EXISTENCE OF STATIONARY SOLUTIONS FOR SOME SYSTEMS OF INTEGRO-DIFFERENTIAL EQUATIONS

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Abstract: The article deals with the existence of solutions of a system of nonlocal reaction-diffusion equations which appears in population dynamics. The proof relies on a fixed point technique. Solvability conditions for elliptic operators in unbounded domains which fail to satisfy the Fredholm property are being used.

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1. Introduction

In the present article we establish the existence of stationary solutions of the system of $N \geq 2$ nonlocal reaction-diffusion equations

$$\frac{\partial u_s}{\partial t} = D_s \Delta u_s + \int_{\mathbb{R}^d} K_s(x-y) g_s(u(y,t)) dy + f_s(x), \quad 1 \leq s \leq N, \tag{1.1}$$

which appears in cell population dynamics. The space variable x is correspondent to the cell genotype, $u_s(x,t)$ are densities for the various groups of cells as functions of their genotype and time and $u(x,t) = (u_1(x,t), u_2(x,t), ..., u_N(x,t))^T$. The right side of (1.1) describes the evolution of cell densities caused by cell proliferation, mutations and cell influx. In this context, the diffusion terms are correspondent to the change of genotype by means of small random mutations, while the integral terms describe large mutations. Here $g_s(u)$ are the rates of cell birth dependent upon u (density dependent proliferation), and the functions $K_s(x-y)$ show the proportions of newly born cells changing their genotype from y to x. Let us assume that they depend on the distance between the genotypes. Finally, the last term in the right side of (1.1) describes the influxes of cells for different genotypes. Note that the single equation analogous to (1.1) has been studied recently in [22] and the case of the superdiffusion has been treated in [23].

Let us assume further down that all $D_s = 1$ and will investigate the existence of solutions of the system of equations

$$\Delta u_s + \int_{\mathbb{R}^d} K_s(x-y) g_s(u(y))dy + f_s(x) = 0, \quad 1 \le s \le N. \tag{1.2}$$

Let us consider the situation when the linear part of this operator fails to satisfy the Fredholm property and conventional methods of nonlinear analysis may not be applied. We will use solvability conditions for non Fredholm operators along with the method of contraction mappings.

Let us consider the problem

$$-\Delta u + V(x)u - au = f \tag{1.3}$$

with $u \in E = H^2(\mathbb{R}^d)$ and $f \in F = L^2(\mathbb{R}^d)$, $d \in \mathbb{N}$, a is a constant and the scalar potential function $V(x)$ either vanishes or tends to 0 at infinity. When $a \ge 0$, the essential spectrum of the operator $A : E \to F$ correspondent to the left side of problem (1.3) contains the origin. Consequently, this operator fails to satisfy the Fredholm property. Its image is not closed, for $d > 1$ the dimension of its kernel and the codimension of its image are not finite. The present work is devoted to the studies of some properties of the operators of this kind. Note that elliptic problems which contain non Fredholm operators were treated actively in recent years. Approaches in weighted Sobolev and Hölder spaces were developed in [2], [3], [4], [5], [6]. The Schrödinger type operators without Fredholm property were studied using the methods of the spectral and the scattering theory in [11], [13], [14], [15], [17]. The Laplacian operator with drift from the point of view of the non Fredholm operators was studied in [16] and linearized Cahn-Hilliard equations in [18] and [20]. Nonlinear non Fredholm elliptic problems were treated in [19] and [21]. Significant applications to the theory of reaction-diffusion equations were developed in [8], [9]. Operators without Fredholm property arise also when studying wave systems with an infinite number of localized traveling waves (see [1]). In particular, when $a = 0$ the operator A is Fredholm in some properly chosen weighted spaces (see [2], [3], [4], [5], [6]). But the situation when $a \neq 0$ is significantly different such that the approach developed in these works cannot be applied.

Let us set $K_s(x) = \varepsilon_s \mathcal{K}_s(x)$ with $\varepsilon_s \ge 0$, such that

$$\varepsilon := \max_{1 \le s \le N} \varepsilon_s$$

and suppose that the following assumption holds.

Assumption 1. *Let* $1 \le s \le N$, *such that* $f_s(x) : \mathbb{R}^5 \to \mathbb{R}$, $\quad f_s(x) \in L^1(\mathbb{R}^5)$ *and* $\nabla f_s(x) \in L^2(\mathbb{R}^5)$. *Moreover,* $f_s(x)$ *is nontrivial for some* s. *Assume also that*

$\mathcal{K}_s(x) : \mathbb{R}^5 \to \mathbb{R}$, *such that* $\mathcal{K}_s(x) \in L^1(\mathbb{R}^5)$ *and* $\nabla \mathcal{K}_s(x) \in L^2(\mathbb{R}^5)$. *Furthermore,*

$$\mathcal{K}^2 := \sum_{s=1}^{N} \|\mathcal{K}_s(x)\|^2_{L^1(\mathbb{R}^5)} > 0$$

and

$$Q^2 := \sum_{s=1}^{N} \|\nabla \mathcal{K}_s(x)\|^2_{L^2(\mathbb{R}^5)} > 0.$$

Note that as distinct from the preceding work [22] dealing with a single integro-differential equation, we assume here for the technical reason the square integrability of the gradients of kernels involved in the nonlocal terms of our system of equations.

The way we choose the space dimension is related to the solvability conditions for linear elliptic problems in unbounded domains (see [21]). There are certain solvability conditions for $d < 5$ but solvability conditions are not required for $d \geq 5$ (see the Appendix). Let us study here only the case of $d = 5$. We will not consider the problem in dimensions $d > 5$ to avoid extra technicalities since the proof will rely on similar ideas and no orthogonality conditions for the solvability of equations (1.8) are required analogously to $d = 5$ (see Lemma 7 of [21]). From the perspective of applications, the space dimension is not limited to $d = 3$ because the space variable is correspondent to cell genotype and not the usual physical space.

By virtue of the Sobolev inequality (see e.g. p.183 of [10]) under the assumption above we have

$$f_s(x) \in L^2(\mathbb{R}^5), \quad 1 \leq s \leq N.$$

We consider the Sobolev space $H^3(\mathbb{R}^5, \mathbb{R}^N)$ of vector functions

$$\{u(x) : \mathbb{R}^5 \to \mathbb{R}^N \mid u_s(x) \in L^2(\mathbb{R}^5),\ (-\Delta)^{\frac{3}{2}} u_s \in L^2(\mathbb{R}^5),\ 1 \leq s \leq N\}$$

equipped with the norm

$$\|u\|^2_{H^3(\mathbb{R}^5,\mathbb{R}^N)} := \sum_{s=1}^{N} \|u_s\|^2_{H^3(\mathbb{R}^5)} = \sum_{s=1}^{N} \{\|u_s\|^2_{L^2(\mathbb{R}^5)} + \|(-\Delta)^{\frac{3}{2}} u_s\|^2_{L^2(\mathbb{R}^5)}\}. \quad (1.4)$$

Also,

$$\|u\|^2_{L^2(\mathbb{R}^5,\mathbb{R}^N)} := \sum_{s=1}^{N} \|u_s\|^2_{L^2(\mathbb{R}^5)}.$$

The operator $(-\Delta)^{\frac{3}{2}}$ is defined by virtue of the spectral calculus. By means of the Sobolev embedding we have

$$\|\phi\|_{L^\infty(\mathbb{R}^5)} \leq c_e \|\phi\|_{H^3(\mathbb{R}^5)}. \quad (1.5)$$

Here $c_e > 0$ is the constant of the embedding. The hat symbol will stand for the standard Fourier transform, namely

$$\widehat{\phi}(p) = \frac{1}{(2\pi)^{\frac{5}{2}}} \int_{\mathbb{R}^5} \phi(x) e^{-ipx} dx. \tag{1.6}$$

This enables us to express the Sobolev norm of a function as

$$\|\phi\|^2_{H^3(\mathbb{R}^5)} = \int_{\mathbb{R}^5} (1 + |p|^6) |\widehat{\phi}(p)|^2 dp. \tag{1.7}$$

When the nonnegative parameters ε_s vanish, we arrive at the standard Poisson equations

$$-\Delta u_s = f_s(x), \quad 1 \le s \le N. \tag{1.8}$$

Assumption 1 via Lemma 7 of [21] implies that problem (1.8) admits a unique solution $u_{0,s}(x) \in H^2(\mathbb{R}^5)$ and no orthogonality relations are required. Clearly,

$$\nabla(-\Delta u_s) = \nabla f_s(x) \in L^2(\mathbb{R}^5).$$

Thus, for the unique solution of the linear problem (1.8) we arrive at $u_{0,s}(x) \in H^3(\mathbb{R}^5)$, such that

$$u_0(x) = (u_{0,1}(x), u_{0,2}(x), ..., u_{0,N}(x))^T \in H^3(\mathbb{R}^5, \mathbb{R}^N).$$

We look for the resulting solution of the nonlinear problem (1.2) as

$$u(x) = u_0(x) + u_p(x), \tag{1.9}$$

where

$$u_p(x) = (u_{p,1}(x), u_{p,2}(x), ..., u_{p,N}(x))^T.$$

Evidently, we obtain the perturbative system of equations

$$-\Delta u_{p,s} = \varepsilon_s \int_{\mathbb{R}^5} \mathcal{K}_s(x-y) g_s(u_0(y) + u_p(y)) dy, \quad 1 \le s \le N. \tag{1.10}$$

Let us define a closed ball in the Sobolev space

$$B_\rho := \{u(x) \in H^3(\mathbb{R}^5, \mathbb{R}^N) \mid \|u\|_{H^3(\mathbb{R}^5, \mathbb{R}^N)} \le \rho\}, \quad 0 < \rho \le 1. \tag{1.11}$$

We seek the solution of (1.10) as the fixed point of the auxiliary nonlinear system of equations

$$-\Delta u_s = \varepsilon_s \int_{\mathbb{R}^5} \mathcal{K}_s(x-y) g_s(u_0(y) + v(y)) dy, \quad 1 \le s \le N \tag{1.12}$$

in ball (1.11). For a given vector function $v(y)$ it is an equation with respect to $u(x)$. The left side of (1.12) contains the operator without Fredholm property

$-\Delta : H^2(\mathbb{R}^5) \to L^2(\mathbb{R}^5)$, due to the fact that its essential spectrum fills the nonnegative semi-axis $[0, +\infty)$ and therefore, such operator has no bounded inverse. The analogous situation appeared in [19] and [21] but as distinct from the present work, the equations treated there required orthogonality conditions. The fixed point technique was used in [12] to evaluate the perturbation to the standing solitary wave of the Nonlinear Schrödinger (NLS) equation when either the external potential or the nonlinear term in the NLS were perturbed but the Schrödinger type operator involved in the nonlinear equation possessed the Fredholm property (see Assumption 1 of [12], also [7]). Let us define a closed ball in the space of N dimensions

$$I := \{z \in \mathbb{R}^N \mid |z| \le c_e \|u_0\|_{H^3(\mathbb{R}^5, \mathbb{R}^N)} + c_e\}. \tag{1.13}$$

For technical purposes we will use following quantities with $1 \le s, j \le N$

$$a_{2,s,j} := \sup_{z \in I} \Big| \nabla \frac{\partial g_s}{\partial z_j} \Big|, \quad a_{2,s} := \sqrt{\sum_{j=1}^N a_{2,s,j}^2}, \quad a_2 := \max_{1 \le s \le N} a_{2,s}.$$

Also,

$$a_{1,s} := \sup_{z \in I} |\nabla g_s(z)|, \quad a_1 := \max_{1 \le s \le N} a_{1,s}.$$

We make the following assumption about the nonlinear parts of the system of equations (1.2).

Assumption 2. *Let $1 \le s \le N$, such that $g_s(z) : \mathbb{R}^N \to \mathbb{R}$ with $g_s(z) \in C_2(\mathbb{R}^N)$. We also assume that $g_s(0) = 0, \quad \nabla g_s(0) = 0$ and $a_2 > 0$.*

Evidently, a_1 defined above is positive as well, otherwise all the functions $g_s(z)$ will be constants in the ball I and a_2 will vanish. For instance, $g_s(z) = z^2, \ z \in \mathbb{R}^N$ clearly satisfies our assumption above.

Let us introduce the operator T_g such that $u = T_g v$, where u is a solution of the system of equations (1.12). Our main proposition is as follows.

Theorem 3. *Let Assumptions 1 and 2 hold. Then system (1.12) defines the map $T_g : B_\rho \to B_\rho$, which is a strict contraction for all $0 < \varepsilon < \varepsilon*$ for some $\varepsilon* > 0$. The unique fixed point $u_p(x)$ of this map T_g is the only solution of the system of equations (1.10) in B_ρ.*

Clearly, the resulting solution of system (1.2) given by (1.9) will be nontrivial due to the fact that the source terms $f_s(x)$ are nontrivial for some $s = 1, ..., N$ and all $g_s(z)$ vanish at the origin due to our assumptions. We will make use of the elementary technical lemma below.

Lemma 4. *Let $\varphi(R) := \alpha R + \dfrac{\beta}{R^4}$ with $R \in (0, +\infty)$ and the constants $\alpha, \beta > 0$. It achieves the minimal value at $R^* = \left(\dfrac{4\beta}{\alpha}\right)^{\frac{1}{5}}$, which is given by $\varphi(R^*) = \dfrac{5}{4^{\frac{4}{5}}} \alpha^{\frac{4}{5}} \beta^{\frac{1}{5}}$.*

Let us proceed to the proof of our main result.

2. The existence of the perturbed solution

Proof of Theorem 3. Let us choose arbitrarily $v(x) \in B_\rho$ and denote the terms involved in the integral expressions in right side of system (1.12) as $G_s(x) := g_s(u_0(x)+v(x))$. We apply the standard Fourier transform (1.6) to both sides of the system of equations (1.12). This yields

$$\widehat{u}_s(p) = \varepsilon_s (2\pi)^{\frac{5}{2}} \frac{\widehat{\mathcal{K}}_s(p)\widehat{G}_s(p)}{p^2}, \quad 1 \leq s \leq N.$$

Hence for the norm we obtain

$$\|u_s\|^2_{L^2(\mathbb{R}^5)} = (2\pi)^5 \varepsilon_s^2 \int_{\mathbb{R}^5} \frac{|\widehat{\mathcal{K}}_s(p)|^2 |\widehat{G}_s(p)|^2}{|p|^4} dp. \tag{2.1}$$

Clearly, for any $\phi(x) \in L^1(\mathbb{R}^5)$

$$\|\widehat{\phi}(p)\|_{L^\infty(\mathbb{R}^5)} \leq \frac{1}{(2\pi)^{\frac{5}{2}}} \|\phi(x)\|_{L^1(\mathbb{R}^5)}. \tag{2.2}$$

Note that as distinct from articles [19] and [21] containing results in lower dimensions, in the present work we do not try to control the norms

$$\left\| \frac{\widehat{\mathcal{K}}_s(p)}{p^2} \right\|_{L^\infty(\mathbb{R}^5)}.$$

Let us estimate the right side of (2.1) by virtue of (2.2) with $R \in (0, +\infty)$ as

$$(2\pi)^5 \varepsilon_s^2 \int_{|p|\leq R} \frac{|\widehat{\mathcal{K}}_s(p)|^2 |\widehat{G}_s(p)|^2}{|p|^4} dp + (2\pi)^5 \varepsilon_s^2 \int_{|p|>R} \frac{|\widehat{\mathcal{K}}_s(p)|^2 |\widehat{G}_s(p)|^2}{|p|^4} dp \leq$$

$$\leq \varepsilon_s^2 \|\mathcal{K}_s\|^2_{L^1(\mathbb{R}^5)} \left\{ \frac{1}{(2\pi)^5} \|G_s(x)\|^2_{L^1(\mathbb{R}^5)} |S_5| R + \frac{1}{R^4} \|G_s(x)\|^2_{L^2(\mathbb{R}^5)} \right\}. \tag{2.3}$$

Here and further down S_5 stands for the unit sphere in the space of five dimensions centered at the origin and $|S_5|$ for its Lebesgue measure (see e.g. p.6 of [10]). Since $v(x) \in B_\rho$, we arrive at

$$\|u_0 + v\|_{L^2(\mathbb{R}^5, \mathbb{R}^N)} \leq \|u_0\|_{H^3(\mathbb{R}^5, \mathbb{R}^N)} + 1$$

and by virtue of the Sobolev embedding (1.5)

$$|u_0 + v| \leq c_e \|u_0\|_{H^3(\mathbb{R}^5, \mathbb{R}^N)} + c_e.$$

Let us use the formula

$$G_s(x) = \int_0^1 \nabla g_s(t(u_0(x)+v(x))).(u_0(x)+v(x))dt, \quad 1 \le s \le N.$$

Here and below the dot symbol denotes the scalar product of two vectors in $\mathbb{R}^N$. With the ball I defined in (1.13), we easily obtain

$$|G_s(x)| \le \sup_{z\in I}|\nabla g_s(z)||u_0(x)+v(x)| \le a_1|u_0(x)+v(x)|.$$

Thus

$$\|G_s(x)\|_{L^2(\mathbb{R}^5)} \le a_1\|u_0+v\|_{L^2(\mathbb{R}^5,\mathbb{R}^N)} \le a_1(\|u_0\|_{H^3(\mathbb{R}^5,\mathbb{R}^N)}+1).$$

Clearly, for $t \in [0,1]$ and $1 \le j \le N$

$$\frac{\partial g_s}{\partial z_j}(t(u_0(x)+v(x))) = \int_0^t \nabla \frac{\partial g_s}{\partial z_j}(\tau(u_0(x)+v(x))).(u_0(x)+v(x))d\tau.$$

This yields

$$\Big|\frac{\partial g_s}{\partial z_j}(t(u_0(x)+v(x)))\Big| \le \sup_{z\in I}\Big|\nabla\frac{\partial g_s}{\partial z_j}\Big||u_0(x)+v(x)| = a_{2,s,j}|u_0(x)+v(x)|,$$

such that by means of the Schwarz inequality

$$|G_s(x)| \le |u_0(x)+v(x)|\sum_{j=1}^N a_{2,s,j}|u_{0,j}(x)+v_j(x)| \le a_2|u_0(x)+v(x)|^2$$

and

$$\|G_s(x)\|_{L^1(\mathbb{R}^5)} \le a_2\|u_0+v\|^2_{L^2(\mathbb{R}^5,\mathbb{R}^N)} \le a_2(\|u_0\|_{H^3(\mathbb{R}^5,\mathbb{R}^N)}+1)^2. \tag{2.4}$$

This enables us to derive the upper bound for the right side of (2.3) as

$$\varepsilon^2\|\mathcal{K}_s(x)\|^2_{L^1(\mathbb{R}^5)}(\|u_0\|_{H^3(\mathbb{R}^5,\mathbb{R}^N)}+1)^2\Big\{\frac{a_2^2}{(2\pi)^5}(\|u_0\|_{H^3(\mathbb{R}^5,\mathbb{R}^N)}+1)^2|S_5|R+\frac{a_1^2}{R^4}\Big\}$$

with $R \in (0,+\infty)$. By virtue of Lemma 4 we derive the minimal value of the expression above. Therefore,

$$\|u\|^2_{L^2(\mathbb{R}^5,\mathbb{R}^N)} \le \varepsilon^2\mathcal{K}^2\frac{|S_5|^{\frac{4}{5}}}{(2\pi)^4}a_2^{\frac{8}{5}}(\|u_0\|_{H^3(\mathbb{R}^5,\mathbb{R}^N)}+1)^{3\frac{3}{5}}a_1^{\frac{2}{5}}\frac{5}{4^{\frac{4}{5}}}. \tag{2.5}$$

Clearly, (1.12) implies

$$\nabla(-\Delta u_s) = \varepsilon_s\nabla\int_{\mathbb{R}^5}\mathcal{K}_s(x-y)G_s(y)dy, \quad 1 \le s \le N,$$

such that by means of (2.2) along with (2.4)

$$\|\nabla(-\Delta u_s)\|^2_{L^2(\mathbb{R}^5)} \le \varepsilon_s^2 \|G_s(x)\|^2_{L^1(\mathbb{R}^5)} \|\nabla \mathcal{K}_s(x)\|^2_{L^2(\mathbb{R}^5)} \le$$

$$\le \varepsilon^2 a_2^2 (\|u_0\|_{H^3(\mathbb{R}^5,\mathbb{R}^N)}+1)^4 \|\nabla \mathcal{K}_s(x)\|^2_{L^2(\mathbb{R}^5)}.$$

Hence

$$\sum_{s=1}^N \|(-\Delta)^{\frac{3}{2}} u_s\|^2_{L^2(\mathbb{R}^5)} \le \varepsilon^2 a_2^2 (\|u_0\|_{H^3(\mathbb{R}^5,\mathbb{R}^N)}+1)^4 Q^2. \tag{2.6}$$

By virtue of the definition of the norm (1.4) along with upper bounds (2.5) and (2.6) we arrive at

$$\|u\|_{H^3(\mathbb{R}^5,\mathbb{R}^N)} \le \varepsilon(\|u_0\|_{H^3(\mathbb{R}^5,\mathbb{R}^N)}+1)^2 a_2^{\frac{4}{5}} \sqrt{\frac{|S_5|^{\frac{4}{5}}}{(2\pi)^4} a_1^{\frac{2}{5}} \frac{5\mathcal{K}^2}{4^{\frac{4}{5}}} + a_2^{\frac{2}{5}} Q^2} \le \rho$$

for all values of $\varepsilon > 0$ small enough, such that $u(x) \in B_\rho$ as well. If for a certain $v(x) \in B_\rho$ there exist two solutions $u_{1,2}(x) \in B_\rho$ of system (1.12), each component $u_s(x)$ of their difference $u(x) := u_1(x) - u_2(x) \in L^2(\mathbb{R}^5, \mathbb{R}^N)$ satisfies Laplace's equation. Since there are no nontrivial square integrable harmonic functions, $u(x) = 0$ a.e. in $\mathbb{R}^5$. Thus, system (1.12) defines a map $T_g : B_\rho \to B_\rho$ for all $\varepsilon > 0$ sufficiently small.

Thus our goal is to prove that this map is a strict contraction. We choose arbitrarily $v_{1,2}(x) \in B_\rho$. Via the argument above $u_{1,2} = T_g v_{1,2} \in B_\rho$ as well. System (1.12) gives us

$$-\Delta u_{1,s} = \varepsilon_s \int_{\mathbb{R}^5} \mathcal{K}_s(x-y) g_s(u_0(y)+v_1(y))dy, \quad 1 \le s \le N, \tag{2.7}$$

$$-\Delta u_{2,s} = \varepsilon_s \int_{\mathbb{R}^5} \mathcal{K}_s(x-y) g_s(u_0(y)+v_2(y))dy, \quad 1 \le s \le N. \tag{2.8}$$

Let us define

$$G_{1,s}(x) := g_s(u_0(x)+v_1(x)), \quad G_{2,s}(x) := g_s(u_0(x)+v_2(x)).$$

We apply the standard Fourier transform (1.6) to both sides of systems (2.7) and (2.8), which yields

$$\widehat{u_{1,s}}(p) = \varepsilon_s (2\pi)^{\frac{5}{2}} \frac{\widehat{\mathcal{K}}_s(p)\widehat{G_{1,s}}(p)}{p^2}, \quad \widehat{u_{2,s}}(p) = \varepsilon_s (2\pi)^{\frac{5}{2}} \frac{\widehat{\mathcal{K}}_s(p)\widehat{G_{2,s}}(p)}{p^2}$$

and express the norm

$$\|u_{1,s} - u_{2,s}\|^2_{L^2(\mathbb{R}^5)} = \varepsilon_s^2 (2\pi)^5 \int_{\mathbb{R}^5} \frac{|\widehat{\mathcal{K}}_s(p)|^2 |\widehat{G_{1,s}}(p) - \widehat{G_{2,s}}(p)|^2}{|p|^4} dp.$$

Evidently, it can be estimated from above using (2.2) by

$$\varepsilon^2\|\mathcal{K}_s(x)\|^2_{L^1(\mathbb{R}^5)}\Big\{\frac{|S_5|}{(2\pi)^5}\|G_{1,s}(x)-G_{2,s}(x)\|^2_{L^1(\mathbb{R}^5)}R+\frac{\|G_{1,s}(x)-G_{2,s}(x)\|^2_{L^2(\mathbb{R}^5)}}{R^4}\Big\}$$

with $R\in(0,+\infty)$. For $1\le s\le N$, let us make use of the formula

$$G_{1,s}(x)-G_{2,s}(x)=\int_0^1\nabla g_s(u_0(x)+tv_1(x)+(1-t)v_2(x)).(v_1(x)-v_2(x))dt.$$

Obviously, for $v_{1,2}(x)\in B_\rho$ and $t\in[0,1]$ we have

$$\|v_2(x)+t(v_1(x)-v_2(x))\|_{H^3(\mathbb{R}^5,\mathbb{R}^N)}\le t\|v_1(x)\|_{H^3(\mathbb{R}^5,\mathbb{R}^N)}+$$

$$+(1-t)\|v_2(x)\|_{H^3(\mathbb{R}^5,\mathbb{R}^N)}\le\rho,$$

such that $v_2(x)+t(v_1(x)-v_2(x))\in B_\rho$ as well. We obtain

$$|G_{1,s}(x)-G_{2,s}(x)|\le\sup_{z\in I}|\nabla g_s(z)||v_1(x)-v_2(x)|=a_{1,s}|v_1(x)-v_2(x)|,$$

such that

$$\|G_{1,s}(x)-G_{2,s}(x)\|_{L^2(\mathbb{R}^5)}\le a_{1,s}\|v_1(x)-v_2(x)\|_{H^3(\mathbb{R}^5,\mathbb{R}^N)}.$$

Apparently, for $1\le j\le N$

$$\frac{\partial g_s}{\partial z_j}(u_0(x)+tv_1(x)+(1-t)v_2(x))=\int_0^1\nabla\frac{\partial g_s}{\partial z_j}(\tau[u_0(x)+tv_1(x)+(1-t)v_2(x)]).$$

$$.(u_0(x)+tv_1(x)+(1-t)v_2(x))d\tau,$$

such that

$$\Big|\frac{\partial g_s}{\partial z_j}(u_0(x)+tv_1(x)+(1-t)v_2(x))\Big|\le$$

$$\le\sup_{z\in I}\Big|\nabla\frac{\partial g_s}{\partial z_j}\Big|\{|u_0(x)|+t|v_1(x)|+(1-t)|v_2(x)|\}$$

with $t\in[0,1]$. Therefore, by means of the Schwarz inequality

$$|G_{1,s}(x)-G_{2,s}(x)|\le\sum_{j=1}^N a_{2,s,j}\{|u_0(x)|+\frac{1}{2}|v_1(x)|+\frac{1}{2}|v_2(x)|\}|v_{1,j}(x)-v_{2,j}(x)|\le$$

$$\le a_{2,s}\{|u_0(x)|+\frac{1}{2}|v_1(x)|+\frac{1}{2}|v_2(x)|\}|v_1(x)-v_2(x)|.$$

The Schwarz inequality yields the upper bound for $\|G_{1,s}(x)-G_{2,s}(x)\|_{L^1(\mathbb{R}^5)}$ as

$$a_{2,s}\{\|u_0(x)\|_{L^2(\mathbb{R}^5,\mathbb{R}^N)}+\frac{1}{2}\|v_1(x)\|_{L^2(\mathbb{R}^5,\mathbb{R}^N)}+\frac{1}{2}\|v_2(x)\|_{L^2(\mathbb{R}^5,\mathbb{R}^N)}\}\times$$

$$\times\|v_1(x)-v_2(x)\|_{L^2(\mathbb{R}^5,\mathbb{R}^N)} \le a_2\{\|u_0(x)\|_{H^3(\mathbb{R}^5,\mathbb{R}^N)}+1\}\|v_1(x)-v_2(x)\|_{H^3(\mathbb{R}^5,\mathbb{R}^N)}.$$

This enables us to estimate from above $\|u_1(x)-u_2(x)\|^2_{L^2(\mathbb{R}^5,\mathbb{R}^N)}$ by

$$\varepsilon^2\mathcal{K}^2\|v_1(x)-v_2(x)\|^2_{H^3(\mathbb{R}^5,\mathbb{R}^N)}\Big\{\frac{{a_2}^2}{(2\pi)^5}(\|u_0\|_{H^3(\mathbb{R}^5,\mathbb{R}^N)}+1)^2|S_5|R+\frac{{a_1}^2}{R^4}\Big\}.$$

We use Lemma 4 to minimize the expression above over $R\in(0,+\infty)$ to prove that $\|u_1(x)-u_2(x)\|^2_{L^2(\mathbb{R}^5,\mathbb{R}^N)}$ has an upper bound given by

$$\varepsilon^2\mathcal{K}^2\|v_1-v_2\|^2_{H^3(\mathbb{R}^5,\mathbb{R}^N)}\frac{5}{4^{\frac{4}{5}}}\frac{a_2^{\frac{8}{5}}}{(2\pi)^4}(\|u_0\|_{H^3(\mathbb{R}^5,\mathbb{R}^N)}+1)^2|S_5|^{\frac{4}{5}}a_1^{\frac{2}{5}}. \tag{2.9}$$

By means of (2.7) and (2.8)

$$\nabla(-\Delta)(u_{1,s}(x)-u_{2,s}(x))=\varepsilon_s\nabla\int_{\mathbb{R}^5}\mathcal{K}_s(x-y)[G_{1,s}(y)-G_{2,s}(y)]dy.$$

Hence via (2.2)

$$\|\nabla(-\Delta)(u_{1,s}(x)-u_{2,s}(x))\|^2_{L^2(\mathbb{R}^5)} \le \varepsilon^2\|G_{1,s}(x)-G_{2,s}(x)\|^2_{L^1(\mathbb{R}^5)}\|\nabla\mathcal{K}_s(x)\|^2_{L^2(\mathbb{R}^5)}.$$

As a results, the norm $\|(-\Delta)^{\frac{3}{2}}(u_{1,s}(x)-u_{2,s}(x))\|^2_{L^2(\mathbb{R}^5)}$ is bounded above by

$$\varepsilon^2a_2^2(\|u_0\|_{H^3(\mathbb{R}^5,\mathbb{R}^N)}+1)^2\|v_1(x)-v_2(x)\|^2_{H^3(\mathbb{R}^5,\mathbb{R}^N)}\|\nabla\mathcal{K}_s(x)\|^2_{L^2(\mathbb{R}^5)},$$

such that

$$\sum_{s=1}^N\|(-\Delta)^{\frac{3}{2}}(u_{1,s}(x)-u_{2,s}(x))\|^2_{L^2(\mathbb{R}^5)} \le$$

$$\le \varepsilon^2a_2^2(\|u_0\|_{H^3(\mathbb{R}^5,\mathbb{R}^N)}+1)^2Q^2\|v_1(x)-v_2(x)\|^2_{H^3(\mathbb{R}^5,\mathbb{R}^N)}. \tag{2.10}$$

Inequalities (2.9) and (2.10) imply that the norm $\|u_1-u_2\|_{H^3(\mathbb{R}^5,\mathbb{R}^N)}$ has an upper bound given by

$$\varepsilon(\|u_0\|_{H^3(\mathbb{R}^5,\mathbb{R}^N)}+1)a_2^{\frac{4}{5}}\Big[\frac{5\mathcal{K}^2}{4^{\frac{4}{5}}}\frac{a_1^{\frac{2}{5}}}{(2\pi)^4}|S_5|^{\frac{4}{5}}+Q^2a_2^{\frac{2}{5}}\Big]^{\frac{1}{2}}\|v_1-v_2\|_{H^3(\mathbb{R}^5,\mathbb{R}^N)}.$$

Therefore, the map $T_g: B_\rho\to B_\rho$ defined by system (1.12) is a strict contraction for all values of $\varepsilon>0$ sufficiently small. Its unique fixed point $u_p(x)$ is the only solution of system (1.10) in B_ρ and the resulting $u(x)\in H^3(\mathbb{R}^5,\mathbb{R}^N)$ given by (1.9) is a solution of the system of equations (1.2). ■

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3. Appendix

In the present article we used solvability conditions for linear elliptic problems in $\mathbb{R}^d$ derived in [21]. Let us state them below for the convenience of the readers. We study the existence of solutions of the linear problem

$$-\Delta\phi - \omega\phi = -h(x), \ \omega \geq 0 \tag{3.1}$$

in the space $H^2(\mathbb{R}^d), \ d \in \mathbb{N}$ equipped with the standard norm

$$\|u\|^2_{H^2(\mathbb{R}^d)} := \|u\|^2_{L^2(\mathbb{R}^d)} + \|\Delta u\|^2_{L^2(\mathbb{R}^d)}. \tag{3.2}$$

The right side of (3.1) is assumed to be square integrable.

Lemma 5. *Let $h(x) \in L^2(\mathbb{R})$. The the following assertions hold:*

a) When $\omega > 0$ and $xh(x) \in L^1(\mathbb{R})$ problem (3.1) admits a unique solution in $H^2(\mathbb{R})$ if and only if

$$\left(h(x), \frac{e^{\pm i\sqrt{\omega}x}}{\sqrt{2\pi}}\right)_{L^2(\mathbb{R})} = 0. \tag{3.3}$$

b) When $\omega = 0$ and $x^2h(x) \in L^1(\mathbb{R})$ problem (3.1) admits a unique solution in $H^2(\mathbb{R})$ if and only if

$$(h(x), 1)_{L^2(\mathbb{R})} = 0, \ (h(x), x)_{L^2(\mathbb{R})} = 0. \tag{3.4}$$

Lemma 6. *Let $h(x) \in L^2(\mathbb{R}^d), \ d \geq 2$. The the following assertions hold:*

a) When $\omega > 0$ and $xh(x) \in L^1(\mathbb{R}^d)$ problem (3.1) admits a unique solution in $H^2(\mathbb{R}^d)$ if and only if

$$\left(h(x), \frac{e^{ipx}}{(2\pi)^{\frac{d}{2}}}\right)_{L^2(\mathbb{R}^d)} = 0, \ p \in S^d_{\sqrt{\omega}} \ a.e., \ d \geq 2. \tag{3.5}$$

b) When $\omega = 0$ and $|x|^2h(x) \in L^1(\mathbb{R}^2)$ problem (3.1) admits a unique solution in $H^2(\mathbb{R}^2)$ if and only if

$$(h(x), 1)_{L^2(\mathbb{R}^2)} = 0, \ (h(x), x_k)_{L^2(\mathbb{R}^2)} = 0, \ 1 \leq k \leq 2. \tag{3.6}$$

c) When $\omega = 0$ and $|x|h(x) \in L^1(\mathbb{R}^d), \ d = 3, 4$ problem (3.1) admits a unique solution in $H^2(\mathbb{R}^d)$ if and only if

$$(h(x), 1)_{L^2(\mathbb{R}^d)} = 0, \ d = 3, 4. \tag{3.7}$$

d) When $\omega = 0$ and $|x|h(x) \in L^1(\mathbb{R}^d), \ d \geq 5$ problem (3.1) possesses a unique solution in $H^2(\mathbb{R}^d)$.

Lemma 7. *Let $\omega = 0$ and $h(x) \in L^1(\mathbb{R}^d) \cap L^2(\mathbb{R}^d)$ with $d \geq 5$. Then problem (3.1) admits a unique solution in $H^2(\mathbb{R}^d)$.*