information into (12) and we conclude that

$$\begin{aligned} \left[\iint_{\mathbb{R}^2 \times \mathbb{R}^2} (v(x) - v(y))^2 \tau^2(x) u_2(x) u_2(y) k(x - y) dx dy \right]^2 &= J_1^2 \\ &\leq J_2^2 \leq C \iint_{\mathcal{R}_R} (v(x) - v(y))^2 \tau^2(x) u_2(x) u_2(y) k(x - y) dx dy, \end{aligned} \quad (17)$$

for some $C > 0$.

Since clearly $\mathcal{R}_R \subseteq \mathbb{R}^2 \times \mathbb{R}^2$, we can simplify the estimate in (17) by writing

$$\iint_{\mathbb{R}^2 \times \mathbb{R}^2} (v(x) - v(y))^2 \tau^2(x) u_2(x) u_2(y) k(x - y) dx dy \leq C.$$

In particular, since $\tau = 1$ in B_R ,

$$\iint_{B_R \times B_R} (v(x) - v(y))^2 u_2(x) u_2(y) k(x - y) dx dy \leq C.$$

Since C is independent of R , we can send $R \rightarrow +\infty$ in this estimate and obtain that the map

$$\mathbb{R}^2 \times \mathbb{R}^2 \ni (x, y) \mapsto (v(x) - v(y))^2 u_2(x) u_2(y) k(x - y)$$

belongs to $L^1(\mathbb{R}^2 \times \mathbb{R}^2)$.

Using this and the fact that \mathcal{R}_R approaches the empty set as $R \rightarrow +\infty$, we conclude that

$$\lim_{R \rightarrow +\infty} \iint_{\mathcal{R}_R} (v(x) - v(y))^2 u_2(x) u_2(y) k(x - y) dx dy = 0.$$

Therefore, going back to (17),

$$\begin{aligned} &\left[\iint_{\mathbb{R}^2 \times \mathbb{R}^2} (v(x) - v(y))^2 u_2(x) u_2(y) k(x - y) dx dy \right]^2 \\ &\leq \lim_{R \rightarrow +\infty} \left[\iint_{\mathbb{R}^2 \times \mathbb{R}^2} (v(x) - v(y))^2 \tau^2(x) u_2(x) u_2(y) k(x - y) dx dy \right]^2 \\ &\leq \lim_{R \rightarrow +\infty} C \iint_{\mathcal{R}_R} (v(x) - v(y))^2 \tau^2(x) u_2(x) u_2(y) k(x - y) dx dy \\ &= 0. \end{aligned}$$

This and (5) imply that $(v(x) - v(y))^2 k(x - y) = 0$ for any $(x, y) \in \mathbb{R}^2 \times \mathbb{R}^2$. Hence, recalling (2), we have that $v(x) = v(y)$ for any $x \in \mathbb{R}^2$ and any $y \in B_{r_0}(x)$.

As a consequence, the set $\{y \in \mathbb{R}^2 \text{ s.t. } v(y) = v(0)\}$ is open and closed in \mathbb{R}^2 , and so, by connectedness, we obtain that v is constant, say $v(x) = a$ for some $a \in \mathbb{R}$. So we define $\omega := \frac{(a, 1)}{\sqrt{a^2 + 1}}$ and we observe that

$$\nabla u(x) = u_2(x) (v(x), 1) = u_2(x) \sqrt{a^2 + 1} \omega.$$

Thus, if $\omega \cdot y = 0$ then

$$u(x + y) - u(x) = \int_0^1 \nabla u(x + ty) \cdot y dt = \int_0^1 u_2(x + ty) \sqrt{a^2 + 1} \omega \cdot y dt = 0.$$

Therefore, if we set $u_\star(t) := u(t\omega)$ for any $t \in \mathbb{R}$, and we write any $x \in \mathbb{R}^2$ as

$$x = (\omega \cdot x) \omega + y_x$$

with $\omega \cdot y_x = 0$, we conclude that

$$u(x) = u((\omega \cdot x) \omega + y_x) = u((\omega \cdot x) \omega) = u_\star(\omega \cdot x).$$

This completes the proof of Theorem 2. □

For completeness, we observe that a more general version of Theorem 2 holds true, namely if we replace assumption (5) with a “stability assumption” in the sense of [2]: the precise statement goes as follows:

Theorem 3. *Let u be a solution of (3) in the whole of \mathbb{R}^2 , with $\|u\|_{C^1(\mathbb{R}^2)} < +\infty$ and $f \in C^1(\mathbb{R})$. Assume that there exists $\psi > 0$ which solves*

$$\mathcal{L}\psi(x) = f'(u(x)) \psi(x) \text{ for any } x \in \mathbb{R}^2.$$

Then, u is necessarily one-dimensional.

Notice that, in this setting, Theorem 2 is a particular case of Theorem 3, choosing $\psi := \partial u_2$ and recalling (8).

The proof of Theorem 3 is like the one of Theorem 2, with only a technical difference: instead of (7), one has to define, for $i \in \{1, 2\}$,

$$v(x) := \frac{u_i(x)}{\psi(x)}.$$

Then the proof of Theorem 2 goes through (replacing u_2 with ψ when necessary) and implies that v is constant, i.e. $u_i = a_i \psi$, for some $a_i \in \mathbb{R}$. This gives that $\nabla u(x) = \psi(x) (a_1, a_2)$, which in turn implies the one-dimensional symmetry of u .

Also, we think that it is an interesting open problem to investigate if symmetry results in the spirit of Theorems 2 and 3 hold true in higher dimension.

References

- [1] G. ALBERTI, L. AMBROSIO AND X. CABRÉ, *On a long-standing conjecture of E. De Giorgi: symmetry in 3D for general nonlinearities and a local minimality property*. Acta Appl. Math. 65, No. 1-3, 9-33 (2001).
- [2] L. AMBROSIO AND X. CABRÉ, *Entire solutions of semilinear elliptic equations in \mathbb{R}^3 and a conjecture of De Giorgi*. J. Am. Math. Soc. 13, No. 4, 725-739 (2000).
- [3] M. T. BARLOW, R. F. BASS AND C. GUI, *The Liouville property and a conjecture of De Giorgi*. Commun. Pure Appl. Math. 53, No. 8, 1007-1038 (2000).
- [4] P. W. BATES, P. C. FIFE, X. REN AND X. WANG, *Traveling waves in a convolution model for phase transitions*. Arch. Ration. Mech. Anal. 138, No. 2, 105-136 (1997).
- [5] H. BERESTYCKI, L. CAFFARELLI AND L. NIRENBERG, *Further qualitative properties for elliptic equations in unbounded domains*. Ann. Sc. Norm. Super. Pisa, Cl. Sci., IV. Ser. 25, No. 1-2, 69-94 (1997).
- [6] H. BERESTYCKI, F. HAMEL AND R. MONNEAU, *One-dimensional symmetry of bounded entire solutions of some elliptic equations*. Duke Math. J. 103, No. 3, 375-396 (2000).
- [7] X. CABRÉ AND J. SOLÀ-MORALES, *Layer solutions in a half-space for boundary reactions*. Commun. Pure Appl. Math. 58, No. 12, 1678-1732 (2005).
- [8] X. CABRÉ AND E. CINTI, *Energy estimates and 1-D symmetry for nonlinear equations involving the half-Laplacian*. Discrete Contin. Dyn. Syst. 28, No. 3, 1179-1206 (2010).
- [9] X. CABRÉ AND E. CINTI, *Sharp energy estimates for nonlinear fractional diffusion equations*. Calc. Var. Partial Differ. Equ. 49, No. 1-2, 233-269 (2014).
- [10] X. CABRÉ AND Y. SIRE, *Nonlinear equations for fractional Laplacians. II: Existence, uniqueness, and qualitative properties of solutions*. Trans. Am. Math. Soc. 367, No. 2, 911-941 (2015).
- [11] J. COVILLE, *Harnack type inequality for positive solution of some integral equation*. Ann. Mat. Pura Appl. (4) 191, No. 3, 503-528 (2012).
- [12] E. DE GIORGI, *Proceedings of the international meeting on recent methods in nonlinear analysis. Rome, May 8-12, 1978*. Collana Atti di Congressi. Bologna: Pitagora Editrice. 669 p. (1979).
- [13] R. DE LA LLAVE AND E. VALDINOCI, *Symmetry for a Dirichlet-Neumann problem arising in water waves*. Math. Res. Lett. 16, No. 5-6, 909-918 (2009).
- [14] M. DEL PINO, M. KOWALCZYK, J. WEI, *On De Giorgi's conjecture in dimension $N \geq 9$* . Ann. Math. (2) 174, No. 3, 1485-1569 (2011).

- [15] A. FARINA, *Symmetry for solutions of semilinear elliptic equations in \mathbb{R}^N and related conjectures*. Atti Accad. Naz. Lincei, Cl. Sci. Fis. Mat. Nat., IX. Ser., Rend. Lincei, Mat. Appl. 10, No. 4, 255-265 (1999).
- [16] A. FARINA AND E. VALDINOCI, *The state of the art for a conjecture of de Giorgi and related problems. Recent progress on reaction-diffusion systems and viscosity solutions. Based on the international conference on reaction-diffusion systems and viscosity solutions, Taichung, Taiwan, January 36, 2007*. Hackensack, NJ: World Scientific (ISBN 978-981-283-473-7/hbk). 74-96 (2009).
- [17] P. C. FIFE, *Mathematical aspects of reacting and diffusing systems*. Lecture Notes in Biomathematics. 28. Berlin-Heidelberg-New York: Springer-Verlag. IV, 185 p. (1979).
- [18] P. C. FIFE, *An integrodifferential analog of semilinear parabolic PDEs*. Partial differential equations and applications. Collected papers in honor of Carlo Pucci on the occasion of his 70th birthday. New York, NY: Marcel Dekker. Lect. Notes Pure Appl. Math. 177, 137-145 (1996).
- [19] N. GHOUSSOUB AND C. GUI, *On a conjecture of De Giorgi and some related problems*. Math. Ann. 311, No. 3, 481-491 (1998).
- [20] V. HUTSON, S. MARTINEZ, K. MISCHAIKOW AND G. T. VICKERS, *The evolution of dispersal*. J. Math. Biol. 47, No. 6, 483-517 (2003).
- [21] J. MEDLOCK AND M. KOT, *Spreading disease: Integro-differential equations old and new*. Math. Biosci. 184, No. 2, 201-222 (2003).
- [22] O. SAVIN, *Regularity of flat level sets in phase transitions*. Ann. Math. (2) 169, No. 1, 41-78 (2009).
- [23] Y. SIRE AND E. VALDINOCI, *Fractional Laplacian phase transitions and boundary reactions: a geometric inequality and a symmetry result*. J. Funct. Anal. 256, No. 6, 1842-1864 (2009).