CONSTRUCTION OF QUASI-PERIODIC SOLUTIONS OF STATE-DEPENDENT DELAY DIFFERENTIAL EQUATIONS BY THE PARAMETERIZATION METHOD I: FINITELY DIFFERENTIABLE, HYPERBOLIC CASE

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ABSTRACT. In this paper, we use the parameterization method to construct quasi-periodic solutions of state-dependent delay differential equations. For example

$$\begin{cases} \dot{x}(t) = f(\theta, x(t), \epsilon x(t - \tau(x(t)))) \\ \dot{\theta}(t) = \omega. \end{cases}$$

Under the assumption of exponential dichotomies for the $\epsilon = 0$ case, we use a contraction mapping argument to prove the existence and smoothness of the quasi-periodic solution. Furthermore, the result is given in an *a − posteriori* format. The method is very general and applies also to equations with several delays, distributed delays etc.

1. INTRODUCTION

Differential equations with state-dependent delay(SD-DDEs) appear as natural models in several scientific disciplines, such as Physics, automatic control, neural networks, infectious diseases, population growth and cell production. One pioneering work on models of electrodynamics is [Dri63]. A recent survey with many references is [HKWW06]. Such problems fall outside the scope of the theory of constant delay equations, which are treated as dynamical systems with semi-flows on an infinite-dimensional functional space(see, e.g., [DvGVLW95, GW13, HVL93]). The complicated structure of SD-DDEs gives rise to challenging problems both in mathematical analysis and in numerical computation.

From an abstract point of view, the state dependent delay equations involve the composition operator. The lack of regularity of the evaluation operator *ev* on many of the spaces makes unavailable some familiar results

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of solutions, such as existence, uniqueness, smooth dependence on the initial data and parameters. Several papers [Wal03a, Wal03b, HKWW06] have contributed to constructing a well-defined phase space and to formulating the fundamental theory as an infinite dimensional dynamical system. In the abstract framework, one introduces the so called solution manifold, a smooth submanifold of finite co-dimension of continuous function space, and proves under mild hypothesis that the initial value problem is well-posed on the solution manifold, and the solutions define a semiflow of continuous differential solution operators.

The paper [HKWW06] develops the so called *sun − star* calculus (see [DvGVLW95] for details) to prove the existence of Lipschitz local center manifold at a stationary point. Later, [Kri06] improves the regularity of the obtained center manifold to continuously differentiable. Furthermore, a local Hopf bifurcation theorem has been proved in [Eic06] for SD-DDEs. [Sie12] provides an alternative proof by showing the periodic boundary-value problems for delay differential equations are locally equivalent to finite dimensional algebraic systems of equations. Applying the S^1-equivalent degree, [HW10] develops a global Hopf bifurcation theory. By the Poincaré maps, [MPN11] studies the stability of periodic solutions of SD-DDEs. There are also some numerical methods contributed to the study for these classes of differential equations, such as [HDMU12, MKW14].

Then standard program for dynamical systems would go from the existence and uniqueness to the qualitative property of solutions. The purpose of this paper is to propose a different approach- the parameterization method, which, in general, formulates a functional equation for a parameterization of the invariant manifold as well as the dynamics on it. Rather than trying to seek all the solutions, we construct solutions with good recurrence properties, such as periodic, quasi-periodic solutions, stable/unstable manifolds together with those asymptotic to them. The parameterization method lends itself to very efficient computer implementations since it provides a global representation of the manifold, and it also allows a very efficient discussion of dependence on parameters. An introduction on the parameterization method is presented in [CFdlL03, CFdlL05, HCF+]. Some extensions of this method to quasi-periodic systems and numerical implementations are developed in [HdlL06, LdlL09, HdlL13].

In this paper, we focus on the construction of quasi-periodic solutions of SD-DDEs via the parameterization method. To this end, the evolution problem and the existence of quasi-periodic solution are transformed into solving a functional equation on a Banach space, which allows us to apply various methods in nonlinear analysis. As our motivation, we show the persistence of quasi-periodic solutions under some hyperbolic hypothesis. Furthermore, an *a posteriori* result is obtained. More precisely, given a function that solves the functional equation approximately and that satisfies some non-degeneracy conditions, then there is a true solution. Moreover, the distance from the true solution to the approximate one is bounded by the residual of the approximate solution in the functional. One can, for example, take as an approximate solution the result of a numerical computation. To verify the reliability of the computed solutions, it suffices to check that they satisfy the equation approximately and that they satisfy the non-degeneracy conditions. See [HdlL06] for example.

We do not aim to present technically optimal results nor discuss the most general models, but only to present some significant results that illustrate the main idea of our method. We mention that the work in [LdlL09] also uses the parameterization method to construct invariant tori when considering a quasi-periodic perturbation on the autonomous linear delay equations. Since the spectrum of the linear system may intersect the imaginary axis, the small divisors problem appears inevitably, which causes difficulties. Nevertheless, under the assumption of exponential dichotomy, we will not encounter the small divisor problem in this paper.

In this paper, we will consider the following quasi-periodic differential equation with state-dependent delay

$$\begin{cases} \dot{x}(t) = f(\theta, x(t), \epsilon x(t - \tau(x(t)))) \\ \dot{\theta}(t) = \omega \end{cases} \tag{1.1}$$

where $\theta \in \mathbb{T}^d = \mathbb{R}^d/2\pi\mathbb{Z}^d$, f is now defined on $\mathbb{T}^d \times \mathbb{R}^n \times \mathbb{R}^n$, and the frequency $\omega \in \mathbb{R}^d$ is rationally independent, i.e. $\omega \cdot k \neq 0$ for all $k \in \mathbb{Z}^d - \{0\}$. Our method is to find a function $K : \mathbb{T}^d \to \mathbb{R}^n$ solving the functional equation

$$\partial K \cdot \omega = f(Id, K, \epsilon K \circ (Id - \omega\tau \circ K))$$

in such a way that $x(t) = K(\theta + \omega t)$ is a solution of (1.1)

It should be clear that the method developed here can be adapted to other problems, such as several delays. In Section 4, we give some of these extensions, which are technically not too complicated.

Our main results(see Theorem 3.1) are based on a perturbation setting, i.e. the positive parameter ϵ in (1.1) is assumed to be small enough. We also need to assume the unperturbed systems satisfy some non-degeneracy assumptions, which correspond to hyperbolicity. Similar results have already been obtained for constant delay differential equations using the evolution operators (see [HVL93]). However, by our methods, we provide an efficient and direct way to discuss the existence and regularities of the invariant objects, without involving the evolution operator or variation equations. We note that in this paper, we consider finitely differentiable solutions. The analytic regularity require different considerations.

The paper is organized as follows. In Section 2, we give some definitions and properties of function spaces and exponential dichotomies, which are well known and can be skipped at a first reading. The new results are given in Section 3. In Section 3, we present and prove an *a posteriori* theorem for the quasi-periodic model by the contraction mapping theorem. See Theorem 3.1. As a corollary, we show the persistence of quasi-periodic solutions when the unperturbed system has a quasi-periodic solution. We conclude by the last section explaining some improvements and further work.

2. Preliminaries

In this section, we collect some standard definitions and recall some well-known properties. This will serve to set the notations. This section could be skipped in a first reading.

2.1. **Function spaces.** In this paper, we will work on a scale of Hölder spaces $\{C^r(\mathbb{T}^d, \mathbb{R}^n)\}_{r\geq 0}$. More precisely, for the integer r, we refer to the set of continuously differentiable periodic functions u of order r, with the norm

$$\|u\|_r = \sup_{0\leq k\leq r} \sup_{\theta\in\mathbb{T}^d} \|D^k u(\theta)\| < \infty.$$

For the non-integer r, we mean the Hölder spaces

$$\left\{ u \in C^{[r]}(\mathbb{T}^d, \mathbb{R}^n) : \|u\|_r = \max\{ \sup_{\theta\neq\xi\in\mathbb{T}^d} \frac{\|D^{[r]}u(\theta) - D^{[r]}u(\xi)\|}{|\theta-\xi|^{r-[r]}}, \|u\|_{[r]} \} < \infty \right\},$$

where $[r]$ is the integer part of positive r. Obviously, for $0 < r < s$ one has

$$C^0 \hookrightarrow C^r \hookrightarrow C^s \supseteq C^\infty \equiv \bigcap_{r>0} C^r$$

where " $\hookrightarrow$ " represents existence of a continuous embedding.

We recall some results on the regularity of the composition of two functions expressed in the scale of spaces introduced above. It is well known that a composition on the right is an operator that causes loss of derivatives. The composition operator appears naturally in section 3, because of the state-dependent delay terms. Thus we cite several theorems on these arguments in a concrete form. For more details, we refer the reader to [dlLO99].

Lemma 2.1. *Let real numbers* $r \geq 1, s \geq 1, l \geq 0$ *and* $t = \min(r, s)$.

(i) If $f \in C^l(\mathbb{T}^d, \mathbb{R}^n)$ *and* $g \in C^l(\mathbb{T}^d, \mathcal{L}(\mathbb{R}^n, \mathbb{R}^n))$*, then* $g \cdot f \in C^l(\mathbb{T}^d, \mathbb{R}^n)$ *and*

$$\|g \cdot f\|_l \leq 2^l \|g\|_l \cdot \|f\|_l \; . \tag{2.1}$$

(ii) If $f \in C^r(\mathbb{T}^d, \mathbb{R}^n)$ *and* $g \in C^s(\mathbb{T}^d, \mathbb{R}^n)$*, then* $g \circ f \in C^t(\mathbb{T}^d, \mathbb{R}^n)$ *and there is a constant* M_t *such that*

$$\|g \circ f\|_t \leq M_t \|g\|_t (1 + \|f\|_t^t). \tag{2.2}$$

The above lemma is mainly used to estimate the norms when composing with other functions. For simplicity, we will always denote the constant on the right hand sides by C when proving our main theorems. The next lemma is on some sufficient conditions for the continuity of composition operators, which is used in Theorem 3.1.

Lemma 2.2. *(Continuity) Let $r \geq 1, s \geq 1, \min(r,s) > t \geq 0$ and $g \in C^s(\mathbb{T}^d, \mathbb{R}^n)$. Then the map*

$$g_* : C^r(\mathbb{T}^d, \mathbb{R}^n) \ni f \mapsto g \circ f \in C^t(\mathbb{T}^d, \mathbb{R}^n)$$

is continuous. Furthermore, if t is an integer, g_* is also continuous for $min\{r,s\} \geq t$.

Another important property of the scale of Banach spaces $\{C^r(\mathbb{T}^d, \mathbb{R}^n)\}_{r\geq 0}$ is the Hadamard interpolation inequality. The interpolation inequalities allow us to control the norm of a function in the scale of Hölder spaces.

Lemma 2.3. *For all $0 \leq r \leq t$, $\alpha \in [0,1]$ and $u \in C^t(\mathbb{T}^d, \mathbb{R}^n)$, one has*

$$\begin{aligned} ||u||_s \leq& C||u||_r^{1-\alpha}||u||_t^{\alpha}, \\ s =& (1-\alpha)r + \alpha t, \end{aligned} \tag{2.3}$$

where C depends only on r, s and α.

The Lemma 2.3 can be easily proved by using the modern $C^\infty-$*smoothing* techniques, see [Zeh75]. We also refer to [dlLO99] for some general situations in which functions are defined on Banach space.

The interpolation inequalities in Lemma 2.3 lead to the following result, which is very similar to proofs of the center manifold theorem(Notably [Lan73]). We consider an operator that preserves some set which is bounded in a high regularity space and which is a contraction in a low regularity norm. We conclude that it has a fixed point in the intermediate norms. As a corollary of the argument we obtain an a-posteriori format with explicit bounds.

Lemma 2.4. *Let $\{C^r(\mathbb{T}^d, \mathbb{R}^n)\}_{r\geq 0}$ be a scale of Hölder spaces and $\mathscr{A}$ be an operator defined on the scale of spaces.*

We assume:

(i) There is an index r such that the operator $\mathscr{A}$ maps the ball $B^r_\delta(u_0)$ into itself, where u_0 is in $\mathbb{C}^r(\mathbb{T}^d, \mathbb{R}^n)$ and $B^r_\delta(u_0) = \{u \in \mathbb{C}^r(\mathbb{T}^d, \mathbb{R}^n) : ||u - u_0||_r < \delta\}$;

(ii) For any $0 \leq q \leq r$, the operator $\mathscr{A}$ is continuous from $C^q(\mathbb{T}^d, \mathbb{R}^n)$ to itself;

(iii) There is constant $0 < \kappa < 1$ such that

$$||\mathscr{A}[\hat{u}] - \mathscr{A}[u]||_0 \leq \kappa||\hat{u} - u||_0$$

for any $\hat{u}, u \in B^r_\delta(u_0)$*,*

Then, for any $0 < q < r$*, there exists a unique fixed point* u^* *of* $\mathscr{A}$ *in* $\overline{B^r_\delta(u_0)}_{C^0}$ *such that*

$$\|u^*\|_q \leq \frac{C(2\delta)^{q/r}}{1-\kappa^{1-q/r}}\|\mathscr{A}[u_0]-u_0\|_0^{1-q/r}. \tag{2.4}$$

where $\overline{B^r_\delta(u_0)}_{C^0}$ *denotes the closure of* $B^r_\delta(u_0)$ *in* $C^0(\mathbb{T}^d, \mathbb{R}^n)$*.*

Proof: Since $\mathscr{A}^n[u_0] \in B^r_\delta(u_0)$ for any n, one easily obtain

$$\|\mathscr{A}^{n+1}[u_0] - \mathscr{A}^n[u_0]\|_0 \leq \kappa^n \|\mathscr{A}[u_0] - u_0\|_0$$

and

$$\|\mathscr{A}^{n+1}[u_0] - \mathscr{A}^n[u_0]\|_r \leq 2\delta.$$

The interpolation inequality yields

$$\|\mathscr{A}^{n+1}[u_0] - \mathscr{A}^n[u_0]\|_q \leq C(2\delta)^{q/r}\|\mathscr{A}[u_0] - u_0\|_0^{(1-q/r)}\kappa^{n(1-q/r)},$$

which is a Cauchy sequence in C^q. Denoting the limit by u^*, the continuity of $\mathscr{A}$ proves it is exactly the desired fixed point and (2.4) holds. Obviously, $u^* \in \overline{B^r_\delta(u_0)}_{C^q} \subset \overline{B^r_\delta(u_0)}_{C^0}$.

If there are two fixed points $u^{(1)}$ and $u^{(2)}$ of $\mathscr{A}$ in $\overline{B^r_\delta(u_0)}_{C^0}$, we have

$$\begin{aligned}
\|u^{(1)} - u^{(2)}\|_0 &= \|\mathscr{A}[u^{(1)}] - \mathscr{A}[u^{(2)}]\|_0 \\
&\leq \|\mathscr{A}[u^{(1)}] - \mathscr{A}[u^{(1)}_n]\|_0 + \|\mathscr{A}[u^{(1)}_n] - \mathscr{A}[u^{(2)}_n]\|_0 + \|\mathscr{A}[u^{(2)}] - \mathscr{A}[u^{(2)}_n]\|_0 \\
&\leq \|\mathscr{A}[u^{(1)}] - \mathscr{A}[u^{(1)}_n]\|_0 + \kappa\|u^{(1)}_n - u^{(2)}_n\|_0 + \|\mathscr{A}[u^{(2)}] - \mathscr{A}[u^{(2)}_n]\|_0 \\
&\leq \|\mathscr{A}[u^{(1)}] - \mathscr{A}[u^{(1)}_n]\|_0 + \|\mathscr{A}[u^{(2)}] - \mathscr{A}[u^{(2)}_n]\|_0 \\
&\quad + \kappa(\|u^{(1)} - u^{(1)}_n)\|_0 + \|u^{(1)} - u^{(2)}\|_0 + \|u^{(2)} - u^{(2)}_n\|_0)
\end{aligned}$$

where $u^{(1)}_n$ and $u^{(2)}_n$ are sequences in $B^r_\delta(u_0)$ converging to $u^{(1)}$ and $u^{(2)}$ in the C^0-norm respectively. Then the continuity of $\mathscr{A}$ yields

$$\|u^{(1)} - u^{(2)}\|_0 \leq \kappa\|u^{(1)} - u^{(2)}\|_0,$$

which implies the uniqueness of $\mathscr{A}$ in $\overline{B^r_\delta(u_0)}_{C^0}$. □

Remark 2.1. *(Parameter dependence) If* $\mathscr{A}$ *is a uniform contraction on an open set* V *in a Banach space* Y*, then the fixed point mapping* $u^* : V \to C^q(\mathbb{T}^d, \mathbb{R}^n)$ *has the same regularity as the parameterized operator* $\mathscr{A} : C^q(\mathbb{T}^d, \mathbb{R}^n) \times V \to C^q(\mathbb{T}^d, \mathbb{R}^n)$*. See the uniform contraction theorem in* [Chi06].

2.2. **Exponential dichotomy.** When studying non-autonomous systems, exponential dichotomy is a fundamental tool for studying their asymptotic behavior (boundedness, stability, etc). See [CL95, SS74]. In this paper, we restricts ourselves on the quasi-periodic case.

Definition 2.1. *(Linear Skew-product flow) Assume X and Y is the Hausdorff space. Then a flow $\pi : X \times Y \times \mathbb{R} \to X \times Y$ is said to be skew-product flow if*

$$\pi(x, y, t) = (\varphi(x, y, t), \sigma(y, t)),$$

where $\sigma : Y \times \mathbb{R} \to Y$ is a flow on Y. If, in addition, $\varphi(x, y, t)$ is linear in x for every (y, t) in $Y \times \mathbb{R}$, then π is said to be a linear skew-product flow.

Definition 2.2. *(Exponential Dichotomy) Assume π is a linear skew-product flow on the product space $X\times Y$, where X is a finite dimensional vector space and $\varphi(x, y, t) = \Phi(y, t)x$. We shall say that π admits a exponential dichotomy at y with positive constants B, λ if there exists a projection P on X such that*

$$\begin{aligned} & \|\Phi(y,t)P\Phi^{-1}(y,s)\| \leq Be^{-\lambda(t-s)} && for\ t \geq s \\ \text{and} \quad & \|\Phi(y,t)(Id-P)\Phi^{-1}(y,s)\| \leq Be^{\lambda(t-s)} && for\ t \leq s. \end{aligned} \tag{2.5}$$

In this paper, we pay attention to an elementary linear skew-product flow $\pi = (\Phi, \sigma)$ on the trivial bundle $\mathbb{R}^n\times\mathbb{T}^d$. More precisely, for a quasi-periodic coefficient linear differential equation with frequency ω

$$\dot{\xi}(t) = \hat{M}(t)\xi,$$

one can extend it to the autonomous equations with phase space $\mathbb{R}^n \times \mathbb{T}^d$,

$$\begin{cases} \dot{\xi}(t) = M(\theta)\xi \\ \dot{\theta}(t) = \omega \end{cases} \tag{2.6}$$

where M is the hull of quasi-periodic function $\hat{M}$. Then $\sigma(\theta, t) = \theta + \omega t$ and $\Phi(\theta, t)$ is the principle fundamental matrix solution to

$$\dot{\xi}(t) = M(\theta + \omega t)\xi.$$

Since ω is rational independent, the closure of the flow $\sigma(\theta, t)$ starting at any base point θ is the whole torus $\mathbb{T}^d$, or equivalently, the flow is dense on the torus.

Remark 2.2. *(Smoothness) From* [JS81] *and* [SS74]*, one deduces that the existence of exponential dichotomy at any $\theta \in \mathbb{T}^d$ leads to the existence of exponential dichotomies with uniform constants B and λ. Furthermore, the associated projections $P(\theta)$ have the same regularity as M and satisfy*

$$P(\theta + \omega t) = \Phi(\theta, t)P(\theta)\Phi^{-1}(\theta, t).$$

Remark 2.3. *(Persistence) The theorem 4.3 in* [CL95] *implies the persistence of exponential dichotomy, i.e. the exponential dichotomy is not destroyed by a sufficient small perturbation of the cocycle Φ. Moreover, the exponent λ and the constant B is uniform for all systems which are sufficient small perturbations. See* [CL95] *for a more detailed description.*

2.3. **Solutions of non-homogeneous equations with dichotomy.** In the analysis on the model (1.1) in section 3, we are led to considering the solution u of the equation

$$\partial u(\theta)\cdot\omega - M(\theta)u(\theta) = g(\theta) \tag{2.7}$$

where $g \in C^r(\mathbb{T}^d, \mathbb{R}^n)$ and $M \in C^r(\mathbb{T}^d, \mathscr{L}(\mathbb{R}^n, \mathbb{R}^n))$ are given and $u \in \mathcal{B}(\mathbb{T}^d, \mathbb{R}^n)$ (the bounded functions space) is the unknown. We will be concerned with the existence and regularity properties of u solving (2.7) as well as quantitative estimates for several norms on u.

Differentiating $\Phi^{-1}(\theta, t)u(\theta + \omega t)$ with respect to t, we have that (2.7) implies

$$\frac{d}{dt}\Phi^{-1}(\theta, t)u(\theta + \omega t) = \Phi^{-1}(\theta, t)g(\theta + \omega t)$$

which yields

$$\Phi^{-1}(\theta, t)u(\theta + \omega t) = u(\theta) + \int_0^t \Phi^{-1}(\theta, s)g(\theta + \omega s)ds. \tag{2.8}$$

Next, we use the assumption of the exponential dichotomy to decompose $\mathbb{R}^n$ into $\mathcal{R}anP(\theta) \oplus \mathcal{K}erP(\theta)$ for any $\theta \in \mathbb{T}^d$. On the subspace $\mathcal{K}erP(\theta)$, we see that

$$\begin{aligned}
(Id-&P(\theta))u(\theta)\\
=&(Id - P(\theta))\Phi^{-1}(\theta, t)u(\theta + \omega t) - \int_0^t (Id - P(\theta))\Phi^{-1}(\theta, s)g(\theta + \omega s)ds\\
=&(Id - P(\theta))\Phi^{-1}(\theta, t)u(\theta + \omega t) - \int_0^t \Phi^{-1}(\theta, s)(Id - P(\theta + \omega s))g(\theta + \omega s)ds.
\end{aligned}$$

Noticing that, by Definition 2.2, we have for any θ the following estimates

$$\|(Id - P(\theta))\Phi^{-1}(\theta, t)u(\theta + \omega t)\| \le Be^{-\lambda t}\|u\|_\infty \to 0$$

as $t \to +\infty$ and

$$\int_0^{+\infty} \|(Id - P(\theta))\Phi^{-1}(\theta, s)g(\theta + \omega s)\|ds \le \int_0^{+\infty} Be^{-\lambda s}\|g\|_0 ds < +\infty.$$

Thus,

$$(Id - P(\theta))u(\theta) = -\int_0^{+\infty} \Phi^{-1}(\theta, s)(Id - P(\theta + \omega s))g(\theta + \omega s)ds. \tag{2.9}$$

The same arguments on the subspace $\mathcal{R}anP(\theta)$ lead to

$$P(\theta)u(\theta) = \int_{-\infty}^{0} \Phi^{-1}(\theta, s)P(\theta + \omega s)g(\theta + \omega s)ds. \tag{2.10}$$

Combining (2.9) and (2.10), the unique bounded solution to (2.7) is

$$\begin{aligned} u(\theta) = &\int_{-\infty}^{0} \Phi^{-1}(\theta, s)P(\theta + \omega s)g(\theta + \omega s)ds \\ &- \int_{0}^{+\infty} \Phi^{-1}(\theta, s)(Id - P(\theta + \omega s))g(\theta + \omega s)ds \end{aligned} \tag{2.11}$$

and satisfies $u \in C^0(\mathbb{T}^d, \mathbb{R}^n)$ with $||u||_0 \leq \frac{2B}{\lambda}||g||_0$.

To establish the higher regularity of the solution u, we formally calculate the derivative of (2.7) with respect to θ, which results in

$$\partial(Du) \cdot \omega - M(\theta)Du(\theta) = DM(\theta)u(\theta) + Dg(\theta) \tag{2.12}$$

Equation (2.12) is essentially the same as (2.7). Hence, we can find a unique solution Du for (2.12) which belongs to $C^0(\mathbb{R}^d, \mathcal{L}(\mathbb{R}^n, \mathbb{R}^n))$ with

$$\begin{aligned} ||Du||_0 &\leq 2B(||DM||_0 \cdot ||u||_0 + ||Dg||_0)/\lambda \\ &\leq [(2B\lambda^{-1})^2||M||_1 + 2B\lambda^{-1}] \cdot ||g||_1. \end{aligned}$$

Now we show the formal derivative is the true derivative of u ,which implies $u \in C^1(\mathbb{T}^d, \mathbb{R}^n)$. Denoting

$$Q(\theta, \xi) = u(\theta + \xi) - u(\theta) - Du(\theta) \cdot \xi$$

and taking its directional derivative, (2.7) and (2.12) yields

$$\partial Q(\theta, \xi) \cdot \omega = M(\theta)Q(\theta, \xi) + \hat{g}(\theta, \xi),$$

where

$$\begin{aligned} \hat{g}(\theta, \xi) =&[g(\theta + \xi) - g(\theta) - Dg(\theta) \cdot \xi] + [M(\theta + \xi) - M(\theta) - DM(\theta) \cdot \xi]u(\theta) \\ &+ (M(\theta + \xi) - M(\theta)) \cdot (u(\theta + \xi) - u(\theta)). \end{aligned}$$

By the mean value theorem,

$$\frac{||M(\theta + \xi) - M(\theta)|| \cdot ||u(\theta + \xi) - u(\theta)||}{|\xi|} \leq ||DM||_0 \cdot ||u(\theta + \xi) - u(\theta)||,$$

which implies $\frac{||\hat{g}(\cdot, \xi)||_0}{|\xi|}$ approaches zero as $|\xi| \to 0$ if $g, M \in C^1$. Then,

$$\frac{||Q(\cdot, \xi)||_0}{|\xi|} \leq \frac{2B||\hat{g}(\cdot, \xi)||_0}{\lambda|\xi|} \to 0.$$

This completes the argument for $r = 1$.

By induction, we assume that if $g, M \in C^{k-1}$ with $2 \leq k \leq r$, the solution u of (2.7) is C^{k-1}. Then, for (2.12), we see the inhomogeneous term is C^{k-1}

when $g, M \in C^k$, which implies the exact derivative Du is C^{k-1} and therefore u is C^k. This completes the induction.

Furthermore, we still need to show the inverse is bounded. Taking k-th derivative to (2.7), it is readily seen that $D^k u$ satisfies

$$\partial D^k u \cdot \omega - M D^k u = \mathscr{L}_k(DM \cdot D^{k-1}u, D^2 M \cdot D^{k-2}u, \cdots, D^k M \cdot u) + D^k g$$

where $\mathscr{L}_k$ is a linear combination of its arguments with integer coefficients. A trivial induction argument also yields

$$\|u\|_r \leq C\|g\|_r, \tag{2.13}$$

where $C = \mathscr{P}(\frac{2B}{\lambda}, \|M\|_r)$ and $\mathscr{P}$ is a polynomial with positive coefficients.

We conclude the above results as the following lemma.

Lemma 2.5. *Given $M \in C^r(\mathbb{T}^d, \mathbb{R}^{n\times n})$ and $r \in \mathbb{N}$. Assume that the linear skew-product flow of* (2.6) *admits exponential dichotomy with positive constants λ and B. Then, for any $g \in C^r(\mathbb{T}^d, \mathbb{R}^n)$, there exists a unique bounded solution u of* (2.7). *More precisely, u belongs to $C^r(\mathbb{T}^d, \mathbb{R}^n)$ and satisfies* (2.13).

We denote by η the operator that, given $g \in C^r(\mathbb{T}^d, \mathbb{R}^n)$, associates the unique u which solves (2.7). *We have that η is linear and bounded.*

Remark 2.4. *Although Fourier analysis plays an important role in studying invariant equation on torus, it may lead to the loss of regularities even when M is constant. However, by the method of variation of parameters developed above, we avoid this problem.*

3. Functional equation for quasi-periodic solutions

In this section, we formulate the problem of existence of quasi-periodic solutions as a functional equation (3.2) and state our main result Theorem 3.1.

3.1. Formulation of invariance equation.

Following the parameterization method, we seek a function

$$K : \mathbb{T}^d \to \mathbb{R}^n \tag{3.1}$$

in such a way that $x(t) = K(\theta+\omega t)$ is a solution of (1.1). Since ω is rationally independent, it is sufficient to solve the invariance equation

$$\partial K(\theta) \cdot \omega = f(\theta, K(\theta), \epsilon K(\theta - \omega\tau(K(\theta)))) \tag{3.2}$$

where the left hand side is the directional derivative. When $\epsilon = 0$, equation (3.2) reduces to the invariance equation without delay

$$\partial K(\theta) \cdot \omega = f(\theta, K(\theta), 0). \tag{3.3}$$

Assume $K_0(\theta)$ is an approximate solution of (3.2) and denote the error by

$$E(K_0)(\theta) = \partial K_0(\theta)\cdot\omega - f(\theta, K_0(\theta), \epsilon K_0(\theta-\omega\tau(K_0(\theta)))). \tag{3.4}$$

In the perturbation framework, we look for some Δ such that $K_0+\Delta$ is a solution of (3.2). To this end, we give some formal calculations to clarify the main idea.

Given a function $S:\mathbb{T}^d\to\mathbb{R}^n$, we denote $\widetilde{S}$ by $\widetilde{S}(\theta) = S(\theta-\omega\tau(S(\theta)))$. Substituting $K = K_0+\Delta$ into (3.2) yields

$$\begin{aligned}\partial\Delta(\theta)\cdot\omega - D_2 f(\theta, K_0(\theta), \epsilon\widetilde{K_0}(\theta))\Delta(\theta)&\\ =&f(\theta, K_0(\theta)+\Delta(\theta), \epsilon\widetilde{K_0+\Delta}(\theta)) - f(\theta, K_0(\theta), \epsilon\widetilde{K_0}(\theta))\\ &- D_2 f(\theta, K_0(\theta), \epsilon\widetilde{K_0}(\theta))\Delta(\theta) - E(K_0)(\theta).\end{aligned} \tag{3.5}$$

Let $M_\epsilon(\theta) = D_2 f(\theta, K_0(\theta), \epsilon\widetilde{K_0}(\theta))$ and assume that for M_0, the corresponding linear skew-product flow (2.6) admits exponential dichotomy at some base point $\theta_0\in\mathbb{T}^d$. Then, by the Remark (2.3), all the linear skew product flows for M_ϵ admit exponential dichotomy with the uniform exponent λ and constant B for sufficient small ϵ. Formally from Lemma 2.5, equation (3.5) reads

$$\begin{aligned}\Delta(\theta) =&\eta_\epsilon[f(Id, K_0+\Delta, \epsilon\widetilde{K_0+\Delta}) - f(Id, K_0, \epsilon\widetilde{K_0})\\ &- D_2 f(Id, K_0, \epsilon\widetilde{K_0})\Delta - E(K_0)](\theta)\\ \equiv&\mathscr{A}[\Delta](\theta)\end{aligned} \tag{3.6}$$

where η_ϵ is defined in Lemma 2.5. The left is to choose suitable spaces to satisfy the assumptions in Lemma 2.4.

3.2. **Formulation of the result.** Recalling that $B^r_\delta = \{u\in C^r(\mathbb{T}^d,\mathbb{R}^n) : \|u\|_r < \delta\}$, we give our main result on the existence and smoothness of quasi-periodic solutions in an a-posteriori format.

Theorem 3.1. *Let $r\in\mathbb{N}$, $\rho>0$, $f\in C^{r+3}(\mathbb{T}^d\times\mathbb{R}^n\times\mathbb{R}^n,\mathbb{R}^n)$ and $\tau\in C^{r+2}(\mathbb{T}^d,\mathbb{R}^n)$. Assume that there is an approximate solution $K_0\in B^{r+2}_\rho$ of* (3.2) *such that , for $M_0(\theta) = D_2 f(\theta, K_0(\theta), 0)$, the corresponding linear skew product flow over the rotation admits exponential dichotomy in the sense of Definition 2.2.*

Then, there exist small positive constants ϵ_0 and δ such that, for any $0<\epsilon<\epsilon_0$, *if the error $\|E(K_0)\|_{r+1}$ defined in* (3.4) *is sufficient small, there is a solution $K\in C^r(\mathbb{T}^d,\mathbb{R}^n)$ of* (3.2) *and*

$$\|K-K_0\|_r \le C(\delta\cdot\|E(K_0)\|_0)^{1-1/(1+r)},$$

where the constant C depends only on r, $\|f\|_{r+3}, \|\tau\|_{r+2}, \rho, \omega$ and the constants B, λ in exponential dichotomy.

Furthermore, the solution K of (3.2) *is locally unique. More precisely, K is unique in $\overline{B^{r+1}_{\delta}(K_0)}_{C^0}$.*

Proof: Under the assumptions on the regularities of f, τ and K_0, the formal discussions above are indeed true derivatives. Now we concentrate the operator $\mathscr{A}$ acting on the ball B^{r+1}_{δ}. Using the results on the regularity of composite operators, for $\Delta \in B^{r+1}_{\delta}$ we obtain from (3.6) that

$$
\begin{aligned}
&\|\mathscr{A}[\Delta]\|_{r+1} \\
\le& \|\eta_\epsilon\| \cdot \Big\{ \|f(Id, K_0+\Delta, \epsilon\widetilde{K_0}) - f(Id, K_0, \epsilon\widetilde{K_0}) - D_2 f(Id, K_0, \epsilon\widetilde{K_0})\Delta\|_{r+1} \\
&+ \|f(Id, K_0+\Delta, \epsilon\widetilde{K_0+\Delta}) - f(Id, K_0+\Delta, \epsilon\widetilde{K_0})\|_{r+1} + \|E(K_0)\|_{r+1} \Big\} \\
\le& \|\eta_\epsilon\| \cdot \Bigg\{ \|E(K_0)\|_{r+1} + \int_0^1 \int_0^1 \mu \|D_{22} f(Id, K_0 + s\mu\Delta, \epsilon\widetilde{K_0})\Delta^{\otimes 2}\|_{r+1} d\mu ds \\
&+ \epsilon \int_0^1 \|D_3 f(Id, K_0+\Delta, \epsilon\widetilde{K_0} + \epsilon s(\widetilde{K_0+\Delta} - \widetilde{K_0})) \cdot (\widetilde{K_0+\Delta} - \widetilde{K_0})\|_{r+1} ds \Bigg\}.
\end{aligned} \tag{3.7}
$$

Now we deduce from Lemma 2.1 that

$$
\begin{aligned}
&\|D_3 f(Id, K_0+\Delta, \epsilon\widetilde{K_0} + \epsilon s(\widetilde{K_0+\Delta} - \widetilde{K_0})) \cdot (\widetilde{K_0+\Delta} - \widetilde{K_0})\|_{r+1} \\
&\quad \le 2^{r+1} \|D_3 f(Id, K_0+\Delta, \epsilon\widetilde{K_0} + \epsilon s(\widetilde{K_0+\Delta} - \widetilde{K_0}))\| \cdot \|\widetilde{K_0+\Delta} - \widetilde{K_0}\|_{r+1} \\
&\quad \le 2^{r+1} M_{r+1} \|D_3 f\|_{r+1} (1 + \|(Id, K_0+\Delta, \epsilon(1-s)\widetilde{K_0} + \epsilon s\widetilde{K_0+\Delta})\|^{r+1}_{r+1}) \\
&\quad \le 2^{r+1} M_{r+1} \|D_3 f\|_{r+1} [1 + (1 + \|K_0\|_{r+1} + \|\Delta\|_{r+1} + \epsilon(1-s)\|\widetilde{K_0}\|_{r+1} \\
&\qquad + \epsilon s \|\widetilde{K_0+\Delta}\|_{r+1})^{r+1}] \\
&\quad \le C(r, \|f\|_{r+2}, \rho, \delta, \|\tau\|_{r+2})
\end{aligned} \tag{3.8}
$$

since $\|\widetilde{K_0}\|_{r+1}$ and $\|\widetilde{K_0+\Delta}\|_{r+1}$ are also bounded by the similar arguments.

In the following, we will not give the detail estimates as in (3.7) and (3.8) and will use C denoting the universal constant. It is worth noticing that the constant C would be different from line to line.

We finally get the bound from (3.7) that

$$\|\mathscr{A}[\Delta]\|_{r+1} \le C(\|\Delta\|^2_{r+1} + \|\widetilde{K_0+\Delta} - \widetilde{K_0}\|_{r+1} + \|E(K_0)\|_{r+1}).$$

Observing that

$$
\begin{aligned}
\|\widetilde{K_0+\Delta} - \widetilde{K_0}\|_{r+1} \le& \|K_0(Id - \omega\tau(K_0+\Delta)) - K_0(Id - \omega\tau(K_0))\|_{r+1} \\
&+ \|\Delta(Id - \omega\tau(K_0+\Delta))\|_{r+1} \\
\le& C\delta
\end{aligned} \tag{3.9}
$$

and a trivial estimate yields $\|\mathscr{A}[\Delta]\|_{r+1} < \delta$ if

$$C(\epsilon + \delta^2 + \|E(K_0)\|_{r+1}) < \delta < 1, \tag{3.10}$$

where the constant C depends only on $\|f\|_{r+3}, \|\tau\|_{r+2}, \rho, \omega, B, \lambda$ and r. Thus $\mathscr{A}$ maps B^r_δ into itself.

The rest of this proof is to show $\mathscr{A}$ is a contraction in the C^0-norm. For $\Delta, \Delta' \in B^{r+1}_\delta$, we arrange the terms as

$$\begin{aligned}
\mathscr{A}[\Delta']-&\mathscr{A}[\Delta]\\
&=\eta_\epsilon\Big\{[f(Id, K_0+\Delta', \epsilon\widetilde{K_0+\Delta'}) - f(Id, K_0+\Delta, \epsilon\widetilde{K_0+\Delta'})\\
&\quad -D_2 f(Id, K_0+\Delta, \epsilon\widetilde{K_0+\Delta'})(\Delta'-\Delta)]\\
&\quad +[f(Id, K_0+\Delta, \epsilon\widetilde{K_0+\Delta'}) - f(Id, K_0+\Delta, \epsilon\widetilde{K_0+\Delta})]\\
&\quad +[D_2 f(Id, K_0+\Delta, \epsilon\widetilde{K_0+\Delta'}) - D_2 f(Id, K_0+\Delta, \epsilon\widetilde{K_0})]\cdot(\Delta'-\Delta)\\
&\quad + [D_2 f(Id, K_0+\Delta, \epsilon\widetilde{K_0}) - D_2 f(Id, K_0, \epsilon\widetilde{K_0})]\cdot(\Delta'-\Delta)\Big\}.
\end{aligned}$$

Similar but much simpler arguments yield

$$\begin{aligned}
\|\mathscr{A}[\Delta'] - \mathscr{A}[\Delta]\|_0 \le{}& C\{\|f\|_2\cdot\|\Delta'-\Delta\|_0^2 + \epsilon\|f\|_1\cdot\|\widetilde{K_0+\Delta'} - \widetilde{K_0+\Delta}\|_0\\
&+ \epsilon\|f\|_2\cdot\|\Delta'-\Delta\|_0\cdot\|\widetilde{K_0+\Delta'} - \widetilde{K_0}\|_0 + \|f\|_2\cdot\|\Delta\|_0\cdot\|\Delta'-\Delta\|_0\}\\
\le{}& C(3\delta+\epsilon)\|\Delta'-\Delta\|_0 + C\epsilon\cdot\|\widetilde{K_0+\Delta'} - \widetilde{K_0+\Delta}\|_0.
\end{aligned}$$

By the mean value theorem, we show that

$$\begin{aligned}
\|\widetilde{K_0+\Delta'} - \widetilde{K_0+\Delta}\|_0 \le{}& \|K_0(Id-\omega\tau(K_0+\Delta')) - K_0(Id-\omega\tau(K_0+\Delta))\|_0\\
&+ \|\Delta'(Id-\omega\tau(K_0+\Delta')) - \Delta'(Id-\omega\tau(K_0+\Delta))\|_0\\
&+ \|\Delta'(Id-\omega\tau(K_0+\Delta)) - \Delta(Id-\omega\tau(K_0+\Delta))\|_0\\
\le{}& (\,\|K_0\|_1\cdot|\omega|\cdot\|\tau\|_1 + \delta\,|\omega|\cdot\|\tau\|_1 + 1\,)\cdot\|\Delta'-\Delta\|_0\\
\le{}& C\|\Delta'-\Delta\|_0,
\end{aligned} \tag{3.11}$$

which implies

$$\|\mathscr{A}[\Delta'] - \mathscr{A}[\Delta]\|_0 \le C(3\delta+2\epsilon)\|\Delta'-\Delta\|_0 < \|\Delta'-\Delta\|_0$$

if

$$C(3\delta+2\epsilon) < 1. \tag{3.12}$$

Thus, we can always choose a small δ and ϵ_0 such that, for any $0 < \epsilon < \epsilon_0$, the conditions (3.10) and (3.12) hold if $\|E(K_0)\|_{r+1}$ is sufficient small. The continuity arguments of $\mathscr{A}$ on C^r are direct application of Lemma 2.2

and thus omitted. Theorem 2.4 immediately implies the existence of $\Delta^* \in C^r(\mathbb{T}^d, \mathbb{R}^n)$ such that $K = K_0 + \Delta^*$ is the exact solution of (3.2) and

$$\|K - K_0\|_r \le \frac{C(2\delta)^{1-1/(r+1)}}{1-\kappa^{1-1/(r+1)}} \|E(K_0,\epsilon)\|_0^{1-1/(r+1)}.$$

The local uniqueness is also an immediate result of Theorem 2.4. □

Remark 3.1. *We remark that the operator $\mathscr{A}$ may not be a contraction on a closed ball in the space $C^r(\mathbb{T}^d, \mathbb{R}^n)$. To see this, let us revisit the contraction arguments. When processing the expressions* (3.11)*, one has to show, abstractly, the composition operator, which is defined from $C^q(\mathbb{T}^d, \mathbb{R}^n)$ to itself and maps Δ to $\Lambda \circ \Delta$ for a given $\Lambda \in C^q(\mathbb{T}^d, \mathbb{R}^n)$, is differentiable, or at least Lipschitz. However, this is usually impossible for any positive q. See* [dlLO99] *for the general discussions.*

The above theorem gives *a posteriori* result on the state dependent delay differential equation. In particular, if K_0 is a solution of (3.3) , then we can free one order regularity on the assumption of K_0 to obtain the exact solution.

Corollary 3.1. *Under the same assumptions in Theorem 3.1 except that the approximate solution K_0 belongs to $C^{r+1}(\mathbb{T}^d, \mathbb{R}^n)$, then, for sufficient small ϵ, there exists a locally unique solution $K \in C^r(\mathbb{T}^d, \mathbb{R}^n)$ nearby of* (3.2)*, which is continuous in ϵ together with its any partial derivatives up to r.*

Proof: Now we can modify equation (3.6) into

$$\begin{aligned}\Delta(\theta) = \eta[&f(\theta, K_0(\theta) + \Delta(\theta), \epsilon\widetilde{K_0 + \Delta}(\theta)) - f(\theta, K_0(\theta), 0) \\ &- D_2 f(\theta, K_0(\theta), 0)\Delta(\theta))]\end{aligned}$$

and replace the error term by zero. Then the proof is essentially the same as before. However, the argument on (3.9) is also reduced into the estimate of $\|\widetilde{K_0 + \Delta}\|_{r+1}$, which would not need one more order regularity of K_0. □

By the contraction arguments, we actually avoid the difficulties described in Remark 3.1. However, it is at the price of losing derivatives of the exact solution and lack of the C^1-smoothness in the parameter ϵ, although we have endowed the original system high regularities.

4. Further discussion

Although we have assumed that f and τ are defined globally, actually one can weaken this restriction since the discussions are in the vicinity of the approximate solution.

For the quasi-periodic differential system with multiple state-dependent time lags

$$\begin{cases} \dot{x}(t) = f(\theta, x(t), \epsilon x(t-\tau_1(x(t))), \cdots, \epsilon x(t-\tau_l(x(t)))) \\ \dot{\theta}(t) = \omega \end{cases}$$

one also obtain the operator $\mathscr{A}$ defined in (3.6) with adding some components in the functions f and its derivatives. However, when verifying the conditions (i) and (iii) in lemma 2.4, the estimates of $\mathscr{A}[\Delta]$ and $\mathscr{A}[\Delta] - \mathscr{A}[\Delta']$ involves more terms by duplication and subtraction arguments. For example, when $l = 2$, one has

$$\begin{aligned}
&\mathscr{A}[\Delta] \\
=&\eta_\epsilon \Big[f(Id, K_0 + \Delta, \epsilon\widetilde{K_0}, \epsilon\widehat{K_0}) - f(Id, K_0, \epsilon\widetilde{K_0}, \epsilon\widehat{K_0}) - D_2 f(Id, K_0, \epsilon\widetilde{K_0}, \epsilon\widehat{K_0})\Delta \\
&\qquad + f(Id, K_0 + \Delta, \epsilon\widetilde{K_0 + \Delta}, \epsilon\widehat{K_0 + \Delta}) - f(Id, K_0 + \Delta, \epsilon\widetilde{K_0 + \Delta}, \epsilon\widehat{K_0}) \\
&\qquad + f(Id, K_0 + \Delta, \epsilon\widetilde{K_0 + \Delta}, \epsilon\widehat{K_0}) - f(Id, K_0 + \Delta, \epsilon\widetilde{K_0}, \epsilon\widehat{K_0}) \Big] \\
=&\eta_\epsilon \Bigg[\int_0^1 \int_0^1 \mu D_{22} f(Id, K_0 + s\mu\Delta, \epsilon\widetilde{K_0}, \epsilon\widehat{K_0}) \cdot \Delta^{\otimes 2} ds d\mu \\
&\quad + \epsilon \int_0^1 D_3 f(Id, K_0 + \Delta, \epsilon\widetilde{K_0} + \epsilon s(\widetilde{K_0 + \Delta} - \widetilde{K_0}), \epsilon\widehat{K_0 + \Delta}) \\
&\quad \times (\widetilde{K_0 + \Delta} - \widetilde{K_0}) ds \\
&\quad + \epsilon \int_0^1 D_4 f(Id, K_0 + \Delta, \epsilon\widetilde{K_0}, \epsilon\widehat{K_0} + \epsilon s(\widehat{K_0 + \Delta} - \widehat{K_0})) \\
&\quad \times (\widehat{K_0 + \Delta} - \widehat{K_0}) ds - E(K_0) \ \Bigg]
\end{aligned}$$

where the hat of function, say S, is defined as $\widehat{S}(\theta) = S(\theta - \omega\tau_2(S(\theta)))$. Then the estimates are essentially the same as the single delay case.

One interesting respect is the study of subharmonic solutions of (1.1). Thus we are looking for an embedding $K : \mathbb{T} \to \mathbb{R}^n$ such that $x(t) = K(t/n)$ is the desired solution. The most interesting thing is to eliminate the perturbation setting. However, as remarked in [HVL93], results on almost periodic solutions without assumptions of smallness of the perturbed vector field are very difficult to obtain.

Another problem deserving consideration is in what sense the exponential dichotomy present for $\epsilon = 0$ persists for the solutions constructed. Of course, the ever notion of exponential dichotomy is problematic. In a forthcoming paper we formulate an appropriate notion of dichotomy(also based on the parameterization method) to establish its persistence and develop a theory of stable and unstable manifolds.

At last, we remark that, without the assumption of exponential dichotomy, the small divisor problem would appear. For a very particular case, [SB03] considers a differential-difference equation on the torus and reduce the perturbed system into a pure rotation. However, for the general analytic SD-DDEs, a prime difficulty is to determine the analytic domain for the expression $x(t - \tau(x(t)))$ involving the unknowns. In particular, for the quasi-periodic solutions, it refers to $K(\theta - \omega\tau(K(\theta)))$. From [MPN14], we know that the dynamic properties of the map $\theta \to \theta - \omega\tau(K(\theta))$ play a crucial role in determining the analyticity and non-analyticity of the solutions. We will consider these problems in a forthcoming paper [HdlL15].

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