

EQUILIBRIUM QUASI-PERIODIC CONFIGURATIONS WITH RESONANT FREQUENCIES IN QUASI-PERIODIC MEDIA II: KAM THEORY

LEI ZHANG, XIFENG SU, AND RAFAEL DE LA LLAVE

ABSTRACT. We develop an a-posteriori KAM theory for the equilibrium equations for quasi-periodic solutions in a quasi-periodic Frenkel-Kontorova model when the frequency of the solutions resonates with the frequencies of the substratum.

The KAM theory we develop is very different both in the methods and in the conclusions from the more customary KAM theory for Hamiltonian systems or from the KAM theory in quasi-periodic media for solutions with frequencies Diophantine with respect to the frequencies of the media. The main difficulty is that we cannot use transformations (as in the Hamiltonian case) nor Ward identities (as in the case of frequencies Diophantine with those of the media).

The technique we use is to add an extra equation to make the linearization of the equilibrium equation factorize. This requires an extra counterterm. We compare this phenomenon with other phenomena in KAM theory. It seems that this technique could be used in several other problems.

As a conclusion, we obtain that the perturbation expansions developed in the previous paper [SZdL14] converge when the potentials are in a codimension one manifold in a space of potentials. The method of proof also leads to efficient (low storage requirements and low operation count) algorithms to compute the quasi-periodic solutions.

Quasi-periodic Frenkel-Kontorova models, resonant frequencies, equilibria, quasicrystals, Lindstedt series, counterterms, KAM theory
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1. INTRODUCTION

The goal of this paper is to develop a KAM theory for the functional equation:

$$(1) \quad v(\psi + \Omega) + v(\psi - \Omega) - 2v(\psi) + W((\psi, \eta) + \beta v(\psi)) + \lambda = 0$$

where $W : \mathbb{T}^d \rightarrow \mathbb{R}$, $\Omega \in \mathbb{R}^{d-1}$, $\beta \in \mathbb{R}^d$ are given, ψ is a variable in \mathbb{T}^{d-1} and we can think of $\eta \in \mathbb{T}$ as a parameter. We are to find $v : \mathbb{T}^{d-1} \times \mathbb{T}^1 \rightarrow \mathbb{R}$ as a function of (ψ, η) and $\lambda \in \mathbb{R}$ as a function of the parameter η . We will refer to (1) as the “equilibrium equation”.

The equation (1) was derived in the companion paper [SZdLL14] as the equation satisfied by hull functions of quasi-periodic equilibria in Frenkel-Kontorova models of deposition on quasi-crystals when the frequency of the equilibrium solution is resonant with the frequencies of the substratum. The variable η is an angle which has the meaning of a transversal phase.

Roughly, the model describes particles interacting with their neighbors and with a substratum which is quasi-periodic. The configuration describing the state of the system is parameterized by the hull function v . We try

to place the particles with a frequency (inverse of the density of particles) which resonates with the frequencies of the medium. Ω represents the *intrinsic frequencies*. Since the medium is resonating with the frequency of the configuration, the positions of the particles are parameterized by $d - 1$ angles, i.e. they cover densely a $d - 1$ dimensional torus which is indexed by η . The W represents the forces of the particles with the substratum and the λ is an external force.

We refer to the paper [SZdlL14] for the discussion of the physical motivations (there are several physical motivations for the model) and for a formal analysis. From the strictly logical point of view, this paper and [SZdlL14] are completely independent even if they are motivated by the same physical problem. They also rely on very different techniques. To avoid repetition, we refer to [SZdlL14] for references to the previous literature on the problem as well as for physical motivations.

The main goal of this paper is to develop a KAM theory for the equation (1), but we will have to add a one dimensional extra parameter to it.

The main source of the difficulty to implement a Newton method – as needed in KAM theory – is that the equation (36) (the linearization of the equilibrium equation (1)) is not easily analyzable in a way that leads to tame estimates. We will deal with this problem by adding an extra auxiliary equation which implies that (36) can be solved with tame estimates. The addition of an extra equation that allows to solve the linearization is similar in spirit to the introduction of the reducibility in KAM theory [Mos67]. Nevertheless, our auxiliary equation is very different from the one in reducibility.

1.1. The method of adding extra parameters to equations. The main observation that allows us to develop a KAM theory is that if we are allowed to adjust a one dimensional parameter in the potential, then the linearized equilibrium equation admits a very nice structure (it can be factorized into two first order equations).

Hence, in Section 1.1.1, we will add an extra parameter to the left hand side of (1) to obtain the modified equilibrium equation (2) such that its linearization can be factorized. Then, we supplement (2) with another equation (4) in Section 1.1.2 (we call it *the factorization equation*) which encodes that the linearization of the equilibrium equation can be solved.

1.1.1. The modified equilibrium equation. For each fixed $\eta \in \mathbb{T}^1$ we will be looking for a function $v : \mathbb{T}^{d-1} \rightarrow \mathbb{R}$ and for numbers λ, σ in such a way

that we have

$$\begin{aligned}
 & \mathcal{E}[v, \sigma, \lambda](\psi, \eta) \\
 (2) \quad & \equiv v(\psi + \Omega) + v(\psi - \Omega) - 2v(\psi) + W((\psi, \eta) + \beta v(\psi)) + v(\psi)\sigma + \lambda \\
 & = 0.
 \end{aligned}$$

In the rest of the paper, we will call this modified equation (2) the equilibrium equation. The equation (2) has a symmetry that makes the solutions not unique (this corresponds to a gauge symmetry related to the choice of origin of the phase in the original problem). Hence, to obtain local uniqueness, we supplement (2) with the following normalization:

$$(3) \quad \int_{\mathbb{T}^{d-1}} v(\psi) d\psi = 0.$$

We consider \mathcal{E} as a functional that given a function v and two numbers σ, λ produces another function given by the second line in (2). And we treat the equation (2) as searching for zeros of the functional \mathcal{E} .

Since v, σ, λ all depend on the parameter η , for convenience, we will write v_η when we need to emphasize the fact that v depends on η , and similarly for all the other functions.

The hard analysis study will obtain σ, λ as a function of η (and may be frequency Ω , but we will not discuss dependence on Ω in this paper).

Depending on the physical solutions, we may impose the value of one of these variables and determine the others. For example, if we are imposing an external force we may want to fix λ or if the material is constrained to have a certain density, we may fix Ω .

It is important to know that, once we have established the hard theorem, eliminating some variables in terms of the others is just an application of the finite dimensional implicit function theorem.

Remark 1. *Variants of the idea of adding external parameters and then setting them to zero, has appeared in many guises. In perturbative expansions in Physics, it is called the method of counterterms [BS80, Gal85]. In differential equations, it is called Cesari's alternative method (Chapter IX of [Hal80],[CH82]). Closer to us, in KAM theory, it was introduced in [Mos67]. It was realized in [dlL86, BHS96, Yoc92, Sev99] that it provided a good way to deal with degenerate problems. A very systematic treatment of dependence on parameters (including parameters taking values in nowhere dense sets) appears in [Van02].*

Remark 2. *The exact form of the counterterm added is not that important. We could have put other counterterms $\sigma F(v)$ for almost any function $F \not\equiv 0$.*

The way of thinking geometrically of the counterterms is that there is codimension 1 set of potentials for which the solutions move differentiably

with respect to parameters. The counterterm is a projection that moves to keep the problem in this manifold. Of course, we could have taken any other family of corrections to the codimension one manifold where the solutions are found. We refer to Section 5 for a precise formulation and more details.

Remark 3. Even if we could consider (2) as a functional equation for each value of η , we will show in the next section that the symmetries of the equation involve mixing the ψ and η dependence. Relatedly, we note that the equation (2) for a fixed value of η does not have a variational principle.

It will be important to mention that, because β has components both in the ψ and the η directions, the equation (2) cannot be considered just as a parameterized version of the equations considered in [SdlL12b].

1.1.2. *The factorization equation.* For the KAM treatment of the equilibrium equation (2), we will find it useful to supplement (2) with another equation which we call the factorization equation

$$(4) \quad \mathcal{F}[v, \sigma, c](\psi, \eta) \equiv [-c(\psi) + 2 - \partial_\beta W((\psi, \eta) + \beta v_\eta(\psi)) - \sigma] c(\psi + \Omega) - 1 = 0.$$

This should be considered as a functional equation for functions $v, c : \mathbb{T}^{d-1} \rightarrow \mathbb{R}$ and number σ when η is fixed. Of course, v is also supposed to be part of the solution of (2).

The equation (4) gives a condition which ensures that the linearization of the equilibrium equation (2) has a nice structure which allows to implement a Newton method with tame estimates (namely, that it factorizes into two first order difference equations). See Section 4.2 for a discussion of (4) as a tool for solving (2).

Of course, the equation (4) is hard to solve exactly, but we will be able to develop a quasi-Newton method for (4).

Even if we do not know how to carry out a KAM theory for the equilibrium equation (2) alone, we can carry out a KAM procedure for the pair of equations (2) and (4). As discussed in Section 1.1.3, similar things (a functional equation supplemented by another auxiliary one that makes the associated linearized equation solvable) have happened in classical problems in KAM theory [Mos67, Eli88].

Remark 4. Notice that both the equilibrium equation (2) and the factorization equations (4) are coupled because v is an unknown in both equations.

Nevertheless, the effect of the factorization equation on the equilibrium equation appears only through the counterterm σ and is a mild effect (linear in σ). (See Section 4.3 for a detailed treatment.) On the other hand, the effect of the variables of the equilibrium equation is very strong. Hence, one can think of the pair (2) and (4) almost as an upper triangular system of equations. Indeed, the perturbative treatment of the pair (2) and (4) has a skew product structure. See Section 5.1.

1.1.3. *Comparison with the application of reducibility.* The method of adding an extra equation so that the linearization of the equilibrium equation is solvable, has already appeared in KAM theory.

In the theory of perturbation of lower dimensional elliptic tori, the classical treatment is to try to reduce the linearized equation to an equation with constant coefficients [Mos67, Mel63]. This requires extra non-resonance conditions and, in principle extra parameters. See, in particular [CW99, HLY06, Tre91], which study the problem of breakdown of resonant tori in Hamiltonian systems.

Of course, in the present case, the situation is completely different in the details (since we do not seek reducibility but rather factorization into two second order equations) as well as in the concepts (in [Mos67], the parameters are related to initial conditions or the characteristic numbers of the linearized equation). Hence the parameter count of the present method is very different from what one could expect from reducibility and the method is very different. Also the reducibility equations have a very different geometric meaning from the factorization equations.

However, based on the analogy of reducibility and its further developments, e.g. [Eli88, Pös89, JS96], one could conjecture that there should be a direct analysis of the linearized equation based, for example, on [CW93, Bou97]. We do not know how to do that yet. The paper [dlLH10] presents a comparison of the numerical methods based on reducibility and the methods based on using a direct Newton iteration.

The factorization method has analogies in higher dimensional systems and in elliptic PDE [Koz83, Mos88]. One can think of factorization as an analogue of putting the PDE in divergence form. The transformation to divergence form is achieved in [Koz83, Mos88] for elliptic operators taking advantage of an identity (which is analogous to the Ward identities in gauge theory). Here, on the other hand, we have to do a KAM theorem to obtain an auxiliary function that gives the factorization.

The present method gives smooth dependence on parameters, justifies the perturbation theory and leads to efficient numerical implementations.

1.2. Properties of the equilibrium equations (2) and its associated factorization equation (4). Before embarking on the hard analysis, in this section, we derive some identities and symmetries of the equations which are only soft analysis. This section can be skipped by readers interested only in the KAM methods.

Many of the symmetries and elementary properties derived for the equilibrium equation in [SZdlL14] lift straightforwardly to the factorization equation. Surprisingly, the formal perturbation theory developed here for the pair (2) and (4) is more efficient than the perturbation theory for (1)

alone developed in [SZdlL14]. The perturbation theory developed in [SZdlL14] was only for perturbation around integrable solutions, but the perturbation theory for the pair (2) and (4) is developed around any solution of both equations (2) and (4). See Section 5.

Of course, the expansions around zero found in [SZdlL14] are a particular case of the expansions found here since the linearization of the equilibrium equation around zero admits a trivial factorization.

1.2.1. *The symmetries of the equilibrium equations (2).* We note that the symmetries for the equilibrium equation found in [SZdlL14] extend to the factorization equation.

We have that if $v_\eta, \sigma(\eta), \lambda(\eta)$ is a solution of (2), for any function $\iota(\eta)$ so is:

$$(5) \quad \begin{aligned} \tilde{v}_\eta(\psi) &= v_{\eta+\iota(\eta)\beta_\eta}(\psi + \iota(\eta)\beta_\psi) + \iota(\eta), \\ \tilde{\sigma}(\eta) &= \sigma(\eta + \iota(\eta)\beta_\eta), \\ \tilde{\lambda}(\eta) &= \lambda(\eta + \iota(\eta)\beta_\eta) - \iota(\eta) \sigma(\eta + \iota(\eta)\beta_\eta). \end{aligned}$$

Since the symmetry (5) involves changes of arguments, giving a v_η , finding the $\iota(\eta)$ that accomplishes the normalization is not trivial and involves solving the implicit equation

$$(6) \quad I(\eta + \beta_\eta \iota(\eta)) + \iota(\eta) = 0$$

where $I(\eta) \equiv \int_{\mathbb{T}^{d-1}} v_\eta(\psi) d\psi$.

Applying the finite dimensional implicit function theorem, we can solve (6) if I and its derivative with respect to $\iota(\eta)$ are both small. In contrast, in the non-resonant case treated in [SdlL12b], the normalization we imposed of the function could always be solved explicitly.

As we will prove in Section 4, the solutions of (2) that satisfy the normalization (3) will be locally unique.

2. FUNCTION SPACES AND LINEAR ESTIMATES

To formulate the KAM results (as well as to make quantitative the Lindstedt series) we need to define precisely the norms of analytic functions. In this section, we collect the definitions and some standard properties of spaces of analytic functions. Of course, to understand the statements of the results it is enough to read the definitions.

In Section 2.1, we collect several standard definitions of spaces and present some preliminary results on these spaces. In Section 2.2 we present definitions of the Diophantine properties we will use in this paper. In Section 2.3 we present well known estimates for cohomology equations, which are the basis of the KAM procedure. Besides the customary constant coefficient

equations, we study first order cohomology equations with non-constant coefficients in section 2.4, which were also studied in [Her83].

2.1. Spaces of functions we will use. We will use a variation on the same spaces of analytic functions which have been used very often in KAM theory since [Mos67]. We will use the same notations as in [dLL08, CdLL10b, SdLL12b, SdLL12a].

We denote by

$$D_\rho \equiv \{ \psi \in \mathbb{C}^{d-1} / \mathbb{Z}^{d-1} \mid |\operatorname{Im}(\psi_j)| < \rho \}.$$

We denote the Fourier expansion of a periodic function $v(\psi)$ on D_ρ by

$$v(\psi) = \sum_{k \in \mathbb{Z}^{d-1}} \hat{v}_k e^{2\pi i k \cdot \psi},$$

where \cdot is the Euclidean scalar product in \mathbb{C}^{d-1} and \hat{v}_k are the Fourier coefficients of v .

We denote by \mathcal{A}_ρ the space of analytic functions on D_ρ which are real for real argument and extend continuously to $\overline{D_\rho}$. We make \mathcal{A}_ρ a Banach space by endowing it with the supremum norm:

$$\|v\|_\rho = \sup_{\psi \in \overline{D_\rho}} |v(\psi)|.$$

The spaces of analytic functions \mathcal{A}_ρ are the same spaces as in [Mos67] and that some of their elementary properties used in the argument were discussed in [CdLL10a, SdLL12b]. Notably:

- Interpolation inequalities (Hadamard three circle theorem):

$$(7) \quad \|v\|_{\theta\rho + (1-\theta)\rho'} \leq \|v\|_\rho^\theta \|v\|_{\rho'}^{1-\theta}.$$

- Cauchy inequalities:

$$\|D^l v\|_{\rho-\delta} \leq C(l, d) \delta^{-l} \|v\|_\rho,$$

$$|\hat{v}_k| \leq e^{-2\pi |k| \rho} \|v\|_\rho.$$

- The regularity of the composition:

Proposition 1. *Let f be an analytic function in a domain $\mathcal{D} \subseteq \mathbb{C}$, $v \in \mathcal{A}_\rho$. Assume $v(\mathbb{T}_\rho^d) \subseteq \mathcal{D}$, $\operatorname{dist}(v(\mathbb{T}_\rho^d), \mathbb{C} - \mathcal{D}) \geq \xi > 0$. Then,*

- (1) $f \circ v \in \mathcal{A}_\rho$;
- (2) If $\|\tilde{v}\|_\rho < \frac{\xi}{2}$, we have

$$\|f(v + \tilde{v}) - f(v) - f'(v)\tilde{v}\|_\rho \leq C\|\tilde{v}\|_\rho^2.$$

The cohomology equations we will have to consider are different from those studied before and we will present the results in Section 2.4.

Remark 5. *The method of proof works also for spaces of functions with finite differentiability. Indeed, KAM theory is often formulated as an abstract implicit function theorem for the functional \mathcal{E} acting on spaces of functions that satisfy some mild properties [Sch60, Zeh75]. (The paper [CdLL10a] presents an implicit function theorem well suited for the method in this paper.)*

In particular, the method in this paper works as well when v is considered in Sobolev spaces of high enough regularity. For simplicity, we will not formulate the finite differentiable version of the results.

2.2. Diophantine condition. We will assume that $\alpha \in \mathbb{R}^d$ is non-resonant, i.e.

$$(8) \quad |\alpha \cdot k| \neq 0 \quad \forall k \in \mathbb{Z}^d - \{0\}.$$

We are interested in the resonant rotation number $\omega \in \mathbb{R}$ such that for some $k \in \mathbb{Z}^d - \{0\}, m \in \mathbb{Z}$, we have

$$(9) \quad k \cdot \omega \alpha - m = 0.$$

If (9) holds we can find a matrix $B \in SL(d, \mathbb{Z}), \Omega \in \mathbb{R}^{d-1}, L \in \mathbb{Z}^d$ such that

$$(10) \quad B\omega\alpha = (\Omega, 0) + L \quad \text{with} \quad \Omega \cdot \tilde{k} \notin \mathbb{Z} \text{ for } \tilde{k} \in \mathbb{Z}^{d-1} - \{0\}.$$

We also require that Ω satisfies the Diophantine condition in \mathbb{R}^{d-1} :

$$(11) \quad |\tilde{k} \cdot \Omega - m| \geq \nu |\tilde{k}|^{-\tau} \quad \forall \tilde{k} \in \mathbb{Z}^{d-1} - \{0\}, m \in \mathbb{Z}.$$

Here ν, τ are positive numbers.

We denote by $\mathcal{D}(\nu, \tau)$ the set of Ω which satisfy (11). We also denote $\mathcal{D}(\tau) = \cup_{\nu > 0} \mathcal{D}(\nu, \tau)$.

It is well known that the set $\mathcal{D}(\tau)$ with $\tau > d-1$ is of full $d-1$ dimensional Lebesgue measure in \mathbb{R}^{d-1} . Of course, if we fix α the frequencies of the media, the ω 's that lead to a resonance as in (11) form a countable set. Further discussion of resonances appear in [SZdLL14].

2.3. Cohomology equations. It is standard in KAM theory to solve for v given ϕ with zero average in such a way that:

$$(12) \quad v(\psi + \Omega) - v(\psi) = \phi(\psi),$$

where $\Omega \in \mathcal{D}(\nu, \tau)$.

We will use equation similar to (12) frequently with some fixed frequency Ω . To simplify our notations, we will denote $v(\psi + \Omega)$ and $v(\psi - \Omega)$ as $v_+(\psi)$ and $v_-(\psi)$, respectively. Similar notations will be used for other functions. We also use T to represent the translation operators, i.e. $[T_\Omega v](\psi) = v(\psi + \Omega)$.

Estimates for (12) for Diophantine frequencies were proved by [Rüs75, Rüs76]. The crucial point of these estimates is that the solution is bounded in smaller domains and that there are bounds on the solution in the smaller domains. These estimates have large constants if the loss of domain is small, but the constants can be chosen to be a power of the domain loss (tame estimates). See (15).

The proof of these tame estimates with worse exponents is elementary and was known much earlier than [Rüs75, Rüs76].

Tame estimates (and hence Diophantine conditions) are used in the convergence proofs and in the KAM theory developed here.

Lemma 1. *Let $\phi \in \mathcal{A}_\rho(\mathbb{T}^{d-1})$ be such that*

$$(13) \quad \int_{\mathbb{T}^{d-1}} \phi(\psi) d\psi = 0.$$

Assume that $\Omega \in \mathcal{D}(\nu, \tau)$.

Then, there exists a unique solution v of (12) which satisfies

$$(14) \quad \int_{\mathbb{T}^{d-1}} v(\psi) d\psi = 0.$$

The solution $v \in \mathcal{A}_{\rho'}$ for any $0 < \rho' < \rho$

$$(15) \quad \|v\|_{\rho'} \leq C(d, \tau) \nu^{-1} (\rho - \rho')^{-\tau} \|\phi\|_\rho.$$

Furthermore, any distribution solution of (12) differs from the solution claimed before by a constant.

If ϕ is such that it takes real values for real arguments, so does v .

Similarly if we consider analytic functions $\tilde{\phi} \in \mathcal{A}_\rho(\mathbb{T}^d)$ satisfying $\int_{\mathbb{T}^{d-1}} \tilde{\phi}_\eta(\psi) d\psi = 0$, we can solve

$$(16) \quad \tilde{v}(\psi + \Omega) - \tilde{v}(\psi) = \tilde{\phi}(\psi).$$

Then the solution \tilde{v} is also analytic in (ψ, η) and we have

$$(17) \quad \|\tilde{v}\|_{\rho'} \leq C(d, \tau) \nu^{-1} (\rho - \rho')^{-\tau} \|\tilde{\phi}\|_\rho.$$

We note that, as it is well known that obtaining v solving (12) for given ϕ is very explicit in terms of Fourier coefficients. If

$$\phi(\psi) = \sum_{k \neq 0} \hat{\phi}_k e^{2\pi i(k \cdot \psi)}$$

then, v is given by

$$v(\psi) = \sum_{k \neq 0} \hat{\phi}_k (e^{2\pi i k \cdot \Omega} - 1)^{-1} e^{2\pi i(k \cdot \psi)}.$$

The above formula for the solution makes it clear that if ϕ is real for real values of its arguments, so is v . Note also that if the function ϕ is

discretized in terms of N Fourier coefficients, the computation of the Fourier coefficients of v takes only N operations. It also makes it clear how to obtain the tame estimates (with a worse exponent) for Diophantine frequencies or the results for (86).

The well known and elementary argument for the estimates with worst exponent observes that for any $0 < \rho' < \rho$, we have

$$\begin{aligned} \sup_{\psi \in \overline{D}_{\rho'}} |v_{\eta}(\psi)| &\leq \sum_{k \neq 0} |\hat{\phi}_k(\eta)| |e^{2\pi i k \cdot \Omega} - 1|^{-1} e^{2\pi |k| \rho'} \\ &\leq \sum_{k \neq 0} C \|\phi_{\eta}\|_{\rho} e^{-2\pi |k| \rho} \sup_{m \in \mathbb{Z}} |k \cdot \Omega - m|^{-1} e^{2\pi |k| \rho'} \\ &\leq \sum_{l > 0} C \|\phi_{\eta}\|_{\rho} l^{d-2} \sup_{|k|=l} \sup_{m \in \mathbb{Z}} |k \cdot \Omega - m|^{-1} e^{-2\pi |k|(\rho - \rho')} \end{aligned}$$

where we have used just Cauchy estimates for the Fourier coefficients, and calculus estimates for the exponential function (we bound the distance of the exponential from 1 by the distance of its argument to $2\pi\mathbb{Z}$). If we assume (86) we obtain that the series converges by the root test. If we assume (11) we obtain the tame estimates but with a worse exponent than the one obtained in [Rüs75, Rüs76] by a subtler argument.

2.4. Cohomology equations with non-constant coefficients. In this section we consider a generalization of the above theory for equations with non-constant coefficients. These equations have also been called *twisted cohomology equations*. They play an important role in [Her83]. Our strategy is similar to those in [Her83].

We will be considering equations of the form

$$(18) \quad a(\psi)v(\psi + \Omega) - b(\psi)v(\psi) = \lambda + \phi(\psi)$$

where $\psi \in \mathbb{T}^{d-1}$ and a, b are fixed functions in \mathcal{A}_{ρ} , taking real values for real inputs.

We consider that $\phi \in \mathcal{A}_{\rho}$ is the known data of the equation and $\lambda \in \mathbb{R}$, $v \in \mathcal{A}_{\rho-\delta}$ are the unknowns we seek for. We will refer to (18) as “twisted cohomology equations”.

We will show that, given some non-degeneracy conditions in a and b (It suffices that a and b are moderately away from 0.) we can obtain estimates for λ and v . The idea, already present in [Her83], is that we can rewrite the equation (18) into a constant coefficient equation. We will go through the procedure in details because we will need rather detailed estimates on the effect of a and b , but the treatment is in [Her83].

Lemma 2. *Suppose $\|a - 1\|_{\rho} < r < 1$, $\|b - 1\|_{\rho} < r < 1$ and that Ω satisfies the Diophantine condition (11). Then there exist positive real valued*

functions $\gamma_a, \gamma_b \in \mathcal{A}_{\rho-\delta/2}$ and real numbers \bar{a}, \bar{b} such that

$$(19) \quad \begin{aligned} a(\psi) &= \bar{a} \frac{\gamma_a(\psi + \Omega)}{\gamma_a(\psi)}, \\ b(\psi) &= \bar{b} \frac{\gamma_b(\psi)}{\gamma_b(\psi + \Omega)}. \end{aligned}$$

In addition, for the twisted cohomology equation (18), there exists a unique solution $\lambda \in \mathbb{R}$ and $v \in \mathcal{A}_{\rho-\delta}$ for (18) satisfying the normalization

$$(20) \quad \int_{\mathbb{T}^{d-1}} v d\psi = 0$$

such that

$$(21) \quad \begin{aligned} |\lambda| &\leq \frac{\|\phi\|_\rho \|\gamma_a(\gamma_b)_+\|_{\rho-\delta/2}}{\int_{\mathbb{T}^{d-1}} \gamma_a(\gamma_b)_+ d\psi} \\ \|\gamma_a \gamma_b v\|_{\rho-\delta} &\leq C \delta^{-\tau} \|(\lambda + \phi) \gamma_a(\gamma_b)_+\|_\rho \quad \forall \delta > 0. \end{aligned}$$

Therefore,

$$(22) \quad \|v\|_{\rho-\delta} \leq C \|\gamma_a\|_\rho \|1/\gamma_a\|_{\rho-\delta} \|\gamma_b\|_\rho \|1/\gamma_b\|_{\rho-\delta} \delta^{-\tau} \|\lambda + \phi\|_\rho \quad \forall \delta > 0.$$

Proof. To show (19), it suffices to observe that, taking logarithms, (19) is equivalent to:

$$(23) \quad \begin{aligned} (\log a)(\psi) &= \log(\bar{a}) + \log(\gamma_a)(\psi + \Omega) - \log(\gamma_a)(\psi) \\ (\log b)(\psi) &= \log(\bar{b}) + \log(\gamma_b)(\psi) - \log(\gamma_b)(\psi + \Omega) \end{aligned}$$

which are cohomology equations with constant coefficients. Applying Lemma 1, we get solutions $\gamma_a, \gamma_b \in \mathcal{A}_{\rho-\delta/2}$, $\bar{a}, \bar{b} \in \mathbb{R}$ and estimates

$$(24) \quad \begin{aligned} \|\log \gamma_a\|_{\rho-\delta/2} &\leq C(d, \tau, \nu) \delta^{-\tau} \|\log a - \log \bar{a}\|_\rho, \\ \|\log \gamma_b\|_{\rho-\delta/2} &\leq C(d, \tau, \nu) \delta^{-\tau} \|\log b - \log \bar{b}\|_\rho. \end{aligned}$$

Once we solve the constant coefficient cohomology equation (23) for $\log a, \log b, \log \gamma_a, \log \gamma_b$ we obtain a, b, γ_a, γ_b by taking exponentials. This ensures that they are positive for real values of the argument. We refer to \bar{a}, \bar{b} as the average coefficients of the cohomology equation.

Once we have the solution of (19), we realize that the equation (18) is equivalent to

$$(25) \quad \bar{a}(\gamma_a \gamma_b v)_+ - \bar{b}(\gamma_a \gamma_b v) = (\lambda + \phi) \gamma_a(\gamma_b)_+$$

which is a cohomology equation with constant coefficients. Let us denote $m = \gamma_a \gamma_b v$ for simplicity.

When $\bar{a} = \bar{b}$, we can solve (25) using previous methods and get estimates (21).

Note if $\bar{a} \neq \bar{b}$, the equation (25) is easier to solve since no small divisors appear.

- If $\bar{a} > \bar{b}$, then (25) is equivalent to

$$m_+ - \frac{\bar{b}}{\bar{a}}m = \frac{\lambda + \phi}{\bar{a}}\gamma_a(\gamma_b)_+,$$

namely, $m_+ = \frac{\lambda + \phi}{\bar{a}}\gamma_a(\gamma_b)_+ + \frac{\bar{b}}{\bar{a}}m$.

Therefore,

$$m(\psi) = \sum_{n=0}^{\infty} \left(\frac{\bar{b}}{\bar{a}}\right)^n \frac{\lambda + \phi(\psi - (n+1)\Omega)}{\bar{a}} \gamma_a(\psi - (n+1)\Omega) \gamma_b(\psi - n\Omega)$$

is a solution of (25). Hence, $\|m\|_\rho \leq C(\bar{a}, \bar{b}) \|(\lambda + \phi)\gamma_a(\gamma_b)_+\|_\rho$.

- If $\bar{a} < \bar{b}$, then (25) is equivalent to

$$\frac{\bar{a}}{\bar{b}}m_+ - m = \frac{\lambda + \phi}{\bar{b}}\gamma_a(\gamma_b)_+,$$

namely, $m = \frac{\bar{a}}{\bar{b}}m_+ - \frac{\lambda + \phi}{\bar{b}}\gamma_a(\gamma_b)_+$.

Therefore,

$$m(\psi) = - \sum_{n=0}^{\infty} \left(\frac{\bar{a}}{\bar{b}}\right)^n \frac{\lambda + \phi(\psi + n\Omega)}{\bar{b}} \gamma_a(\psi + n\Omega) \gamma_b(\psi + (n+1)\Omega)$$

is a solution of (25). Hence, $\|m\|_\rho \leq C(\bar{a}, \bar{b}) \|(\lambda + \phi)\gamma_a(\gamma_b)_+\|_\rho$.

Note that when $\bar{a} \neq \bar{b}$ we do not have any loss of domain, but the estimates depend on \bar{a}, \bar{b} . To get estimates uniformly in a, b , we need to use the Fourier method. \square

The choice of the parameter λ so as to achieve the normalization deserves some discussions. In the case $\bar{a} = \bar{b}$ we see that we have to choose λ in such a way that the right hand side of (25) has zero average. In this case, however, we can choose the average of the solution of (25) arbitrarily.

In the case $\bar{a} \neq \bar{b}$, given any λ , the solution of (25) will be unique. Furthermore, the solution will be an affine function of λ and, hence, so will be its average.

We see that the derivative of the average of the solution with respect to λ is

$$\frac{d}{d\lambda} \langle m \rangle = \frac{\langle \gamma_a(\gamma_b)_+ \rangle}{\bar{a} - \bar{b}}.$$

In summary, in the equal average coefficient case, the equation requires adjusting one parameter to be solvable, but it gives back one free parameter of solutions. In the different average coefficient case, we do not require any parameter to ensure the solutions, but the solution is unique.

In both cases, the solutions of twisted cohomology equations require as many parameters as they give back. This allows us to discuss the solutions of the equations which factorize into twisted cohomology equations and they require as many parameters as they give back.

Remark 6. *It will be important for subsequent applications that the estimates (21) are formulated in terms of γ . We could have attempted to formulate them in terms of a , but this is not practical for subsequent applications. On the other hand, we have formulated the normalization condition (20) in a way that it is independent of γ . This will be important in our iterative scheme because γ will change from step to step.*

In subsequent applications, we will be using Lemma 2 when the a, b are changing. It will be important to track the changes of γ_a, γ_b when we change a, b . See estimates (72).

3. THE KAM THEOREM

In this section, we will state precisely the main result of this paper.

3.1. Statement of the main result. For the following theorem, we will fix the parameter η and omit the subscript η .

We denote $\langle g \rangle$ the average of some function g and $\tilde{g} = g - \langle g \rangle$. We also use the notation:

$$\mathcal{L} = T_\Omega + T_{-\Omega} - 2 + \partial_\beta W((\psi, \eta) + \beta v) + \sigma.$$

Theorem 1. *Let $\alpha \in \mathbb{R}^d$ such that $\alpha \cdot j \neq 0$, $j \in \mathbb{Z}^d \setminus \{0\}$ and $\omega \in \mathbb{R}$ be such that $\omega\alpha$ is resonant. Let W be an analytic function defined in a domain $\mathcal{D} \subset \mathbb{C}^d / \mathbb{Z}^d$. Take $\rho > 0$ and $0 < s < \rho/2$.*

Denote:

$$(26) \quad \mathcal{E}(v, \sigma, \lambda) = v_+ + v_- - 2v + W((\psi, \eta) + \beta v) + \sigma v + \lambda,$$

$$(27) \quad \mathcal{F}(v, \sigma, c) = (-c + 2 - \partial_\beta W((\psi, \eta) + \beta v) - \sigma)c_+ - 1.$$

We assume:

(H1) *Initial guesses: Let $(v^0, c^0, \sigma^0, \lambda^0)$ be an approximate solution such that $\|\mathcal{E}[v^0, \sigma^0, \lambda^0]\|_\rho \leq \epsilon$ and $\|\mathcal{F}[v^0, \sigma^0, c^0]\|_\rho \leq \epsilon$.*

(H2) *Diophantine properties: There exists a matrix $B \in SL(d, \mathbb{Z})$, $\Omega \in \mathbb{R}^{d-1}$, $L \in \mathbb{Z}^d$ such that $B\omega\alpha = (\Omega, 0) + L$ with*

$$|l \cdot \Omega - n| \geq \nu |l|^{-\tau} \quad \forall l \in \mathbb{Z}^{d-1} - \{0\}, n \in \mathbb{Z}.$$

(H3) *Non-degeneracy conditions:*

$$\|c^0 - 1\|_\rho < M_1 < 1,$$

$$|\sigma^0| < M_2 < 1,$$

$$\|W_{v^0}\|_{C^2(\mathcal{D})} < M_3 < 1$$

$$\|v^0\|_\rho < m < 1,$$

where W_{v^0} is short for $W((\psi, \eta) + \beta v^0)$, $M_1 + M_2 + M_3 < 1$ and m depends on d, τ, ν, M_i, s .

(H4) *Composition condition: Denote*

$$\mathcal{R}_{v^0} = \{(\psi, \eta) + \beta v^0(\psi), (\psi, \eta) \in \mathbb{T}_\rho^d\}.$$

We assume $\mathcal{R}_{v^0} \subset \mathcal{D}$ and $\text{dist}(\mathcal{R}_{v^0}, \partial\mathcal{D}) \geq 2\xi > 0$, where ξ depends on $d, \tau, \nu, M_i, s, |\beta|_1$.

Let $\rho' = \rho - s - \delta < \rho - s$. Let ϵ be such that

$$\epsilon \leq \epsilon^* \delta^{4\tau},$$

where ϵ^* depends on $d, \tau, \nu, M, s, |\beta|_1$ and will be specified in the proof.

Then, there exist functions $v^*, c^* \in \mathcal{A}_{\rho'}$ and numbers $\sigma^*, \lambda^* \in \mathbb{R}$ such that

$$(28) \quad \begin{aligned} \mathcal{E}[v^*, \sigma^*, \lambda^*] &= 0, \\ \mathcal{F}[v^*, \sigma^*, c^*] &= 0. \end{aligned}$$

In addition,

$$(29) \quad \begin{aligned} \|v^0 - v^*\|_{\rho'} &\leq C\delta^{-4\tau}\epsilon, \\ |\lambda^0 - \lambda^*| &\leq C\delta^{-2\tau}\epsilon, \\ |\sigma^0 - \sigma^*| &\leq C\delta^{-2\tau}\epsilon, \\ \|c^0 - c^*\|_{\rho'} &\leq C\delta^{-4\tau}\epsilon, \end{aligned}$$

where C depends on $d, \tau, \nu, M, s, |\beta|_1$. In the rest of the paper, we will use C to denote any constant depending on these parameters.

We also have local uniqueness: Suppose $(v_1, \sigma_1, \lambda_1, c_1)$ and $(v_2, \sigma_2, \lambda_2, c_2)$ satisfying

$$(30) \quad \mathcal{E}[v_1, \sigma_1, \lambda_1] = \mathcal{E}[v_2, \sigma_2, \lambda_2] = 0,$$

$$(31) \quad \mathcal{F}[v_1, \sigma_1, c_1] = \mathcal{F}[v_2, \sigma_2, c_2] = 0,$$

and the normalization condition

$$(32) \quad \int_{\mathbb{T}^{d-1}} v_1 d\psi = \int_{\mathbb{T}^{d-1}} v_2 d\psi = 0.$$

If

$$(33) \quad \max\{\|v_2 - v_1\|_{3\rho/2}, |\sigma_2 - \sigma_1|, |\lambda_2 - \lambda_1|\} < \epsilon^* \rho,$$

then, we have

$$(34) \quad (v_1, c_1, \sigma_1, \lambda_1) = (v_2, c_2, \sigma_2, \lambda_2).$$

Finally, we have that the solution depends on the parameter η analytically.

The Theorem 1 is in an ‘‘a-posteriori’’ format. Given an approximate solution which satisfies some non-degeneracy condition, we establish that there is a true solution nearby.

For the experts in KAM theory, we note that we have two parameters s, δ measuring the domain loss. The parameter s measures the domain loss in

the first step and the parameter δ measures the domain loss in the subsequent steps. The first step sets up several quantities affect subsequent steps and are estimated in a different way than the other steps. The steps after the first are all very similar, and the estimates follow the standard Nash-Moser scheme, even if the solution of the linearized equation is not standard.

To produce the approximate solution, we can use a variety of methods. In the case that W is small, we can take as approximate solution $(v^0, c^0, \sigma^0, \lambda^0) = (0, 1, 0, 0)$ (the solution corresponding to $W = 0$) and then (H1) just becomes smallness condition on W .

As we will see later, we can use as approximate solutions, the result of a Lindstedt series and then, we obtain a validation of the Lindstedt procedure, and a consequence for the (complex) differentiability of the solutions and the convergence of Lindstedt series (see Section 5).

Even if we will not discuss it in this paper, one could take as an approximate solution the outcome of a numerical computation and Theorem 1 gives a validation of the numerical result (provided that we check a few “condition numbers”).

4. PROOF OF THEOREM 1

4.1. Outline of the proof. The proof of Theorem 1 is based on a quadratically convergent iterative method. The convergence of the method will be established by a Nash-Moser argument. Here we give an outline of the proof.

The linearization of the equilibrium equation is a linear second order difference equation. We will solve this linearized equilibrium equation using that it factorizes into two first order difference equations. In our case, we cannot write down the factorization in closed form (as in [SdIL12b, dLL08]). So we will impose an auxiliary equation for the coefficients of the factorized equation. We will call this equation the factorization equation. We will not need to solve exactly the factorization equation at each step, but we will require that there is an approximate solution of the factorization equation with an accuracy comparable to the accuracy of the approximate solution of the equilibrium equation. Then, we will derive an iterative method that improves both of the equilibrium equation and the factorization equation. This is reminiscent of the procedure to study an elliptic torus and its reducibility at the same time. See [Mos67, Pös89]. Note without the parameter σ , we can not find coefficients satisfying the factorization equation.

Even with this novelty, we still have difficulty that the equilibrium equation and the factorization equation are coupled. We do not know how to establish convergence if we solve one equation after another. Therefore

we will derive a way to solve the two coupled linearized equations simultaneously. After we factorize the linearized equilibrium equation, we will face three twisted cohomology equations (difference equations with non-constant coefficients), two for the linearized equilibrium equation and one for the linearized factorization equation. We will solve them using the technique in Section 4.2. This completes one step of iteration.

One small technical complication is that the first step is estimated in a different way than the subsequent steps. In the first step of the iteration, we compute some auxiliary function related to factorization (the auxiliary functions γ related to the twisted cohomology equation) and estimate them from the original data. In subsequent steps, even if we compute the same auxiliary functions, we do not estimate them from the data but we estimate the change on the auxiliary functions induced by the (rather small) changes in the approximate solution.

Under some minor non-degeneracy conditions, we can repeat the process indefinitely. Finally, we will prove that the iterative procedure converges with suitably chosen domains under the analytic norm. This is very standard in KAM theory.

The local uniqueness will be obtained by showing that the linearized equation admits unique solutions using a standard technique in Section 4.8. See [CdLL10b].

We also note that the main part of Theorem 1 has what is called “a-posteriori” format in numerical analysis. We show that if there is an approximate enough solution that satisfies some non-degeneracy conditions, then, there is a true solution nearby. It is well known that this a-posteriori format leads to smooth dependence on parameters, bootstrap of regularity, etc.

Finally, the analyticity with respect to parameters (we will use either ϵ or η as parameters) will be a corollary of the existence of the perturbative expansions (see Section 5.1). Using the a-posteriori format of Theorem 1 and the local uniqueness, it follows that the solution is complex differentiable in the parameters. Hence, it is analytic on the parameters. Furthermore, since the perturbative expansions are the Taylor expansions of an analytic function they converge.

We note that the iterative process described above leads to an efficient algorithm, which we have formulated in Section 4.4.

4.2. Motivation for the iterative step. Our goal in this section is to devise a procedure that given an approximate solution produces another approximate solution with much smaller error and not much worse non-degeneracy conditions. This procedure is done for the equilibrium and factorization equations simultaneously.

4.2.1. *The equilibrium equation.* We consider the initial guess $(v^0, \sigma^0, \lambda^0)$ which solves equation (2) with a small error e , where $\|e\|_\rho < \epsilon$, i.e.

$$(35) \quad v_+^0 + v_-^0 - 2v^0 + W((\psi, \eta) + \beta v^0) + \sigma^0 v^0 + \lambda^0 = e.$$

The Newton procedure for the approximate solutions of the equilibrium equation (35) requires to find an update $(\hat{v}, \hat{\sigma}, \hat{\lambda})$ satisfying

$$(36) \quad \hat{v}_+ + \hat{v}_- - 2\hat{v} + \partial_\beta W_{v^0} \hat{v} + \hat{\sigma} v^0 + \sigma^0 \hat{v} + \hat{\lambda} = -e,$$

where we denote $\partial_\beta W = \beta \cdot \nabla W$ for simplicity and we use W_v to indicate $W((\psi, \eta) + \beta v)$.

4.2.2. *The factorization equation.* The equation (36) is not easy to solve directly. Nevertheless, we impose that it can be factorized into two first order difference equations with non-constant coefficients. If we accomplish this, we can solve (36) using the theory of twisted cohomology equations developed in Section 2.4.

Therefore we want that the operator \mathcal{L} (recall that we denote $\mathcal{L} = T_\Omega + T_{-\Omega} - 2 + \partial_\beta W_{v^0} + \sigma$) is factorized into two first order operators:

$$(37) \quad \mathcal{L} = \mathcal{A}_+ \mathcal{A}_-,$$

where

$$(38) \quad \mathcal{A}_+ = a(\psi)T_\Omega - b(\psi)$$

$$(39) \quad \mathcal{A}_- = c(\psi) - d(\psi)T_{-\Omega}$$

and a, b, c, d are the new unknowns.

A direct calculation shows

$$(40) \quad \mathcal{A}_+ \mathcal{A}_- \hat{v} = a(\psi)c_+(\psi)\hat{v}_+ - [a(\psi)d_+(\psi) + b(\psi)c(\psi)]\hat{v} + b(\psi)d(\psi)\hat{v}_-.$$

Hence we want to choose a, b, c, d satisfying the following equations

$$(41) \quad \begin{aligned} a(\psi)c_+(\psi) &= 1, \\ -[a(\psi)d_+(\psi) + b(\psi)c(\psi)] &= -2 + \partial_\beta W_{v^0} + \sigma^0, \\ b(\psi)d(\psi) &= 1. \end{aligned}$$

Note that the problem of factorization has always many solutions. If $\mathcal{A}_+ \mathcal{A}_-$ is a solution and g is any invertible function, then $\tilde{\mathcal{A}}_+ \tilde{\mathcal{A}}_-$ is also a solution, where

$$(42) \quad \begin{aligned} \tilde{\mathcal{A}}_+ &= \mathcal{A}_+ g, \\ \tilde{\mathcal{A}}_- &= g^{-1} \mathcal{A}_-. \end{aligned}$$

We can use this non-uniqueness to impose some extra normalization condition. In this paper, we will take the normalization

$$(43) \quad b(x) = 1.$$

With condition (43) we can simplify the system (41) to (after eliminating a):

$$(44) \quad (-c + 2 - \partial_\beta W_{v^0} - \sigma^0)c_+ = 1.$$

This is how the operator \mathcal{F} comes into play. Note it is a non-linear, non-local equation which we will have to solve iteratively. By assumption (H1), we can solve this equation with an error f using some initial guess c^0 , i.e.

$$(45) \quad (-c^0 + 2 - \partial_\beta W_{v^0} - \sigma^0)c_+^0 - 1 = f$$

and we have $\|f\|_p < \epsilon$.

Then the Newton procedure for (45) requires solving the following equation for \hat{c} :

$$(46) \quad -c_+^0 \hat{c} + (-c^0 + 2 - \partial_\beta W_{v^0} - \sigma^0)\hat{c}_+ - c_+^0 \hat{\sigma} - \partial_\beta \partial_\beta W_{v^0} c_+^0 \hat{v} = -f.$$

Then $c + \hat{c}$ would be a more accurate solution.

We replace \mathcal{L} with $\mathcal{A}_+ \mathcal{A}_-$ in the original equilibrium equation (36) to get

$$(47) \quad \mathcal{A}_+ \mathcal{A}_- \hat{v} + \hat{\sigma} v^0 + \hat{\lambda} = -e.$$

Remark 7. *Note that the equation (47) is slightly different from (36), so the $(\hat{v}, \hat{\sigma}, \hat{\lambda}, \hat{c})$ we find is not exactly the solution of the linearized equation of (2). But the only difference between (47) and (36) is a term $f\hat{v}(c_+^0)^{-1}$, which we will show is quadratic in e, f (since \hat{v} will be of the order of e). To obtain quadratic convergence, the full Newton step will have to improve both the invariance and the factorization equation at the same time. (A moment's reflection shows that improving one equation and then the other does not lead to quadratic convergence.) We will prove our Newton procedure using (47) instead of (36) and improving the factorization still converges quadratically to a solution of (2).*

In the following section, we will develop a method to solve (47) and (46) simultaneously. In Section 4.6 we will obtain estimates for the size of the error. Then, we will show in Section 4.7 that they lead to an improved solution and that the process can be iterated and converges to a solution.

4.3. Solving the linearized equations. Our goal in this section is to solve (47) and (46) simultaneously.

The main difficulty arises because the equations (47) and (46) are coupled (the unknown $\hat{\sigma}$ appears in both of them). The main observation is that since $\hat{\sigma}$ appears in an affine way, we can guess that the solutions will be affine in $\hat{\sigma}$. This allows to uncouple the equations.

When we find the corrections $(\hat{v}, \hat{\sigma}, \hat{\lambda}, \hat{c})$, we can write down the updated solution of the equilibrium equation (2) and the factorization equation (45):

$$(48) \quad (v^1, \sigma^1, \lambda^1, c^1) = (v^0 + \hat{v}, \sigma^0 + \hat{\sigma}, \lambda^0 + \hat{\lambda}, c^0 + \hat{c}),$$

which can be used in the next iterative step. We will show that the improved solution (48) is indeed more approximate (in a smaller domain). Then the procedure can be repeated indefinitely and converges to a solution if the errors are small enough.

Now we write

$$(49) \quad \hat{v} = A + \hat{\sigma}B,$$

$$(50) \quad \hat{\lambda} = G + \hat{\sigma}D,$$

where A and B are functions, G and D are numbers. All these quantities will be determined shortly.

Then the equation (47) becomes

$$(51) \quad \mathcal{A}_+ \mathcal{A}_- A + G = -e,$$

$$(52) \quad \mathcal{A}_+ \mathcal{A}_- B + D = -v^0.$$

The Newton equation for factorization (46) after substitution (51) and (52) becomes

$$(53) \quad -c_+^0 \hat{c} + (-c^0 + 2 - \partial_\beta W_{v^0} - \sigma^0) \hat{c}_+ + (-c_+^0 - \partial_\beta \partial_\beta W_{v^0} c_+^0 B) \hat{\sigma} = \partial_\beta \partial_\beta W_{v^0} c_+^0 A - f.$$

Hence the Newton step improving simultaneously the invariance and factorization equations consists in solving (51), (52), (53). Notice that this system has an upper triangular structure, we can solve (51), (52) for A, B, G, D and then solve (53) for \hat{c} and $\hat{\sigma}$. Once we have $\hat{\sigma}$, we can determine \hat{v} and $\hat{\lambda}$ using (49) and (50).

To solve (51) and (52), we observe that each of them can be obtained by solving two cohomology equations with non-constant coefficients. We can find the zero average solution of (51) and (52). In fact, G is determined so that A has zero average and D is determined so that B has zero average. This ensures that $\langle \hat{v} \rangle = 0$.

We note that the choice of the constants in the solution of two consecutive twisted cohomology equations is a simple extension of the arguments developed at the end of Section 2.4.

In the case that $\mathcal{A}_+, \mathcal{A}_-$ both have different average coefficients, we see that the average of the solution as a function of G is the composition of two affine functions, hence affine and if the linear part of each of them is not zero, we obtain that the composition of the two affine functions has non-zero derivative.

In the case that \mathcal{A}_+ has equal average coefficient, but \mathcal{A}_- does not, we choose G so that \mathcal{A}_+ is solvable and choose the average of its solution so that the solution of \mathcal{A}_- satisfies the normalization.

In the case that \mathcal{A}_+ has different average coefficient but \mathcal{A}_- has the same average coefficient, we choose G so that the solution produced by \mathcal{A}_+ satisfies the compatibility conditions for \mathcal{A}_- . Then, the solution of \mathcal{A}_- is defined up to a constant which can be chosen in a unique way to satisfy the normalization condition.

In the case that both \mathcal{A}_+ , \mathcal{A}_- have the same average coefficients—which is the case that appears in standard KAM theory—we choose G so that \mathcal{A}_+ is solvable. Then, we use the free parameter of the solution of \mathcal{A}_- to ensure the solvability of \mathcal{A}_- and use the free constant in the solution of \mathcal{A}_- to adjust the normalization.

Therefore, we obtain the solution of (51). Similar procedure can be done for (52).

Now that we have solved (51) and (52) we turn to solving (53). As long as

$$(54) \quad \int_{\mathbb{T}^{d-1}} \left(-c_+^0 - \partial_\beta \partial_\beta W_{v^0} c_+^0 B \right) d\psi \neq 0,$$

which can be proved under some non-degeneracy conditions on the initial guesses, equation (53) is a twisted cohomology equation. Provided that $-c_+^0$ and $-c^0 + 2 - \partial_\beta W_{v^0} - \sigma^0$ are away from zero, we can apply Lemma 2 to solve (53) after A and B are found.

To make the procedure clear, we will write the full algorithm in Section 4.4. In Section 4.5, we will carefully estimate the errors after one iterative step to show that the errors of the equilibrium and the factorization equation are both reduced quadratically after performing the corrections in the iterative step. Finally, we will prove the convergence rigorously in Section 4.7.

4.4. Formulation of the iterative step.

Algorithm 1. (1) Given initial guesses $v^0, \sigma^0, \lambda^0, c^0$, set $a^0 = (c_+^0)^{-1}$.

(2) Calculate the errors $e = \mathcal{E}(v^0, \sigma^0, \lambda^0)$ and $f = \mathcal{F}(v^0, \sigma^0, c^0)$.

(3) Find γ_{a^0} and \bar{a}^0 satisfying

$$\log a^0 = \log \bar{a}^0 + \log(\gamma_{a^0})_+ - \log(\gamma_{a^0})_-.$$

Also find γ_{c^0} and \bar{c}^0 satisfying

$$\log c^0 = \log \bar{c}^0 + \log(\gamma_{c^0}) - \log(\gamma_{c^0})_-.$$

(4) Compute $\bar{a}_+, \bar{b}_+, \gamma_{a_+}, \gamma_{b_+}$ (the average coefficients and the auxiliary functions) for \mathcal{A}_+ and $\bar{a}_-, \bar{b}_-, \gamma_{a_-}, \gamma_{b_-}$ for \mathcal{A}_- .

(5a) If $\bar{a}_+ \neq \bar{b}_+, \bar{a}_- \neq \bar{b}_-$, compute

$$\alpha = -\mathcal{A}_+^{-1} \mathcal{A}_-^{-1} e, \quad \beta = -\mathcal{A}_+^{-1} \mathcal{A}_-^{-1} 1.$$

Set $G = -\frac{\langle \alpha \rangle}{\langle \beta \rangle}$. Set $A = \alpha + G\beta$.

- (5b) If $\bar{a}_+ = \bar{b}_+, \bar{a}_- \neq \bar{b}_-$, choose $G = -\langle e \rangle$. Set $\tilde{\alpha} = -\mathcal{A}_+^{-1}(e + G)$ and $\langle \tilde{\alpha} \rangle = 0$. Set $\beta_1 = \mathcal{A}_-^{-1}\tilde{\alpha}, \beta_2 = -\mathcal{A}_-^{-1}1$. Set $A = \beta_1 - \beta_2 \frac{\langle \beta_1 \rangle}{\langle \beta_2 \rangle}$.
- (5c) If $\bar{a}_+ \neq \bar{b}_+, \bar{a}_- = \bar{b}_-$, choose $\alpha_1 = -\mathcal{A}_+^{-1}e, \alpha_2 = -\mathcal{A}_+^{-1}1$. Set $G = \frac{\langle \alpha_1 \rangle}{\langle \alpha_2 \rangle}$. Set $A = \mathcal{A}_-^{-1}(\alpha_1 - G\alpha_2)$ and $\langle A \rangle = 0$.
- (5d) If $\bar{a}_+ = \bar{b}_+, \bar{a}_- = \bar{b}_-$, choose $G = -\langle e \rangle$. Set $\alpha = \mathcal{A}_+^{-1}(e - \langle e \rangle)$ and $\langle \alpha \rangle = 0$. Set $A = \mathcal{A}_-^{-1}\alpha$ and $\langle A \rangle = 0$.
- (6) Find B and D in a similar way as we found A and G . Also set $\langle B \rangle = 0$.
- (7) Find $\hat{\sigma}$ and \hat{c} by solving (53).
- (8) Set $\hat{v} = A + \hat{\sigma}B$ and $\hat{\lambda} = G + \hat{\sigma}D$.
- (9) Update u, λ, σ and c and repeat the steps.

Remark 8. When we apply repeatedly the iterative steps to obtain estimates, it would be advantageous in Step 3 to use the information we have on γ_a computed in the previous steps, because it leads to better estimates. In the first step, of course we can only use Cauchy estimates on the initial data. See Section 4.1.

Remark 9. The iterative method described above achieves quadratic convergence (see Section 4.7). It only entails performing algebraic operations among functions, composing them, taking derivatives and solving the cohomology equations.

If we discretize a function by the values of the point in a grid of N points and (redundantly) N Fourier coefficients, we note that each of the operations above requires N operations either in the grid representation or in the Fourier space representation. Of course, if we obtain either the Fourier or the real space representation for a function, we can obtain the other representation using the FFT algorithm that requires $N \log(N)$ operations.

Hence the method in Section 4 achieves quadratic convergence but no matrix inversion (or storage) is required. It only requires $O(N)$ storage and $O(N \log(N))$ operations per step.

This has already been observed in [CdIL09] in the periodic case (both for short range and for long range interactions) and [SdIL12b] for the quasi-periodic non-resonant case. Numerical implementations in the non-resonant periodic case were carried out in [BdIL13]. We have not implemented this method, but we think it would be interesting to do so.

4.5. Estimates on the corrections. Denote $V^0 = -c^0 + 2 - \partial_\beta W_{v^0} - \sigma^0$. In the first step, we argue as follows. From (24), it is clear that all the quantities $\|\gamma_{c^0}\|_{\rho-\delta/4}, \|\gamma_{c^0}^{-1}\|_{\rho-\delta/4}, \|\gamma_{(c^0)^{-1}}\|_{\rho-\delta/4}, \|\gamma_{(c^0)^{-1}}^{-1}\|_{\rho-\delta/4}, \|\gamma_{V^0}\|_{\rho-\delta/4}$ and $\|\gamma_{V^0}^{-1}\|_{\rho-\delta/4}$ are bounded by some constant which depends only on d, τ, v, M, s . We denote the bound by E .

In subsequent steps, we will just assume that we have the bounds E for the above auxiliary quantities. These bounds will be derived from the bounds in the first step.

Apply the estimate (22) of twisted cohomology equations for equations (51) and (52), also recall the averages of A and B both vanish we have

$$(55) \quad \|A\|_{\rho-\delta/2} \leq CE^4 \delta^{-2\tau} \|\tilde{e}\|_{\rho} = C\delta^{-2\tau} \|\tilde{e}\|_{\rho},$$

$$(56) \quad \|B\|_{\rho-\delta/2} \leq CE^4 \delta^{-2\tau} \|\tilde{v}^0\|_{\rho} = C\delta^{-2\tau} \|\tilde{v}^0\|_{\rho},$$

$$(57) \quad |G| \leq CE^4 \|e\|_{\rho} \leq C\epsilon,$$

$$(58) \quad |D| \leq CE^4 \|v^0\|_{\rho} \leq C.$$

Choose m small, we can make $\|B\|_{\rho-\delta/2} < 1$. Then

$$(59) \quad \|(-c_+^0 - \partial_{\beta}\partial_{\beta}W_{v^0}c_+^0B)^{-1}\|_{\rho-\delta/4} \leq (1 - M_1)^{-1}(1 - M_2)^{-1}.$$

This guarantees $\int_{\mathbb{T}^{d-1}} (-c_+^0 - \partial_{\beta}\partial_{\beta}W_{v^0}c_+^0B) d\psi \neq 0$. Therefore the cohomology equation (53) is solvable.

The estimate for (53) gives

$$(60) \quad \begin{aligned} |\hat{\sigma}| &\leq C(\|A\|_{\rho-\delta/2} + \|f\|_{\rho}) \\ &\leq C\delta^{-2\tau}\epsilon + C\epsilon \\ &\leq C\delta^{-2\tau}\epsilon, \end{aligned}$$

and

$$(61) \quad \begin{aligned} \|\hat{c}\|_{\rho-\delta} &\leq CE^4 \delta^{-\tau} (\|A\|_{\rho-\delta/2} + \|f\|_{\rho}) \\ &\leq C\delta^{-3\tau}\epsilon. \end{aligned}$$

From (49), we have the estimate

$$(62) \quad \begin{aligned} \|\hat{v}\|_{\rho-\delta/2} &\leq \|A\|_{\rho-\delta/2} + |\hat{\sigma}|\|B\|_{\rho-\delta/2} \\ &\leq C\delta^{-4\tau}\epsilon. \end{aligned}$$

Also, from the estimates for G, D and σ , we have

$$(63) \quad |\lambda| \leq C\epsilon + C\delta^{-2\tau}\epsilon \leq C\delta^{-2\tau}\epsilon.$$

4.6. Estimates on the improved error. Next we estimate the new error for both equilibrium and factorization equation after one iteration step.

We need the Assumption (H4) to make sure the compositions can be done in appropriate domains. Recall that $\|W_{v^0}\|_{\mathcal{D}} < M$. Also note \hat{v} is relatively smaller than v^0 . From (63), we see that as long as ϵ is small enough the range of $v + \hat{v}$ is inside the domain of W . So we can choose proper ξ such that $\text{dist}(\mathcal{R}_{v^0}, \partial\mathcal{D}) \geq \xi > 0$.

For the subsequent iterative step, we note that if $\delta^{-4\tau}\epsilon$ is small enough, $\|\hat{v}\|_{\rho-\delta}$ will also be small. So that under smallness condition on $\delta^{-4\tau}\epsilon$, we can ensure that $v + \hat{v}$ is well inside the domain of definition of W . The linear

estimates are valid for all δ 's, but in order to ensure that we can apply the non-linear estimates, we have to choose δ 's (the domain loss) in such a way that $\delta^{-4\tau}\epsilon$ is small.

It is standard in KAM theory (and we will do it later in section 4.7) that one can choose domain losses δ_n in such a way that the composition condition is met, such that δ_n 's go to zero fast enough, so there is still a domain left. Therefore, the composition will remain in the proper domain for all iterative steps and the procedure will converge in a non-trivial domain.

Using the Taylor expansions (see Proposition 1) and the equations (35) and (45) for the initial guesses, we have:

$$\begin{aligned}
& \mathcal{E}[v^0 + \hat{v}, \sigma^0 + \hat{\sigma}, \lambda^0 + \hat{\lambda}] \\
&= \mathcal{E}[v^0, \sigma^0, \lambda^0] + \hat{v}_+ + \hat{v}_- - 2\hat{v} + \hat{\sigma}v^0 + \sigma^0\hat{v} + \hat{\sigma}\hat{v} + \hat{\lambda} - W_{v^0} + W_{v^0+\hat{v}} \\
(64) \quad &= e + \mathcal{A}_+\mathcal{A}_-\hat{v} - f\hat{v} - (\partial_\beta W_{v^0} + \sigma^0)\hat{v} + \hat{\sigma}v^0 + \sigma^0\hat{v} + \hat{\sigma}\hat{v} + \hat{\lambda} \\
&\quad - W_{v^0} + W_{v^0+\hat{v}} \\
&= -f\hat{v} + \hat{\sigma}\hat{v} + (W_{v^0+\hat{v}} - W_{v^0} - \partial_\beta W_{v^0}\hat{v}),
\end{aligned}$$

and

$$\begin{aligned}
& \mathcal{F}[v^0 + \hat{v}, \sigma^0 + \hat{\sigma}, c^0 + \hat{c}] \\
&= \mathcal{F}[v^0, \sigma^0, c^0] + [-(c^0 + \hat{c}) + 2 - \partial_\beta W_{v^0} - (\sigma^0 + \hat{\sigma})]\hat{c}_+ \\
(65) \quad &+ [-\hat{c} - \partial_\beta W_{v^0+\hat{v}} + \partial_\beta W_{v^0} - \hat{\sigma}]c_+^0 \\
&= [-\hat{c} - (\partial_\beta W_{v^0+\hat{v}} - \partial_\beta W_{v^0}) - \hat{\sigma}]\hat{c}_+ \\
&\quad - [(\partial_\beta W_{v^0+\hat{v}} - \partial_\beta W_{v^0})c_+^0 - \partial_\beta \partial_\beta W_{v^0}c_+^0\hat{v}].
\end{aligned}$$

Take $0 < \delta < \rho$, we have

$$\begin{aligned}
(66) \quad & \|\mathcal{E}[v^0 + \hat{v}, \sigma^0 + \hat{\sigma}, \lambda^0 + \hat{\lambda}]\|_{\rho-\delta} \\
&\leq \|f\hat{v}\|_{\rho-\delta} + \|\hat{\sigma}\hat{v}\|_{\rho-\delta} + \|W_{v^0+\hat{v}} - W_{v^0} - \partial_\beta W_{v^0}\hat{v}\|_{\rho-\delta} \\
&\leq \|f\|_{\rho-\delta}\|\hat{v}\|_{\rho-\delta} + \|\hat{\sigma}\|_{\rho-\delta}\|\hat{v}\|_{\rho-\delta} + C\|\hat{v}\|_{\rho-\delta}^2.
\end{aligned}$$

Also,

$$\begin{aligned}
(67) \quad & \|\mathcal{F}[v^0 + \hat{v}, \sigma^0 + \hat{\sigma}, c^0 + \hat{c}]\|_{\rho-\delta} \\
&\leq (\|\hat{c}\|_{\rho-\delta} + C\|\hat{v}\|_{\rho-\delta} + \|\hat{\sigma}\|_{\rho-\delta})\|\hat{c}\|_{\rho-\delta} + C\|\hat{v}\|_{\rho-\delta}^2\|c^0\|_{\rho-\delta}.
\end{aligned}$$

Now it is clear the new errors are quadratic in $f, \hat{v}, \hat{\sigma}, \hat{c}$, multiplied by the domain loss to a negative power. This is what are called ‘‘tame estimates’’ in KAM theory. The final estimates for both equilibrium and factorization equations on the updated solutions are

$$(68) \quad \|\mathcal{E}[v^0 + \hat{v}, \sigma^0 + \hat{\sigma}, \lambda^0 + \hat{\lambda}]\|_{\rho-\delta} \leq C\delta^{-8\tau}\epsilon^2$$

and

$$(69) \quad \|\mathcal{F}[v^0 + \hat{v}, \sigma^0 + \hat{\sigma}, c^0 + \hat{c}]\|_{\rho-\delta} \leq C\delta^{-8\tau}\epsilon^2.$$

Therefore we get updates for c, v, σ and λ which reduce the error of both the equilibrium equation and the factorization equation quadratically under the condition that all the bounds we derived before still hold for all iterative steps. We can prove the convergence following standard procedure, which we give in the following.

4.7. Proof of convergence. The main effect of the iterative step is to reduce (very much) the error in (47) and (46). But it could also deteriorate the constants in the non-degeneracy assumptions when it improves the solution.

Therefore the first goal of this section is to show that the constants in non-degeneracy assumptions of the solution do not deteriorate much and that the deterioration can be estimated by the error. In this case, as we will see, the convergence can be proved by choosing suitable domains (this is the choice we will do first) for each iterative step as is standard in KAM theory.

We will use subscript n to denote the quantities ρ, δ and ϵ after application of the iterative step n times, while we use superscript n for v, c, σ, λ and B . We take

$$(70) \quad \rho_0 = \rho, \rho_1 = \rho - s - \delta_0, \delta_n = \delta_0 \cdot 2^{-n} \quad \text{and} \quad \rho_{n+1} = \rho_n - \delta_n.$$

Denote $\epsilon_n = \|\mathcal{E}(v^n, \delta^n, \lambda^n)\|_{\rho_n}$ and $\tilde{\epsilon}_n = \|\mathcal{F}(v^n, \sigma^n, c^n)\|_{\rho_n}$. We will prove the following holds for all iterative steps by induction.

(B1): All the quantities $\|v^n\|_{\rho_n}, \|c^n - 1\|_{\rho_n}, |\sigma^n|, \|W_{v^0}\|_{\rho_n}, \|B^n\|_{\rho_n}, \|\gamma_{c^n}\|_{\rho_n}, \|\gamma_{c^n}^{-1}\|_{\rho_n}, \|\gamma_{v^n}\|_{\rho_n}, \|\gamma_{v^n}^{-1}\|_{\rho_n}$, and $\|(-c_+^0 - \partial_\beta \partial_\beta W_{v^0} c_+^0 B)^{-1}\|_{\rho_n}$ are still bounded uniformly in n . To be precise, for any quantity $A^0 < E$, we will prove $|A^n - A^{n-1}| < E \cdot 2^{-n}$, therefore, $A^n < 2E$ for any n .

(B2): Denote $\mathcal{R}_{v^n} = \{(\psi, \eta) + \beta v^n, (\psi, \eta) \in \mathbb{T}_{\rho_n}^d\}$. Then $\text{dist}(\mathcal{R}_{v^n}, \partial\mathcal{D})$ is also bounded by ξ^* .

(B3): $\epsilon_{n+1} \leq (C\epsilon_0)^{2^{n+1}}, \quad \tilde{\epsilon}_{n+1} \leq (C\tilde{\epsilon}_0)^{2^{n+1}}$.

Note that as a consequence of (B3), and the choices of δ_n , we obtain that the composition assumption holds if ϵ_0 is small enough.

The first step is already shown in Section 4.5. Now we assume the first n steps are proved.

We first prove (B1). We only prove the bound

$$(71) \quad \|\gamma_{c^n} - \gamma_{c^{n-1}}\|_{\rho_n - \delta/4} < E \cdot 2^{-n}.$$

Proof of the other bounds is similar (up to changing the symbols).

From (24), we have

$$(72) \quad \|\log \gamma_{c^n} - \log \gamma_{c^{n-1}}\|_{\rho_n - \delta_n/4} \leq C\delta_n^{-\tau} \|\log c^n - \log c^{n-1}\|_{\rho_n}.$$

From estimate (61) for \hat{c} , we have

$$(73) \quad \|\hat{c}^n\|_{\rho_n} \leq C\delta_n^{-3\tau}\epsilon_n.$$

As a result, we have

$$(74) \quad \|\log c^n - \log c^{n-1}\|_{\rho_n} \leq C\delta_n^{-3\tau}\epsilon_n.$$

Combining (72) and (74), we have

$$(75) \quad \|\log \gamma_{c^n} - \log \gamma_{c^{n-1}}\|_{\rho_n - \delta_n/4} \leq C\delta_n^{-4\tau}\epsilon_n.$$

Therefore we have (using that γ_{c^n} are uniformly bounded)

$$(76) \quad \|\gamma_{c^n} - \gamma_{c^{n-1}}\|_{\rho_n - \delta_n/4} \leq C\delta_n^{-4\tau}\epsilon_n.$$

Note for (B2), we only need bounds for v^n , which is easy to prove.

Now recall $\epsilon_n < (C\epsilon_0)^{2^n}$. Therefore if we choose ϵ_0 such that $\delta^{-4\tau}\epsilon_0$ is small enough, we can guarantee (71) (i.e. (B2)). (B3) can be shown as follows

$$(77) \quad \begin{aligned} \epsilon_n &\leq C\delta_{n-1}^{-8\tau}\epsilon_{n-1}^2 \\ &= C\delta_0^{-8\tau}(2^{8\tau})^{n-1}\epsilon_{n-1}^2 \\ &\leq (C\delta_0^{-8\tau})^{1+2+\dots+2^{n-1}}(2^{8\tau})^{(n-1)+(n-2)\cdot 2+\dots+1\cdot(n-1)}\epsilon_0^{2^n} \\ &\leq (C\delta_0^{-8\tau}2^{8\tau}\epsilon_0)^{2^n} \\ &= (C2^{8\tau}\epsilon_0)^{2^n}. \end{aligned}$$

Similar estimates also hold for $\tilde{\epsilon}_n$.

4.8. Proof of local uniqueness. Suppose we have two solutions $(v_1, \sigma_1, \lambda_1, c_1)$ and $(v_2, \sigma_2, \lambda_2, c_2)$ of both the equilibrium and the factorization equations, which also satisfy the non-degeneracy conditions. We have

$$(78) \quad \mathcal{E}[v_1, \sigma_1, \lambda_1] = \mathcal{E}[v_2, \sigma_2, \lambda_2] = 0.$$

Then we can write

$$(79) \quad \begin{aligned} \mathcal{E}[v_2, \sigma_2, \lambda_2] &= \mathcal{E}[v_1, \sigma_1, \lambda_1] + D\mathcal{E}[v_1, \sigma_1, \lambda_1] \cdot (v_2 - v_1, \sigma_2 - \sigma_1, \lambda_2 - \lambda_1) + R^2 \\ &= D\mathcal{E}[v_1, \sigma_1, \lambda_1] \cdot (v_2 - v_1, \sigma_2 - \sigma_1, \lambda_2 - \lambda_1) + R^2 = 0, \end{aligned}$$

where R^2 is the Taylor remainder for \mathcal{E} .

Another way to read this identity (79) is to say that $v_1 - v_1, \lambda_2 - \lambda_1, \sigma_2 - \sigma_1$ is a solution of the equation

$$(80) \quad \mathcal{A}_+\mathcal{A}_-(v_2 - v_1) + (\sigma_2 - \sigma_1)v^1 + (\lambda_2 - \lambda_1) = -R^2.$$

Similarly, we have

$$(81) \quad -c_+^1(c_2 - c_1) + (-c^1 + 2 - \partial_\beta W_{v^1 - \sigma^1})(c_2 - c_1)_+ - c_+^1(\sigma_2 - \sigma_1) - \partial_\beta \partial_\beta W_{v^1 c_+^1}(v_2 - v_1) = -\tilde{R}^2.$$

We have shown in Section 4.5 that the solutions of (80) and (81) satisfying the normalization condition are unique and satisfy the bounds (60) and (62).

So we have

$$(82) \quad \|v_2 - v_1\|_{\frac{\rho}{2}} + \|c_2 - c_1\|_{\frac{\rho}{2}} + |\sigma_2 - \sigma_1| \leq C\rho^{-4\tau} \|R^2\|_{\rho} \leq C\rho^{-4\tau} (\|v_2 - v_1\|_{\rho} + \|c_2 - c_1\|_{\rho} + |\sigma_2 - \sigma_1|)^2.$$

Since $\|v_2 - v_1\|_{\rho}$, $\|c_2 - c_1\|_{\rho}$ and $|\sigma_2 - \sigma_1|$ are all small, we have

$$(83) \quad \begin{aligned} \|v_2 - v_1\|_{\frac{\rho}{2}} &\leq C\rho^{-4\tau} \|v_2 - v_1\|_{\rho}^2 \\ &\leq C\rho^{-4\tau} \|v_2 - v_1\|_{\frac{\rho}{2}} \|v_2 - v_1\|_{\frac{3\rho}{2}}. \end{aligned}$$

The last inequality uses the interpolation inequality (7).

Therefore, as long as $\rho^{-4\tau} \|v_2 - v_1\|_{\frac{3\rho}{2}}$ is sufficiently small (depending on the properties of the auxiliary function c in the factorization equation), we have $\|v_1 - v_2\|_{\frac{\rho}{2}} = 0$, which also implies $\sigma_2 = \sigma_1$ and $\lambda_2 = \lambda_1$.

5. CONSEQUENCE OF THEOREM 1 AND ITS PROOF

5.1. Perturbative series around any solution. In this section we present methods to compute formal power series expansions. In Section 5.2 we will show that, under Diophantine conditions on the frequency, they converge on a sufficiently small domain.

Assume:

A0: We are given a family of interaction potentials W^{μ} indexed by an external parameter μ which can be complex. We assume that W^{μ} is analytic in both its arguments ψ, η and the parameter μ .

A1: For some parameter μ_0 we have a solution v^0, σ^0, λ^0 of (2). We assume that $v^0 \in \mathcal{A}_{\rho}$ for some $\rho > 0$.

A2: The operator \mathcal{L}_{v^0} , the linearization of the equilibrium equation factorizes at μ_0 into two first order operators.

Our goal is to find a perturbative expansion of the solutions of (2) in formal power series of $\mu - \mu_0$. Later, we will show that these perturbative expansions are convergent following an argument of [Mos67] which is made much easier by the a-posteriori format of Theorem 1. See Section 5.2. We seek

$$(84) \quad \begin{aligned} v^{\mu} &= v^0 + \sum_{n>0} (\mu - \mu_0)^n v^n \\ \sigma^{\mu} &= \sigma^0 + \sum_{n>0} (\mu - \mu_0)^n \lambda^n \\ \lambda^{\mu} &= \lambda^0 + \sum_{n>0} (\mu - \mu_0)^n \lambda^n \end{aligned}$$

in such a way that, when we substitute it in (2) and expand (formally) in powers of $(\mu - \mu_0)^n$ we obtain that the coefficients of same powers match.

Note that this generalizes the standard Lindstedt series, which is a particular case of the expansion in the case that the family is just $W^\mu = \mu W$ and that we expand near $\mu_0 = 0$.

If we substitute (84) in (2), expand in powers of $\mu - \mu_0$, the coefficient of order n has the form:

$$(85) \quad v_\eta^n(\psi + \Omega) + v_\eta^n(\psi - \Omega) + v_\eta^n(\psi)(-2 + \partial_\beta W^{\mu_0}((\psi, \eta))) + \sigma^0 v^n + \sigma^n v^0 \lambda^n = R_n$$

where R_n is a polynomial expression in v^0, \dots, v^{n-1} .

The main observation is that the equation (85) is precisely the Quasi-Newton equations (47) which can be solved by factorization. Note that we get the perturbative series to all orders only assume that W^μ factorizes at $\mu = \mu_0$ and the expansion series we get has $\sigma = 0$ to all orders.

We note that to solve the cohomology equations without any quantitative estimates, as shown in [SZdlL14], it suffices to assume

$$(86) \quad \lim_{N \rightarrow \infty} \frac{1}{N} \sup_{|\tilde{k}| \leq N, m \in \mathbb{Z}} \left| \ln |\tilde{k} \cdot \Omega - m| \right| = 0,$$

which is weaker than Diophantine and, indeed weaker than Brjuno conditions.

We have, therefore established the following:

Lemma 3. *Under the assumptions **A0**, **A1** and that Ω satisfies the quantitative conditions.*

Then, we can find formal power series as in (84) such that for any $N \in \mathbb{N}$ and any $\rho', 0 < \rho' < \rho$, we have

$$\left\| \mathcal{E}_\mu \left[\sum_{n \leq N} v^n (\mu - \mu_0)^n, \sum_{n \leq N} \lambda^n (\mu - \mu_0)^n \right] \right\|_{\rho'} \leq C_{N, \rho'} |\mu - \mu_0|^{N+1}.$$

Furthermore, it is possible to find a formal power series that satisfies the normalization

$$\langle v_\eta^n \rangle = 0 \quad n = 1, \dots, N.$$

We refer to it as the “normalized perturbative expansion”.

This normalized perturbative expansion is unique.

Similarly, we can obtain existence of perturbation expansions for both the equilibrium and the factorization equations. In analogy with (84), we seek expansions

$$(87) \quad \begin{aligned} v^\mu &= v^0 + \sum_{n>0} (\mu - \mu_0)^n v^n & \lambda^\mu &= \lambda^0 + \sum_{n>0} (\mu - \mu_0)^n \lambda^n \\ c^\mu &= c^0 + \sum_{n>0} (\mu - \mu_0)^n c^n & \sigma^\mu &= \sigma^0 + \sum_{n>0} (\mu - \mu_0)^n \sigma^n \end{aligned}$$

in such a way that the equilibrium and factorization equations are solved.

The equations for order n are

$$(88) \quad \begin{aligned} v_+^n + v_-^n + (-2 + \partial_\beta W_{v^0}^{\mu_0} + \sigma^0) v^n + v^0 \sigma^n + \lambda^n &= R_n \\ -c_+^0 c^n + (-c^0 + 2 - \partial_\beta W_{v^0}^{\mu_0} - \sigma^0) c_+^n - c_+^0 \sigma^n - \partial_\beta \partial_\beta W_{v^0}^{\mu_0} c_+^0 v^n &= \tilde{R}_n. \end{aligned}$$

From the proof of the KAM theorem, it's clear these can be solved to all orders assuming that W^μ factorizes at $\mu = \mu_0$. The series we get do not have $\sigma = 0$, which is different from the series when we do not require the factorization for all orders.

We have, therefore established the following:

Lemma 4. *Under the assumptions **A0**, **A1**, **A2** and that Ω satisfies the quantitative conditions.*

Then, we can find formal power series as in (87) such that for any $N \in \mathbb{N}$ and any ρ' , $0 < \rho' < \rho$, we have

$$\left\| \mathcal{E}_\mu \left[\sum_{n \leq N} v^n (\mu - \mu_0)^n, \sum_{n \leq N} \sigma^n (\mu - \mu_0)^n, \sum_{n \leq N} \lambda^n (\mu - \mu_0)^n \right] \right\|_{\rho'} \leq C_{N, \rho'} |\mu - \mu_0|^{N+1}$$

and

$$\left\| \mathcal{F}_\mu \left[\sum_{n \leq N} v^n (\mu - \mu_0)^n, \sum_{n \leq N} \sigma^n (\mu - \mu_0)^n, \sum_{n \leq N} c^n (\mu - \mu_0)^n \right] \right\|_{\rho'} \leq C_{N, \rho'} |\mu - \mu_0|^{N+1}.$$

Furthermore, it is possible to find a formal power series that satisfies the normalization

$$\langle v_\eta^n \rangle = 0 \quad n = 1, \dots, N.$$

We refer to it as the “normalized perturbative expansion”.

This normalized perturbative expansion is unique.

The above results could be interpreted in a more geometric way. We have proved convergence for the perturbative expansion involving also the counterterms that ensure the factorization. Given a potential W small, the main result shows that we can find a locally unique counterterm σ , which is a functional on W so that the equation factorizes. The set $\{W(v) + \sigma(W)v\}$ can be interpreted as a codimension 1 manifold (the factorization manifold) in the space of potentials.

The perturbative expansions in Lemma 2 can be interpreted as the perturbative expansions for a path of potentials indexed by ϵ but requiring that W stays on the factorization manifold.

In this geometric interpretation, the condition (54) that among σ we can achieve the factorization can be interpreted geometrically as saying that the direction (in the space of potentials) given by σv are transversal to the factorization manifold F . Note that this factorization manifold may depend on ω .

5.2. Convergence of Lindstedt series for the equilibrium and factorization equations.

Theorem 2. *Assume that the conditions of Lemma 4 hold and that Ω is Diophantine. Then, the normalized formal series obtained in Lemma 4 is convergent.*

Note that a particular case of the above result is the convergence of the Lindstedt series when the frequency is Diophantine.

We need the frequency to be Diophantine because, as we will see, the proof uses repeatedly the KAM theorem. It seems quite possible that for the frequencies that satisfy (86) but not (11), for many perturbations, it is possible to obtain perturbative expansions to all orders, which nevertheless do not converge.

Proof. Without loss of generality, we assume that v^0 satisfies (3).

Using the KAM theorem (Theorem 1), we obtain that for all μ in a small ball centered on μ_0 there exists a (unique) normalized solution. (It suffices to note that v^0 is a sufficiently approximate solution for all μ close to μ_0 .)

Now, given any $\tilde{\mu}$ in this small ball, we can obtain a perturbative expansion in powers of $\mu - \tilde{\mu}$.

We consider the first term of the expansion $v_{\tilde{\mu}}^1$. We remark that it will be uniformly bounded in $\|\cdot\|_{\rho'}$.

We note that because

$$\|\mathcal{E}_{\mu}[v_{\tilde{\mu}}^0 + (\mu - \tilde{\mu})v_{\tilde{\mu}}^1, \lambda_{\tilde{\mu}}^0 + (\mu - \tilde{\mu})\lambda_{\tilde{\mu}}^1]\|_{\rho'} \leq C|\mu - \tilde{\mu}|^2$$

we can apply the KAM theorem (note that the non-degeneracy conditions of the KAM theorem are satisfied with uniform bounds when the ball is considered small enough) and obtain that there is a normalized solution $v_{\mu}^*, \lambda_{\mu}^*$ of (2) for any value of μ in a ball around $\tilde{\mu}$ and that it satisfies

$$(89) \quad \begin{aligned} \|v_{\mu}^* - v_{\tilde{\mu}}^0 + (\mu - \tilde{\mu})v_{\tilde{\mu}}^1\|_{\rho''} &\leq C|\mu - \tilde{\mu}|^2 \\ \|\lambda_{\mu}^* - \lambda_{\tilde{\mu}}^0 + (\mu - \tilde{\mu})\lambda_{\tilde{\mu}}^1\|_{\rho''} &\leq C|\mu - \tilde{\mu}|^2. \end{aligned}$$

Using the uniqueness obtained in the KAM we obtain that $v_\mu^* = v_\mu^0$, $\lambda_\mu^* = \lambda_\mu^0$. Hence, (89) means that $v_{\tilde{\mu}}^1$ is the derivative at $\mu = \tilde{\mu}$ of the mapping that to μ associates v_μ^0, λ_μ^0 if we give v the topology in $\mathcal{A}_{\rho''}$.

We recall that the Cauchy-Goursat theorem shows that any complex function which is differentiable at every point, is analytic [Ahl78]. This argument also works for functions taking values in Banach spaces. Alternatively, it is not difficult to show that the mapping $\mu \mapsto v_\mu^1$ is continuous (a quick way is to show that the graph of the map is closed because of the uniqueness and that, since it uniformly bounded, it is compact by Montel's theorem [Ahl78]).

Once we have that the function is analytic, we know its Taylor series converges, but the Taylor series has to be the one given by the formal series expansion. \square

Notice as a corollary of the dependence on parameters we can obtain that the solution is analytic in the parameter η .

Since in the physical applications $\eta \in \mathbb{T}^1$ is important to discuss the periodicity in η of the solutions thus obtained, we remark that, if we start with an approximate solution which is periodic in η , we will obtain a solution which is also periodic in η .

This can be seen in two ways. One can observe that, applying Theorem 1 we can obtain solutions in small enough intervals of η . They will be analytic in these small intervals and therefore they give a global analytic solution. Furthermore, we observe that if the approximate solution corresponding to $\eta = 1$ is close to the solution corresponding to $\eta = 0$, they have to agree because of uniqueness. Hence, we obtain that the solution is periodic.

We could also argue that, the proof is based on an iterative step and that, by examining the proof, all the steps preserve the periodicity in η of the approximate solutions. Hence, the KAM procedure that we describe for a fixed η lifts to a procedure for periodic functions of η .

The lifting of the problem to a space of functions of η also gives a direct proof of smooth dependence on parameters.

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SCHOOL OF MATHEMATICS, GEORGIA INSTITUTE OF TECHNOLOGY, 686 CHERRY ST., ATLANTA
GA 30332, USA

E-mail address: lzhang98@math.gatech.edu

SCHOOL OF MATHEMATICAL SCIENCES, BEIJING NORMAL UNIVERSITY, No. 19, XINJIEKOUWAI
ST., HAI DIAN DISTRICT, BEIJING 100875, P. R. CHINA

E-mail address: billy3492@gmail.com, xfsu@bnu.edu.cn

SCHOOL OF MATHEMATICS, GEORGIA INSTITUTE OF TECHNOLOGY, 686 CHERRY ST., ATLANTA
GA 30332, USA

JLU-GT JOINT INSTITUTE FOR THEORETICAL SCIENCE, JILIN UNIVERSITY, CHANGCHUN, 130012,
CHINA

E-mail address: rafael.delallave@math.gatech.edu