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Qualitative Properties of Solutions of Nonlinear Anisotropic PDEs in Local and Nonlocal Settings

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Academic Year 2014/2015

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Summary

This thesis is concerned with the study of several qualitative properties shared by the solutions of elliptic equations set in the Euclidean space $\mathbb{R}^n$. The main focus of the work is on entire solutions of anisotropic/heterogeneous equations that show some kind of symmetric properties and, in particular, that possess one-dimensional symmetry.

The dissertation is divided into two parts. The first part deals with *local* partial differential equations, while the second one addresses a class of less familiar *nonlocal* equations driven by integral operators. As these two major subjects require rather different formalizations, we attach to each part a detailed introduction on the framework under analysis. Nevertheless, we present below a brief sketch of the main contributions contained in each chapter.

- In Chapter 1 we consider the anisotropic equation

$$\operatorname{div}\left(B'(H(\nabla u))\nabla H(\nabla u)\right) + F'(u) = 0 \quad \text{in } \mathbb{R}^n, \tag{1}$$

  where B is a positive, increasing, convex mapping, H is a positive homogeneous function of degree 1 and F is a potential. Important examples of settings included in our analysis are given by anisotropic versions of the p-Laplace and minimal surfaces equations.
  Under these hypotheses, we prove that the solutions of (1) satisfy a pointwise gradient bound in the spirit of Modica ([M85]). Thanks to this estimate, we are able to obtain various one-dimensional symmetry and rigidity results.

- In the following Chapter 2 we continue our analysis of the qualitative properties enjoyed by the solutions of (1). For any fixed solution u, we introduce the family of scaled energies

$$\mathscr{W}(R) := \frac{1}{R^{n-1}} \int_{W_R} B(H(\nabla u(x))) + G(u(x))\, dx,$$

  where R is positive, the domain W_R is the so-called Wulff ball of *radius* R corresponding to H and $-G := F - c_u$ is an appropriate choice of potential equivalent to F. Under an additional geometrical assumption on H, we show that $\mathscr{W}$ is monotone non-decreasing. Then, we deduce a Liouville-type result for solutions having finite total *mass*.

- Chapter 3 contains several regularity results for the integral equation

$$-L_K u = f \quad \text{in } \Omega, \tag{2}$$

  where Ω is a domain of $\mathbb{R}^n$, f is a measurable function and

$$-L_K u(x) := \text{P.V.} \int_{\mathbb{R}^n} (u(x) - u(y))\, K(x, y)\, dy. \tag{3}$$

Here, K denotes a symmetric kernel subject to suitable ellipticity/growth conditions. The most common example of operator in the form (3) is given by the so-called fractional Laplacian $(-\Delta)^s$ of order $2s$, which corresponds to $K(x,y) = |x-y|^{-n-2s}$, with $s \in (0,1)$. However, our results apply to more general choices of K, such as non-homogeneous and truncated kernels.
We show the validity of various estimates in Hölder spaces for the solutions of (2), in dependence of the properties of f. The results presented here are mostly not original, but form a useful regularity toolbox for the subsequent chapters.

- In Chapter 4 we consider equation (2) with right-hand side f in $L^2(\Omega)$ and K satisfying a suitable *joint* regularity condition in the variables x and y. By extending the well-known translation method of Nirenberg to this nonlocal setting, we obtain interior estimates in higher-order Sobolev and Nikol'skii spaces for the solution u. In particular, we prove that u belongs to the fractional Sobolev space $H^{2s-\varepsilon}_{\text{loc}}(\Omega) = W^{2s-\varepsilon,2}_{\text{loc}}(\Omega)$, for any $\varepsilon > 0$.

- In Chapter 5 we focus on the Ginzburg-Landau-type energy
$$\mathscr{E}_K(u) := \frac{1}{4} \iint |u(x) - u(y)|^2 \, K(x,y) \, dx dy + \int W(x, u(x)) \, dx, \tag{4}$$
where W is a space-dependent double-well potential with zeros at $u = \pm 1$ and K is the previously discussed kernel. We consider the case of a periodic environment - modeled by requiring K and W to be periodic with respect to the action of $\mathbb{Z}^n$ - and study the existence of *plane-like* minimizers. For any given direction $\omega \in \mathbb{R}^n \setminus \{0\}$, we indeed prove that there exists a minimizer $u_\omega : \mathbb{R}^n \to [-1,1]$ of (4) such that
$$\Big\{ |u_\omega(x)| < 9/10 \Big\} \subset \Big\{ \omega \cdot x \in [0, M_0 |\omega|] \Big\},$$
for some universal constant $M_0 > 0$. This amounts to saying that, although there may not exist one-dimensional minimizers of (4), we are still able to produce examples of minimizers whose intermediate level sets lie in strips of universal width.

- Chapter 6 is devoted to a construction similar to the one just presented. In this chapter we consider a translation-invariant setting, obtained by taking W independent of the space variable x and $K(x,y) = \bar{K}(x-y)$, for some $\bar{K}$.
By adapting to this framework the techniques of [PSV13] - where the result was obtained for the fractional Laplacian - we show the existence of monotone, one-dimensional solutions to the nonlocal Allen-Cahn equation
$$-L_K u + W'(u) = 0 \quad \text{in } \mathbb{R}^n, \tag{5}$$
which connect the pure phases -1 and 1 at infinity. The solutions constructed are indeed minimizers of an appropriate energy functional associated to (5). On top of the existence result, we also obtain estimates for the behaviour of this energy when restricted to balls of increasing radius.

- In the conclusive Chapter 7 we address a rather different result related to a quantity introduced in [CRS10]: the fractional mean curvature. Given an open set E with regular boundary and a smooth global diffeomorphism Ψ of $\mathbb{R}^n$, we estimate the difference between the fractional mean curvature of the sets E and $\Psi(E)$ in terms of the derivatives of the perturbation $\Psi - I$, where I stands for the identity map.

The results presented have been obtained through the three years of my doctoral program. They are the outcome of several scientific collaborations and are contained in the research papers [C15, C15b, CFV14, CFV15, CP15, CV15].

Riassunto

Questa tesi è dedicata allo studio di varie proprietà qualitative possedute dalle soluzioni di equazioni ellittiche poste nello spazio euclideo $\mathbb{R}^n$. L'attenzione principale del lavoro è rivolta a soluzioni intere di equazioni anisotrope/eterogenee che mostrano qualche genere di proprietà di simmetria e, in particolare, che posseggono simmetria unidimensionale.

L'elaborato è diviso in due parti. La prima parte è riservata ad equazioni alle derivate parziali *locali*, mentre la seconda si concentra su di una classe meno usuale di equazioni *non locali*, determinate da operatori integrali. Poiché questi due ambiti richiedono formalizzazioni alquanto differenti l'una dall'altra, anteponiamo a ciascuna parte un'introduzione dettagliata al contesto esaminato. Ciononostante, è riportato qui di seguito un breve sunto dei principali contributi contenuti in ciascun capitolo.

- Nel Capitolo 1 consideriamo l'equazione anisotropa

$$\operatorname{div}\big(B'(H(\nabla u))\nabla H(\nabla u)\big) + F'(u) = 0 \quad \text{in } \mathbb{R}^n, \tag{1}$$

  dove B è un'applicazione positiva, crescente e convessa, H è una funzione positivamente omogenea di grado 1 e F è un potenziale. Esempi importanti inclusi nella nostra analisi sono rappresentati da versioni anisotrope del p-laplaciano e dell'equazione delle superficie minime.
  Sotto queste ipotesi, dimostriamo che le soluzioni di (1) soddisfano una stima puntuale del gradiente nello spirito di Modica ([M85]). Grazie a questa disuguaglianza, siamo in grado di ottenere alcuni risultati di simmetria unidimensionale e di rigidità.

- Nel successivo Capitolo 2 continuiamo la nostra analisi delle proprietà qualitative delle soluzioni di (1). Fissata una soluzione u, introduciamo la famiglia di energie pesate

$$\mathscr{W}(R) := \frac{1}{R^{n-1}} \int_{W_R} B(H(\nabla u(x))) + G(u(x))\, dx,$$

  dove R è positivo, il dominio W_R è la cosiddetta bolla di Wulff di *raggio* R corrispondente a H e $-G := F - c_u$ è un'opportuna scelta di potenziale equivalente a F. Sotto un'ulteriore assunzione geometrica su H, mostriamo che $\mathscr{W}$ è monotona non decrescente. Da ciò ne deduciamo un risultato di tipo Liouville per soluzioni aventi *massa* totale finita.

- Il Capitolo 3 contiene vari risultati di regolarità per l'equazione integrale

$$-L_K u = f \quad \text{in } \Omega, \tag{2}$$

  dove Ω è un dominio di $\mathbb{R}^n$, f è una funzione misurabile e

$$-L_K u(x) := \text{P.V.} \int_{\mathbb{R}^n} (u(x) - u(y))\, K(x,y)\, dy. \tag{3}$$

Qui denotiamo con K un nucleo simmetrico soggetto a condizioni di ellitticità/crescita appropriate. Il più comune esempio di operatore nella forma (3) è costituito dal cosiddetto laplaciano frazionario $(-\Delta)^s$ di ordine $2s$, corrispondente alla scelta $K(x,y) = |x-y|^{-n-2s}$, con $s \in (0,1)$. Tuttavia, i nostri risultati si applicano ad una più vasta gamma di nuclei K, quali ad esempio nuclei non omogenei e troncati.
In rapporto alle caratteristiche di f, mostriamo la validità di svariate stime in spazi di Hölder per le soluzioni di (2). I risultati presentati qui sono per lo più non originali, ma formano un utile compendio di teoria della regolarità per i capitoli seguenti.

- Nel Capitolo 4 consideriamo l'equazione (2) avente lato destro f in $L^2(\Omega)$ e nucleo K soddisfacente un'opportuna condizione di regolarità *congiunta* nelle due variabili x e y. Estendendo il ben noto metodo delle traslazioni di Nirenberg a questo contesto non locale, otteniamo stime interiori per la soluzione u in spazi di Sobolev e Nikol'skii di ordine superiore. In particolare, proviamo che u appartiene allo spazio di Sobolev frazionario $H^{2s-\varepsilon}_{\mathrm{loc}}(\Omega) = W^{2s-\varepsilon,2}_{\mathrm{loc}}(\Omega)$, per ogni $\varepsilon > 0$.

- Nel Capitolo 5 ci concentriamo sull'energia di tipo Ginzburg-Landau

$$\mathscr{E}_K(u) := \frac{1}{4} \iint |u(x)-u(y)|^2 \, K(x,y) \, dx dy + \int W(x,u(x)) \, dx, \tag{4}$$

  dove W è un potenziale a doppio pozzo con dipendenza spaziale, annullantesi in $u = \pm 1$ e K è il nucleo discusso in precedenza. Consideriamo il caso di un ambiente periodico - realizzato imponendo su K e W periodicità rispetto all'azione di $\mathbb{Z}^n$ - e studiamo l'esistenza di minimi *similplanari*. Per ogni direzione assegnata $\omega \in \mathbb{R}^n \setminus \{0\}$, mostriamo infatti l'esistenza di un minimo $u_\omega : \mathbb{R}^n \to [-1,1]$ di (4) tale che

$$\Big\{ |u_\omega(x)| < 9/10 \Big\} \subset \Big\{ \omega \cdot x \in [0, M_0|\omega|] \Big\},$$

  per qualche costante universale $M_0 > 0$. Con ciò possiamo concludere che, nonostante possano non esistere minimi unidimensionali di (4), siamo comunque in grado di produrre esempi di minimi i cui insiemi di livello intermedi giacciano in strisce di larghezza universale.

- Il Capitolo 6 è dedicato ad una costruzione assai simile a quella appena presentata. In questo capitolo consideriamo un funzionale invariante per traslazioni, ottenuto tramite la scelta di un potenziale W indipendente dalla variabile spaziale x e di un nucleo nella forma $K(x,y) = \bar{K}(x-y)$, per qualche $\bar{K}$.
  Adattando a questo contesto le tecniche di [PSV13] - dove il risultato è stato ottenuto per il laplaciano frazionario - mostriamo l'esistenza di soluzioni unidimensionali e monotone dell'equazione di Allen-Cahn nonlocale

$$-L_K u + W'(u) = 0 \quad \text{in } \mathbb{R}^n, \tag{5}$$

  congiungenti le fasi pure -1 e 1 all'infinito. Le soluzioni costruite sono inoltre minimi di un opportuno funzionale di energia associato a (5). In aggiunta al risultato di esistenza, otteniamo anche alcune stime sul comportamento della restrizione di questa energia a bolle di raggio crescente.

- Nel conclusivo Capitolo 7 ci dedichiamo ad un risultato piuttosto differente, relativo ad una quantità introdotta in [CRS10]: la curvatura media frazionaria. Dato un insieme aperto E avente bordo regolare ed un diffeomorfismo globale liscio Ψ di $\mathbb{R}^n$, forniamo una stima della differenza tra la curvatura media frazionaria degli insiemi E e $\Psi(E)$ in termini delle derivate della perturbazione $\Psi - I$, dove I rappresenta l'identità.

I risultati qui presentati sono stati ottenuti nel corso dei tre anni del mio programma di dottorato. Essi sono il prodotto di varie collaborazioni scientifiche e sono contenuti negli articoli di ricerca [C15, C15b, CFV14, CFV15, CP15, CV15].

Résumé

Cette thèse est consacrée à l'étude de diverses propriétés qualitatives des solutions d'équations elliptiques posées dans l'espace euclidien $\mathbb{R}^n$. L'objectif principal du travail est celui d'étudier les solutions d'équations anisotropes et ou hétérogénes qui montrent un certain degré de symétrie, avec un intêret particulier pour celles qui possèdent une symétrie unidimensionnelle.

Le mémoire est divisée en deux parties. La prèmiere partie est consacrée aux équations aux dérivées partielles de type *local*, tandis que la seconde partie se concentre sur l'étude des équations de type *non local* déterminées par des opérateurs intégraux. Dans le deux cas de figure, il s'agit de modèles de grande actualité. Étant donné que ces deux sujets exigent une formalisation plutôt différente, nous donnons une introduction détaillée pour chacune des deux parties.

Néanmois, nous présentons ci-dessous un bref aperçu des principales contributions contenues dans chaque chapitre.

- Dans le Chapitre 1, nous considérons l'équation anisotrope

$$\operatorname{div}\big(B'(H(\nabla u))\nabla H(\nabla u)\big) + F'(u) = 0 \quad \text{dans } \mathbb{R}^n, \tag{1}$$

  où B est une application positive, croissante et convexe, H est une fonction positivement homogène de degré 1 et F est un potentiel. Des exemples important inclus dans notre analyse sont représentés par des versions anisotropes du p-laplacien et de l'équation de surface minimale.
  Sous ces hypothèses, nous montrons que les solutions de (1) satisfont une estimation ponctuelle du gradient à la Modica ([M85]). Grâce à cette inegalité, nous obtenons divers résultats de symétrie unidimensionnelle et de rigidité géométrique.

- Dans le Chapitre 2, nous poursuivons notre analyse concernant les propriétés qualitatives des solutions de l'équation (1). Pour toute solution fixée u, nous introduisons la famille des énergies pondérées

$$\mathscr{W}(R) := \frac{1}{R^{n-1}} \int_{W_R} B(H(\nabla u(x))) + G(u(x))\, dx,$$

  où $R > 0$, le domaine W_R est la boule de Wulff, de *rayon* R, correspondant à H et $-G := F - c_u$ est un choix approprié de potentiel équivalent à F. Sous une hypothése géométrique supplémentaire sur H, nous démontrons que $\mathscr{W}$ est monotone non décroissante. Ensuite, on en déduit un résultat de type Liouville pour toutes les solutions qui ont une *masse* totale finie.

- Le Chapitre 3 contient plusieurs résultats de régularité pour l'équation intégrale

$$-L_K u = f \quad \text{dans } \Omega, \tag{2}$$

où Ω est un domaine de $\mathbb{R}^n$, f est une fonction mesurable et

$$-L_K u(x) := \text{P.V.} \int_{\mathbb{R}^n} (u(x) - u(y))\, K(x,y)\, dy. \tag{3}$$

Ici, K désigne un noyau symétrique satisfaisant des conditions d'ellipticité/croissance appropriées (et naturelles). L'exemple le plus commun d'opérateur sous la forme (3) est constitué par laplacien fractionnaire $(-\Delta)^s$ d'ordre $2s$, ce qui correspond au choix $K(x,y) = |x-y|^{-n-2s}$, avec $s \in (0,1)$. Cependant, nos résultats s'appliquent à une plus vaste gamme de noyaux K, tels que les noyaux non homogènes et tronqués. Nous montrons la validité des différentes estimations dans des espaces de Hölder pour les solutions de (2), en fonction des propriétés de f. Les résultats présentés ici constituent un condensé de la théorie de régularité qui sera utile dans les chapitres suivants.

- Dans le Chapitre 4, nous considérons l'équation (2) avec second membre f dans $L^2(\Omega)$ et noyau K satisfaisant une condition convenable de régularité *conjointe* dans les deux variables x et y. En étendant la célèbre méthode des translations de Nirenberg à ce cadre non local, nous obtenons des estimations locales pour la solution u dans des espaces de Sobolev et de Nikol'skii d'ordre supérieur. En particulier, nous prouvons que u appartient à l'espace de Sobolev fractionnaire $H^{2s-\varepsilon}_{\text{loc}}(\Omega) = W^{2s-\varepsilon,2}_{\text{loc}}(\Omega)$, pour tout $\varepsilon > 0$.

- Dans le Chapitre 5, nous nous concentrons sur l'énergie de type Ginzburg-Landau

$$\mathscr{E}_K(u) := \frac{1}{4} \iint |u(x) - u(y)|^2\, K(x,y)\, dx dy + \int W(x, u(x))\, dx, \tag{4}$$

où W est un potentiel à double puits, qui dépend aussi de la variable spatiale x, ayant des zéros pour $u = \pm 1$ et K est le noyau discuté précédemment. Nous considérons le cas d'un milieu périodique - obtenu en imposant à K et W la périodicité déterminée par l'action de $\mathbb{Z}^n$ - et nous étudions l'existence des minimiseurs *apres-planaires*. Pour toutes directions $\omega \in \mathbb{R}^n \setminus \{0\}$, nous montrons qu'il ya effectivement un minimiseur $u_\omega : \mathbb{R}^n \to [-1,1]$ de (4) tel que

$$\Big\{ |u_\omega(x)| < 9/10 \Big\} \subset \Big\{ \omega \cdot x \in [0, M_0 |\omega|] \Big\},$$

pour une constante universelle $M_0 > 0$. Avec cela, nous pouvons conclure que, même s'il peut ne pas exister de minimiseurs à symétrie unidimensionnelle de (4), nous sommes toujours en mesure de produire des exemples de minimiseurs dont les ensemble de niveau intermédiaires demeurent confinés dans des bandes de largeur universelle.

- Le Chapitre 6 est consacré à une construction très similaire à celle présentée dans la partie précédente de ce travail. Dans ce chapitre, nous examinons une fonctionnelle invariante par translations, obtenue par le choix d'un potentiel W indépendant de la variable spatiale x et d'un noyau de la forme $K(x,y) = \bar{K}(x-y)$, pour un certain $\bar{K}$. En adaptant à ce cadre les techniques de [PSV13] - où le résultat a été obtenu pour le laplacien fractionnaire - nous montrons l'existence des solutions unidimensionnelles et monotones de l'équation d'Allen-Cahn non locale

$$-L_K u + W'(u) = 0 \quad \text{in } \mathbb{R}^n, \tag{5}$$

reliant les phases pures -1 et 1 à l'infini. Les solutions construites sont en effet des minimiseurs d'une appropriée fonctionnelle d'énergie associée à (5). En plus du résultat d'existence, nous obtenons aussi des estimations sur le comportement de la restriction de cette énergie aux boules de rayon croissant.

- Dans le dernier Chapitre 7, nous abordons un problème assez différent, lié à une quantité introduite dans [CRS10]: la courbure moyenne fractionnaire. Étant donné un ensemble ouvert E à frontière régulière et un difféomorphisme global lisse Ψ de $\mathbb{R}^n$, nous estimons la différence entre les courbures moyennes fractionnaires des ensembles E et $\Psi(E)$ en fonction des dérivées de la perturbation $\Psi - I$, où I représente l'identité.

Les résultats présentés ici ont été obtenus pendant les trois années de mon de doctorat de recherche. Ils sont le produit de plusieurs collaborations scientifiques et sont contenus dans les articles de recherche [C15, C15b, CFV14, CFV15, CP15, CV15].

Part I

Symmetry and rigidity results for entire solutions of singular, degenerate, anisotropic PDEs

Introduction and formulation of the setting

In this first part, we consider a variational problem set in an anisotropic medium. The physical motivation we have in mind comes from some well-established models of surface energy, see for instance [T78, G06] and references therein for a classical introduction to the topic.

Surface energy arises since the microscopic environment of the interface of a medium is different from the one in the bulk of the substance. In many concrete cases, such as for the common cooking salt, the different behavior depends significantly on the space direction and so these anisotropic surface energies have now become very popular in metallurgy and crystallography, see e.g. [W01, D44, AC77]. Applications to crystal growth and thermodynamics are discussed in [M-KBK77, C84, TCH92] and in [G93], respectively.

Other applications of related anisotropic models occur in noise-removal procedures in digital image processing, crystalline mean curvature flows and crystalline fracture theory, see e.g. [NP99, BNP01a, BNP01b, EO04, OBGXY05] and references therein. See also [FM91, C04] for anisotropic problems related to the Willmore functional and [CiaS09, WX11] for elliptic anisotropic systems inspired by fluidodynamics.

Of course, besides this surface energy, the medium may also be subject to exterior forces and the total energy functional is in this case the sum of an anisotropic surface energy plus a potential term. More precisely, the mathematical framework we work in is inspired by the Wulff crystal construction (see pages 571–573 in [T78]) and it may be formally introduced as follows.

Given a domain $\Omega \subseteq \mathbb{R}^n$, with $n \geqslant 2$, consider the functional

$$\mathscr{W}_\Omega(u) := \int_\Omega B(H(\nabla u(x))) - F(u(x))\, dx. \tag{6}$$

and the associated Euler-Lagrange equation

$$\operatorname{div}\big(B'(H(\nabla u))\nabla H(\nabla u)\big) + F'(u) = 0. \tag{7}$$

Here, B denotes a mapping of class $C^{3,\beta}_{\rm loc}((0,+\infty)) \cap C^1([0,+\infty))$, with $\beta \in (0,1)$, such that $B(0) = B'(0) = 0$ and

$$B(t), B'(t), B''(t) > 0 \text{ for any } t \in (0,+\infty). \tag{8}$$

On the other hand, $H : \mathbb{R}^n \to \mathbb{R}$ is a positive homogeneous function of degree 1, that is

$$H(t\xi) = tH(\xi) \text{ for any } \xi \in \mathbb{R}^n \setminus \{0\} \text{ and any } t > 0.$$

Also, we require H to be of class $C^{3,\beta}_{\rm loc}(\mathbb{R}^n \setminus \{0\})$, with

$$H(\xi) > 0 \text{ for any } \xi \in \mathbb{R}^n \setminus \{0\}. \tag{9}$$

Notice that, being H homogeneous and defined at the origin, it necessarily holds $H(0) = 0$. Finally, we let $F \in C^{2,\beta}_{\mathrm{loc}}(\mathbb{R})$.

The function H is a norm (although, possibly not even) in $\mathbb{R}^n$ which encodes the anisotropy of the underlying space. Associated to H are two important geometrical loci, namely the open *unit ball*

$$B_1^H := \Big\{\xi \in \mathbb{R}^n : H(\xi) < 1\Big\}, \tag{10}$$

and the *Wulff shape* W_1^H, which is simply the unit ball of the dual function

$$H^*(x) := \sup_{\xi \in S^{n-1}} \frac{\langle x, \xi\rangle}{H(\xi)}, \tag{11}$$

that is

$$W_1^H := B_1^{H^*} = \Big\{x \in \mathbb{R}^n : H^*(x) < 1\Big\}. \tag{12}$$

We refer to [CiaS09, WX11] for some basic properties of this set and to [T78] for a nice geometrical construction. See also Appendix A for a physical interpretation of the Wulff shape.

In the following we will always consider anisotropies H having uniformly convex unit ball, i.e. such that the principal curvatures of its boundary are bounded away from zero. Every H having uniformly convex unit ball will be called *uniformly elliptic*. We remark that, since the second fundamental form of ∂B_1^H at a point $\xi \in \partial B_1^H$ is given by

$$\mathrm{II}_\xi(\zeta, \upsilon) = \frac{H_{ij}(\xi)\zeta_i \upsilon_j}{|\nabla H(\xi)|} \text{ for any } \zeta, \upsilon \in \nabla H(\xi)^\perp,$$

and being ∂B_1^H compact, the uniform ellipticity of H is equivalent to ask

$$H_{ij}(\xi)\zeta_i\zeta_j \geqslant \lambda|\zeta|^2 \text{ for any } \xi \in \partial B_1^H,\ \zeta \in \nabla H(\xi)^\perp, \tag{13}$$

for some $\lambda > 0$. Any positive λ for which (13) is satisfied will be said to be an *ellipticity constant* for H. Notice that, by homogeneity, (13) actually extends to

$$H_{ij}(\xi)\zeta_i\zeta_j \geqslant \lambda|\xi|^{-1}|\zeta|^2 \text{ for any } \xi \in \mathbb{R}^n \setminus \{0\},\ \zeta \in \nabla H(\xi)^\perp. \tag{14}$$

Of course, here above and throughout the remainder of the dissertation, the summation convention over repeated subscripts is adopted, unless differently specified.

In addition to the already stated hypothesis on B, H and F, we assume that either (A) or (B) is satisfied, where:

(A) There exist $p > 1$, $\kappa \in [0, 1)$ and $\gamma, \Gamma, \lambda > 0$ such that

$$H \text{ is uniformly elliptic with constant } \lambda,$$

and

$$\begin{aligned} \gamma(\kappa + t)^{p-2}t \leqslant{}& B'(t) \leqslant \Gamma(\kappa + t)^{p-2}t, \\ \gamma(\kappa + t)^{p-2} \leqslant{}& B''(t) \leqslant \Gamma(\kappa + t)^{p-2}, \end{aligned}$$

for any $t > 0$.

(B) The function B is of class $C^{3,\beta}_{\mathrm{loc}}([0, +\infty))$, with $B'''(0) = 0$,

$$B''(0) > 0, \tag{15}$$

and H is in the form

$$H_M(\xi) = \sqrt{\langle M\xi, \xi\rangle}, \tag{16}$$

for some $M \in \mathrm{Mat}_n(\mathbb{R})$ symmetric and positive definite.

The model we consider is indeed very general and it allows at the same time an anisotropic dependence on the space variable and a possible singularity or degeneracy of the diffusion operator. For instance, we can take into account the following examples of B:

$$B(t) = \frac{\left(\kappa^2 + t^2\right)^{p/2} - \kappa^p}{p} \qquad \text{and} \qquad B(t) = \sqrt{1+t^2} - 1, \tag{17}$$

with $p > 1$, and $\kappa \geqslant 0$.

Such choices are related to the *anisotropic p-Laplace equation*

$$\operatorname{div}\left(H^{p-1}(\nabla u)\nabla H(\nabla u)\right) + F'(u) = 0, \tag{18}$$

obtained by taking $\kappa = 0$ in the first example proposed in (17), and the *anisotropic minimal surface equation*

$$\operatorname{div}\left(\frac{H(\nabla u)\nabla H(\nabla u)}{\sqrt{1+H^2(\nabla u)}}\right) + F'(u) = 0. \tag{19}$$

In particular, when $H(\xi) = |\xi|$, equations (18) and (19) reduce respectively to the classical p-Laplace and minimal surface equations.

As a concrete realization of the anisotropy H, one may consider any perturbation of the standard Euclidean norm of the type

$$H(\xi) = |\xi|_q + \lambda|\xi| = \left(\sum_{j=1}^{n} |\xi|^q\right)^{1/q} + \lambda \left(\sum_{j=1}^{n} |\xi|^2\right)^{1/2},$$

with q large and $\lambda > 0$. We stress that the combination of such a H along with B as in the first example of (17), fulfill hypothesis (A).

On the other hand, the choice of H in the form (16) together with B as in (17), with $\kappa > 0$ in the p-Laplacian case, actually produces an operator that satisfies hypothesis (B).

In Section 1.3 we discuss why (16) is the only kind of anisotropy allowed in the framework of assumption (B). Namely, we prove that (B) is equivalent to another condition which is the one typically adopted in order to apply some well-known regularity results to the solution u.

In the following two chapters we will establish some qualitative properties shared by the bounded entire solutions of (7). In particular, we will be interested in rigidity properties, such as Liouville-type theorems and one-dimensionality results. For us, $u : \mathbb{R}^n \to \mathbb{R}$ is said to be *one-dimensional* in a region $\Omega \subseteq \mathbb{R}^n$ if it can be written as

$$u(x) = u_0(\omega \cdot x) \text{ for any } x \in \Omega, \tag{20}$$

for some function $u_0 : \mathbb{R} \to \mathbb{R}$ and a unit vector $\omega \in S^{n-1}$.

Before advancing to the statements of these results, we need some more definitions. First of all, we specify which is the kind of solutions we are dealing with. We say that u is a *bounded weak solution* of equation (7) in the whole space $\mathbb{R}^n$ if either

(i) $u \in L^\infty(\mathbb{R}^n) \cap W^{1,p}_{\text{loc}}(\mathbb{R}^n)$, if (A) holds, or

(ii) $u \in W^{1,\infty}(\mathbb{R}^n)$, if (B) holds,

and u satisfies

$$\int_{\mathbb{R}^n} B'(H(\nabla u(x)))\nabla H(\nabla u(x))) \cdot \nabla\varphi(x)\,dx = \int_{\mathbb{R}^n} F'(u(x))\varphi(x)\,dx,$$

for any test function $\varphi \in C_c^\infty(\mathbb{R}^n)$. Of course, in view of the structural hypothesis (A) or (B), the standard regularity theory for quasilinear elliptic PDEs applies and the solution u enjoys stronger differentiability features. We will comment more diffusely on such regularity properties later in Section 1.4.

Furthermore, we associate to any bounded solution u the finite quantities

$$u_* := \inf_{\mathbb{R}^n} u \quad \text{and} \quad u^* := \sup_{\mathbb{R}^n} u, \tag{21}$$

and the gauge

$$c_u := \sup \left\{ F(t) : t \in [u_*, u^*] \right\}. \tag{22}$$

For $t \in \mathbb{R}$ we thus set

$$G(t) := c_u - F(t). \tag{23}$$

Notice that such G is a non-negative function on the range of u and that putting it in place of $-F$ in (6) does not alter at all the setting - and, in particular, equation (7) -, once u is fixed.

Now that all this terminology has been introduced, we can move on and state our results. Basically, we obtain a pointwise gradient bound for the solutions of (7) and a monotonicity formula for an appropriate rescaling of the functional (6). With the aid of these two powerful instruments, we then deduce the aforementioned rigidity results for u. We point out that both these two contributions are generalization of the results first obtained by Modica in the important works [M85, M89]. After the publications of these two papers, further extensions of the results of Modica were proved. We will give more detailed informations on the existing literature later on, in Chapters 1 and 2.

Although our results form a single, uniform treatise on the qualitative properties enjoyed by the entire solutions of equation (7), we preferred to split their exposition into separate chapters. We mainly do this in order to keep the presentation more adherent to the two papers [CFV14, CFV15], written in collaboration with Alberto Farina and Enrico Valdinoci, where the material collected here has first appeared.

Chapter 1

Gradient bounds and rigidity results

1.1 Statement of the main results

The main results discussed in this chapter are a pointwise estimate on the gradient of the solution of (7), from which we deduce some rigidity and symmetry properties (in particular, we obtain one-dimensional Euclidean symmetry and Liouville-type results).

The first result we present is a pointwise bound on the gradient in terms of the effective potential G. Notice that classical elliptic estimates provide bounds of the gradient in either Hölder or Lebesgue norms, but do not give any pointwise information in general. In dimension 1, the pointwise estimate that we present reduces to the classical Energy Conservation Law.

In higher dimension, estimates of this kind were given first by [M85] for the semilinear equation

$$\Delta u + F'(u) = 0$$

with $F \leqslant 0$. Observe that this case is comprised in our setting by choosing $H(\xi) = |\xi|$, $B(t) = t^2/2$. Then, [CGS94] extended such estimates to the quasilinear case

$$\operatorname{div}(\Phi'(|\nabla u|^2)\nabla u) + F'(u) = 0$$

with $F \leqslant 0$. This is a particular case in our framework given by $H(\xi) = |\xi|$, $B(t) = \Phi(t^2)/2$.

Recently, some attention has been given to the case of anisotropic media and the first pointwise estimate in this setting was given in [FV14] for equations of the type

$$\operatorname{div}(H(\nabla u)\nabla H(\nabla u)) + F'(u) = 0.$$

Again, this is a particular case for us, obtained by taking $B(t) = t^2/2$.

Our purpose is to extend the previous results to the general case of anisotropic media with possible nonlinearities, singularities and nondegeneracies in the diffusion operator. Indeed, the function H encodes the anisotropy of the medium and the function B the possible degeneracies of the operator. The precise statement of our pointwise bound is the following

Theorem 1.1. *Let u be a bounded weak solution of equation (7) in the whole of $\mathbb{R}^n$. Then, for any $x \in \mathbb{R}^n$,*

$$B'(H(\nabla u(x)))H(\nabla u(x)) - B(H(\nabla u(x))) \leqslant c_u - F(u(x)). \tag{1.1.1}$$

Moreover, if there exists $x_0 \in \mathbb{R}^n$ such that

$$\nabla u(x_0) \neq 0$$

and

$$B'(H(\nabla u(x_0)))H(\nabla u(x_0)) - B(H(\nabla u(x_0))) = c_u - F(u(x_0)), \tag{1.1.2}$$

then

$$B'(H(\nabla u))H(\nabla u) - B(H(\nabla u)) = c_u - F(u). \tag{1.1.3}$$

on the whole connected component of $\{\nabla u \neq 0\}$ containing x_0.

Now we state the main symmetry result of the chapter, according to which the equality in (1.1.2) implies that the solution only depends on one Euclidean variable (in particular, the classical and anisotropic curvatures of the level sets vanish identically).

Theorem 1.2. *Let u be a bounded weak solution of (7) in $\mathbb{R}^n$. Suppose that there exists $x_0 \in \mathbb{R}^n$ such that $\nabla u(x_0) \neq 0$ and (1.1.2) holds true. Then, u is one-dimensional in the connected component $\mathscr{S}$ of $\{\nabla u \neq 0\}$ containing x_0, and the level sets of u in $\mathscr{S}$ are affine hyperplanes.*

We observe that one-dimensional solutions $u(x) = u_0(\omega \cdot x)$ of (7) satisfy the ordinary differential equation

$$B''(H(\omega \dot{u}_0))\, H^2(\omega)\, \ddot{u}_0 + F'(u_0) = 0. \tag{1.1.4}$$

Equivalently, (1.1.3) reduces in this case to the Energy Conservation Law

$$b(H(\omega \dot{u}_0)) = c_{u_0} - F(u_0),$$

where $b(t) := B'(t)t - B(t)$.

Theorem 1.2 was proved in the isotropic setting in [CGS94] under the additional assumption that $F \leqslant 0$, and in the planar, anisotropic setting in [FV14]. Therefore Theorem 1.2 is new in the anisotropic setting even for cases that are not singular or degenerate (e.g. for $B(t) = t^2/2$). We stress in particular that the proof of this result is different from the ones in [CGS94, FV14] since we exploit for the first time the consequences of the vanishing of the P-function by taking into account explicitly an appropriate remainder term: indeed, such P-function is not only a subsolution of a suitable equation, but it is also a solution of an equation with a term that has a sign and that vanishes when P is constant (see the forthcoming equation (1.5.3) for details).

Under some further (but natural) assumptions, Theorem 1.2 holds globally in the whole of the space, as next results point out.

Theorem 1.3. *Suppose that hypothesis (B) is in force and let u be bounded weak solution of (7) in $\mathbb{R}^n$. Assume that there exists $x_0 \in \mathbb{R}^n$ such that $\nabla u(x_0) \neq 0$ and (1.1.2) holds true. Then, u is one-dimensional in the full space $\mathbb{R}^n$.*

We observe that the assumptions of Theorem 1.3 are satisfied by many cases of interest, such as the minimal surface and the regularized p-Laplace equations (e.g. with B as in (17), with $\kappa > 0$ in the p-Laplacian case). A global version of Theorem 1.3 which encompasses all the cases under consideration is given by the following result.

Theorem 1.4. *Let u be a bounded weak solution of equation (7) in $\mathbb{R}^n$. Then, u is one-dimensional in the full space $\mathbb{R}^n$.*

Differently from [CGS94], in which results similar to Theorems 1.3 and 1.4 were obtained in the isotropic setting with a different method, we do not need to require any sign assumption on F.

Next is a Liouville-type result that shows that the solution is constant if the effective potential and its derivative vanish at some point. The isotropic case was dealt with in [CGS94, CFV12].

Theorem 1.5. *Let u be a bounded weak solution of (7) in $\mathbb{R}^n$. If hypothesis (B) is in force, with $\kappa = 0$ and $p > 2$, assume in addition that, given a value $r \in \mathbb{R}$ such that $F(r) = c_u$ and $F'(r) = 0$, we have*

$$|F'(\sigma)| = O(|\sigma - r|^{p-1}) \text{ as } \sigma \to r. \tag{1.1.5}$$

If there exists a point $x_0 \in \mathbb{R}^n$ for which $F(u(x_0)) = c_u$ and $F'(u(x_0)) = 0$, then u is constant.

Notice that condition (1.1.5) cannot be removed from Theorem 1.5, since, without such assumption, one can construct smooth, non-constant, one-dimensional solutions: see Proposition 7.2 in [FSV08] for an explicit, non-constant example in which (1.1.5) is not satisfied and

$$F\left(\min_{\mathbb{R}^n} u\right) = F\left(\max_{\mathbb{R}^n} u\right) = c_u \quad \text{and} \quad F'\left(\min_{\mathbb{R}^n} u\right) = F'\left(\max_{\mathbb{R}^n} u\right) = 0.$$

We also remark that, in principle, to obtain c_u one is expected to know all the values of the solution u and to compute the potential out of them. The next result shows in fact that this is not necessary, and that c_u may be computed once we know only the infimum and the supremum of the solution. Take note that the isotropic case was dealt with in [FV10].

Theorem 1.6. *Let u be a bounded weak solution of (7) in $\mathbb{R}^n$ and F be possibly subjected to the additional conditions discussed in Theorem 1.5. Then,*

$$c_u = \max\left\{ F\left(\inf_{\mathbb{R}^n} u\right), F\left(\sup_{\mathbb{R}^n} u\right)\right\}.$$

Furthermore, if there exists $y_0 \in \mathbb{R}^n$ such that $F(u(y_0)) = c_u$, then

$$\text{either } u(y_0) = \inf_{\mathbb{R}^n} u \text{ or } u(y_0) = \sup_{\mathbb{R}^n} u.$$

Finally, we present a general result, which focuses on the investigation of hypotheses (A) and (B). Although the formulations of these conditions are simple and convey rather effectively which kind of functions B and H are allowed in our framework, sometimes it is more useful to know the requirements that the composition $B \circ H$ is asked to fulfill (see e.g. [CGS94]). This is for instance the case where one is interested in applying the known elliptic regularity theory to the equation (7). For such necessities, in the following result we provide another set of equivalent conditions.

Theorem 1.7. *Assumption (A) and (B) are respectively equivalent to*

(A)′ There exist $p > 1$, $\bar{\kappa} \in [0, 1)$ and $\bar{\gamma}, \bar{\Gamma} > 0$ such that, for any $\xi \in \mathbb{R}^n \setminus \{0\}$, $\zeta \in \mathbb{R}^n$,

$$[\mathrm{Hess}\,(B \circ H)(\xi)]_{ij}\, \zeta_i \zeta_j \geqslant \bar{\gamma}(\bar{\kappa} + |\xi|)^{p-2}|\zeta|^2,$$

$$\sum_{i,j=1}^{n} \left| [\mathrm{Hess}\,(B \circ H)(\xi)]_{ij} \right| \leqslant \bar{\Gamma}(\bar{\kappa} + |\xi|)^{p-2},$$

and

(B)′ *The composition $B \circ H$ is of class $C^{3,\beta}_{\mathrm{loc}}(\mathbb{R}^n)$ and for any $K > 0$ there exist a positive constant $\bar{\gamma}$ such that, for any $\xi, \zeta \in \mathbb{R}^n$, with $|\xi| \leqslant K$, we have*

$$[\mathrm{Hess}\,(B \circ H)(\xi)]_{ij}\, \zeta_i \zeta_j \geqslant \bar{\gamma}\, |\zeta|^2.$$

We subdivide the remaining part of the chapter as follows.

First, in Section 1.2 we collect some technical and ancillary results. In the subsequent Section 1.3, we establish the equivalence of the two sets of hypotheses (A)-(B) and (A)′-(B)′, thus proving Theorem 1.7. The regularity of the solutions is briefly tackled in Section 1.4.

The proof of Theorem 1.1 relies on a P-function argument that is discussed in Section 1.5 (roughly speaking, one has to check that a suitable energy function is a subsolution of a partial differential equation and to use the Maximum Principle to obtain the desired bound).

Finally, the proofs of the main results are collected in Section 1.6.

1.2 Some preliminary results

The first part of this section is mainly devoted to some elementary facts about positive homogeneous functions. We mostly provide only the statements, referring to [FV14] for the omitted proofs.

We recall that a function $H : \mathbb{R}^n \setminus \{0\} \to \mathbb{R}$ is said to be positive homogeneous of degree $d \in \mathbb{R}$ if

$$H(t\xi) = t^d H(\xi) \text{ for any } \xi \in \mathbb{R}^n \setminus \{0\} \text{ and } t > 0.$$

Lemma 1.8. *If $H \in C^m(\mathbb{R}^n \setminus \{0\})$ is positive homogeneous of degree d and $\alpha \in \mathbb{N}^n$ with $\alpha_1 + \cdots + \alpha_n = m$, then $\partial^\alpha H$ is positive homogeneous of degree $d - m$.*

Notice that the corresponding result proved in [FV14], Lemma 2, only deals with integer degrees. Nevertheless, the proof works the same way considering a real degree d.

Next, we establish the identities commonly used in the course of the main proofs.

Lemma 1.9. *If $H \in C^3(\mathbb{R}^n \setminus \{0\})$ is positive homogeneous of degree 1, we have that*

$$H_i(\xi)\xi_i = H(\xi), \tag{1.2.1}$$

$$H_{ij}(\xi)\xi_i = 0, \tag{1.2.2}$$

$$H_{ijk}(\xi)\xi_i = -H_{jk}(\xi). \tag{1.2.3}$$

Now, we justify the smoothness of H needed to write (7) and to use the regularity theory:

Lemma 1.10. *Let $H \in C^1(\mathbb{R}^n \setminus \{0\})$ be a positive homogeneous function of degree d admitting non-negative values and $B \in C^1([0, +\infty))$, with $B(0) = 0$. Assume that either $d > 1$ or $d = 1$ and $B'(0) = 0$. Then H can be extended by setting $H(0) := 0$ to a continuous function, such that $B \circ H \in C^1(\mathbb{R}^n)$ and*

$$\partial_i (B \circ H)(0) = 0 = \lim_{x \to 0} B'(H(x)) H_i(x).$$

Proof. Setting $H(0) := 0$ clearly transforms H into a continuous function on the whole of $\mathbb{R}^n$, since $|H(\xi)| \leqslant |\xi|^d \sup_{\mathbb{S}^{n-1}} |H|$, for any $\xi \in \mathbb{R}^n \setminus \{0\}$. Moreover, $B \circ H \in C^1(\mathbb{R}^n \setminus \{0\})$, and

$$\begin{aligned}
\partial_i (B \circ H)(0) &= \lim_{t \to 0} \frac{B(H(te_i))}{t} = \lim_{t \to 0^{\pm}} \frac{B(H(\pm|t|e_i))}{t} \\
&= \lim_{t \to 0^{\pm}} \frac{B(|t|^d H(\pm e_i))}{t} = \pm H(\pm e_i)^{\frac{1}{d}} \lim_{s \to 0^+} \frac{B(s)}{s} s^{\frac{d-1}{d}} \\
&= \pm H(\pm e_i)^{\frac{1}{d}} B'(0) \lim_{s \to 0^+} s^{\frac{d-1}{d}} = 0.
\end{aligned}$$

On the other hand, by Lemma 1.8, $H_i(x) = |x|^{d-1} H_i(x/|x|)$ for any $x \in \mathbb{R}^n \setminus \{0\}$, and so

$$\begin{aligned}\lim_{x\to 0} |B'(H(x))H_i(x)| &\leqslant \sup_{S^{n-1}} |H_i| \lim_{x\to 0} |x|^{d-1} \left| B'\left(|x|^d H\left(\frac{x}{|x|}\right)\right)\right| \\ &= |B'(0)| \sup_{S^{n-1}} |H_i| \lim_{x\to 0} |x|^{d-1} = 0,\end{aligned}$$

as desired. □

Then, we have the following characterization of the positive definiteness of the composition $B \circ H$.

Lemma 1.11. *Let $B \in C^2((0,+\infty))$ be a function satisfying (8) and $H \in C^2(\mathbb{R}^n \setminus \{0\})$ be positive homogeneous of degree 1 satisfying (9). Then, the following two statements are equivalent:*

(i) $\operatorname{Hess}(B \circ H)$ *is positive definite in* $\mathbb{R}^n \setminus \{0\}$*;*

(ii) The restriction of $\operatorname{Hess}(H)(\xi)$ *to $\xi^\perp$ is a positive definite endomorphism $\xi^\perp \to \xi^\perp$, for all $\xi \in S^{n-1}$.*

Proof. Our argument is an adaptation of the proof of Proposition 2 on page 102 of [WX11]. The case covered there is the one with $B(t) = t^2$.

First, we prove that (i) implies (ii). Fix $\xi \in S^{n-1}$. Assumption (i) is equivalent to

$$\left[B''(H(\xi))H_i(\xi)H_j(\xi) + B'(H(\xi))H_{ij}(\xi)\right]\zeta_i\zeta_j > 0 \text{ for any } \zeta \in \mathbb{R}^n \setminus \{0\}. \tag{1.2.4}$$

Observe now that $\nabla H(\xi)$ cannot be orthogonal to ξ, since, by (1.2.1), $H_i(\xi)\xi_i = H(\xi) > 0$. Therefore, $\nabla H(\xi)^\perp$ and ξ span the whole of $\mathbb{R}^n$. Letting now $V \in \xi^\perp$, we write

$$V = \zeta + \lambda\xi, \qquad \text{for some } \zeta \in \nabla H(\xi)^\perp \setminus \{0\}, \lambda \in \mathbb{R}.$$

Applying (1.2.4) with $\zeta = V - \lambda\xi$ and using (1.2.2), we get

$$\begin{aligned}0 &< \left[B''(H(\xi))H_i(\xi)H_j(\xi) + B'(H(\xi))H_{ij}(\xi)\right]\zeta_i\zeta_j = B'(H(\xi))H_{ij}(\xi)\zeta_i\zeta_j \\ &= B'(H(\xi))H_{ij}(\xi)(V_i - \lambda\xi_i)(V_j - \lambda\xi_j) = B'(H(\xi))H_{ij}(\xi)V_iV_j,\end{aligned}$$

which, by (8), gives (ii).

Conversely, assume that (ii) holds. Let $V \in \mathbb{R}^n \setminus \{0\}$ and decompose it into $V = \eta + \lambda\xi$, for $\eta \in \xi^\perp$, $\lambda \in \mathbb{R}$. By (1.2.2), (8) and (ii) we obtain

$$\begin{aligned}&\left[B''(H(\xi))H_i(\xi)H_j(\xi) + B'(H(\xi))H_{ij}(\xi)\right]V_iV_j \\ &\quad = B''(H(\xi))H_i(\xi)H_j(\xi)V_iV_j + B'(H(\xi))H_{ij}(\xi)(\eta_i + \lambda\xi_i)(\eta_j + \lambda\xi_j) \\ &\quad = B''(H(\xi))[V \cdot \nabla H(\xi)]^2 + B'(H(\xi))H_{ij}(\xi)\eta_i\eta_j \geqslant B'(H(\xi))H_{ij}(\xi)\eta_i\eta_j > 0,\end{aligned}$$

if $\eta \neq 0$. If on the other hand $\eta = 0$, i.e. $V = \lambda\xi$ with $\lambda \neq 0$, then, using (1.2.1) and (1.2.2),

$$\begin{aligned}&\left[B''(H(\xi))H_i(\xi)H_j(\xi) + B'(H(\xi))H_{ij}(\xi)\right]V_iV_j \\ &\quad = \lambda^2\left[B''(H(\xi))H_i(\xi)H_j(\xi)\xi_i\xi_j + B'(H(\xi))H_{ij}(\xi)\xi_i\xi_j\right] = \lambda^2 B''(H(\xi))H^2(\xi) > 0,\end{aligned}$$

so that (i) is proved. □

Next, we have a result ensuring the convexity of H. We point out that this actually comes as a corollary of Lemma 1.11 and Lemma 1.9 together.

Lemma 1.12. *Let $H \in C^2(\mathbb{R}^n \setminus \{0\})$ be a positive homogeneous function of degree 1 satisfying (9) and $B \in C^2((0,+\infty))$ be such (8) holds. Assume also* Hess $(B \circ H)$ *to be positive definite in $\mathbb{R}^n \setminus \{0\}$. Then H is convex and*

$$H_{ij}(\xi)\,\eta_i\eta_j \geqslant 0 \qquad \text{for any } \xi \in \mathbb{R}^n \setminus \{0\} \text{ and } \eta \in \mathbb{R}^n. \tag{1.2.5}$$

Following is a linear algebra result that is crucial for the subsequent proofs of Proposition 1.22 and Theorem 1.2.

Proposition 1.13. *Let H and B as in the statement of Lemma 1.12. Then, given any matrix $\{c_{ij}\}_{i,j\in\{1,\dots,n\}}$, we have*

$$H_{ij}(\xi)H_{k\ell}(\xi)c_{ik}c_{j\ell} \geqslant 0 \qquad \text{for any } \xi \in \mathbb{R}^n \setminus \{0\}. \tag{1.2.6}$$

Moreover, assume that equality holds in (1.2.6) for a vector $\xi = (\xi_1, \dots, \xi_n) \in \mathbb{R}^n \setminus \{0\}$ such that

$$\xi_1 = \dots = \xi_{n-1} = 0. \tag{1.2.7}$$

Then[1]

$$c_{i'j'} = 0 \text{ for any } i', j' \in \{1, \dots, n-1\}. \tag{1.2.8}$$

Proof. We follow the argument given at the end of the proof of Proposition 1 of [FV14]. By point (ii) in Lemma 1.11 and (1.2.2), we know that

$$\begin{array}{l}\text{Hess}(H)(\xi) \text{ has } n-1 \text{ strictly positive eigenvalues and one null eigenvalue}\\ \text{(the latter corresponding to the eigenvector } \xi).\end{array} \tag{1.2.9}$$

Therefore, we can diagonalize it via an orthogonal matrix $\{M_{ij}\}_{i,j\in\{1,\dots,n\}}$, by writing

$$H_{ij} = M_{pi}\lambda_p M_{pj}, \text{ with } \lambda_1 \geqslant \ldots \geqslant \lambda_{n-1} > \lambda_n = 0. \tag{1.2.10}$$

So, setting

$$\vartheta_{pr} := M_{pi}M_{rm}c_{im}, \tag{1.2.11}$$

for fixed p and r, we have that

$$0 \leqslant (\vartheta_{pr})^2 = (M_{pi}M_{rk}c_{ik})(M_{pj}M_{r\ell}c_{j\ell}) = M_{pi}M_{pj}M_{rk}M_{r\ell}c_{ik}c_{j\ell}.$$

Now, multiply by $\lambda_p\lambda_r$ and sum over p and r. We get

$$0 \leqslant \lambda_p\lambda_r(\vartheta_{pr})^2 = M_{pi}\lambda_p M_{pj}M_{rk}\lambda_r M_{r\ell}c_{ik}c_{j\ell} = H_{ij}H_{k\ell}c_{ik}c_{j\ell}, \tag{1.2.12}$$

which proves (1.2.6).

Now we assume (1.2.7) and we suppose that equality holds in (1.2.6). We claim that

$$M_{ni'} = 0 \text{ for any } i' \in \{1, \dots, n-1\}. \tag{1.2.13}$$

For this, we use a classical linear algebra procedure: we define $w_i := M_{ni}$ and we consider the vector $w = (w_1, \dots, w_n)$. We exploit (1.2.10) and we have, for any $j \in \{1, \dots, n\}$,

$$\begin{aligned}\big(\text{Hess}(H)(\xi)w\big)_j = H_{jk}w_k = M_{ij}\lambda_i M_{ik}w_k &= M_{ij}\lambda_i M_{ik}M_{nk}\\ &= M_{ij}\lambda_i\delta_{in} = M_{nj}\lambda_n = 0 = (0\,w)_j.\end{aligned}$$

[1]To avoid confusion, we use indices like i ranging in $\{1, \dots, n\}$ and like i' ranging in $\{1, \dots, n-1\}$.

That is, w is an eigenvector for $\mathrm{Hess}(H)(\xi)$ and so, by (1.2.9), w is parallel to ξ. Thus, by (1.2.7), w is parallel to $(0,\dots,0,1)$ and so $w_{i'}=0$ for any $i'\in\{1,\dots,n-1\}$, proving (1.2.13).

Now, if equality holds in (1.2.6), then (1.2.12) gives that

$$0=\lambda_p\lambda_r(\vartheta_{pr})^2.$$

Consequently, by (1.2.10), we obtain that

$$\vartheta_{p'r'}=0 \text{ for any } p',r'\in\{1,\dots,n-1\}. \tag{1.2.14}$$

Hence, we invert (1.2.11) and we obtain that

$$M_{pj}M_{rk}\vartheta_{pr}=M_{pj}M_{pi}M_{rk}M_{rm}c_{im}=\delta_{ij}\delta_{mk}c_{im}=c_{jk}$$

for any $j,k\in\{1,\dots,n\}$. So, recalling (1.2.13) and (1.2.14), we have, for any $j',k'\in\{1,\dots,n-1\}$,

$$c_{j'k'}=M_{pj'}M_{rk'}\vartheta_{pr}=M_{p'j'}M_{r'k'}\vartheta_{p'r'}=0,$$

where the indices p',r' are summed over $\{1,\dots,n-1\}$. □

Now we collect two technical inequalities concerning function B which will be used in the proofs of Theorems 1.4 and 1.5.

Lemma 1.14. *Let $B\in C^2((0,+\infty))\cap C^0([0,+\infty))$ be a function satisfying $B(0)=0$ and (8). Then,*

$$B'(t)t-B(t)>0, \tag{1.2.15}$$

for any $t>0$.

Proof. For any $t>0$ set

$$b(t):=B'(t)t-B(t). \tag{1.2.16}$$

Clearly, $b\in C^1((0,+\infty))$. By differentiation we get, for $t>0$,

$$b'(t)=B''(t)t+B'(t)-B'(t)=B''(t)t>0,$$

since $B''(t)$ is positive. Thus, b is strictly increasing and so

$$b(t)>b(0^+)=0, \qquad \text{for any } t>0,$$

which proves the lemma. □

Lemma 1.15. *Let $B\in C^2((0,+\infty))\cap C^1([0,+\infty))$ be such $B(0)=0$. Assume that B either satisfies the requirements displayed in (A) or (B). Then, for any $M>0$, there exists $\varepsilon>0$ such that*

$$B'(t)t-B(t)\geqslant\varepsilon t^{p^*} \textit{ for any } t\in[0,M], \tag{1.2.17}$$

where

$$p^*=\begin{cases} p & \textit{if (A) holds with } \kappa=0\\ 2 & \textit{otherwise.}\end{cases}$$

Proof. Let $M > 0$, b be as in (1.2.16) and $\varepsilon > 0$ to be determined later. Define, for any non-negative t,

$$E(t) := b(t) - \varepsilon t^{p^*} = B'(t)t - B(t) - \varepsilon t^{p^*}.$$

If we prove that

$$E'(t) \geqslant 0 \text{ for any } t \in (0, M], \tag{1.2.18}$$

is true, then we are done, since in this case we have

$$E(t) \geqslant E(0) = 0 \text{ for any } t \in (0, M],$$

which leads directly to (1.2.17). In order to obtain the validity of (1.2.18), we first claim that

$$B''(t)t \geqslant ct^{p^*-1} \text{ for any } t \in (0, M], \tag{1.2.19}$$

for some $c > 0$.

To see that (1.2.19) is valid, we deal separately with the three possibilities: hypothesis (A) is in force with $\kappa > 0$, (A) is in force with $\kappa = 0$ and (B) is in force.

First, we assume (A) to hold, with $\kappa > 0$. In this case we have

$$B''(t)t \geqslant \gamma\,(\kappa + t)^{p-2}\, t \geqslant \gamma\kappa^{p-2} t,$$

if $p \geqslant 2$, and

$$B''(t)t \geqslant \gamma\,(\kappa + t)^{p-2}\, t \geqslant \gamma\,(\kappa + M)^{p-2}\, t,$$

if $1 < p < 2$. Recalling the definition of p^*, we understand that in both situations (1.2.19) is true.

When $\kappa = 0$ in hypothesis (A), then (1.2.19) easily follows - with $c = \gamma$ - by writing down the lower bound for B''.

Finally, if (B) is in force, we know that B'' is continuous in the whole half-line $[0, +\infty)$, with $B''(t) > 0$ for any $t \geqslant 0$. Consequently, there exists $\gamma_* > 0$ for which $B''(t) \geqslant \gamma_*$ for any $t \in [0, M]$ and, again, (1.2.19) follows.

By (1.2.19) and choosing $\varepsilon > 0$ small enough, we conclude that

$$E'(t) = B''(t)t - \varepsilon p^* t^{p^*-1} \geqslant (c - \varepsilon p^*)t^{p^*-1} \geqslant 0,$$

which gives (1.2.18). □

Notice that, in the setting of the chapter, Lemma 1.14 actually comes as a corollary of Lemma 1.15. Nevertheless, we preferred to state them independently one to the other, as the first do not involve the additional structural conditions (A)-(B) at all.

Next is a lemma which gathers some results on H and its dual H^*.

Lemma 1.16. *Let $B \in C^2((0, +\infty))$ and $H \in C^2(\mathbb{R}^n \setminus \{0\})$. Assume B to satisfy (8), the function H to be positive homogeneous of degree 1 satisfying (9) and* $\mathrm{Hess}(B \circ H)$ *to be positive definite in $\mathbb{R}^n \setminus \{0\}$. Then, the ball B_1^H defined by (10) is strictly convex.*

Furthermore, the dual function H^* defined by (11) is of class $C^2(\mathbb{R}^n \setminus \{0\})$, the formulae

$$H^*(\nabla H(\xi)) = H(\nabla H^*(\xi)) = 1, \tag{1.2.20}$$

hold true for any $\xi \in \mathbb{R}^n \setminus \{0\}$ and the map $\Psi_H : \mathbb{R}^n \to \mathbb{R}^n$, defined by setting

$$\Psi_H(\xi) := H(\xi)\nabla H(\xi),$$

for any $\xi \in \mathbb{R}^n$, is a global homeomorphism of $\mathbb{R}^n$, with inverse Ψ_{H^*}.

Proof. Notice that $B \circ H \in C^2(\mathbb{R}^n \setminus \{0\}) \cap C^0(\mathbb{R}^n)$ and its Hessian is positive definite in $\mathbb{R}^n \setminus \{0\}$. Hence $B \circ H$ is strictly convex in the whole of $\mathbb{R}^n$. Moreover, being B' positive by (8), the ball B_1^H is also a sublevel set of $B \circ H$ and thus strictly convex.

The other claims are valid by virtue of [CiaS09, Lemma 3.1]. Note that H is assumed to be even in [CiaS09], but this assumption is not used in the proof of Lemma 3.1 there. Hence this result is valid also in our setting.

Moreover, H^* is of class C^2 outside of the origin, since so is the diffeomorphism Ψ_H. □

Finally, we present a lemma ensuring the continuity of the second derivative of B at the origin starting from some regularity assumptions on the composition $B \circ H$. The framework in which this result is meant to be set is that of hypothesis (B)′ and, in fact, explicit use of it will be made in the following Section 1.3.

Lemma 1.17. *Let $H \in C^2(\mathbb{R}^n \setminus \{0\})$ be a positive homogeneous function of degree 1 satisfying (9) and $B \in C^1([0, +\infty)) \cap C^2((0, +\infty))$, with $B(0) = B'(0) = 0$. Assume in addition that $B \circ H$ has some pure second derivative, say, the first, continuous at the origin. Then, $B \in C^2([0, +\infty))$ with*

$$B''(0) = H^{-2}(e_1) \frac{\partial^2 (B \circ H)}{\partial \xi_1^2}(0). \tag{1.2.21}$$

In particular, this holds if $B \circ H \in C^2(\mathbb{R}^n)$.

Proof. Since, for every $\xi \neq 0$,

$$\frac{\partial^2 (B \circ H)}{\partial \xi_1^2}(\xi) = B''(H(\xi)) H_1^2(\xi) + B'(H(\xi)) H_{11}(\xi),$$

by choosing $\xi = te_1$, with $t > 0$, and the homogeneity properties of H we obtain

$$\frac{\partial^2 (B \circ H)}{\partial \xi_1^2}(te_1) = B''(tH(e_1)) H_1^2(e_1) + \frac{B'(tH(e_1))}{t} H_{11}(e_1). \tag{1.2.22}$$

Now, observe that

$$H_1(e_1) = \nabla H(e_1) \cdot e_1 = H(e_1) > 0,$$

by (1.2.1) and

$$H_{11}(e_1) = \nabla H_1(e_1) \cdot e_1 = 0,$$

by (1.2.2). Therefore, by (1.2.22) we get

$$B''(tH(e_1)) = H^{-2}(e_1) \frac{\partial^2 (B \circ H)}{\partial \xi_1^2}(te_1),$$

which yields (1.2.21) by passing to the limit as $t \to 0^+$. □

1.3 Equivalence between assumptions (A)-(B) and (A)′-(B)′

In this second preliminary section we prove the equivalence of the two couples of structural conditions previously stated. We show that both (A) and (B) respectively boil down to the more traditional (A)′ and (B)′. First, we have

Proposition 1.18. *Let $B \in C^2((0, +\infty))$ be a function satisfying (8) and $H \in C^2(\mathbb{R}^n \setminus \{0\})$ be positive homogeneous of degree 1, such that (9) is true. Then, assumptions (A) and (A)′ are equivalent. Moreover, we may take*

$$\bar{\kappa} = \kappa, \tag{1.3.1}$$

and the constants $\bar{\gamma}, \bar{\Gamma}, \lambda$ and γ, Γ to be independent of κ.

Proof. First of all, denote with $C \geqslant 1$ a constant for which

$$C^{-1}|\xi| \leqslant H(\xi) \leqslant C|\xi|, \; |\nabla H(\xi)| \leqslant C \text{ and } |\mathrm{Hess}(H)| \leqslant C|\xi|^{-1},$$

hold for any $\xi \in \mathbb{R}^n \setminus \{0\}$. Then, observe that the ellipticity and growth conditions displayed in (A)$'$ are respectively equivalent to

$$\left[B''(H(\xi))H_i(\xi)H_j(\xi) + B'(H(\xi))H_{ij}(\xi)\right]\zeta_i\zeta_j \geqslant \bar{\gamma}\left(\bar{\kappa} + |\xi|\right)^{p-2}|\zeta|^2, \tag{1.3.2}$$

$$\sum_{i,j=1}^{n}\left|B''(H(\xi))H_i(\xi)H_j(\xi) + B'(H(\xi))H_{ij}(\xi)\right| \leqslant \bar{\Gamma}\left(\bar{\kappa} + |\xi|\right)^{p-2}, \tag{1.3.3}$$

for any $\xi \in \mathbb{R}^n \setminus \{0\}$ and $\zeta \in \mathbb{R}^n$.

We start by showing that (A) implies (A)$'$, in its above mentioned equivalent form. First, we check that (1.3.3) is true. We have

$$\begin{aligned}\sum_{i,j=1}^{n}\left|B''(H(\xi))H_i(\xi)H_j(\xi) + B'(H(\xi))H_{ij}(\xi)\right| &\leqslant \Gamma(\kappa + H(\xi))^{p-2}\left[C^2 + CH(\xi)|\xi|^{-1}\right] \\ &\leqslant 2\Gamma C^2(\kappa + c_*|\xi|)^{p-2} \\ &= 2\Gamma C^2 c_*^{p-2}(c_*^{-1}\kappa + |\xi|)^{p-2},\end{aligned}$$

with

$$c_* := \begin{cases} C & \text{if } p \geqslant 2, \\ 1/C & \text{if } 1 < p < 2. \end{cases} \tag{1.3.4}$$

The proof of (1.3.2) is a bit more involved. We write

$$\zeta = \alpha\xi + \eta, \tag{1.3.5}$$

for some $\alpha \in \mathbb{R}$ and $\eta \in \nabla H(\xi)^\perp$. We stress that ξ and $\nabla H(\xi)^\perp$ span the whole $\mathbb{R}^n$ in view of (1.2.1). Thus, decomposition (1.3.5) is admissible. We distinguish between the two cases: $2|\alpha\xi| \leqslant |\zeta|$ and $2|\alpha\xi| > |\zeta|$. In the first situation, we have

$$|\eta|^2 = |\zeta - \alpha\xi|^2 = |\zeta|^2 - 2\alpha\langle\zeta,\xi\rangle + \alpha^2|\xi|^2 \geqslant (|\zeta| - |\alpha\xi|)^2 \geqslant \frac{|\zeta|^2}{4}.$$

Therefore, by applying (1.2.1), (1.2.2) and (14), we get

$$\begin{aligned}&\left[B''(H(\xi))H_i(\xi)H_j(\xi) + B'(H(\xi))H_{ij}(\xi)\right]\zeta_i\zeta_j \\ &\quad = B''(H(\xi))(H_i(\xi)\zeta_i)^2 + B'(H(\xi))H_{ij}(\xi)\eta_i\eta_j \geqslant 0 + \gamma(\kappa + H(\xi))^{p-2}H(\xi)\lambda|\xi|^{-1}|\eta|^2 \\ &\quad \geqslant 4^{-1}\gamma\lambda C^{-1}(\kappa + c_*^{-1}|\xi|)^{p-2}|\zeta|^2 = 4^{-1}\gamma\lambda C^{-1}c_*^{2-p}(c_*\kappa + |\xi|)^{p-2}|\zeta|^2 \\ &\quad \geqslant 4^{-1}\gamma\lambda C^{-1}c_*^{2-p}(c_*^{-1}\kappa + |\xi|)^{p-2}|\zeta|^2,\end{aligned}$$

where in last line we recognized that, for every $p > 1$,

$$(c_*\kappa + s)^{p-2} \geqslant (c_*^{-1}\kappa + s)^{p-2} \text{ for any } s > 0, \tag{1.3.6}$$

being $C \geqslant 1$. On the other hand, if the opposite inequality occurs we deduce that, by (1.2.1),

$$|\langle\nabla H(\xi),\zeta\rangle| = |\langle\nabla H(\xi),\alpha\xi + \eta\rangle| = |\alpha|H(\xi) \geqslant \frac{|\alpha||\xi|}{C} \geqslant \frac{|\zeta|}{2C},$$

so that, we compute

$$\begin{aligned}&\left[B''(H(\xi))H_i(\xi)H_j(\xi)+B'(H(\xi))H_{ij}(\xi)\right]\zeta_i\zeta_j\\&\quad=B''(H(\xi))(H_i(\xi)\zeta_i)^2+B'(H(\xi))H_{ij}(\xi)\eta_i\eta_j\geqslant\gamma(\kappa+H(\xi))^{p-2}(2C)^{-2}|\zeta|^2+0\\&\quad\geqslant 4^{-1}\gamma C^{-2}(\kappa+c_*^{-1}|\xi|)^{p-2}|\zeta|^2=4^{-1}\gamma C^{-2}c_*^{2-p}(c_*\kappa+|\xi|)^{p-2}|\zeta|^2\\&\quad\geqslant 4^{-1}\gamma C^{-2}c_*^{2-p}(c_*^{-1}\kappa+|\xi|)^{p-2}|\zeta|^2,\end{aligned}$$

and thus the proof of (1.3.2) is complete.

Now, we focus on the opposite implication, i.e. that (A)′ implies (A). Let $t>0$ and take $\xi\in\mathbb{R}^n\setminus\{0\}$ such that $t=H(\xi)$. Plugging $\zeta=\xi$ in (1.3.2), by (1.2.1) and (1.2.2) we obtain

$$\bar\gamma\,(\bar\kappa+|\xi|)^{p-2}\,|\xi|^2\leqslant\left[B''(t)H_i(\xi)H_j(\xi)+B'(t)H_{ij}(\xi)\right]\xi_i\xi_j=B''(t)H^2(\xi),$$

and hence that

$$B''(t)\geqslant\bar\gamma C^{-2}(\bar\kappa+c_*^{-1}t)^{p-2}=\bar\gamma C^{-2}c_*^{2-p}(c_*\bar\kappa+t)^{p-2}.$$

On the other hand, the choice $\zeta\in\nabla H(\xi)^\perp$ in (1.3.2) leads to

$$\begin{aligned}\bar\gamma\,(\bar\kappa+|\xi|)^{p-2}\,|\zeta|^2&\leqslant\left[B''(t)H_i(\xi)H_j(\xi)+B'(t)H_{ij}(\xi)\right]\zeta_i\zeta_j=B'(t)H_{ij}(\xi)\zeta_i\zeta_j\\&\leqslant CB'(t)|\xi|^{-1}|\zeta|^2\leqslant C^2B'(t)t^{-1}|\zeta|^2.\end{aligned}\tag{1.3.7}$$

As before we deduce

$$B'(t)\geqslant\bar\gamma C^{-2}c_*^{2-p}(c_*\bar\kappa+t)^{p-2}t.$$

The remaining inequalities involving B' and B'' in (A) can be similarly deduced from (1.3.3). Indeed, notice that (1.2.1) and (1.2.2) respectively yield

$$\begin{aligned}H_1(e_1)&=\langle\nabla H(e_1),e_1\rangle=H(e_1),\\H_{11}(e_1)&=\langle\nabla^2H(e_1)e_1,e_1\rangle=0.\end{aligned}$$

Hence, if we take $\mu>0$ such that $t=H(\mu e_1)$, setting $\xi=\mu e_1$ in (1.3.3) we get

$$\begin{aligned}\bar\Gamma\,(\bar\kappa+|\xi|)^{p-2}&\geqslant\sum_{i,j=1}^n\left|B''(t)H_i(\mu e_1)H_j(\mu e_1)+B'(t)H_{ij}(\mu e_1)\right|\\&\geqslant B''(t)H_1(e_1)H_1(e_1)+B'(t)\mu^{-1}H_{11}(e_1)\\&=B''(t)H^2(e_1).\end{aligned}$$

Consequently, recalling (1.3.6) we obtain

$$B''(t)\leqslant\bar\Gamma C^2(\bar\kappa+c_*t)^{p-2}=\bar\Gamma C^2c_*^{p-2}(c_*^{-1}\bar\kappa+t)^{p-2}\leqslant\bar\Gamma C^2c_*^{p-2}(c_*\bar\kappa+t)^{p-2}.$$

As a byproduct, the previous inequality implies in particular that

$$B'(1)=\int_0^1B''(t)\,dt\leqslant\frac{\bar\Gamma C^2c_*^{p-2}}{p-1}(c_*\bar\kappa+1)^{p-1}.$$

Hence, by taking $t=1$ in the first line of (1.3.7) we see that H is uniformly elliptic, with constant

$$\lambda=\frac{(p-1)c_*^{2(2-p)}\bar\gamma}{2C^2(c_*+1)\Gamma}.\tag{1.3.8}$$

Note that we took advantage of the fact that $\bar{\kappa} < 1$, along with definition (1.3.4), to deduce this bound. Finally, the growth condition on B' can be obtained as follows. Select $\xi \in \mathbb{R}^n \setminus \{0\}$ in a way that $e_1 \in \nabla H(\xi)^{\perp}$ and $H(\xi) = t$. This can be easily done for instance by taking $\xi = t\nabla H^*(e_2)$. Indeed, by Lemma 1.16, together with the homogeneity properties of H and ∇H, we have

$$\begin{aligned} 0 = \langle e_2, e_1 \rangle &= H(H^*(e_2)\nabla H^*(e_2))\langle \nabla H(H^*(e_2)\nabla H^*(e_2)), e_1\rangle \\ &= H^*(e_2)H(\nabla H^*(e_2))\langle \nabla H(\nabla H^*(e_2)), e_1\rangle = H^*(e_2)\langle \nabla H(\nabla H^*(e_2)), e_1\rangle. \end{aligned}$$

Such a choice implies that

$$\langle \nabla H(\xi), e_1\rangle = \langle \nabla H(\nabla H^*(e_2)), e_1\rangle = 0.$$

Moreover, it is easy to see that $H(\xi) = t$. From (1.3.3) we may then compute

$$\begin{aligned} \bar{\Gamma}\,(\bar{\kappa} + |\xi|)^{p-2} &\geqslant \sum_{i,j=1}^{n} \left|B''(t)H_i(\xi)H_j(\xi) + B'(t)H_{ij}(\xi)\right| \\ &\geqslant B''(t)H_1(\xi)H_1(\xi) + B'(t)H_{11}(\xi) \\ &= B'(t)H_{11}(\xi) \geqslant B'(t)\lambda|\xi|^{-1}, \end{aligned}$$

from which we get, as before,

$$B'(t) \leqslant \bar{\Gamma}\lambda^{-1}Cc_*^{p-2}(c_*\bar{\kappa} + t)^{p-2}t,$$

with λ as in (1.3.8). This conclude the proof of the second part of our claim.

The fact that we may assume (1.3.1) to hold - up to relabeling the constants γ, Γ or $\bar{\gamma}, \bar{\Gamma}$ in dependence of C - is a consequence of the inequalities

$$(\bar{\kappa} + t)^{p-2} \leqslant (c_*\bar{\kappa} + t)^{p-2} \leqslant c_*^{p-2}(\bar{\kappa} + t)^{p-2},$$

and

$$c_*^{2-p}(\kappa + |\xi|)^{p-2} \leqslant (c_*^{-1}\kappa + |\xi|)^{p-2} \leqslant (\kappa + |\xi|)^{p-2}. \quad \square$$

On the other hand, the characterization of (B) in terms of (B)$'$ is the content of the following

Proposition 1.19. *Let $B \in C^3((0, +\infty)) \cap C^1([0, +\infty))$ be a function satisfying (8) along with $B(0) = B'(0) = 0$ and $H \in C^3(\mathbb{R}^n \setminus \{0\})$ be positive homogeneous of degree 1, such that (9) is true. Then, hypotheses (B) and (B)$'$ are equivalent.*

Proof. We begin by showing that (B) implies (B)$'$. First, we deal with the regularity of the composition $B \circ H$. By the general assumptions on B and H, it is clear that $B \circ H \in C^{3,\beta}_{\mathrm{loc}}(\mathbb{R}^n \setminus \{0\}) \cap C^1(\mathbb{R}^n)$. Thus, we only need to check the second and third derivatives of $B \circ H$ at the origin. For any $e \in S^{n-1}$ and $t > 0$, by the homogeneity of H we have

$$\begin{aligned} (B \circ H)_{ij}(te) &= B''(H(te))H_i(te)H_j(te) + B'(H(te))H_{ij}(te) \\ &= B''(tH(e))H_i(e)H_j(e) + \frac{B'(tH(e))}{tH(e)}H(e)H_{ij}(e). \end{aligned}$$

Hence, taking the limit as $t \to 0^+$

$$\lim_{t\to 0^+} (B \circ H)_{ij}(te) = B''(0)\left[H_i(e)H_j(e) + H(e)H_{ij}(e)\right]. \tag{1.3.9}$$

Now, observe that, being H of the special form (16), we may explicitly compute

$$M_{ij} = \partial_{ij}\left(\frac{H^2}{2}\right)(\xi) = H_i(\xi)H_j(\xi) + H(\xi)H_{ij}(\xi), \tag{1.3.10}$$

for any $\xi \in \mathbb{R}^n$. As a consequence of (1.3.10), the right hand side of (1.3.9) does not depend on $e \in S^{n-1}$ and so

$$(B \circ H)_{ij}(0) = B''(0)M_{ij}.$$

Now we focus on the third derivative. First, by differentiating (1.3.10) we deduce the identity

$$H_i(\xi)H_{jk}(\xi) + H_j(\xi)H_{ik}(\xi) + H_k(\xi)H_{ij}(\xi) = -H(\xi)H_{ijk}(\xi).$$

With this in hand we compute

$$\begin{aligned}(B \circ H)_{ijk}(te) &= B'''(H(te))H_i(te)H_j(te)H_k(te) \\ &\quad + B''(H(te))\left[H_i(te)H_{jk}(te) + H_j(te)H_{ik}(te) + H_k(te)H_{ij}(te)\right] \\ &\quad + B'(H(te))H_{ijk}(te) \\ &= B'''(tH(e))H_i(e)H_j(e)H_k(e) \\ &\quad + \frac{1}{tH(e)}\left[\frac{B'(tH(e))}{tH(e)} - B''(tH(e))\right]H^2(e)H_{ijk}(e).\end{aligned} \tag{1.3.11}$$

Now, we claim that

$$\lim_{s \to 0^+} \frac{1}{s}\left[\frac{B'(s)}{s} - B''(s)\right] = 0. \tag{1.3.12}$$

Indeed, since $B'(0) = B'''(0) = 0$, the Taylor expansions of B' and B'' are

$$B'(s) = B''(0)s + o(s^2) \quad \text{and} \quad B''(s) = B''(0) + o(s),$$

as $s \to 0^+$. Therefore

$$\frac{B'(s)}{s} - B''(s) = \frac{B''(0)s}{s} - B''(0) + o(s) = o(s),$$

and (1.3.12) follows. Thus, letting $t \to 0^+$ in (1.3.11), we get

$$\lim_{t \to 0^+}(B \circ H)_{ijk}(te) = B'''(0)H_i(e)H_j(e)H_k(e) + 0 \cdot H^2(e)H_{ijk}(e) = 0,$$

for any $e \in S^{n-1}$. We may thence conclude that $B \circ H \in C^{3,\beta}_{\text{loc}}(\mathbb{R}^n)$. Finally, we prove that $\operatorname{Hess}(B \circ H)$ is uniformly elliptic on compact subsets, as required in (B)′. Let

$$C := \max_{\xi \in S^{n-1}} H(\xi). \tag{1.3.13}$$

By (8) and (15), for any $K > 0$, there exists $\gamma_* > 0$ such that

$$B''(t) \geqslant \gamma_*, \tag{1.3.14}$$

for any $t \in [0, CK]$. Since $B'(0) = 0$, we also infer that

$$B'(t) = \int_0^t B''(s)\,ds \geqslant \gamma_* t, \tag{1.3.15}$$

for any $t \in [0, CK]$. Let $\xi, \eta \in \mathbb{R}^n$, with $|\xi| \leqslant K$. Observe that, by (1.3.13), it holds

$$H(\xi) \leqslant |\xi| H\left(\frac{\xi}{|\xi|}\right) \leqslant CK.$$

Then, by (1.3.14), (1.3.15) and (1.3.10),

$$\begin{aligned}(B\circ H)_{ij}(\xi)\eta_i\eta_j &= \left[B''(H(\xi))H_i(\xi)H_j(\xi)+B'(H(\xi))H_{ij}(\xi)\right]\eta_i\eta_j\\ &\geqslant \gamma_*\left[H_i(\xi)H_j(\xi)+H(\xi)H_{ij}(\xi)\right]\eta_i\eta_j\\ &= \gamma_* M_{ij}\eta_i\eta_j,\end{aligned}$$

and the result follows from the positive definiteness of M.

Now, we turn to the converse implication, i.e. that (B)$'$ implies (B). First, we address the regularity of B and the validity of (15). In view of Lemma 1.17 we already know that $B\in C^2([0,+\infty))$ with, say,

$$B''(0)=\frac{(B\circ H)_{11}(0)}{H^2(e_1)}.$$

Hence, by the definite positiveness of the Hessian of $B\circ H$, we deduce that (15) holds true. On the other hand, by (1.2.1) and (1.2.2) we compute

$$\begin{aligned}(B\circ H)_{111}(0) &= \lim_{t\to 0}\frac{(B\circ H)_{11}(te_1)-(B\circ H)_{11}(0)}{t}\\ &= \lim_{t\to 0}\frac{B''(H(te_1))H_1(te_1)H_1(te_1)+B'(H(te_1))H_{11}(te_1)-B''(0)H^2(e_1)}{t}\\ &= \pm H^3(e_1)\lim_{t\to 0^\pm}\frac{B''(|t|H(e_1))-B''(0)}{|t|H(e_1)}\\ &= \pm H^3(e_1)\lim_{s\to 0^+}\frac{B''(s)-B''(0)}{s}.\end{aligned}$$

Since the left hand side exists finite, the same should be true for the right one, too. Thus, we obtain that $B\in C^3([0,+\infty))$ with $B'''(0)=0$. This concludes the proof, as the local Hölderianity of the third derivative of B up to 0 may be easily deduced from that of $B\circ H$.

Finally, we show that H needs to be of the type (16). By Lemma 1.10, we know that

$$\partial_i(B\circ H)(\xi)=\begin{cases}B'(H(\xi))H_i(\xi) & \text{if } \xi\neq 0\\ 0 & \text{if } \xi=0.\end{cases}$$

Thus, we can proceed to compute the second partial derivatives of $B\circ H$ at the origin. We have

$$\begin{aligned}\partial^2_{ij}(B\circ H)(0) &= \lim_{t\to 0}\frac{B'(H(te_j))H_i(te_j)-0}{t}=\lim_{t\to 0^\pm}\frac{B'(|t|H(\pm e_j))H_i(\pm e_j)}{t}\\ &= \pm\lim_{t\to 0^\pm}\frac{B'(|t|H(\pm e_j))}{|t|H(\pm e_j)}H_i(\pm e_j)H(\pm e_j)\\ &= \pm B''(0)H_i(\pm e_j)H(\pm e_j).\end{aligned}\tag{1.3.16}$$

Therefore, recalling (15), we may conclude that the limit exists if and only if

$$H_i(e_j)H(e_j)=-H_i(-e_j)H(-e_j),\qquad \text{for any } i,j\in\{1,\dots,n\},$$

or, equivalently,

$$\partial_i\left(H^2\right)(e_j)=-\partial_i\left(H^2\right)(-e_j),\qquad \text{for any } i,j\in\{1,\dots,n\}.\tag{1.3.17}$$

Knowing this, we can check the continuity of the derivatives at the origin. Since

$$\partial^2_{ij}(B\circ H)(\xi)=B''(H(\xi))H_i(\xi)H_j(\xi)+B'(H(\xi))H_{ij}(\xi),$$

for any $\xi \neq 0$, we compute

$$\begin{aligned}\lim_{\xi\to 0}\partial^2_{ij}(B\circ H)(\xi) &= \lim_{\xi\to 0}\left[B''(H(\xi)) - \frac{B'(H(\xi))}{H(\xi)}\right]H_i(\xi)H_j(\xi)\\ &\quad + \lim_{\xi\to 0}\frac{B'(H(\xi))}{H(\xi)}\left[H_i(\xi)H_j(\xi) + H(\xi)H_{ij}(\xi)\right]\\ &=: L_1 + L_2.\end{aligned}$$

We observe that $L_1 = 0$, since H_iH_j is homogeneous of degree 0, and thus bounded, and B is of class C^2 at the origin. Therefore we get

$$\lim_{\xi\to 0}\partial^2_{ij}(B\circ H)(\xi) = L_2 = B''(0)\lim_{\xi\to 0}\left[H_i(\xi)H_j(\xi) + H(\xi)H_{ij}(\xi)\right],$$

so that, recalling (1.3.16), the continuity of the second derivatives is equivalent to

$$\lim_{\xi\to 0}\left[H_i(\xi)H_j(\xi) + H(\xi)H_{ij}(\xi)\right] = H_i(e_j)H(e_j).$$

Rewriting last identity as

$$\lim_{\xi\to 0}\partial^2_{ij}\left(\frac{H^2}{2}\right)(\xi) = \partial_i\left(\frac{H^2}{2}\right)(e_j), \tag{1.3.18}$$

we notice that, since $\partial^2_{ij}\left(H^2/2\right)$ is a homogeneous function of degree 0, by (1.3.18) it has limit at the origin and so it is necessarily constant. This means that H^2 is a polynomial of degree 2 and thus

$$H(\xi) = H_M(\xi) := \sqrt{\langle M\xi,\xi\rangle}, \qquad \text{for any } \xi\in\mathbb{R}^n,$$

with $M\in \mathrm{Mat}_n(\mathbb{R})$ symmetric and positive definite. The function H_M thus defined is clearly positive homogeneous of degree 1 and it satisfies (1.3.17), since it is even. □

We remark that Theorem 1.7 now follows easily from Propositions 1.18 and 1.19 combined.

1.4 Regularity of the solutions

In this short section we point out some regularity properties of the weak solutions of (7).

Proposition 1.20. *Let u be a bounded weak solution of (7) in the whole of $\mathbb{R}^n$. Then, given any $x_0\in\mathbb{R}^n$ and $R\in(0,1)$, there exist $\alpha\in(0,1)$ and $C>0$, depending only on n, R, $\|u\|_{L^\infty(\mathbb{R}^n)}$ and the constants involved in (A) or (B), so that*

$$\|\nabla u\|_{L^\infty(\mathbb{R}^n)} \leqslant C, \tag{1.4.1}$$

$$|\nabla u(x)-\nabla u(y)| \leqslant CR^{-\alpha}|x-y|^\alpha, \tag{1.4.2}$$

for any $x,y\in B_R(x_0)$. In particular, $u\in C^{1,\alpha}_{\mathrm{loc}}(\mathbb{R}^n)$, for such α.

Proof. Under hypothesis (A), we can apply Theorem 1 on page 127 of [T84]. Notice that the ellipticity and growth conditions there required are satisfied, since assumption (A) is equivalent to (A)′ by Theorem 1.7. Condition (1.7) of [T84] is also valid, due to the fact that F' is continuous and u bounded. Finally, the locally boundedness of the gradient in [T84] could be easily extended to the whole of $\mathbb{R}^n$, giving (1.4.1). See also [DB83].

If on the other hand (B) is in force, then (1.4.1) is already satisfied, as $u\in W^{1,\infty}(\mathbb{R}^n)$ by definition. To obtain (1.4.2), the uniform ellipticity of $\mathrm{Hess}(B\circ H)(\nabla u)$ - which is ensured by condition (B)′, using again Theorem 1.7 - allows us to appeal to Theorem 1.1 on page 339 of [LU68]. Notice that we know in addition that $u\in W^{2,2}_{\mathrm{loc}}(\mathbb{R}^n)$ in this case, thanks to Proposition 1 in [T84], the boundedness of ∇u and the structural conditions in (B)′. □

If we stay far from the points on which ∇u vanishes, then we can obtain even more regularity for u, as displayed by the following result.

Proposition 1.21. *Let u be a bounded weak solution of (7) in $\mathbb{R}^n$. Then, for any $x \in \mathbb{R}^n$ with $\nabla u(x) \neq 0$ there exists $R > 0$ and $\alpha \in (0,1)$ such that $u \in C^{3,\alpha}(B_R(x))$.*

In particular, we have that $u \in C^3(\{\nabla u \neq 0\})$.

Moreover, under assumption (B) we have the stronger conclusion that $u \in C^{3,\alpha}_{\mathrm{loc}}(\mathbb{R}^n)$.

Proof. If (A) holds and x is as in the statement, then we may apply Theorem 6.4 on page 284 of [LU68] in some neighborhood of x contained in $\{\nabla u \neq 0\}$, which exists due to the continuity of ∇u granted by Proposition 1.20, to obtain the thesis.

The same result also holds if condition (B) is valid, relying instead on Theorem 6.3 on page 283 of [LU68]. Note that, in this case, the non-degeneracy of ∇u is no longer required, obtaining that u is actually of class $C^{3,\alpha}_{\mathrm{loc}}$ on the whole of $\mathbb{R}^n$. □

1.5 P-function computations

Now we perform a P-function argument, by showing that a suitable energy functional is a subsolution of a partial differential equation (in fact, it is a solution, with a remainder term which has a sign). Classical computations of this kind can be found in [P76, S81].

For the sake of briefness, in the following we will often drop the argument of various functions, adopting for instance the notations $H = H(\nabla u)$, $H_i = (\partial_i H)(\nabla u)$, $B = B(H(\nabla u))$, $B' = B'(H(\nabla u))$, etc.

Proposition 1.22. *Let u be a bounded weak solution of (7) in the whole of $\mathbb{R}^n$. Set*

$$a_{ij} := B'' H_i H_j + B' H_{ij}, \qquad d_{ij} := a_{ij}/H, \tag{1.5.1}$$

and

$$P(u;x) := B'(H(\nabla u(x)))H(\nabla u(x)) - B(H(\nabla u(x))) - G(u(x)), \tag{1.5.2}$$

where G is as in (23). Then,

$$(d_{ij}P_i)_j - b_k P_k = \mathscr{R} \geqslant 0 \text{ on } \{\nabla u \neq 0\}, \tag{1.5.3}$$

where

$$\begin{aligned} b_k &:= \frac{B'''}{B''} H^{-2} H_\ell P_\ell H_k + \left[\frac{B'''}{B''} + \frac{B''}{B'}\right] G' H^{-1} H_k + \left[\frac{B'B'''}{(B'')^2} + 1\right] H^{-2} H_{k\ell} P_\ell \\ \text{and } \mathscr{R} &:= B'B'' H_{ij} H_{k\ell} u_{ik} u_{j\ell}. \end{aligned} \tag{1.5.4}$$

Proof. First of all, we point out that, by Proposition 1.21, u is $C^3(\{\nabla u \neq 0\})$. We will therefore implicitly assume every calculation to be performed on $\{\nabla u \neq 0\}$. The computation is quite long and somehow delicate, but we provide full details of the argument for the facility of the reader.

By differentiating (1.5.2), we get for any $i \in \{1, \ldots, n\}$

$$P_i = B'' H H_k u_{ki} + B' H_k u_{ki} - B' H_k u_{ki} - G' u_i = B'' H H_k u_{ki} - G' u_i. \tag{1.5.5}$$

Thus, recalling (1.5.1),

$$\begin{aligned} (d_{ij}P_i)_j &= (B'' H H_k d_{ij} u_{ki})_j - (G' d_{ij} u_i)_j \\ &= (B'' H_k)_j a_{ij} u_{ki} + B'' H_k (a_{ij} u_{ki})_j - (G' d_{ij} u_i)_j. \end{aligned} \tag{1.5.6}$$

Next, observe that from (7) we have

$$a_{ij}u_{ij} = G'. \tag{1.5.7}$$

Being u of class C^3, we compute for any k

$$\begin{aligned}(a_{ij}u_{ki})_j - (a_{ij}u_{ij})_k &= (a_{ij})_j u_{ki} - (a_{ij})_k u_{ij} \\ &= \left[B'''H_iH_jH_\ell + B''H_{i\ell}H_j + B''H_iH_{j\ell} + B''H_{ij}H_\ell + B'H_{ij\ell}\right] u_{j\ell}u_{ki} \\ &\quad - \left[B'''H_iH_jH_\ell + B''H_{i\ell}H_j + B''H_iH_{j\ell} + B''H_{ij}H_\ell + B'H_{ij\ell}\right] u_{k\ell}u_{ij} = 0,\end{aligned} \tag{1.5.8}$$

by interchanging the indices i and ℓ in the last term. Therefore, using (1.5.7) we obtain

$$(a_{ij}u_{ki})_j = (a_{ij}u_{ij})_k = (G')_k = G''u_k.$$

Plugging this into (1.5.6) we have

$$\begin{aligned}(d_{ij}P_i)_j &= (B''H_k)_j a_{ij}u_{ki} + B''H_kG''u_k - (G'd_{ij}u_i)_j \\ &= (B''H_k)_j a_{ij}u_{ki} + B''H_kG''u_k - G''d_{ij}u_iu_j - G'(d_{ij}u_i)_j.\end{aligned} \tag{1.5.9}$$

Now, we collect the two terms containing G'', getting, by (1.2.1) and (1.2.2),

$$\begin{aligned}B''H_kG''u_k - G''d_{ij}u_iu_j &= G''H^{-1}\left[B''HH_ku_k - a_{ij}u_iu_j\right] \\ &= G''H^{-1}\left[B''H^2 - B''H_iH_ju_iu_j - B'H_{ij}u_iu_j\right] \\ &= G''H^{-1}\left[B''H^2 - B''H^2 - 0\right] = 0.\end{aligned}$$

Hence, (1.5.9) becomes

$$\begin{aligned}(d_{ij}P_i)_j &= (B''H_k)_j a_{ij}u_{ki} - G'(d_{ij}u_i)_j \\ &= (B''H_k)_j a_{ij}u_{ki} - G'(d_{ij})_j u_i - G'd_{ij}u_{ij} \\ &= (B''H_k)_j a_{ij}u_{ki} - G'(d_{ij})_j u_i - (G')^2H^{-1},\end{aligned} \tag{1.5.10}$$

where in the last line we made use of (1.5.7). Appealing to (1.2.1), (1.2.2) and (1.2.3), we compute

$$\begin{aligned}(d_{ij})_j u_i &= \left[B''H^{-1}H_iH_j + B'H^{-1}H_{ij}\right]_j u_i \\ &= \left[B'''H^{-1}H_iH_jH_\ell - B''H^{-2}H_iH_jH_\ell + B''H^{-1}H_{i\ell}H_j + B''H^{-1}H_iH_{j\ell}\right. \\ &\quad \left. + B''H^{-1}H_{ij}H_\ell - B'H^{-2}H_{ij}H_\ell + B'H^{-1}H_{ij\ell}\right] u_{j\ell}u_i \\ &= \left[B'''H_jH_\ell - B''H^{-1}H_jH_\ell + 0 + B''H_{j\ell} + 0 - 0 - B'H^{-1}H_{j\ell}\right] u_{j\ell} \\ &= \left[B''' - B''H^{-1}\right] H_jH_\ell u_{j\ell} + \left[B'' - B'H^{-1}\right] H_{j\ell}u_{j\ell}.\end{aligned} \tag{1.5.11}$$

Writing explicitly (1.5.7)

$$G' = a_{ij}u_{ij} = B''H_iH_ju_{ij} + B'H_{ij}u_{ij},$$

we deduce

$$H_{ij}u_{ij} = (B')^{-1}\left[G' - B''H_iH_ju_{ij}\right]. \tag{1.5.12}$$

By this equation, (1.5.11) becomes

$$\begin{aligned}(d_{ij})_j u_i &= \left[B''' - B''H^{-1}\right] H_jH_\ell u_{j\ell} + (B')^{-1}\left[B'' - B'H^{-1}\right]\left[G' - B''H_jH_\ell u_{j\ell}\right] \\ &= \left[B''' - (B')^{-1}(B'')^2\right] H_jH_\ell u_{j\ell} + G'(B')^{-1}\left[B'' - B'H^{-1}\right].\end{aligned} \tag{1.5.13}$$

Now, inverting (1.5.5), we get

$$H_k u_{ki} = (B''H)^{-1}\left[P_i + G'u_i\right]. \tag{1.5.14}$$

Exploiting (1.5.14) in (1.5.13) and using (1.2.1), we obtain

$$\begin{aligned}(d_{ij})_j u_i &= (B''H)^{-1}\left[B''' - (B')^{-1}(B'')^2\right]\left[P_\ell + G'u_\ell\right]H_\ell + G'(B')^{-1}\left[B'' - B'H^{-1}\right]\\ &= H^{-1}\left[(B'')^{-1}B''' - (B')^{-1}B''\right]\left[P_\ell + G'u_\ell\right]H_\ell + G'(B')^{-1}\left[B'' - B'H^{-1}\right]\\ &= H^{-1}\left[(B'')^{-1}B''' - (B')^{-1}B''\right]H_\ell P_\ell + G'\left[(B'')^{-1}B''' - H^{-1}\right].\end{aligned}$$

By this last equality, (1.5.10) becomes

$$\begin{aligned}(d_{ij}P_i)_j &= (B''H_k)_j a_{ij}u_{ki} - G'H^{-1}\left[(B'')^{-1}B''' - (B')^{-1}B''\right]H_\ell P_\ell\\ &\quad - (G')^2\left[(B'')^{-1}B''' - H^{-1}\right] - (G')^2 H^{-1}\\ &= (B''H_k)_j a_{ij}u_{ki} - G'H^{-1}\left[(B'')^{-1}B''' - (B')^{-1}B''\right]H_\ell P_\ell - (G')^2(B'')^{-1}B'''.\end{aligned} \tag{1.5.15}$$

Now, we use (1.5.14) to write, for any j and k,

$$\begin{aligned}(B''H_k)_j &= B'''H_kH_\ell u_{j\ell} + B''H_{k\ell}u_{j\ell}\\ &= B'''H_k(B''H)^{-1}\left[P_j + G'u_j\right] + B''H_{k\ell}u_{j\ell}\\ &= (B'')^{-1}B'''H^{-1}H_kP_j + G'(B'')^{-1}B'''H^{-1}H_ku_j + B''H_{k\ell}u_{j\ell},\end{aligned}$$

and

$$\begin{aligned}a_{ij}u_{ik} &= \left[B''H_iH_j + B'H_{ij}\right]u_{ik} = B''H_iH_ju_{ik} + B'H_{ij}u_{ik}\\ &= B''H_j(B''H)^{-1}\left[P_k + G'u_k\right] + B'H_{ij}u_{ik}\\ &= H^{-1}H_jP_k + G'H^{-1}H_ju_k + B'H_{ij}u_{ik}.\end{aligned}$$

We put together the two formulae just obtained, getting

$$\begin{aligned}(B''H_k)_j a_{ij}u_{ki} &= \left[(B'')^{-1}B'''H^{-1}H_kP_j + G'(B'')^{-1}B'''H^{-1}H_ku_j + B''H_{k\ell}u_{j\ell}\right]\\ &\quad\times\left[H^{-1}H_jP_k + G'H^{-1}H_ju_k + B'H_{ij}u_{ik}\right]\\ &= (B'')^{-1}B'''H^{-2}H_kP_jH_jP_k + G'(B'')^{-1}B'''H^{-2}H_kP_jH_ju_k\\ &\quad + B'(B'')^{-1}B'''H^{-1}H_kP_jH_{ij}u_{ik} + G'(B'')^{-1}B'''H^{-2}H_ku_jH_jP_k\\ &\quad + (G')^2(B'')^{-1}B'''H^{-2}H_ku_jH_ju_k + G'B'(B'')^{-1}B'''H^{-1}H_ku_jH_{ij}u_{ik}\\ &\quad + B''H^{-1}H_{k\ell}u_{j\ell}H_jP_k + G'B''H^{-1}H_{k\ell}u_{j\ell}H_ju_k + B'B''H_{k\ell}u_{j\ell}H_{ij}u_{ik}.\end{aligned}$$

Making use of (1.5.14), (1.2.1) and (1.2.2), this becomes

$$\begin{aligned}(B''H_k)_j a_{ij}u_{ki} &= (B'')^{-1}B'''H^{-2}(H_\ell P_\ell)^2 + G'(B'')^{-1}B'''H^{-1}H_\ell P_\ell\\ &\quad + B'(B'')^{-1}B'''H^{-1}P_jH_{ij}(B''H)^{-1}\left[P_i + G'u_i\right] + G'(B'')^{-1}B'''H^{-1}H_\ell P_\ell\\ &\quad + (G')^2(B'')^{-1}B''' + 0\\ &\quad + B''H^{-1}H_{k\ell}P_k(B''H)^{-1}\left[P_\ell + G'u_\ell\right] + 0 + B'B''H_{ij}H_{k\ell}u_{ik}u_{j\ell}.\end{aligned}$$

Developing the products and exploiting again (1.2.2), we have

$$\begin{aligned}(B''H_k)_j a_{ij}u_{ki} &= (B'')^{-1}B'''H^{-2}(H_\ell P_\ell)^2 + G'(B'')^{-1}B'''H^{-1}H_\ell P_\ell \\ &\quad + B'(B'')^{-2}B'''H^{-2}H_{ij}P_iP_j + 0 + G'(B'')^{-1}B'''H^{-1}H_\ell P_\ell \\ &\quad + (G')^2(B'')^{-1}B''' + H^{-2}H_{k\ell}P_kP_\ell + 0 + B'B''H_{ij}H_{k\ell}u_{ik}u_{j\ell}.\end{aligned}$$

Simplifying and collecting similar terms, we get

$$\begin{aligned}(B''H_k)_j a_{ij}u_{ki} &= (B'')^{-1}B'''H^{-2}(H_\ell P_\ell)^2 + 2G'(B'')^{-1}B'''H^{-1}H_\ell P_\ell + B'B''H_{ij}H_{k\ell}u_{ik}u_{j\ell} \\ &\quad + H^{-2}\left[B'(B'')^{-2}B''' + 1\right]H_{k\ell}P_kP_\ell + (G')^2(B'')^{-1}B'''.\end{aligned}$$

Plugging this into (1.5.15) we finally obtain

$$\begin{aligned}(d_{ij}P_i)_j &= (B'')^{-1}B'''H^{-2}(H_\ell P_\ell)^2 + 2G'(B'')^{-1}B'''H^{-1}H_\ell P_\ell + B'B''H_{ij}H_{k\ell}u_{ik}u_{j\ell} \\ &\quad + H^{-2}\left[B'(B'')^{-2}B''' + 1\right]H_{k\ell}P_kP_\ell + (G')^2(B'')^{-1}B''' \\ &\quad - G'H^{-1}\left[(B'')^{-1}B''' - (B')^{-1}B''\right]H_\ell P_\ell - (G')^2(B'')^{-1}B''' \\ &= (B'')^{-1}B'''H^{-2}(H_\ell P_\ell)^2 + G'H^{-1}\left[(B'')^{-1}B''' + (B')^{-1}B''\right]H_\ell P_\ell \\ &\quad + H^{-2}\left[B'(B'')^{-2}B''' + 1\right]H_{k\ell}P_kP_\ell + B'B''H_{ij}H_{k\ell}u_{ik}u_{j\ell}.\end{aligned}$$

The last term of the formula above coincides with the remainder $\mathscr{R}$ as defined in (1.5.4) and it is non-negative by (8) and via an application of Proposition 1.13 with $c_{ij} := u_{ij}$. Therefore, inequality (1.5.3) is proved. □

1.6 Proofs of the main results

This section contains the proofs of Theorems 1.1-1.6. We divided it into six subsections, each dealing with one result.

1.6.1 Proof of Theorem 1.1

The proof is a suitable adaptation of the one of Theorem 1 in [FV14], which in turn exploits the techniques developped in [M85, CGS94, FV10].

Consider the family of solutions

$$\mathcal{F} := \left\{ v \text{ bounded weak solution of (7) in } \mathbb{R}^n : u_* \leqslant v(x) \leqslant u^* \text{ for any } x \in \mathbb{R}^n \right\},$$

where u_* and u^* were defined in (21). Note that, by Proposition 1.20, the set $\mathcal{F}$ is compact in the topology of $C^{1,\alpha}_{\mathrm{loc}}(\mathbb{R}^n)$. Moreover,

$$\text{if } v \in \mathcal{F}, \text{ then } G(v(x)) \geqslant 0 \text{ for any } x \in \mathbb{R}^n. \tag{1.6.1}$$

Set now

$$P_0 := \sup\left\{ P(v;x) : v \in \mathcal{F},\ x \in \mathbb{R}^n \right\}, \tag{1.6.2}$$

and observe that if we show that

$$P_0 \leqslant 0, \tag{1.6.3}$$

then the gradient bound (1.1.1) will then be proved. To check that (1.6.3) is true, we argue by contradiction and suppose that

$$P_0 > 0. \tag{1.6.4}$$

Let $\{v_k\}_{k\in\mathbb{N}} \subset \mathcal{F}$ and $\{x_k\}_{k\in\mathbb{N}} \subset \mathbb{R}^n$ be such that

$$\lim_{k\to+\infty} P(v_k; x_k) = P_0. \tag{1.6.5}$$

Define the new sequence of translated functions

$$w_k(x) := v_k(x + x_k) \text{ for any } x \in \mathbb{R}^n.$$

Clearly, $w_k \in \mathcal{F}$ and $P(w_k; 0) = P(v_k; x_k)$. Therefore, w_k converges, up to a subsequence, to a function w in $C^{1,\alpha}_{\mathrm{loc}}(\mathbb{R}^n) \cap \mathcal{F}$. In particular,

$$\begin{aligned}
P(w; 0) &= B'(H(\nabla w(0)))H(\nabla w(0)) - B(H(\nabla w(0))) - G(w(0)) \\
&= \lim_{k\to+\infty} \Big[B'(H(\nabla w_k(0)))H(\nabla w_k(0)) - B(H(\nabla w_k(0))) - G(w_k(0))\Big] \\
&= \lim_{k\to+\infty} P(w_k; 0) = \lim_{k\to+\infty} P(v_k; x_k) \\
&= P_0,
\end{aligned}$$

where the last identity follows from (1.6.5). By (1.6.4), (1.6.1) and the non-negativity of the function B, we deduce that

$$0 < P_0 = P(w; 0) \leqslant B'(H(\nabla w(0)))H(\nabla w(0)).$$

Hence, $\nabla w(0) \neq 0$, as H vanishes at the origin. By the continuity of ∇w we know that there exists a radius $\rho > 0$ such that $|\nabla u|$ is uniformly bounded away from zero in the closed ball $\overline{B_\rho}$. Consequently, we may apply Proposition 1.22 to obtain that $P(w; \cdot)$ satisfies the partial differential inequality

$$\left(d_{ij} P_i(w; \cdot)\right)_j - b_k P_k(w; \cdot) \geqslant 0 \text{ in } B_\rho,$$

where d_{ij} and b_k are respectively defined in (1.5.1) and (1.5.4). Observe that the operator above is uniformly elliptic - thanks to the validity of either (A)$'$ or (B)$'$ - and that $P(w; \cdot)$ attains at 0 a positive global maximum. Accordingly, we may use the Strong Maximum Principle (see e.g. Theorem 8.19 at page 198 of [GT01]) to get that $P(w; \cdot)$ is constant in B_ρ. With a standard connectedness argument we conclude that $P(w; \cdot)$ is actually constant on the whole of $\mathbb{R}^n$, that is

$$B'(H(\nabla w(x)))H(\nabla w(x)) - B(H(\nabla w(x))) - G(w(x)) = P_0 \;\text{ for any } x \in \mathbb{R}^n. \tag{1.6.6}$$

Now, since w is bounded, we can find a sequence of points $\{y_k\}_{k\in N} \subset \mathbb{R}^n$ such that

$$\lim_{k\to+\infty} \nabla w(y_k) = 0.$$

By evaluating identity (1.6.6) along this sequence and then passing to the limit, we get

$$\begin{aligned}
P_0 &= \lim_{k\to+\infty} \Big[B'(H(\nabla w(y_k)))H(\nabla w(y_k)) - B(H(\nabla w(y_k))) - G(w(y_k))\Big] \\
&\leqslant \lim_{k\to+\infty} B'(H(\nabla w(y_k)))H(\nabla w(y_k)) = 0,
\end{aligned}$$

in contradiction with (1.6.4). This concludes the proof of (1.1.1).

Hence, we are only left to show the validity of the second part of Theorem 1.1, i.e. that if the equality in (1.1.1) holds at some non-singular point x_0, then it also holds in the whole connected component of $\{\nabla w \neq 0\}$ which contains x_0.

To see this, first observe that, by (1.1.1), we have

$$\begin{aligned} P(u;x_0) &= B'(H(\nabla u(x_0)))H(\nabla u(x_0)) - B(H(\nabla u(x_0)) - G(u(x_0)) = 0 \\ &\geqslant B'(H(\nabla u(x)))H(\nabla u(x)) - B(H(\nabla u(x)) - G(u(x)) \\ &= P(u;x), \end{aligned}$$

for any $x \in \mathbb{R}^n$. That is, x_0 is a global maximum for $P(u;\cdot)$. Since $P(u;\cdot)$ is a solution of the differential inequality (1.5.3) in a neighbourhood of x_0 and $\nabla u(x_0) \neq 0$, we may argue as before and take advantage of the Strong Maximum Principle to deduce that $P(u;\cdot)$ identically vanishes on the whole of the connected component of $\{\nabla u \neq 0\}$ containing x_0. Thus, the proof is complete.

1.6.2 Proof of Theorem 1.2

Up to a rotation and a translation, we may consider the origin lying in a level set $\{u = c\}$, with

$$\nabla u(0) = |\nabla u(0)|\,(0,\ldots,0,1) \neq 0. \tag{1.6.7}$$

We stress that the equation is not invariant under a rotation R, but the function H would be replaced by $\tilde{H} := H \circ R$. Nevertheless, the new function $\tilde{H}$ satisfies the same structural assumptions of H, thus we take the freedom of identifying $\tilde{H}$ with the original H.

We parameterize the level set of u near the origin with the graph of a C^2 function ϕ, i.e. we write $u(x',\phi(x')) = c$ for $x' \in \mathbb{R}^{n-1}$ near the origin. By taking two derivatives, we obtain that

$$\begin{aligned} & u_{i'} + u_n \phi_{i'} = 0 \\ \text{and} \qquad & u_{i'j'} + u_{i'n}\phi_{j'} + u_{j'n}\phi_{i'} + u_{nn}\phi_{i'}\phi_{j'} + u_n \phi_{i'j'} = 0, \end{aligned}$$

for any $i', j' \in \{1,\ldots,n-1\}$, where the derivatives of u are evaluated at $(x',\phi(x'))$ and the derivatives of ϕ are evaluated at x'. In particular, by taking $x' = 0$, we obtain that $\phi_{i'}(0) = 0$ and $u_{i'j'}(0) = -u_n(0)\phi_{i'j'}(0)$, for any $i', j' \in \{1,\ldots,n-1\}$.

Consequently, we have that all the principal curvatures of the level set at 0 vanish if and only if $\phi_{i'j'}(0) = 0$ for any $i', j' \in \{1,\ldots,n-1\}$, and so, by (1.6.7), if and only if

$$u_{i'j'}(0) = 0 \text{ for any } i', j' \in \{1,\ldots,n-1\}. \tag{1.6.8}$$

Hence, we establish (1.6.8) in order to complete the proof of Theorem 1.2. The proof of (1.6.8) is based on Proposition 1.13. We need to check that the hypotheses of Proposition 1.13 are satisfied in this case. First of all, we have that (1.2.7) is guaranteed by (1.6.7) (here $\xi = \nabla u(0)$). Then, by Theorem 1.1, we know that (1.1.3) holds true in the whole connected component $\mathscr{S}$ that contains 0. As a consequence, P vanishes identically in $\mathscr{S}$, thus we obtain from (1.5.3) and (1.5.4) that

$$0 = \left(d_{ij}P_i\right)_j - b_k P_k = \mathscr{R} = B'B''H_{ij}H_{k\ell}u_{ik}u_{j\ell}.$$

This says that equality holds in (1.2.6) with $\xi = \nabla u(0)$ and $c_{ij} = u_{ij}$. Accordingly, the hypotheses of Proposition 1.13 are fulfilled and we obtain (1.6.8) from (1.2.8). The proof of Theorem 1.2 is therefore complete.

1.6.3 Proof of Theorem 1.3

In this case u is of class C^3 everywhere, due to Proposition 1.21, therefore we can differentiate (7) and write it in non-divergence form as

$$(B \circ H)_{ij}u_{ij} + F'(u) = 0.$$

Notice that the matrix $\{(B\circ H)_{ij}\}_{i,j\in\{1,\dots,n\}}$ is elliptic, since assumption (B) is equivalent to (B)′, by Theorem 1.7.

Now, we observe that, in view of Theorem 1.2, u is one-dimensional and that its profile u_0 satisfies the ordinary differential equation (1.1.4) on an interval. Also recall that u, and consequently u_0, is bounded, with bounded gradient.

Thanks to (15), the linearized equation can be represented as a first order system of ODEs in canonical form, so that u_0 extends to a global solution $\hat{u}_0$ by the standard theory for Cauchy problems with globally Lipschitz nonlinearities (see e.g. page 146 in [PSV84]).

Finally, by the Unique Continuation Principle (see e.g. [H05]), we have that u agrees everywhere with the one-dimensional extension of $\hat{u}_0$ to $\mathbb{R}^n$. This concludes the proof.

1.6.4 Proof of Theorem 1.4

Assume $\mathscr{S}$ to be any connected component of $\{\nabla u\neq 0\}$. We claim that

$$\begin{gathered}\mathscr{S}\text{ is foliated by level sets of } u \text{ which are union of parallel affine hyperplanes}\\ \text{and so } \partial\mathscr{S} \text{ is the union of (at most two) parallel hyperplanes.}\end{gathered}\tag{1.6.9}$$

In order to prove this, fix $x_\star\in\mathscr{S}$ and consider the level set $S_{x_\star}:=\{u=u(x_\star)\}$. Notice that

$$S_{x_\star}\subseteq\{\nabla u\neq 0\}.\tag{1.6.10}$$

Indeed, if $x\in S_{x_\star}$, then from (1.1.3) we deduce that

$$\begin{aligned}B'(H(\nabla u(x)))H(\nabla u(x))-B(H(\nabla u(x)))&=c_u-F(u(x))\\&=c_u-F(u(x_\star))=B'(H(\nabla u(x_\star)))H(\nabla u(x_\star))-B(H(\nabla u(x_\star)))>0,\end{aligned}\tag{1.6.11}$$

because $x_\star$ is in $\{\nabla u\neq 0\}$ and so we can apply (1.2.15) taking $t:=H(\nabla u(x_\star))>0$. But then, also $x\in\{\nabla u\neq 0\}$, since otherwise $H(\nabla u)$ would vanish, in contradiction with (1.6.11). This establishes (1.6.10) so that we are allowed to apply Theorem 1.2, concluding that every connected component of $S_{x_\star}$ is contained in a hyperplane, say $\ell_{x_\star}$. In particular, we point out that

$$\text{the connected component of } S_{x_\star} \text{ which contains } x_\star \text{ is equal to } \ell_{x_\star}.\tag{1.6.12}$$

Indeed, $S_{x_\star}$ is closed in the relative topology of $\ell_{x_\star}$, being u continuous. Furthermore, $S_{x_\star}$ is also relatively open, by (1.6.10) and applying Theorem 1.2 together with the Implicit Function Theorem. Thus, (1.6.12) holds true.

Combining (1.6.12) and (1.6.10) we immediately obtain (1.6.9).

Let now ω denote a vector normal to all the hyperplanes in (1.6.9). We claim that

$$u(x_0)=u(y_0)\text{ if }(x_0-y_0)\cdot\omega=0.\tag{1.6.13}$$

To check this, fix $x_0\in\mathbb{R}^n$. If $\nabla u=0$ on the whole ℓ_{x_0}, then (1.6.13) follows from the Fundamental Theorem of Calculus. Conversely, let $x_\sharp$ be a point in $\ell_{x_0}\cap\{\nabla u\neq 0\}$. By (1.6.9) (applied to $x_\sharp$), we have that u is constant on $\ell_{x_\sharp}$, which, in turn, is equal to ℓ_{x_0}. Thus, (1.6.13) is proved and so is the desired one-dimensional Euclidean symmetry.

1.6.5 Proof of Theorem 1.5

Let $r:=u(x_0)$ and fix a point $x\in\mathbb{R}^n\setminus\{x_0\}$. In order to establish the thesis of Theorem 1.5, we shall show that $u(x)=r$. Consider the C^1 function φ, defined by setting

$$\varphi(t):=u(tx+(1-t)x_0)-r\text{ for any }t\in[0,1].$$

In the following we will sometimes adopt the short notation $x_t := tx + (1-t)x_0$. Notice that, by the regularity of u, the function $t \mapsto |\nabla u(tx + (1-t)x_0)|$ is bounded on $[0,1]$. We may therefore apply Lemma 1.15 (and also recall the notation there introduced) to compute

$$\begin{aligned}
|\dot\varphi(t)|^{p^*} &\leqslant |x-x_0|^{p^*} |\nabla u(tx+(1-t)x_0)|^{p^*} \\
&= |x-x_0|^{p^*} |\nabla u(x_t)|^{p^*} \\
&\leqslant \frac{|x-x_0|^{p^*}}{h^{p^*}} H^{p^*}\left(\nabla u(x_t)\right) \\
&\leqslant \frac{|x-x_0|^{p^*}}{\varepsilon h^{p^*}} \left[B'(H(\nabla u(x_t))H(\nabla u(x_t)) - B(H(\nabla u(x_t)))\right],
\end{aligned}$$

for some $\varepsilon > 0$. Next, recalling (1.1.1) and the assumptions of Theorem 1.5, we have that

$$\begin{aligned}
|\dot\varphi(t)|^{p^*} &\leqslant \frac{|x-x_0|^{p^*}}{\varepsilon h^{p^*}} \left[c_u - F(u(x_t))\right] \\
&= \frac{|x-x_0|^{p^*}}{\varepsilon h^{p^*}} \left[F(r) - F(u(tx+(1-t)x_0))\right] \\
&= -\frac{|x-x_0|^{p^*}}{\varepsilon h^{p^*}} \int_r^{u(tx+(1-t)x_0)} F'(\sigma)\, d\sigma.
\end{aligned} \tag{1.6.14}$$

Then, we employ alternatively the Lipschitz regularity of F' or (1.1.5) to write

$$|F'(\sigma)| \leqslant c\,|r-\sigma|^{p^*-1} \text{ for any } \sigma \in \left[\inf_{\mathbb{R}^n} u, \sup_{\mathbb{R}^n} u\right],$$

for some positive constant c. Using this estimate in (1.6.14), we get

$$\begin{aligned}
|\dot\varphi(t)|^{p^*} &\leqslant \frac{c|x-x_0|^{p^*}}{\varepsilon h^{p^*}} \left| \int_r^{u(tx+(1-t)x_0)} |r-\sigma|^{p^*-1} d\sigma \right| \\
&= \frac{c|x-x_0|^{p^*}}{\varepsilon p^* h^{p^*}} |u(tx+(1-t)x_0) - \sigma|^{p^*} \\
&= \frac{c|x-x_0|^{p^*}}{\varepsilon p^* h^{p^*}} |\varphi(t)|^{p^*},
\end{aligned}$$

which yields, if $\varphi(t) \neq 0$,

$$\left|\frac{\dot\varphi(t)}{\varphi(t)}\right| \leqslant \frac{c^{1/p^*}|x-x_0|}{\varepsilon {p^*}^{1/p^*} h} =: K.$$

Finally, set $\psi(t) := (\varphi(t))^2 e^{-Kt}$, for any $t \in [0,1]$. Differentiating ψ, we obtain

$$\begin{aligned}
\dot\psi(t) &= \varphi(t) e^{-Kt} \left[\dot\varphi(t) - K\varphi(t)\right] \\
&= \begin{cases} (\varphi(t))^2 e^{-Kt} \left[\frac{\dot\varphi(t)}{\varphi(t)} - K\right] & \text{if } \varphi(t) \neq 0 \\ 0 & \text{if } \varphi(t) = 0 \end{cases} \\
&\leqslant 0,
\end{aligned}$$

so that ψ is non-increasing. Hence

$$(u(x)-r)^2 e^{-K} = \varphi(1)^2 e^{-K} = \psi(1) \leqslant \psi(0) = \varphi(0)^2 = (u(x_0)-r)^2 = 0,$$

and therefore $u(x) = r$, which concludes the proof.

1.6.6 Proof of Theorem 1.6

We will suppose, without loss of generality, that

$$u \text{ is not constant.} \tag{1.6.15}$$

Then, assume by contradiction that there exists $r_0 \in \left(\inf_{\mathbb{R}^n} u, \sup_{\mathbb{R}^n} u\right)$ such that

$$\sup\left\{F(r) : r \in \left[\inf_{\mathbb{R}^n} u, \sup_{\mathbb{R}^n} u\right]\right\} = c_u = F(r_0).$$

By the continuity of u, there also exists a point $x_0 \in \mathbb{R}^n$ such that $u(x_0) = r_0$. Moreover, r_0 is a local maximum for F, so that $F'(r_0) = 0$. Thus, we can apply Theorem 1.5, deducing that u is constant, in contradiction to (1.6.15).

Chapter 2

Monotonicity formulae and applications

2.1 Statement of the main results

In the present chapter, we discuss the validity of a monotonicity formula for an energy functional related to (6). With the aid of such formula, we derive a Liouville-type result for a particular class of solutions of (7).

Monotonicity formulae are a classical topic in geometric variational analysis. Roughly speaking, the idea of monotonicity formulae is that a suitably rescaled energy functional in a ball possesses some monotonicity properties with respect to the radius of the ball (in our case, the situation is geometrically more complicated, since the ball is non-Euclidean).

Of course, monotonicity formulae are important, since they provide a quantitative information on the energy of the problem; moreover, they often provide additional information on the asymptotic behaviour of the solutions, also in connection with blow-up and blow-down limits, and they play a special role in rigidity and classification results.

One of the main results of the present chapter consists in a monotonicity formula for a suitable rescaled version of the functional (6), over the family of Wulff balls indexed by $R > 0$,

$$W_R^H = W_R := \{x \in \mathbb{R}^n : H^*(x) < R\}, \tag{2.1.1}$$

where H^* is the dual function of H, introduced in (11). Note that H^* is a positive homogeneous function of degree 1 and that it is at least of class $C^2(\mathbb{R}^n \setminus \{0\})$, as showed in Lemma 1.16 below. We also remark that, due to the homogeneity properties enjoyed by H^*, the sets W_R are dilations of the Wulff shape W_1 defined in (12).

The precise statement is given by

Theorem 2.1. *Let u be a bounded weak solution of (7) in the whole of $\mathbb{R}^n$. Under hypothesis (A), assume in addition that H satisfies, for any $\xi, x \in \mathbb{R}^n$,*

$$\operatorname{sgn}\langle H(\xi)\nabla H(\xi), H^*(x)\nabla H^*(x)\rangle = \operatorname{sgn}\langle \xi, x\rangle. \tag{2.1.2}$$

Then, the rescaled energy defined by

$$\mathscr{W}(u; R) = \mathscr{W}(R) := \frac{1}{R^{n-1}} \int_{W_R} B(H(\nabla u(x))) + G(u(x))\, dx, \tag{2.1.3}$$

for any $R > 0$, is monotone non-decreasing.

Observe that when $B(t) = t^2/2$ and $H(x) = |x|$ - i.e. when the operator is simply the Laplacian and the equation is isotropic -, then the result of Theorem 2.1 reduces to the classical monotonicity formula proved in [M89]. Then, the results of [M89] were extended to the non-linear case in [CGS94], still under the assumption $H(x) = |x|$.

Differently from the existing literature, here we introduce the presence of a general non-Euclidean anisotropy H (also, we remove an unnecessary assumption on the sign of F). We point out that, to the best of our knowledge, these results are new even for $B(t) = t^2/2$, i.e. even in the case in which the elliptic operator is non-singular and non-degenerate, but also non-isotropic.

We remark that the anisotropic term in the monotonicity formula provides a number of geometric complications. Indeed, in our case, the unit ball B_1^H is not Euclidean and it does not coincide with its dual ball W_1^H, and a point on the unit sphere does not coincide in general with the normal to the sphere.

Also, we mention that Theorem 2.1 heavily relies on the pointwise gradient estimate proved in Theorem 1.1 of Chapter 1.

In the statement of the monotonicity formula the new condition (2.1.2) is assumed on H. We plan to shed some light on its origin and to better understand its implications.

First, we point out that this assumption comes as a weaker form of the more restrictive

$$\langle H(\xi)\nabla H(\xi), H^*(x)\nabla H^*(x)\rangle = \langle \xi, x\rangle, \tag{2.1.4}$$

for any $\xi, x \in \mathbb{R}^n$. To the authors' knowledge, this latter condition has been first introduced in [FK09] to recover the validity of the mean value property for Q-harmonic functions, that are the solutions of the equation

$$Qu := \frac{\partial}{\partial x_i}\Big(H(\nabla u)H_i(\nabla u)\Big) = 0. \tag{2.1.5}$$

Notice that such solutions are the counterparts of harmonic functions in the anisotropic framework and that equation (2.1.5) is a particular case of our setting by taking $B(t) = t^2/2$ and $F = 0$.

Examples of homogeneous functions H for which (2.1.4) is valid are the norms displayed in (16), as showed by the forthcoming Lemma 2.12. For this reason, we did not need to mention condition (2.1.2) in Theorem 2.1, in the framework given by hypothesis (B).

In the next result we emphasize that anisotropies as the one in (16) are actually the *only* ones which satisfy (2.1.4).

Theorem 2.2. *Let $H \in C^1(\mathbb{R}^n \setminus \{0\})$ be a positive homogeneous function of degree 1 satisfying (9). Assume that its unit ball B_1^H, as defined by (10), is strictly convex. Then, condition (2.1.4) is equivalent to asking H to be of the form (16), for some symmetric and positive definite matrix $M \in \mathrm{Mat}_n(\mathbb{R})$.*

From Theorem 2.2, it follows that assumption (2.1.4) imposes some severe restrictions on the geometric structure of the unit ball of H, which is always an Euclidean ellipsoid. A natural question is therefore to understand in which sense our condition (2.1.2) is more general. For this scope, we will discuss condition (2.1.2) in detail, by making concrete examples and obtaining a complete characterization in the plane. Roughly speaking, the unit ball in the plane under condition (2.1.2) can be constructed by considering a curve in the first quadrant that satisfies a suitable, explicit differential inequality, and then *reflecting* this curve in the other quadrants (of course, if higher regularity on the ball is required, this gives further conditions on the derivatives of the curve at the reflection points). The detailed characterization of condition (2.1.2) in the plane is given by the following technical but operational result.

Proposition 2.3. *Let $r : [0, \pi/2] \to (0, +\infty)$ be a given C^2 function satisfying*

$$r(\theta)r''(\theta) < 2r'(\theta)^2 + r(\theta)^2 \text{ for a.a. } \theta \in \left[0, \frac{\pi}{2}\right], \tag{2.1.6}$$

and

$$r(0) = 1, \qquad r(\pi/2) = r^*, \qquad r'(0) = r'(\pi/2) = 0, \tag{2.1.7}$$

for some $r^* \geqslant 1$. Consider the π-periodic function $\widetilde{r} : \mathbb{R} \to (0, +\infty)$ defined on $[0, \pi]$ by

$$\widetilde{r}(\theta) := \begin{cases} r(\theta) & \text{if } 0 \leqslant \theta \leqslant \frac{\pi}{2}, \\ \dfrac{r^* \sqrt{r(\tau^{-1}(\theta))^2 + r'(\tau^{-1}(\theta))^2}}{r(\tau^{-1}(\theta))^2} & \text{if } \frac{\pi}{2} \leqslant \theta \leqslant \pi, \end{cases}$$

where $\tau : [0, \pi/2] \to [\pi/2, \pi]$ is the bijective map given by

$$\tau(\eta) = \frac{\pi}{2} + \eta - \arctan \frac{r'(\eta)}{r(\eta)}.$$

Then, $\widetilde{r}$ is of class $C^1(\mathbb{R})$, the set

$$\Big\{(\rho\cos\theta, \rho\sin\theta) : \rho \in [0, \widetilde{r}(\theta)),\ \theta \in [0, 2\pi]\Big\}, \tag{2.1.8}$$

is strictly convex and its supporting function

$$\widetilde{H}(\rho\cos\theta, \rho\sin\theta) := \frac{\rho}{\widetilde{r}(\theta)},,$$

defined for $\rho \geqslant 0$ and $\theta \in [0, 2\pi]$, satisfies (2.1.2).

Furthermore, up to a rotation and a homothety of the plane $\mathbb{R}^2$, any even positive 1-homogeneous function $H \in C^2(\mathbb{R}^2 \setminus \{0\})$ satisfying (9), having strictly convex unit ball B_1^H and for which condition (2.1.2) holds true is such that B_1^H is of the form (2.1.8), for some positive $r \in C^2([0, \pi/2])$ satisfying (2.1.6) and (2.1.7).

In addition, if $H \in C^{3,\alpha}_{\text{loc}}(\mathbb{R}^2 \setminus \{0\})$, for some $\alpha \in (0, 1]$, we have that H is uniformly elliptic and satisfies condition (2.1.2) if and only if $r \in C^{3,\alpha}([0, \pi/2])$, inequality (2.1.6) is satisfied at any $\theta \in [0, \pi/2]$ and

$$r''\left(\frac{\pi}{2}\right) = -\frac{r^* r''(0)}{1 - r''(0)}, \qquad r'''\left(\frac{\pi}{2}\right) = -\frac{r^* r'''(0)}{(1 - r''(0))^3},$$

hold along with (2.1.7).

With this characterization, it is easy to construct examples satisfying condition (2.1.2) whose corresponding ball is not an Euclidean ellipsoid, see Remark 2.18.

As an application of Theorem 2.1 we have the following Liouville-type result.

Theorem 2.4. *Let H and u be as in Theorem 2.1. If*

$$\int_{W_R} G(u(x))\, dx = o(R^{n-1}) \text{ as } R \to +\infty, \tag{2.1.9}$$

then u is constant.

In particular, if $G(u) \in L^1(\mathbb{R}^n)$, then u is constant.

We remark that Theorem 2.4 is a sort of rigidity result. The condition that $G(u)$ has finite mass - or, more generally, that the mass has controlled growth - may be seen as a prescription of the values of the solution at infinity (at least, in a suitably averaged sense): the result of Theorem 2.4 gives that the only solution that can satisfy such prescription is the trivial one. In this spirit, Theorem 2.4 may be seen as a variant of the classical Liouville Theorem for harmonic functions (set here in a nonlinear, anisotropic, singular or degenerate framework).

The rest of the chapter is organized as follows.

The proof of the main result of the chapter, Theorem 2.1, is the content of Section 2.2. In the subsequent Section 2.3 we then deduce Theorem 2.4 as a corollary of the monotonicity formula.

The last two sections deal with the characterizations of conditions (2.1.4) and (2.1.2). In Section 2.4 we address Theorem 2.2, while the following Section 2.5 is devoted to the proof of Proposition 2.3.

2.2 The monotonicity formula

In this section we prove Theorem 2.1. Our argument is similar to that presented in [M89] and [CGS94, Theorem 1.4]. Yet, we develop several technical adjustments in order to cope with the difficulties arising in the anisotropic setting. In particular, in the classical, isotropic setting, the monotonicity formulae implicitly rely on some Euclidean geometric features, such as that a point on the unit sphere coincides with the normal of the sphere at that point, as well as the one of the dual sphere (that in the isotropic setting coincides with the original one). These Euclidean geometric properties are lost in our case, therefore we need some more refined geometrical and analytical studies.

The strategy we adopt to show the monotonicity of $\mathscr{W}$ basically relies on taking its derivative and then checking that it is non-negative. To complete this task, however, we make some integral manipulation involving the Hessian of u. Hence, we need u to be twice differentiable, at least in the weak sense.

If (B) is assumed to hold, this is not an issue, since u is C^3 (see Proposition 1.21 in Chapter 2). Therefore, we only focus on case (A). In this framework the solution u is, in general, no more than $C^{1,\alpha}_{\text{loc}}$. To circumvent this lack of regularity, we introduce a sequence of approximating problems and perform the computation on their solutions. Passing to the limit, we then recover the result for u. If one is interested in the proof under hypothesis (B), he should simply ignore the perturbation argument and directly work with u.

Prior to the proper proof of Theorem 2.1, we present some preparatory results about the above mentioned approximation technique. For the remainder of the section, the functions B and H are assumed to be satisfy the assumptions listed in (A).

2.2.1 Technical preliminaries

Let $\varepsilon \in (0, 1)$ and consider the function B_ε defined by

$$B_\varepsilon(t) := B\left(\sqrt{\varepsilon^2 + t^2}\right) - B(\varepsilon), \tag{2.2.1}$$

for any $t > 0$.

First, we present a result which addresses the regularity and growth properties of B_ε.

Lemma 2.5. *The function B_ε is of class $C^2([0,+\infty))$ and it satisfies $B_\varepsilon(0) = B'_\varepsilon(0) = 0$ and (8). Moreover*

$$\begin{aligned} c_p\gamma\,(\kappa+\varepsilon+t)^{p-2}\,t \leqslant & B'_\varepsilon(t) \leqslant C_p\Gamma\,(\kappa+\varepsilon+t)^{p-2}\,t, \\ c_p\gamma\,(\kappa+\varepsilon+t)^{p-2} \leqslant & B''_\varepsilon(t) \leqslant C_p\Gamma\,(\kappa+\varepsilon+t)^{p-2}\,, \end{aligned} \tag{2.2.2}$$

for any $t > 0$, where γ, Γ, κ are as in (A) and

$$c_p := \min\left\{1, 2^{\frac{2-p}{2}}\right\},\ C_p := \max\left\{1, 2^{\frac{2-p}{2}}\right\}.$$

In addition, the composition $B_\varepsilon \circ H$ is of class $C^{1,1}_{\mathrm{loc}}(\mathbb{R}^n)$ and it holds, for any $\xi \in \mathbb{R}^n$,

$$(B_\varepsilon \circ H)(\xi) \geqslant \frac{\bar\gamma}{2(p-1)p}|\xi|^p - c_\star, \tag{2.2.3}$$

where $\bar\gamma$ is as in (A)′ and $c_\star$ is a non-negative constant independent of ε.

Proof. It is immediate to check from definition (2.2.1) that $B_\varepsilon \in C^2([0,+\infty))$. For any $t > 0$, we compute

$$\begin{aligned} B'_\varepsilon(t) &= B'\left(\sqrt{\varepsilon^2+t^2}\right)\frac{t}{\sqrt{\varepsilon^2+t^2}}, \\ B''_\varepsilon(t) &= B''\left(\sqrt{\varepsilon^2+t^2}\right)\frac{t^2}{\varepsilon^2+t^2} + B'\left(\sqrt{\varepsilon^2+t^2}\right)\frac{\varepsilon^2}{(\varepsilon^2+t^2)^{3/2}}. \end{aligned}$$

Thus, inequalities (8) are valid and $B_\varepsilon(0) = B'_\varepsilon(0) = 0$. Furthermore, formulae (2.2.2) can be recovered from the ellipticity and growth conditions of (A) which B satisfies.

Then, we address the composition $B_\varepsilon \circ H$. Notice that we already know that it is of class C^1 on the whole $\mathbb{R}^n$, by virtue of Lemma 1.10, and C^2 outside of the origin, by definition. Thus we only need to check that its gradient is Lipschitz in a neighbourhood of the origin. By using (2.2.2), for any $0 < |\xi| \leqslant 1$ we get

$$\frac{|\partial_i(B_\varepsilon \circ H)(\xi)|}{|\xi|} = \frac{|B'_\varepsilon(H(\xi))H_i(\xi)|}{|\xi|} \leqslant C_p\bar\Gamma(\bar\kappa+\varepsilon+H(\xi))^{p-2}H_i(\xi)\frac{H(\xi)}{|\xi|} \leqslant c,$$

for some positive c.

Finally, we establish (2.2.3). As a preliminary observation, we stress that the Hessian of $B_\varepsilon \circ H$ satisfies (A)′ with $\bar\kappa = \kappa + \varepsilon$. This can be seen as a consequence of (2.2.2), the uniform ellipticity of H and Proposition 1.18 (recall in particular relation (1.3.1)). We consider separately the two possibilities $p \geqslant 2$ and $1 < p < 2$. In the first case, we simply compute

$$\begin{aligned} (B_\varepsilon \circ H)(\xi) &= \int_0^1\int_0^t (B_\varepsilon \circ H)_{ij}(s\xi)\xi_i\xi_j\,dsdt \geqslant \bar\gamma\int_0^1\int_0^t (\kappa+\varepsilon+s|\xi|)^{p-2}|\xi|^2\,dsdt \\ &\geqslant \bar\gamma|\xi|^p\int_0^1\int_0^t s^{p-2}\,dsdt = \frac{\bar\gamma}{(p-1)p}|\xi|^p. \end{aligned}$$

If, on the other hand, $1 < p < 2$, we have

$$\begin{aligned} (B_\varepsilon \circ H)(\xi) &= \int_0^1\int_0^t (B_\varepsilon \circ H)_{ij}(s\xi)\xi_i\xi_j\,dsdt \geqslant \bar\gamma\int_0^1\int_0^t (\kappa+\varepsilon+s|\xi|)^{p-2}|\xi|^2\,dsdt \\ &= \frac{\bar\gamma}{p-1}\left[\frac{(\kappa+\varepsilon+|\xi|)^p-(\kappa+\varepsilon)^p}{p} - (\kappa+\varepsilon)^{p-1}|\xi|\right] \\ &\geqslant \frac{\bar\gamma}{p-1}\left[\frac{|\xi|^p-(\kappa+\varepsilon)^p}{p} - (\kappa+\varepsilon)^{p-1}|\xi|\right]. \end{aligned} \tag{2.2.4}$$

Notice that, by Young's inequality, we estimate

$$(\kappa+\varepsilon)^{p-1}|\xi| \leqslant \frac{|\xi|^p}{2p} + \frac{p-1}{p} 2^{1/(p-1)}(\kappa+\varepsilon)^p.$$

Plugging this into (2.2.4) finally leads to the desired

$$\begin{aligned}(B_\varepsilon \circ H)(\xi) &\geqslant \frac{\bar{\gamma}}{2(p-1)p}|\xi|^p - \frac{\bar{\gamma}}{(p-1)p}\left(1+(p-1)2^{1/(p-1)}\right)(\kappa+\varepsilon)^p \\ &\geqslant \frac{\bar{\gamma}}{2(p-1)p}|\xi|^p - \frac{\bar{\gamma}}{(p-1)p}\left(1+(p-1)2^{1/(p-1)}\right)(\kappa+1)^p.\end{aligned}$$

Hence, (2.2.3) holds in both cases and the proof of the lemma is complete. □

In the following lemma we compare B_ε to B. We study their modulus of continuity and discuss some uniform convergence properties.

Lemma 2.6. *Introduce, for $t \geqslant 0$, the functions $\beta(t) := B'(t)t$, $\beta_\varepsilon(t) := B_\varepsilon'(t)t$.*

Then, the Lipschitz norms of both B_ε and β_ε on compact sets of $[0,+\infty)$ are bounded by a constant independent of ε. More explicitly, for any $M \geqslant 1$ we estimate

$$\begin{aligned}&\|B_\varepsilon\|_{C^{0,1}([0,M])} \leqslant \|B\|_{C^{0,1}([0,2M])}, \\ &\|\beta_\varepsilon\|_{C^{0,1}([0,M])} \leqslant 2\|B'\|_{C^0([0,2M])} + \|\beta\|_{C^{0,1}([0,2M])}.\end{aligned} \tag{2.2.5}$$

Moreover, $B_\varepsilon \to B$ and $\beta_\varepsilon \to \beta$ uniformly on compact sets of $[0,+\infty)$. Quantitatively, we have

$$\begin{aligned}&\|B_\varepsilon - B\|_{C^0([0,M])} \leqslant 2\|B'\|_{C^0([0,2M])}\varepsilon, \\ &\|\beta_\varepsilon - \beta\|_{C^0([0,M])} \leqslant \left(\|B'\|_{C^0([0,2M])} + \|\beta\|_{C^{0,1}([0,2M])}\right)\varepsilon.\end{aligned} \tag{2.2.6}$$

Proof. First of all, we stress that, while $\beta_\varepsilon \in C^1([0,+\infty))$ in view of Lemma 2.5, the same is true also for β, as one can easily deduce from hypothesis (A).

We begin to establish (2.2.5). It is easy to see that the C^0 norms of B_ε and β_ε are bounded by those of B and β respectively. Thus, we may concentrate on the estimates of their Lipschitz seminorms. Let $M \geqslant 1$ and $0 \leqslant s,t \leqslant M$. We have

$$\begin{aligned}|B_\varepsilon(t) - B_\varepsilon(s)| &= \left|B\left(\sqrt{\varepsilon^2+t^2}\right) - B\left(\sqrt{\varepsilon^2+s^2}\right)\right| \\ &\leqslant \|B\|_{C^{0,1}([0,2M])}\left|\sqrt{\varepsilon^2+t^2} - \sqrt{\varepsilon^2+s^2}\right| \\ &\leqslant \|B\|_{C^{0,1}([0,2M])}|t-s|,\end{aligned}$$

so that the first relation in (2.2.5) is proved. The second inequality needs a little more care. Assuming without loss of generality $s \leqslant t$, we compute

$$\begin{aligned}|\beta_\varepsilon(t) - \beta_\varepsilon(s)| &= \left|B'\left(\sqrt{\varepsilon^2+t^2}\right)\frac{t^2}{\sqrt{\varepsilon^2+t^2}} - B'\left(\sqrt{\varepsilon^2+s^2}\right)\frac{s^2}{\sqrt{\varepsilon^2+s^2}}\right| \\ &\leqslant B'\left(\sqrt{\varepsilon^2+t^2}\right)\sqrt{\varepsilon^2+t^2}\left|\frac{t^2}{\varepsilon^2+t^2} - \frac{s^2}{\varepsilon^2+s^2}\right| \\ &\quad + \frac{s^2}{\varepsilon^2+s^2}\left|\beta\left(\sqrt{\varepsilon^2+t^2}\right) - \beta\left(\sqrt{\varepsilon^2+s^2}\right)\right| \\ &\leqslant \|B'\|_{C^0([0,2M])}\frac{|t^2-s^2|}{\sqrt{\varepsilon^2+t^2}} + \|\beta\|_{C^{0,1}([0,2M])}\left|\sqrt{\varepsilon^2+t^2} - \sqrt{\varepsilon^2+s^2}\right| \\ &\leqslant \left(2\|B'\|_{C^0([0,2M])} + \|\beta\|_{C^{0,1}([0,2M])}\right)|t-s|.\end{aligned}$$

Estimates (2.2.6) are proved in a similar fashion. Indeed, for any $0 \leqslant t \leqslant M$,

$$
\begin{aligned}
|B_\varepsilon(t) - B(t)| &= \left| B\left(\sqrt{\varepsilon^2+t^2}\right) - B(\varepsilon) - B(t) \right| \\
&\leqslant \|B\|_{C^{0,1}([0,2M])} \left(\left| \sqrt{\varepsilon^2+t^2} - t \right| + \varepsilon \right) \\
&\leqslant 2\|B\|_{C^{0,1}([0,2M])}\varepsilon,
\end{aligned}
$$

and

$$
\begin{aligned}
|\beta_\varepsilon(t) - \beta(t)| &\leqslant B'\left(\sqrt{\varepsilon^2+t^2}\right) \left| \frac{t^2}{\sqrt{\varepsilon^2+t^2}} - \sqrt{\varepsilon^2+t^2} \right| + \left| \beta\left(\sqrt{\varepsilon^2+t^2}\right) - \beta(t) \right| \\
&\leqslant \left(\|B'\|_{C^0([0,2M])} + \|\beta\|_{C^{0,1}([0,2M])} \right) \varepsilon.
\end{aligned}
$$

Thus, the proof is complete. □

Finally, the next result shows that B_ε can modified far from the origin to make it satisfy (A) with $p = 2$. We will need such a trick in the following Subsection 2.2, in order to overcome a technical difficulty along the proof of Proposition 2.9.

Lemma 2.7. *Let $M > 0$ be fixed and define*

$$
\hat{B}_\varepsilon(t) := \begin{cases} B_\varepsilon(t), & \text{if } t \in [0, M), \\ a(t-M)^2 + b(t-M) + c, & \text{if } t \geqslant M, \end{cases} \tag{2.2.7}
$$

where $a = B''_\varepsilon(M)/2$, $b = B'_\varepsilon(M)$ and $c = B_\varepsilon(M)$. Then, $\hat{B}_\varepsilon \in C^2([0, +\infty))$ and it satisfies the inequalities in (A) with $p = 2$.

Proof. The function $\hat{B}$ is of class $C^2([0, +\infty))$ by construction and by Lemma 2.5. Moreover, the estimates concerning $\hat{B}'_\varepsilon$ in (A) result from the analogous for $\hat{B}''_\varepsilon$ by integration, since $\hat{B}'_\varepsilon(0) = B'_\varepsilon(0) = 0$. Thus, we only need to check that there exist $\hat{\Gamma}_\varepsilon \geqslant \hat{\gamma}_\varepsilon > 0$ for which

$$
\hat{\gamma}_\varepsilon \leqslant \hat{B}''_\varepsilon(t) \leqslant \hat{\Gamma}_\varepsilon \text{ for any } t > 0.
$$

Notice that when $t \geqslant M$ this fact is obviously true. On the other hand, if $t \in (0, M)$, we take advantage of (2.2.2) to compute

$$
\hat{B}''_\varepsilon(t) = B''_\varepsilon(t) \geqslant c_p\gamma(\kappa + \varepsilon + t)^{p-2} \geqslant c_p\gamma \min\left\{(\kappa+\varepsilon)^{p-2}, (\kappa+1+M)^{p-2}\right\} =: \hat{\gamma}_\varepsilon,
$$

and

$$
\hat{B}''_\varepsilon(t) = B''_\varepsilon(t) \leqslant C_p\Gamma(\kappa + \varepsilon + t)^{p-2} \leqslant C_p\Gamma \max\left\{(\kappa+\varepsilon)^{p-2}, (\kappa+1+M)^{p-2}\right\} =: \hat{\Gamma}_\varepsilon.
$$

This finishes the proof. □

We point out that Lemma 2.7 is easily generalized to any function B that satisfies assumption (B) with $\kappa > 0$. See [CFV15, Lemma 2.4].

2.2.2 An auxiliary Dirichlet problem

In this subsection we take advantage of some of the computations displayed above and establish some results concerning an approximating Dirichlet problem.

In the first result we obtain we prove the boundedness of the minimizers of an energy functional related to $\mathscr{W}$.

Lemma 2.8. *Let Ω be a bounded open set of $\mathbb{R}^n$ with $C^{1,\alpha}$ boundary and $u \in W^{1,p}(\Omega) \cap L^\infty(\Omega)$. Let $u^\varepsilon \in W^{1,p}(\Omega)$ be a minimizer of the functional*

$$\mathcal{F}_\varepsilon(v) := \int_\Omega B_\varepsilon(H(\nabla v(x))) - F'(u(x))v(x)\,dx,$$

within the class $W^{1,p}_u(\Omega)$ made up by the functions $v \in W^{1,p}(\Omega)$ such that $v - u \in W^{1,p}_0(\Omega)$. Then, $u^\varepsilon \in L^\infty(\Omega)$ with L^∞ norm bounded independently of ε.

Proof. Our argument follows that of [S63, Theorem 6.2], simplified in agreement to our setting. See also [LU68, Theorem 3.2, p. 328].

We start by observing that we may restrict ourselves to consider $1 < p \leqslant n$, as in the opposite situation the result is a direct consequence of Morrey's inequality. Notice that in this last case the independence of the L^∞ norm of u^ε from ε follows from the fact that also $\|u^\varepsilon\|_{W^{1,p}(\Omega)}$ can be bounded independently of ε. See e.g. [S63, Theorem 6.1].

In order to prove the lemma, we claim that

$$\|u^\varepsilon\|_{L^\infty(\Omega)} \leqslant \|u\|_{L^\infty(\partial\Omega)} + d, \tag{2.2.8}$$

for some constant $d \geqslant 0$ independent of ε. To check that (2.2.8) is true, we begin by showing that u^ε is a.e. bounded above by the right hand side of (2.2.8). If $u^\varepsilon \leqslant \|u\|_{L^\infty(\partial\Omega)}$, we are done. We thus suppose that $u^\varepsilon > \|u\|_{L^\infty(\partial\Omega)}$ on a set of positive measure.

Let $k \in \mathbb{R}$ be such that $k > k_0 := \|u\|_{L^\infty(\partial\Omega)}$. Define $A_k := \{x \in \Omega : u^\varepsilon(x) > k\}$ and

$$u^\varepsilon_k := \min\{u^\varepsilon, k\}.$$

Notice that $u^\varepsilon_k \in W^{1,p}_u(\Omega)$, $u^\varepsilon_k = u^\varepsilon$ in $\Omega \setminus A_k$ and $u^\varepsilon_k = k$ in A_k. By these three observations, together with the minimality of u^ε and the fact that $B(0) = 0$, we deduce that

$$\int_{A_k} B_\varepsilon(H(\nabla u^\varepsilon(x))) \leqslant \int_{A_k} F'(u(x))\,[u^\varepsilon(x) - k]\,dx.$$

Now, since $f := F'(u) \in L^\infty(\Omega)$, by (2.2.3) we obtain

$$\gamma_\star \int_{A_k} |\nabla u^\varepsilon(x)|^p\,dx \leqslant \|f\|_{L^\nu(\Omega)} \left(\int_{A_k} |u^\varepsilon(x) - k|^{\nu/(\nu-1)}\,dx\right)^{(\nu-1)/\nu} + c_\star |A_k|, \tag{2.2.9}$$

with $\gamma_\star = \bar{\gamma}/(2(p-1)p)$ and for $\nu \in (1, +\infty)$ to be determined later.

Now we introduce a parameter $\delta \in (1/p, 1)$ and set $q := \delta p$. As a result, $1 < q < n$ and thus both cases $1 < p < n$ and $p = n$ can be dealt with at the same time.

An application of Sobolev's and Hölder's inequalities on the left hand side of (2.2.9) yields

$$\left(\int_{A_k} |u^\varepsilon(x) - k|^{q^*}\,dx\right)^{p/q^*} \leqslant C_S^p \left(\int_{A_k} |\nabla u^\varepsilon(x)|^q\,dx\right)^{p/q} \leqslant C_S^p |A_k|^{1-\delta} \int_{A_k} |\nabla u^\varepsilon(x)|^p\,dx,$$

where $C_S = C_S(n, q)$ is the Sobolev's (best) constant and $q^* = nq/(n-q)$ is the Sobolev's critical exponent. On the other hand, we use Hölder's and Young's inequalities on the first summand of the right hand side of (2.2.9) to deduce

$$\begin{aligned}\left(\int_{A_k} |u^\varepsilon(x) - k|^{\nu/(\nu-1)}\,dx\right)^{(\nu-1)/\nu} &\leqslant \left(\int_{A_k} |u^\varepsilon(x) - k|^{q^*}\,dx\right)^{1/q^*} |A_k|^{1-\frac{1}{q^*}-\frac{1}{\nu}} \\ &\leqslant \frac{\mu^p}{p}\left(\int_{A_k} |u^\varepsilon(x) - k|^{q^*}\,dx\right)^{p/q^*} + \frac{\mu^{-p'}}{p'} |A_k|^{\left(1-\frac{1}{q^*}-\frac{1}{\nu}\right)p'},\end{aligned}$$

where $p' = p/(p-1)$ is the conjugate exponent of p and $\mu > 0$ is some parameter to be later decided. Notice that we need to restrict to

$$\nu \in \left(\frac{q^*}{q^*-1}, +\infty \right), \tag{2.2.10}$$

in order to perform the above computation. By combining these last two inequalities with (2.2.9) we get

$$\begin{aligned}\left(\int_{A_k} |u^\varepsilon(x) - k|^{q^*} \, dx \right)^{p/q^*} \leqslant \frac{C_S^p |A_k|^{1-\delta}}{\gamma_\star} \Bigg\{ \|f\|_{L^\nu(\Omega)} \Bigg[\frac{\mu^p}{p} \left(\int_{A_k} |u^\varepsilon(x) - k|^{q^*} \, dx \right)^{p/q^*} \\ + \frac{\mu^{-p'}}{p'} |A_k|^{\left(1-\frac{1}{q^*}-\frac{1}{\nu}\right)p'} \Bigg] + c_\star |A_k| \Bigg\}.\end{aligned}$$

By choosing μ in such a way that $2C_S^p |A_k|^{1-\delta} \|f\|_{L^\nu(\Omega)} \mu^p = \gamma_\star p$, last inequality becomes

$$\begin{aligned}\left(\int_{A_k} |u^\varepsilon(x) - k|^{q^*} \, dx \right)^{p/q^*} \leqslant \frac{2C_S^p}{\gamma_\star} \Bigg[\frac{p-1}{p^{p/(p-1)}} \left(\frac{2C_S^p}{\gamma_\star} \right)^{1/(p-1)} \|f\|_{L^\nu(\Omega)}^{p'} \\ + c_\star |A_k|^{1-\left(1-\frac{1}{q^*}-\frac{1}{\nu}\right)p' - \frac{1-\delta}{p-1}} \Bigg] |A_k|^{\left(2-\frac{1}{q^*}-\frac{1}{\nu}-\delta\right)p'}.\end{aligned} \tag{2.2.11}$$

A computation then shows that

$$1 - \left(1 - \frac{1}{q^*} - \frac{1}{\nu}\right) p' - \frac{1-\delta}{p-1} = \frac{(1-\delta)^2}{\delta(p-1)} + \frac{(n-\nu)p}{n\nu(p-1)} > 0, \tag{2.2.12}$$

if we take $\nu \leqslant n$. Notice that

$$\frac{q^*}{q^*-1} - n = \frac{nq - n^2 q + n^2 - nq}{nq - n + q} = -\frac{n^2(q-1)}{n(q-1)+q} < 0,$$

so that such a choice for ν is compatible with (2.2.10). Hence, we deduce from (2.2.11) that

$$\int_{A_k} |u^\varepsilon(x) - k|^{q^*} \, dx \leqslant D |A_k|^{\left(2-\frac{1}{q^*}-\frac{1}{\nu}-\delta\right)\frac{q^*}{p-1}}, \tag{2.2.13}$$

with

$$D^{\frac{p}{q^*}} = \frac{2C_S^p}{\gamma_\star} \left[\frac{p-1}{p^{p'}} \left(\frac{2C_S^p}{\gamma_\star} \right)^{\frac{1}{p-1}} \|f\|_{L^\nu(\Omega)}^{p'} + c_\star |\Omega|^{1-\left(1-\frac{1}{q^*}-\frac{1}{\nu}\right)p' - \frac{1-\delta}{p-1}} \right].$$

Note that (2.2.12) ensures that we can replace $|A_k|$ with $|\Omega|$ inside the bracket.

Let now $h > k$. Clearly $A_h \subseteq A_k$ and

$$\int_{A_k} |u^\varepsilon(x) - k|^{q^*} \, dx \geqslant \int_{A_h} |u^\varepsilon(x) - k|^{q^*} \, dx \geqslant |A_h| (h-k)^{q^*}.$$

Hence, from (2.2.13) we deduce

$$|A_h| \leqslant D \frac{|A_k|^{\left(2-\frac{1}{q^*}-\frac{1}{\nu}-\delta\right)\frac{q^*}{p-1}}}{(h-k)^{q^*}}. \tag{2.2.14}$$

Notice that if we set $\nu = n$, then

$$\left(2 - \frac{1}{q^*} - \frac{1}{\nu} - \delta\right) \frac{q^*}{p-1} = \frac{n}{n - \delta p} \left[1 - (\delta - 1)^2 p'\right] > 1, \tag{2.2.15}$$

if δ is close enough to 1.

Set

$$\alpha := \left(2 - \frac{1}{q^*} - \frac{1}{\nu} - \delta\right)\frac{q^*}{p-1} - 1 \qquad \text{and} \qquad d := 2^{\frac{\alpha+1}{\alpha}} D^{\frac{1}{q^*}} |\Omega|^{\frac{\alpha}{q^*}}.$$

Observe that, thanks to (2.2.15), $\alpha > 0$. For any $i \in \mathbb{N}$, we now define

$$k_i := k_0 + d - \frac{d}{2^i} \qquad \text{and} \qquad x_i := |A_{k_i}|.$$

Note that $\{k_i\}$ is an increasing sequence. By rewriting (2.2.14) with $k = k_i$, $h = k_{i+1}$, we also find that $\{x_i\}$ satisfies

$$x_{i+1} \leqslant D\left(\frac{2}{d}\right)^{q^*} 2^{q^* i} x_i^{1+\alpha}.$$

By this and the way the constant d is defined, we are then in position to apply Lemma 7.1 at page 220 of [G03] and deduce that

$$|A_{k_0+d}| = \lim_{i\to+\infty} x_i = 0,$$

that is

$$u^\varepsilon \leqslant \|u\|_{L^\infty(\partial\Omega)} + d \ \text{ a.e. in } \Omega.$$

The bound from below on u^ε can be recovered for instance by noticing that $-u^\varepsilon$ minimizes

$$\widetilde{\mathcal{F}}_\varepsilon(v) = \int_\Omega B_\varepsilon(H(-\nabla v(x))) + F'(u(x))v(x)\,dx,$$

between all $v \in W^{1,p}_{-u}(\Omega)$ and applying to it the result obtained above. Then, (2.2.8) follows. □

Next is the key proposition of the approximation argument. Basically, we consider some perturbed problems driven by B_ε. We prove that their solutions are H^2 regular and that they converge to u as $\varepsilon \to 0^+$.

Proposition 2.9. *Let Ω be a bounded open set of $\mathbb{R}^n$ with $C^{1,\alpha}$ boundary. The problem*

$$\begin{cases} \operatorname{div}\big(B_\varepsilon'(H(\nabla u^\varepsilon))\nabla H(\nabla u^\varepsilon)\big) + F'(u) = 0, & \text{in } \Omega, \\ u^\varepsilon = u, & \text{on } \partial\Omega, \end{cases} \tag{2.2.16}$$

admits a strong solution $u^\varepsilon \in C^{1,\alpha'}(\overline{\Omega}) \cap H^2(\Omega)$, for some $\alpha' \in (0,1]$ independent of ε. Furthermore, u^ε converges to u in $C^1(\overline{\Omega})$, as $\varepsilon \to 0^+$.

Proof. By using standard methods - see, for instance, [D07, Theorem 3.30] - we know that the functional $\mathcal{F}_\varepsilon$, introduced in Lemma 2.8, admits the existence of a minimizer $u^\varepsilon \in W^{1,p}(\Omega)$, with $u^\varepsilon - u \in W^{1,p}_0(\Omega)$. Note that $\mathcal{F}_\varepsilon$ is coercive, thanks to (2.2.3), the continuity of F' and the boundedness of u. Clearly, u^ε satisfies (2.2.16) in the weak sense.

In view of Lemma 2.8, we know that u^ε is essentially bounded in Ω and that $\|u^\varepsilon\|_{L^\infty(\Omega)}$ is uniform in ε. With this in hand, we can now verify that $u^\varepsilon \in C^{1,\alpha'}$. For this, we notice that Lemma 2.5 and Proposition 1.18 of Chapter 1 ensure that hypothesis (A)$'$ is verified by $B_\varepsilon \circ H$. Hence, by the uniform L^∞ estimates, we may appeal to [L88, Theorem 1] to deduce that $u^\varepsilon \in C^{1,\alpha'}(\overline{\Omega})$, for some $\alpha' \in (0,1]$. Notice that α' is independent of ε and $\|u^\varepsilon\|_{C^{1,\alpha'}(\overline{\Omega})}$ is uniformly bounded in ε.

Consequently, by Arzelà-Ascoli Theorem, the sequence $\{u^\varepsilon\}$ converges in $C^1(\overline{\Omega})$ to a function v, as $\varepsilon \to 0^+$. With the aid of Lemma 2.6, we see that v is the unique solution of

$$\begin{cases} \operatorname{div}\left(B'(H(\nabla v))\nabla H(\nabla v)\right) + F'(u) = 0, & \text{in } \Omega, \\ v = u, & \text{on } \partial\Omega. \end{cases}$$

Therefore, $v = u$ in the whole $\overline{\Omega}$.

Now we prove the H^2 regularity of u^ε. To this aim we employ [T84, Proposition 1]. Notice that we need to check the validity of condition (2.4) there, in order to apply such result. If $p \geqslant 2$ it is an immediate consequence of the fact that $B_\varepsilon \circ H$ satisfies (A)$'$. Indeed, for any $\eta \in \mathbb{R}^n \setminus \{0\}$, $\zeta \in \mathbb{R}^n$, we deduce that

$$[\operatorname{Hess}(B_\varepsilon \circ H)(\xi)]_{ij}\, \zeta_i \zeta_j \geqslant \bar{\gamma}(\kappa + \varepsilon + |\xi|)^{p-2}|\zeta|^2 \geqslant \bar{\gamma}(\kappa + \varepsilon)^{p-2}|\zeta|^2,$$

In case $1 < p < 2$, we take $M > \|\nabla u^\varepsilon\|_{L^\infty(\overline{\Omega})}$ and modify B_ε accordingly to Lemma 2.7. The new function $\hat{B}_\varepsilon$ obtained this way satisfies assumption (A), and thus (A)$'$, with $p = 2$. Moreover, u^ε is still a weak solution to (2.2.16) with B_ε replaced by $\hat{B}_\varepsilon$. This is enough to conclude that $u^\varepsilon \in H^2(\Omega)$ also when $1 < p < 2$.

From the additional Sobolev regularity we deduce that u^ε is actually a strong solution of (2.2.16). Indeed, it is sufficient to observe that, for any $i = 1, \ldots, n$,

$$B_\varepsilon'(H(\nabla u^\varepsilon))H_i(\nabla u^\varepsilon) = (B_\varepsilon \circ H)_i\,(\nabla u^\varepsilon) \in H^1(\Omega),$$

being $(B_\varepsilon \circ H)_i$ locally uniformly Lipschitz, by Lemma 2.5. □

After all these preliminary results, we may finally prove the monotonicity formula.

2.2.3 Proof of Theorem 2.1

First, using the coarea formula we compute

$$\mathscr{W}'(R) = \frac{1-n}{R}\mathscr{W}(R) + \frac{1}{R^{n-1}}\int_{\partial W_R} [B(H(\nabla u)) + G(u)]\, |\nabla H^*|^{-1}\, d\mathbb{H}^{n-1}.$$

Then, notice that the exterior unit normal vector to ∂W_R at $x \in \partial W_R$ is given by

$$\nu(x) = \frac{\nabla H^*(x)}{|\nabla H^*(x)|}. \tag{2.2.17}$$

Thus, by the homogeneity of H and the second identity in (1.2.20) we have

$$H(\nu(x)) = |\nabla H^*(x)|^{-1} H(\nabla H^*(x)) = |\nabla H^*(x)|^{-1}.$$

As a consequence, the derivative of $\mathscr{W}$ at R becomes

$$\mathscr{W}'(R) = \frac{1-n}{R}\mathscr{W}(R) + \frac{1}{R^{n-1}}\int_{\partial W_R} [B(H(\nabla u)) + G(u)]\, H(\nu)\, d\mathbb{H}^{n-1}. \tag{2.2.18}$$

For any $\varepsilon \in (0,1)$, let now $u^\varepsilon \in C^{1,\alpha'}(\overline{W_R}) \cap H^2(W_R)$ be the strong solutions of (2.2.16), with $\Omega = W_R$. Notice that ∂W_R is of class C^2 in view of Lemma 1.16. Hence, we are allowed to apply Proposition 2.9 to obtain such a u^ε. By the results of Proposition 2.9 and Lemma 2.6, along with the C^2 regularity of G, it is immediate to check that

$$\begin{gathered} B_\varepsilon(H(\nabla u^\varepsilon)) \longrightarrow B(H(\nabla u)), \\ B_\varepsilon'(H(\nabla u^\varepsilon))H(\nabla u^\varepsilon) \longrightarrow B'(H(\nabla u))H(\nabla u), \\ G(u^\varepsilon) \longrightarrow G(u) \text{ and } F'(u^\varepsilon) \longrightarrow F'(u), \end{gathered} \tag{2.2.19}$$

uniformly on $\overline{W_R}$.

In view of Lemma 1.16 the function $H\nabla H$ is bijective and its inverse is given by $H^*\nabla H^*$. Hence, exploiting the homogeneity properties of H and ∇H together with (1.2.20), it follows that the identity

$$\begin{aligned} x &= H(H^*(x)\nabla H^*(x))\nabla H(H^*(x)\nabla H^*(x)) = H^*(x)H(\nabla H^*(x))\nabla H(\nabla H^*(x)) \\ &= H^*(x)\nabla H(\nabla H^*(x)), \end{aligned}$$

is true for any $x \in \mathbb{R}^n \setminus \{0\}$. Consequently, using (1.2.1), (2.2.17), the homogeneity of ∇H, the definition of ∂W_R and the divergence theorem, we compute

$$\begin{aligned} \int_{\partial W_R} B_\varepsilon(H(\nabla u^\varepsilon))H(\nu)\, d\mathbb{H}^{n-1} &= \frac{1}{R}\int_{\partial W_R} B_\varepsilon(H(\nabla u^\varepsilon))H^*\langle \nabla H(\nu),\nu\rangle\, d\mathbb{H}^{n-1} \\ &= \frac{1}{R}\int_{W_R} \operatorname{div}\big(B_\varepsilon(H(\nabla u^\varepsilon))H^*\nabla H(\nabla H^*)\big)\, dx = \frac{1}{R}\int_{W_R} \operatorname{div}\big(B_\varepsilon(H(\nabla u^\varepsilon))x\big)\, dx \\ &= \frac{1}{R}\int_{W_R} B_\varepsilon'(H(\nabla u^\varepsilon))H_j(\nabla u^\varepsilon)u^\varepsilon_{ij}x_i\, dx + \frac{n}{R}\int_{W_R} B_\varepsilon(H(\nabla u^\varepsilon))\, dx. \end{aligned}$$

With a completely analogous argument we also deduce that

$$\int_{\partial W_R} G(u^\varepsilon)H(\nu)\, d\mathbb{H}^{n-1} = -\frac{1}{R}\int_{W_R} F'(u^\varepsilon)u^\varepsilon_i x_i\, dx + \frac{n}{R}\int_{W_R} G(u^\varepsilon)\, dx.$$

Putting these last two identities together we obtain

$$\int_{\partial W_R} \left[B_\varepsilon(H(\nabla u^\varepsilon)) + G(u^\varepsilon)\right] H(\nu)\, d\mathbb{H}^{n-1} = \frac{n}{R}\int_{W_R} B_\varepsilon(H(\nabla u^\varepsilon)) + G(u^\varepsilon)\, dx + \frac{I_\varepsilon}{R}, \tag{2.2.20}$$

where

$$I_\varepsilon := \int_{W_R} \left[B_\varepsilon'(H(\nabla u^\varepsilon))H_j(\nabla u^\varepsilon)u^\varepsilon_{ij} - F'(u^\varepsilon)u^\varepsilon_i\right] x_i\, dx.$$

Recalling that u^ε is a strong solution of (2.2.16), we compute

$$\begin{aligned} I_\varepsilon &= \int_{W_R} \Big[\big(B_\varepsilon'(H(\nabla u^\varepsilon))H_j(\nabla u^\varepsilon)u^\varepsilon_i\big)_j - \big(B_\varepsilon'(H(\nabla u^\varepsilon))H_j(\nabla u^\varepsilon)\big)_j u^\varepsilon_i - F'(u^\varepsilon)u^\varepsilon_i\Big] x_i\, dx \\ &= \int_{W_R} \big(B_\varepsilon'(H(\nabla u^\varepsilon))H_j(\nabla u^\varepsilon)u^\varepsilon_i\big)_j x_i\, dx + \int_{W_R} \left[F'(u) - F'(u^\varepsilon)\right] u^\varepsilon_i x_i\, dx. \end{aligned}$$

By the divergence theorem, formulae (1.2.1), (2.2.17) and condition (2.1.2) we find

$$\begin{aligned} I_\varepsilon &= \int_{W_R} \big(B_\varepsilon'(H(\nabla u^\varepsilon))H_j(\nabla u^\varepsilon)u^\varepsilon_i x_i\big)_j\, dx - \int_{W_R} B_\varepsilon'(H(\nabla u^\varepsilon))H_j(\nabla u^\varepsilon)u^\varepsilon_i\delta_{ij}\, dx \\ &\quad + \int_{W_R} \left[F'(u) - F'(u^\varepsilon)\right] u^\varepsilon_i x_i\, dx \\ &= \int_{\partial W_R} \frac{B_\varepsilon'(H(\nabla u^\varepsilon))}{|\nabla H^*|}\langle \nabla H(\nabla u^\varepsilon), \nabla H^*\rangle \langle \nabla u^\varepsilon, x\rangle\, d\mathbb{H}^{n-1} \\ &\quad - \int_{W_R} B_\varepsilon'(H(\nabla u^\varepsilon))\langle \nabla H(\nabla u^\varepsilon), \nabla u^\varepsilon\rangle\, dx + \int_{W_R} \left[F'(u) - F'(u^\varepsilon)\right]\langle \nabla u^\varepsilon, x\rangle\, dx \\ &\geqslant -\int_{W_R} B_\varepsilon'(H(\nabla u^\varepsilon))H(\nabla u^\varepsilon)\, dx + \int_{W_R} \left[F'(u) - F'(u^\varepsilon)\right]\langle \nabla u^\varepsilon, x\rangle\, dx. \end{aligned} \tag{2.2.21}$$

Taking the limit as $\varepsilon \to 0^+$ in (2.2.20) and (2.2.21), by (2.2.19) we obtain

$$\int_{\partial W_R} [B(H(\nabla u)) + G(u)]\, H(\nu)\, d\mathbb{H}^{n-1} \geqslant \frac{n}{R} \int_{W_R} B(H(\nabla u)) + G(u)\, dx - \frac{1}{R} \int_{W_R} B'(H(\nabla u)) H(\nabla u)\, dx.$$

By plugging this last identity in (2.2.18) and recalling (2.1.3) we finally get

$$\mathscr{W}'(R) \geqslant \frac{1}{R^n} \int_{W_R} B(H(\nabla u)) + G(u) - B'(H(\nabla u)) H(\nabla u)\, dx.$$

The result now follows since the integral on the right hand side is non-negative by virtue of the gradient bound given by Theorem 1.1 of Chapter 1.

2.3 The Liouville-type theorem

Here we prove Theorem 2.4. In order to obtain that u is constant, our first goal is to show that, thanks to the estimate yielded by Theorem 1.1, the gradient term in (2.1.3) is bounded by the potential. Then, the monotonicity formula of Theorem 2.1 and the growth assumption on $G(u)$ will conclude the argument.

The following general result allows us to accomplish the first step.

Lemma 2.10. *Let $B \in C^2(0, +\infty) \cap C^1([0, +\infty))$ be a function satisfying (8) and $B(0) = B'(0) = 0$. Assume in addition that B satisfies either (A) or (B). Then, for any $K > 0$ there exists a constant $\delta > 0$ such that*

$$B'(t)t - B(t) \geqslant \delta B(t) \text{ for any } t \in [0, K]. \tag{2.3.1}$$

In particular, under assumption (A), inequality (2.3.1) holds for any $t \geqslant 0$.

Proof. We begin by proving (2.3.1) when (A) is in force. Since $B(0) = B'(0) = 0$, we have

$$B'(t)t - B(t) = \int_0^t B''(s) s\, ds \geqslant \gamma \int_0^t (\kappa + s)^{p-2}\, s\, ds.$$

On the other hand,

$$B(t) = \int_0^t B'(s)\, ds \leqslant \Gamma \int_0^t (\kappa + s)^{p-2}\, s\, ds.$$

By comparing these two expressions, we see that (2.3.1) holds for any $t \geqslant 0$, with $\delta = \gamma/\Gamma$.

Then, we deal with case (B). Fix $K > 0$. Being $B''(0) > 0$ and $B(0) = B'(0) = 0$, it clearly exist $\Gamma_* \geqslant \gamma_* > 0$ such that $B''(t) \in [\gamma_*, \Gamma_*]$, for any $t \in [0, K]$. Hence, as before we compute

$$B'(t)t - B(t) = \int_0^t B''(s) s\, ds \geqslant \gamma_* \int_0^t s\, ds = \frac{\gamma_*}{2} t^2,$$

for any $t \in [0, K]$. Also,

$$B(t) = \int_0^t \int_0^s B''(\sigma)\, d\sigma ds \leqslant \frac{\Gamma_*}{2} t^2,$$

for any $t \in [0, K]$, and again (2.3.1) is proved. ∎

Proof of Theorem 2.4. By combining Lemma 2.10 and Theorem 1.1 of Chapter 1, we deduce that

$$B(H(\nabla u(x))) \leqslant CG(u(x)) \text{ for any } x \in \mathbb{R}^n, \tag{2.3.2}$$

for some constant $C > 0$. We stress that, under assumption (B), it is crucial that ∇u is globally L^∞ - which is true by definition - in order to profitably apply Lemma 2.10. Recalling the definition (2.1.3) of the rescaled energy functional $\mathscr{W}$, in view of (2.3.2) and (2.1.9) we may conclude that

$$\lim_{R\to+\infty} \mathscr{W}(R) \leqslant (C+1) \lim_{R\to+\infty} \frac{1}{R^{n-1}} \int_{W_R} G(u(x))\, dx = 0.$$

But then, Theorem 2.1 tells that $\mathscr{W}$ is non-decreasing in $R \in (0, +\infty)$ and, hence, for any $r > 0$, we have

$$0 \leqslant \mathscr{W}(r) \leqslant \lim_{R\to+\infty} \mathscr{W}(R) = 0,$$

which yields $\mathscr{W} \equiv 0$. Consequently, $\nabla u \equiv 0$, i.e. u is constant. □

2.4 On conditions (16) and (2.1.4)

In the present section we prove Theorem 2.2, thus establishing a characterization of the anisotropies H which satisfy

$$\langle H(\xi)\nabla H(\xi), H^*(x)\nabla H^*(x)\rangle = \langle \xi, x\rangle, \tag{2.1.4}$$

for any $\xi, x \in \mathbb{R}^n$. Indeed, we show that such requirement is necessary and sufficient for H to assume the form

$$H_M(\xi) = \sqrt{\langle M\xi, \xi\rangle}, \tag{16}$$

for some symmetric and positive definite matrix $M \in \mathrm{Mat}_n(\mathbb{R})$.

We begin by showing the necessity of (2.1.4). As a first step towards this aim, we compute the dual function H_M^*.

Lemma 2.11. *Let $M \in \mathrm{Mat}_n(\mathbb{R})$ be symmetric and positive definite. Then, $H_M^* = H_{M^{-1}}$.*

Proof. Being M positive definite and symmetric, the assignment

$$\langle \xi, \eta\rangle_M := \langle M\xi, \eta\rangle,$$

defines an inner product in $\mathbb{R}^n$. We denote the induced norm by $\|\cdot\|_M$. Also notice that M is invertible, so that $H_{M^{-1}}$ is well defined.

Recalling definition (11) of dual function and applying the Cauchy-Schwarz inequality to the inner product $\langle\cdot,\cdot\rangle_M$, we obtain

$$\begin{aligned} H_M^*(x) &= \sup_{\xi\neq 0} \frac{\langle x, \xi\rangle}{\sqrt{\langle M\xi, \xi\rangle}} = \sup_{\xi\neq 0} \frac{\langle M(M^{-1}x), \xi\rangle}{\sqrt{\langle M\xi, \xi\rangle}} = \sup_{\xi\neq 0} \frac{\langle M^{-1}x, \xi\rangle_M}{\|\xi\|_M} \\ &\leqslant \sup_{\xi\neq 0} \frac{\|M^{-1}x\|_M \|\xi\|_M}{\|\xi\|_M} = \|M^{-1}x\|_M \\ &= \sqrt{\langle M^{-1}x, x\rangle}. \end{aligned}$$

On the other hand, the choice $\xi := M^{-1}x$ yields

$$H_M^*(x) \geqslant \frac{\langle x, M^{-1}x\rangle}{\sqrt{\langle MM^{-1}x, M^{-1}x\rangle}} = \sqrt{\langle M^{-1}x, x\rangle}.$$

Hence, recalling definition (16), the thesis follows. □

With this in hand, we are now able to prove the following

Lemma 2.12. *Let $M \in \mathrm{Mat}_n(\mathbb{R})$ be a symmetric and positive definite matrix. Then, the norm H_M satisfies (2.1.4).*

Proof. The proof is a simple computation. Notice that for any symmetric $A \in \mathrm{Mat}_n(\mathbb{R})$ we have

$$\partial_i \left(H_A^2(\xi)\right) = \partial_i \left(A_{jk}\xi_j\xi_k\right) = A_{jk}\delta_{ji}\xi_k + A_{jk}\xi_j\delta_{ki} = 2A_{ij}\xi_j,$$

for any $\xi \in \mathbb{R}^n$, $i = 1, \dots, n$. Thus, we get

$$H_A(\xi)\partial_i H_A(\xi) = \frac{\partial_i \left(H_A^2(\xi)\right)}{2} = A_{ij}\xi_j.$$

Applying then Lemma 2.11 together with the identity yet obtained with both choices $A = M$ and $A = M^{-1}$, we obtain

$$\begin{aligned}\langle H_M(\xi)\nabla H_M(\xi), H_M^*(\eta)\nabla H_M^*(\eta)\rangle &= \langle H_M(\xi)\nabla H_M(\xi), H_{M^{-1}}(\eta)\nabla H_{M^{-1}}(\eta)\rangle \\ &= M_{ij}\xi_j M_{ik}^{-1}\eta_k \\ &= \delta_{jk}\xi_j\eta_k \\ &= \langle \xi, \eta\rangle,\end{aligned}$$

which is (2.1.4). □

Now, we prove that the converse implication is also true. Hence, Theorem 2.2 will follow. Before addressing the actual proof, we need just another abstract lemma. We believe that the content of the following result will appear somewhat evident to the reader. However, we include both the formal statement and the proof.

Lemma 2.13. *Let $\mathcal{T} : \mathbb{R}^n \to \mathbb{R}^n$ be symmetric with respect to the standard inner product in $\mathbb{R}^n$, that is*

$$\langle \mathcal{T}(v), w\rangle = \langle v, \mathcal{T}(w)\rangle, \tag{2.4.1}$$

for any $v, w \in \mathbb{R}^n$. Then, $\mathcal{T}$ is a linear transformation, i.e.

$$\mathcal{T}(v) = Tv \text{ for any } v \in \mathbb{R}^n,$$

for some symmetric $T \in \mathrm{Mat}_n(\mathbb{R})$

Proof. The conclusion follows by simply plugging $w = e_i$ in (2.4.1), where $\{e_i\}_{i=1,\dots,n}$ is the canonical basis in $\mathbb{R}^n$. Indeed, we have

$$[\mathcal{T}(v)]_i = \langle \mathcal{T}(v), e_i\rangle = \langle v, \mathcal{T}(e_i)\rangle$$

for any $v \in \mathbb{R}^n$, $i = 1, \dots, n$. Thus we may conclude that $\mathcal{T}(v) = Tv$, where $T = [T_{ij}]_{i,j=1,\dots,n}$ is the matrix with entries

$$T_{ij} = [\mathcal{T}(e_i)]_j.$$

The symmetry of T clearly follows by employing (2.4.1) again. □

Proof of Theorem 2.2. In view of Lemma 2.12, it is only left to prove that, under condition (2.1.4), H is forced to be of the form (16).

By Lemma 1.16, we know that the map $\Psi_H : \mathbb{R}^n \to \mathbb{R}^n$, defined for $\xi \in \mathbb{R}^n$ by

$$\Psi_H(\xi) := H(\xi)\nabla H(\xi),$$

is invertible with inverse Ψ_{H^*}. Under this notation identity (2.1.4) may be read as

$$\langle \Psi_H(\xi), \Psi_{H^*}(\eta)\rangle = \langle \xi, \eta\rangle, \tag{2.4.2}$$

for any $\xi, \eta \in \mathbb{R}^n$. Applying (2.4.2) with $\eta = \Psi_H(\zeta)$ we get

$$\langle \Psi_H(\xi), \zeta\rangle = \langle \Psi_H(\xi), \Psi_H^{-1}(\eta)\rangle = \langle \Psi_H(\xi), \Psi_{H^*}(\eta)\rangle = \langle \xi, \eta\rangle = \langle \xi, \Psi_H(\zeta)\rangle,$$

for any $\xi, \zeta \in \mathbb{R}^n$. That is, Ψ_H is symmetric with respect to the standard inner product in $\mathbb{R}^n$ and hence linear, by virtue of Lemma 2.13. Therefore, there exists a symmetric $M \in \mathrm{Mat}_n(\mathbb{R})$ such that

$$\nabla\left(\frac{H^2(\xi)}{2}\right) = H(\xi)\nabla H(\xi) = M\xi.$$

This in turn implies that $H = H_M$ and the proof of the proposition is complete. □

2.5 On the weaker assumption (2.1.2)

In this last section we study the condition

$$\operatorname{sgn}\langle H(\xi)\nabla H(\xi), H^*(x)\nabla H^*(x)\rangle = \operatorname{sgn}\langle \xi, x\rangle, \tag{2.1.2}$$

for any $\xi, x \in \mathbb{R}^n$, which has been introduced in the statement of Theorem 2.1. First, we have the following general result that provides a simpler equivalent form for assumption (2.1.2).

Proposition 2.14. *Let H be a $C^1(\mathbb{R}^n \setminus \{0\})$ be a positive homogeneous function of degree 1 satisfying (9). Assume the unit ball B_1^H, as defined by (10), to be strictly convex. Then, (2.1.2) is equivalent to the condition*

$$\langle H(\xi)\nabla H(\xi), \eta\rangle = 0 \quad \textit{if and only if} \quad \langle \xi, H(\eta)\nabla H(\eta)\rangle = 0, \tag{2.5.1}$$

for any $\xi, \eta \in \mathbb{R}^n$.

Proof. First, we remark that, by arguing as in the proof of Theorem 2.2, it is immediate to check that (2.1.2) can be put in the equivalent form

$$\operatorname{sgn}\langle H(\xi)\nabla H(\xi), \eta\rangle = \operatorname{sgn}\langle \xi, H(\eta)\nabla H(\eta)\rangle, \tag{2.5.2}$$

for any $\xi, \eta \in \mathbb{R}^n$. Thus, we need to show that (2.5.1) is equivalent to (2.5.2).

Notice that (2.5.1) is trivially implied by (2.5.2). Thus, we only need to prove that the converse is also true. To see this, assume (2.5.1) to hold and fix $\xi \in \mathbb{R}^n$. If $\xi = 0$, then both sides of (2.5.2) vanish, in view of Lemma 1.10. Suppose therefore $\xi \neq 0$ and consider the hyperplane

$$\Pi := \{\eta \in \mathbb{R}^n : \langle H(\xi)\nabla H(\xi), \eta\rangle = 0\},$$

together with the two half-spaces

$$\Pi_\pm := \{\eta \in \mathbb{R}^n : \pm\langle H(\xi)\nabla H(\xi), \eta\rangle > 0\}.$$

By virtue of (2.5.1), the function $h : \mathbb{R}^n \to \mathbb{R}$, defined by setting

$$h(\eta) := \langle H(\eta)\nabla H(\eta), \xi\rangle,$$

vanishes precisely on Π. Furthermore, by Lemma 1.10 (with $B(t) = t^2/2$), h is continuous on the whole of $\mathbb{R}^n$ and it satisfies

$$h(\xi) = \langle H(\xi)\nabla H(\xi), \xi\rangle = H^2(\xi) > 0.$$

But $\xi \in \Pi_+$, and so h is positive on Π_+, being it connected. Analogously, it holds $h(-\xi) < 0$ from which we deduce that h is negative on Π_-. Thence, (2.5.2) follows. □

With the aid of Proposition 2.14, we now restrict to the planar case $n = 2$ and show that, in this case, all the even anisotropies satisfying (2.1.2) can be obtained by means of an explicit and operative formula. As a result, it will then become clear that (2.1.2) is a weaker assumption than (2.1.4).

Proposition 2.15. *Let $r : [0, \pi/2] \to (0, +\infty)$ be a given C^2 function satisfying*

$$r(\theta)r''(\theta) < 2r'(\theta)^2 + r(\theta)^2 \text{ for a.a. } \theta \in \left[0, \frac{\pi}{2}\right], \tag{2.5.3}$$

and

$$r(0) = 1, \qquad r(\pi/2) = r^*, \qquad r'(0) = r'(\pi/2) = 0, \tag{2.5.4}$$

for some $r^* \geqslant 1$. Consider the π-periodic function $\widetilde{r} : \mathbb{R} \to (0, +\infty)$ defined on $[0, \pi]$ by

$$\widetilde{r}(\theta) := \begin{cases} r(\theta) & \text{if } 0 \leqslant \theta \leqslant \dfrac{\pi}{2}, \\ \dfrac{r^* \sqrt{r(\tau^{-1}(\theta))^2 + r'(\tau^{-1}(\theta))^2}}{r(\tau^{-1}(\theta))^2} & \text{if } \dfrac{\pi}{2} \leqslant \theta \leqslant \pi, \end{cases} \tag{2.5.5}$$

where $\tau : [0, \pi/2] \to [\pi/2, \pi]$ is the bijective map given by

$$\tau(\eta) = \frac{\pi}{2} + \eta - \arctan \frac{r'(\eta)}{r(\eta)}. \tag{2.5.6}$$

Then, $\widetilde{r}$ is of class $C^1(\mathbb{R})$, the set

$$\{(\rho \cos \theta, \rho \sin \theta) : \rho \in [0, \widetilde{r}(\theta)), \theta \in [0, 2\pi]\}, \tag{2.5.7}$$

is strictly convex and its supporting function

$$\widetilde{H}(\rho \cos \theta, \rho \sin \theta) := \frac{\rho}{\widetilde{r}(\theta)},$$

defined for $\rho \geqslant 0$ and $\theta \in [0, 2\pi]$, satisfies (2.5.1).

Furthermore, up to a rotation and a homothety of the plane $\mathbb{R}^2$, any even positive 1-homogeneous function $H \in C^2(\mathbb{R}^2 \setminus \{0\})$ satisfying (9), having strictly convex unit ball B_1^H and for which (2.5.1) holds true is such that B_1^H is of the form (2.5.7), for some positive $r \in C^2([0, \pi/2])$ satisfying (2.5.3) and (2.5.4).

Before heading to the proof of this proposition, we state the following auxiliary result.

Lemma 2.16. *Let $r : [0, \pi/2] \to (0, +\infty)$ be a C^2 function that satisfies condition (2.5.3) and $r'(0) = r'(\pi/2) = 0$. Then,*

$$-\cot \eta < \frac{r'(\eta)}{r(\eta)} < \tan \eta, \tag{2.5.8}$$

for any $\eta \in (0, \pi/2)$.

Proof. For any $\eta \in (0, \pi/2)$, we set

$$q(\eta) := \frac{r'(\eta)}{r(\eta)}$$

Being the tangent function increasing, we see that the right inequality in (2.5.8) is satisfied if and only if

$$f(\eta) := \arctan q(\eta) < \eta. \tag{2.5.9}$$

Since

$$q'(\eta) = \frac{r(\eta)r''(\eta) - r'(\eta)^2}{r(\eta)^2},$$

we see that, for a.e. $\eta \in (0, \pi/2)$,

$$f'(\eta) = \frac{q'(\eta)}{1 + q(\eta)^2} = \frac{r(\eta)r''(\eta) - r'(\eta)^2}{r(\eta)^2 + r'(\eta)^2} < \frac{r(\eta)^2 + r'(\eta)^2}{r(\eta)^2 + r'(\eta)^2} = 1,$$

by virtue of (2.5.3). Observing that $f(0) = 0$, we then conclude that

$$f(\eta) = \int_0^\eta f'(t)\, dt < \eta,$$

which is (2.5.9). A similar argument shows that also the left inequality in (2.5.8) holds true. □

Proof of Proposition 2.15. Let $H \in C^2(\mathbb{R}^2 \setminus \{0\})$ be a given norm. Notice that the boundary of its unit ball B_1^H may be written in polar coordinates as

$$\partial B_1^H = \{\gamma(\theta) : \theta \in [0, 2\pi]\},$$

where

$$\gamma(\theta) = (r(\theta)\cos\theta, r(\theta)\sin\theta), \tag{2.5.10}$$

for some π-periodic $r \in C^2(\mathbb{R})$. Recall that the curvature of such a curve γ is given by

$$k(\theta) = \frac{2r'(\theta)^2 - r(\theta)r''(\theta) + r(\theta)^2}{[r(\theta)^2 + r'(\theta)^2]^{3/2}}, \tag{2.5.11}$$

for any $\theta \in [0, 2\pi]$. Hence, hypothesis (2.5.3) tells us that γ has positive curvature, outside at most a set of zero measure, and, thus, that B_1^H is strictly convex.

We also remark that condition (2.5.1) is equivalent to saying that, for any $\theta, \eta \in [0, 2\pi]$,

$$\gamma'(\theta) \parallel \gamma(\eta) \quad \text{if and only if} \quad \gamma(\theta) \parallel \gamma'(\eta). \tag{2.5.12}$$

This can be seen by noticing that $\nabla H(\gamma(\theta))$ is orthogonal to ∂B_1^H while $\gamma'(\theta)$ is tangent.

At a point $\theta^* \in [0, 2\pi]$ such that

$$r(\theta^*) = \max_{\theta \in \mathbb{R}} r(\theta) =: r^*,$$

we clearly have $r'(\theta^*) = 0$. Assuming, up to a rotation and a homothety of $\mathbb{R}^2$, that $\theta^* = \pi/2$ and $r(0) = 1$, it is immediate to check, by computing

$$\gamma'(\theta) = \big(r'(\theta)\cos\theta - r(\theta)\sin\theta, r'(\theta)\sin\theta + r(\theta)\cos\theta\big), \tag{2.5.13}$$

that condition (2.5.1), in its form (2.5.12), forces r to satisfy (2.5.4).

Now, take $r \in C^2([0, \pi/2])$ as in the statement of the proposition. We shall show that the function $\widetilde{r}$ defined by (2.5.5) is the only extension of r which determines a curve γ satisfying condition (2.5.12). Notice that, by the periodicity of $\widetilde{r}$, it is enough to prove it for $\theta, \eta \in [0, \pi]$. Moreover, if $\theta, \eta \in \{0, \pi/2, \pi\}$, then (2.5.12) is implied by (2.5.4). Consider now $\eta \in (0, \pi/2)$. We address the problem of finding the unique $\theta =: \tau(\eta) \in (0, \pi)$ such that $\gamma(\theta) \parallel \gamma'(\eta)$. First observe that this condition is equivalent to requiring

$$\cot\theta = \frac{r'(\eta)\cos\eta - r(\eta)\sin\eta}{r'(\eta)\sin\eta + r(\eta)\cos\eta} = \frac{\frac{r'(\eta)}{r(\eta)} - \tan\eta}{\frac{r'(\eta)}{r(\eta)}\tan\eta + 1} = \tan\left(\arctan\frac{r'(\eta)}{r(\eta)} - \eta\right), \tag{2.5.14}$$

in view of (2.5.10) and (2.5.13). Then, we see that, by (2.5.13) and Lemma 2.16, $\gamma'(\eta)$ and, therefore, $\gamma(\theta)$ lie in the second quadrant. Thus, we conclude that $\theta \in (\pi/2, \pi)$. Moreover, with this in hand and using again Lemma 2.16, it is easy to deduce from (2.5.14) that

$$\theta = \tau(\eta) = \frac{\pi}{2} + \eta - \arctan \frac{r'(\eta)}{r(\eta)}, \tag{2.5.15}$$

for any $\eta \in [0, \pi/2]$. Condition (2.5.12) then implies that $\gamma'(\theta) \parallel \gamma(\eta)$, which yields (2.5.14) with η and θ interchanged. Comparing the two formulae, we deduce that $\widetilde{r}$ should satisfy

$$\frac{\widetilde{r}'(\tau(\eta))}{\widetilde{r}(\tau(\eta))} = -\frac{r'(\eta)}{r(\eta)}, \tag{2.5.16}$$

for any $\eta \in [0, \pi/2]$. From this relation it is possible to recover the explicit form of $\widetilde{r}$. In order to do this, we multiply by $\tau'(\eta)$ both sides of (2.5.16) and integrate. The left hand side becomes

$$\int_0^\eta \frac{\widetilde{r}'(\tau(t))}{\widetilde{r}(\tau(t))} \tau'(t)\, dt = \log \frac{\widetilde{r}(\tau(\eta))}{\widetilde{r}(\tau(0))} = \log \frac{\widetilde{r}(\tau(\eta))}{r^*}. \tag{2.5.17}$$

The expansion of the right hand side requires a little bit more care. For simplicity of exposition, we will omit to evaluate r and its derivatives at η. We deduce from (2.5.15) that

$$\tau' = 1 - \frac{rr'' - r'^2}{r^2 + r'^2} = \frac{r^2 + 2r'^2 - rr''}{r^2 + r'^2}. \tag{2.5.18}$$

Then, since

$$\left[\log \left(r \left(r^2 + r'^2 \right) \right) \right]' = \frac{3r^2 r' + r'^3 + 2rr'r''}{r \left(r^2 + r'^2 \right)},$$

we compute

$$\begin{aligned} -\frac{r'}{r}\tau' &= -\frac{r^2 r' + 2r'^3 - rr'r''}{r\left(r^2 + r'^2\right)} \\ &= \frac{1}{2}\left[\log\left(r\left(r^2 + r'^2\right) \right) \right]' - \frac{5}{2} \frac{r^2 r' + r'^3}{r\left(r^2 + r'^2\right)} \\ &= \frac{1}{2}\left[\log\left(r\left(r^2 + r'^2\right) \right) - 5\log r \right]' \\ &= \frac{1}{2}\left[\log \frac{r^2 + r'^2}{r^4} \right]'. \end{aligned}$$

Integrating this last expression we get

$$-\int_0^\eta \frac{r'(t)}{r(t)} \tau'(t)\, dt = \frac{1}{2} \log \left(\frac{r(\eta)^2 + r'(\eta)^2}{r(\eta)^4} \frac{r(0)^4}{r(0)^2 + r'(0)^2} \right) = \frac{1}{2} \log \frac{r(\eta)^2 + r'(\eta)^2}{r(\eta)^4}. \tag{2.5.19}$$

By comparing (2.5.17) and (2.5.19), we immediately obtain that $\widetilde{r}$ satisfies (2.5.5).

Now we show that $\widetilde{r}$ has the desired regularity properties. From its definition and (2.5.16) is immediate to see that $\widetilde{r}$ is continuous on the whole $[0, \pi]$ and differentiable on $(0, \pi/2) \cup (\pi/2, \pi)$. Thus, we only need to check $\widetilde{r}'$ at 0, $\pi/2$ and π. Using (2.5.16) and (2.5.4), we compute

$$\widetilde{r}' \left(\frac{\pi}{2}^+ \right) = -\frac{r'(0)\, \widetilde{r}\left(\frac{\pi}{2}\right)}{r(0)} = 0 = \widetilde{r}' \left(\frac{\pi}{2}^- \right), \tag{2.5.20}$$

and

$$\widetilde{r}'(\pi^-) = -\frac{r'\left(\frac{\pi}{2}\right) \widetilde{r}(\pi)}{r\left(\frac{\pi}{2}\right)} = 0 = \widetilde{r}'(0^+). \tag{2.5.21}$$

Being it π-periodic, it follows that $\widetilde{r} \in C^1(\mathbb{R})$.

Finally, we prove that the set (2.5.7) is strictly convex. To see this, it is enough to show that $\widetilde{r}$ satisfies (2.5.3) for almost any $\theta \in [\pi/2, \pi]$. First, we check that $\widetilde{r}$ possesses almost everywhere second derivative. Indeed, by differentiating (2.5.16) we get

$$\left(\frac{\widetilde{r}''(\tau(\theta))}{\widetilde{r}(\tau(\theta))} - \frac{\widetilde{r}'(\tau(\theta))^2}{\widetilde{r}(\tau(\theta))^2}\right)\tau'(\theta) = -\frac{r''(\theta)}{r(\theta)} + \frac{r'(\theta)^2}{r(\theta)^2}. \tag{2.5.22}$$

Thus, if $\tau'(\theta) \neq 0$, which is true at almost any $\theta \in [0, \pi/2]$ in view of (2.5.18) and (2.5.3), we may solve (2.5.22) for $\widetilde{r}''$ and obtain

$$\begin{aligned}\widetilde{r}''(\tau(\theta)) &= \frac{\widetilde{r}'(\tau(\theta))^2}{\widetilde{r}(\tau(\theta))} - \frac{\widetilde{r}(\tau(\theta))}{\tau'(\theta)}\left(\frac{r''(\theta)}{r(\theta)} - \frac{r'(\theta)^2}{r(\theta)^2}\right)\\ &= \frac{\widetilde{r}'(\tau(\theta))^2}{\widetilde{r}(\tau(\theta))} - \frac{\widetilde{r}(\tau(\theta))\left(r(\theta)^2 + r'(\theta)^2\right)\left(r(\theta)r''(\theta) - r'(\theta)^2\right)}{r(\theta)^2\left(r(\theta)^2 + 2r'(\theta)^2 - r(\theta)r''(\theta)\right)},\end{aligned} \tag{2.5.23}$$

where in last line we made use of (2.5.18). With this in hand and recalling (2.5.16), we are able to compute that

$$\begin{aligned}\widetilde{r}(\tau)\widetilde{r}''(\tau) - 2\widetilde{r}'(\tau)^2 - \widetilde{r}(\tau)^2 &= \widetilde{r}'(\tau)^2 - \frac{\widetilde{r}(\tau)^2(r^2 + r'^2)(rr'' - r'^2)}{r^2(r^2 + 2r'^2 - rr'')} - 2\widetilde{r}'(\tau)^2 - \widetilde{r}(\tau)^2\\ &= -\widetilde{r}(\tau)^2\left(\frac{r'^2}{r^2} + \frac{(r^2 + r'^2)(rr'' - r'^2)}{r^2(r^2 + 2r'^2 - rr'')} + 1\right)\\ &= -\frac{\widetilde{r}(\tau)^2(r^2 + r'^2)^2}{r^2(r^2 + 2r'^2 - rr'')}\\ &< 0,\end{aligned}$$

almost everywhere in $[0, \pi/2]$. Thus, the proof is complete. $\square$

In view of Proposition 2.15, every even anisotropy H satisfying (2.1.2) is uniquely determined by its values on the first quadrant. Conversely, any positive $r \in C^2([0, \pi/2])$ for which (2.5.3) and (2.5.4) are true can be extended to $[0, \pi]$ (in a unique way) to obtain a C^1 norm satisfying (2.1.2).

An example of such an anisotropy, which is not of the trivial type (16), is given by

$$\hat{H}_p(\xi) = \begin{cases} |\xi|_p & \text{if } \xi_1\xi_2 \geqslant 0,\\ |\xi|_q & \text{if } \xi_1\xi_2 < 0,\end{cases}$$

where $|\cdot|_p$ is the standard p-norm in $\mathbb{R}^2$ and $q = p/(p-1)$ is the conjugate exponent of p, for $p \in (2, +\infty)$ (see Figure 2.1 below). It can be easily checked that $\hat{H}_p$ satisfies (2.1.2) from formulation (2.5.1).

Unfortunately, $\hat{H}_p$ is no more than $C^{1,1/(p-1)}_{\mathrm{loc}}(\mathbb{R}^2 \setminus \{0\})$. If one is interested in norms having higher regularity properties, additional hypotheses on the behaviour of the defining function r of its unit ball inside the first quadrant need to be imposed. In particular, assumption (2.5.3) should be strengthened by requiring it to hold at *any* $\theta \in [0, \pi/2]$. As a consequence, the class of norms under analysis is restricted to those being uniformly elliptic.

In order to deal with, say, $C^{3,\alpha}$ anisotropies, we have the following result.

Proposition 2.17. *Let $\alpha \in (0, 1]$ and $H \in C^{3,\alpha}_{\mathrm{loc}}(\mathbb{R}^2 \setminus \{0\})$ be an even positive homogeneous function of degree 1 for which (9) holds true. Then, H is uniformly elliptic and satisfies (2.5.1) if and only if, up to a rotation and a homothety of $\mathbb{R}^2$, its unit ball is of*

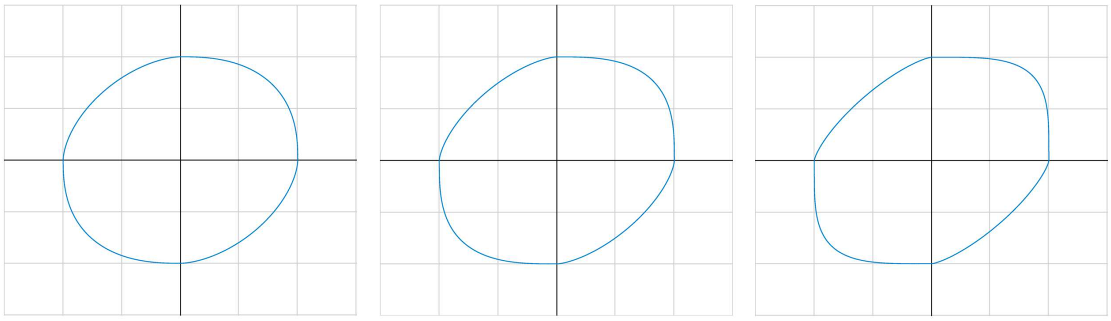

Figure 2.1: The unit circles of $\hat{H}_p$ for the values $p = 5/2$, 3 and 4.

the form (2.5.7)*, where* $\widetilde{r}$ *is given by* (2.5.5) *and* $r \in C^{3,\alpha}([0,\pi/2])$ *is a positive function satisfying*

$$r(\theta)r''(\theta) < 2r'(\theta)^2 + r(\theta)^2 \text{ for any } \theta \in \left[0, \frac{\pi}{2}\right], \tag{2.5.24}$$

$$r''\left(\frac{\pi}{2}\right) = -\frac{r^* r''(0)}{1 - r''(0)}, \qquad r'''\left(\frac{\pi}{2}\right) = -\frac{r^* r'''(0)}{(1 - r''(0))^3}, \tag{2.5.25}$$

and (2.5.4)*.*

Notice that the quantities appearing in both right hand sides of condition (2.5.25) are finite, as one can see by plugging $\theta = 0$ in (2.5.24) and recalling (2.5.4).

Proof of Proposition 2.17. In addition to the regularity properties of the extension $\widetilde{r}$, by Proposition 2.15 we only need to investigate the relation between (2.5.24) and the uniformly convexity of the unit ball of H. Notice that in 2 dimensions this last requirement is just asking the curvature $k(\theta)$, as defined by (2.5.11), to be positive at any angle $\theta \in [0, 2\pi]$. Hence, we see that it implies (2.5.24).

To check that also the converse implication is valid, it is enough to prove that if (2.5.24) is in force, then $\widetilde{r}$ satisfies the same inequality at any $\theta \in [\pi/2, \pi]$. A careful inspection of the proof of Proposition 2.15 - see, in particular, the argument starting below formula (2.5.22) - shows that this is true at any point θ for which $\tau'(\tau^{-1}(\theta)) \neq 0$. But then, comparing formula (2.5.18) with (2.5.24) we have that $\tau' > 0$ on the whole interval $[0, \pi/2]$ and so we are done.

The only thing we still have to verify is that, given $r \in C^{3,\alpha}([0, 2\pi])$, then its extension $\widetilde{r}$ belongs to $C^{3,\alpha}(\mathbb{R})$. Arguing as in the proof of Proposition 2.15, by (2.5.5), (2.5.23) and (2.5.24) we deduce that $\widetilde{r}$ is of class C^1 on the whole of $\mathbb{R}$ and $C^{3,\alpha}$ outside of the points $k\pi/2$, with $k \in \mathbb{Z}$. Moreover, by the periodicity properties of $\widetilde{r}$, we can reduce our analysis to the points 0, $\pi/2$ and π. Using (2.5.18) and (2.5.4), we compute

$$\tau'(0) = 1 - r''(0), \qquad \tau'\left(\frac{\pi}{2}\right) = \frac{r^* - r''\left(\frac{\pi}{2}\right)}{r^*}, \tag{2.5.26}$$

and so, by (2.5.23), (2.5.15), (2.5.4), (2.5.20), (2.5.21) and (2.5.25), we have

$$\begin{aligned}
\widetilde{r}''\left(\frac{\pi}{2}^+\right) &= \frac{\widetilde{r}'\left(\frac{\pi}{2}\right)^2}{\widetilde{r}\left(\frac{\pi}{2}\right)} - \frac{\widetilde{r}\left(\frac{\pi}{2}\right)}{\tau'(0)}\left(\frac{r''(0)}{r(0)} - \frac{r'(0)^2}{r(0)^2}\right) \\
&= -\frac{r^* r''(0)}{1 - r''(0)} = r''\left(\frac{\pi}{2}\right) = \widetilde{r}''\left(\frac{\pi}{2}^-\right),
\end{aligned}$$

and

$$\widetilde{r}''(\pi^-) = \frac{\widetilde{r}'(\pi)^2}{\widetilde{r}(\pi)} - \frac{\widetilde{r}(\pi)}{\tau'\left(\frac{\pi}{2}\right)}\left(\frac{r''\left(\frac{\pi}{2}\right)}{r\left(\frac{\pi}{2}\right)} - \frac{r'\left(\frac{\pi}{2}\right)^2}{r\left(\frac{\pi}{2}\right)^2}\right)$$
$$= -\frac{r''\left(\frac{\pi}{2}\right)}{r^* - r''\left(\frac{\pi}{2}\right)} = r''(0) = \widetilde{r}''(0^+).$$

Hence, $\widetilde{r} \in C^2(\mathbb{R})$. Now we study the third derivative of $\widetilde{r}$. By differentiating (2.5.23) we get

$$\begin{aligned}\widetilde{r}'''(\tau) = &\frac{\widetilde{r}'(\tau)\left(2\widetilde{r}(\tau)\widetilde{r}''(\tau) - \widetilde{r}'(\tau)^2\right)}{\widetilde{r}(\tau)^2} \\ &- \frac{\left(\widetilde{r}'(\tau)\tau'^2 - \widetilde{r}(\tau)\tau''\right)\left(rr'' - r'^2\right)}{r^2\tau'^3} - \frac{\widetilde{r}(\tau)\left(r^2 r''' - 3rr'r'' + 2r'^3\right)}{r^3\tau'^2},\end{aligned} \tag{2.5.27}$$

where every function is meant to be evaluated at θ. Moreover, from (2.5.18) we deduce that

$$\begin{aligned}\tau'' &= -\frac{(r'r'' + rr''' - 2r'r'')\left(r^2 + r'^2\right) - 2\left(rr'' - r'^2\right)(rr' + r'r'')}{\left(r^2 + r'^2\right)^2} \\ &= \frac{3r^2 r' r'' - r'^3 r'' - r^3 r''' - rr'^2 r''' + 2rr'r''^2 - 2rr'^3}{(r^2 + r'^2)^2},\end{aligned}$$

so that, recalling (2.5.4), we have

$$\tau''(0) = -r'''(0), \qquad \tau''\left(\frac{\pi}{2}\right) = -\frac{r'''\left(\frac{\pi}{2}\right)}{r^*}.$$

We plug these identities into (2.5.27) and use (2.5.15), (2.5.4), (2.5.26), (2.5.20), (2.5.21) and (2.5.25). By doing so, we finally conclude that

$$\begin{aligned}\widetilde{r}'''\left(\frac{\pi}{2}^+\right) &= \frac{\widetilde{r}\left(\frac{\pi}{2}\right)\tau''(0)r''(0)}{r(0)\tau'(0)^3} - \frac{\widetilde{r}\left(\frac{\pi}{2}\right)r'''(0)}{r(0)\tau'(0)^2} = -\frac{r^* r''(0)r'''(0)}{(1 - r''(0))^3} - \frac{r^* r'''(0)}{(1 - r''(0))^2} \\ &= -\frac{r^* r'''(0)}{(1 - r''(0))^3} = r'''\left(\frac{\pi}{2}\right) = \widetilde{r}'''\left(\frac{\pi}{2}^-\right),\end{aligned}$$

and

$$\begin{aligned}\widetilde{r}'''(\pi^-) &= \frac{\widetilde{r}(\pi)\tau''\left(\frac{\pi}{2}\right)r''\left(\frac{\pi}{2}\right)}{r\left(\frac{\pi}{2}\right)\tau'\left(\frac{\pi}{2}\right)^3} - \frac{\widetilde{r}(\pi)r'''\left(\frac{\pi}{2}\right)}{r\left(\frac{\pi}{2}\right)\tau'\left(\frac{\pi}{2}\right)^2} = -\frac{r^* r''\left(\frac{\pi}{2}\right)r'''\left(\frac{\pi}{2}\right)}{\left(r^* - r''\left(\frac{\pi}{2}\right)\right)^3} - \frac{r^* r'''\left(\frac{\pi}{2}\right)}{\left(r^* - r''\left(\frac{\pi}{2}\right)\right)^2} \\ &= -\frac{r^{*2} r'''\left(\frac{\pi}{2}\right)}{\left(r^* - r''\left(\frac{\pi}{2}\right)\right)^3} = r'''(0) = \widetilde{r}'''(0^+).\end{aligned}$$

As a result, $\widetilde{r} \in C^{3,\alpha}(\mathbb{R})$ and the proof of the proposition is complete. □

We observe that Proposition 2.3 is a consequence of Propositions 2.15. and 2.17.

Remark 2.18. We point out that it is easy to construct norms which are smooth and satisfy (2.1.2) as small perturbations of those of the form (16). For instance, fix any $\psi \in C^\infty([0, \pi/2])$ having support compactly contained in $(0, \pi/2)$. Then, for $\varepsilon > 0$ define

$$r_\psi(\theta) := 1 + \varepsilon\psi(\theta),$$

for any $\theta \in [0, \pi/2]$. Observe that conditions (2.5.4) and (2.5.25) are satisfied with $r^* = 1$. Moreover, we compute

$$\begin{aligned} r_\psi r_\psi'' - 2r_\psi'^2 - r_\psi^2 &= \varepsilon^2(1+\varepsilon\psi)\psi'' - 2\varepsilon^2\psi'^2 - (1+\varepsilon\psi)^2 \\ &= -1 + \varepsilon\left(-2\psi + \varepsilon\left((1+\varepsilon\psi)\psi'' - 2\psi'^2 - \psi^2\right)\right) \\ &\leqslant -1 + c_\psi\varepsilon, \end{aligned}$$

with c_ψ dependent on the C^2 norm of ψ. Therefore, if we take ε small enough, then r_ψ satisfies (2.5.24) and, by virtue of Proposition 2.17 the associated norm H_ψ is as desired.

Appendix A

The Wulff shape: a physical interpretation

The convex anisotropy H we dealt with all along the previous two chapters is widely considered in the literature. In particular, the Wulff shape associated H is per se the focal point of many studies. Recall that the Wulff shape W_H of H is the 1-sublevel set of the dual function H^* defined in (11), that is

$$W_H := \{x \in \mathbb{R}^n : H^*(x) < 1\}. \tag{A.1}$$

As shown by the classical Wulff theorem (see e.g. Theorem 1.1 in [T78]), W_H is the set which minimizes the anisotropic interfacial energy

$$\Omega \longmapsto \int_{\partial\Omega} H(\nu(x))\, d\mathcal{H}^{n-1}(x),$$

between all sets Ω having the same prescribed volume.

This property is frequently used, for instance, to deduce the equilibrium shape of a crystal, due to the anisotropic nature of the forces there involved. A less common application is described in [T78, Section 2], where the author addresses the problem of determining the closed path a trawler should follow in order to enclose a fixed amount of fish in the shortest time. Assuming the fish to be uniformly distributed in the sea and denoting by $H(\xi)$ the time the sailboat employs to travel, say, one mile in direction $\xi \in \partial B_1$, it is proved that the optimal path is given by following the frontier of a suitable dilation of the Wulff shape of H.

Next we present another physical interpretation of the Wulff shape, which arises quite naturally in a dynamical model related to our framework. Consider equations (7) and (1.1.4) in the case $B(t) = t^2/2$, with no forcing terms (i.e. when $F := 0$) and take the corresponding hyperbolic evolutionary equation

$$u_{tt} = \operatorname{div}\big(H(\nabla u)\nabla H(\nabla u)\big). \tag{A.2}$$

Notice that (A.2) is the classical wave equation when $H(\xi) := |\xi|$. Then, define u^ω to be a one-dimensional travelling wave of velocity $c_\omega > 0$, that is

$$u^\omega(x,t) := u_0(\omega \cdot x - c_\omega t),$$

with u_0 smooth and increasing for simplicity. Using the homogeneity properties of H (e.g. (1.2.1) and (1.2.2)), we see that, if u_0 is not affine, then it is a solution of (A.2) if and only if

$$c_\omega = H(\omega).$$

In this setting, the points reached by the plane wave u^ω in a unit of time form exactly the set $\{x \in \mathbb{R}^n : \omega \cdot x \leqslant H(\omega)\}$. By taking all the possible directions $\omega \in \partial B_1$ we obtain

$$\bigcap_{\omega \in \partial B_1} \{x \in \mathbb{R}^n : \omega \cdot x \leqslant H(\omega)\},$$

which is the Wulff shape of the velocity function $\omega \mapsto c_\omega = H(\omega)$, as one can easily check recalling definition (A.1).

Part II

Existence and regularity results for solutions of nonlocal anisotropic equations

Introduction to nonlocal equations and presentation of the results

In the second part of the dissertation we focus on equations driven by integro-differential operators. Problems related to nonlocal equations arise in many areas of both pure and applied disciplines. In probability theory, stochastic processes with jumps and, in particular, Lévy processes are widely studied. Their main difference from continuous processes resides in the fact that they allow discrete movements and, therefore, discontinuous trajectories. Their analytical counterparts are precisely integro-differential equations.

Nonlocal operators are therefore very common in several applied disciplines, such as physics, biology, ecology and finance. We refer for instance to [S07, DNPV12, V14] for some detailed comments on the existent applications.

The integral operators that we consider are (formally) defined by

$$\begin{aligned} L_K u(x) &:= \mathrm{P.V.} \int_{\mathbb{R}^n} (u(y) - u(x))\, K(y,x)\, dy \\ &= \lim_{\varepsilon \to 0^+} \int_{\mathbb{R}^n \setminus B_\varepsilon(x)} (u(y) - u(x))\, K(y,x)\, dy. \end{aligned} \tag{24}$$

Note that the symbol P.V. stands for the Cauchy principal value and it is defined by the limit appearing on the second line. The kernel $K : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ is a non-negative measurable function that will be usually required to be symmetric, i.e.

$$K(x,y) = K(y,x) \quad \text{for a.a. } x, y \in \mathbb{R}^n, \tag{25}$$

and be subjected to a condition of the type

$$K(x,y) \sim |x-y|^{-n-2s} \quad \text{for a.a. } x, y \in \mathbb{R}^n, \tag{26}$$

for some $s \in (0,1)$.

Observe that L_K is a linear operator. Moreover, L_K is nonlocal, in the sense that the value of $L_K u$ at $x \in \mathbb{R}^n$ depends on the behaviour of the function u at points far from x. This is in sharp contrast with standard operators that lead to PDEs, such as the Laplacian.

The simplest kernel K that we take into consideration is precisely that given by the choice $K(x,y) = |x-y|^{-n-2s}$. In this case, L_K reduces to

$$-(-\Delta)^s u(x) := \mathrm{P.V.} \int_{\mathbb{R}^n} \frac{u(y) - u(x)}{|y-x|^{n+2s}}\, dy,$$

which is, up to a multiplicative constant, typically regarded as the *fractional Laplacian*. We refer to [DNPV12, CS07, B15] and the classical [L72] for more informations on $(-\Delta)^s$. Other than linear, this operator is translation-invariant, homogeneous (of degree $2s$) and rotationally symmetric. For these reasons, the fractional Laplacian is the prototypical example of an operator of the form (24) and the one which is most frequently studied in the literature. However, the results presented here will almost always refer to frameworks dictated by more general choices.

In the following chapters we will describe results related to various problems involving nonlocal operators. In the first two Chapters 3 and 4 we focus on the regularity properties shared by the solutions of equations of the form

$$L_K u = f \quad \text{in } \Omega, \tag{27}$$

where Ω is a domain of $\mathbb{R}^n$, the right-hand side f is a measurable function and L_K is the integral operator defined in (24).

The regularity theory for integro-differential equations has been the object of a great variety of studies in recent years. Fundamental results in what concerns pointwise regularity were achieved by Caffarelli and Silvestre in [CS09, CS11]. The two authors developed there a theory for *viscosity* solutions, in order to deal with general fully non-linear equations. The framework we consider in this thesis is typically that of *weak* (or *energy*) solutions, which are more natural when dealing with variational problems. Note that these two notions of solutions - weak and viscosity ones - are indeed very close, as it is discussed in [R-OS14] and [SerV14].

The literature on the regularity theory for weak solutions is very developed and we do not aim to provide here an exhaustive account of the many contributions. Just to name a few, Kassmann addressed the validity of a Harnack inequality and established interior Hölder regularity for *nonlocal harmonic functions* through the language of Dirichlet forms (see [K07, K09, K11]). In [R-OS14] the authors obtained Hölder regularity up to the boundary for a Dirichlet problem driven by the fractional Laplacian. Concerning regularity results in Sobolev spaces, H^{2s} estimates are proved in [DK12] for entire translation invariant equations. Also, the very recent [KMS15] provides higher differentiability/integrability in a nonlinear setting quite similar to the one considered in Chapter 4.

In dependence of the properties enjoyed by f, a bounded solution u of the linear equation (27) exhibits different regularity features. Under the symmetry assumption (25) and suitable ellipticity/growth conditions in the spirit of (26), we show for instance that $f \in L^\infty(\Omega)$ implies $u \in C^\alpha_{\text{loc}}(\Omega)$, for some $\alpha > 0$. This is the starting point of Chapter 3, where, after a detailed introduction on the notions of solution undertaken and the required functional setting, we deal with several statements regarding the pointwise behaviour of the solutions of (27). We address interior Hölder regularity properties (of various orders) and the continuity up to the boundary of the solution of an appropriate Dirichlet problem associated to (27). At a later stage, we also extend such results to semilinear equations.

On the other hand, in Chapter 4 we restrict ourselves to consider a datum $f \in L^2(\Omega)$. With such little regularity on f, we can not expect in general to infer anything on the pointwise differentiability of u. Therefore, we investigate the regularity of u in Sobolev spaces. In parallel to the classical H^2 estimates for second-order equations, we show that the solutions u of (27) belong to the fractional Sobolev space $H^{2s-\varepsilon}_{\text{loc}}(\Omega)$, for any $\varepsilon > 0$ - at least, under some mild regularity assumptions on the kernel K.

The result just described is, to the best of our knowledge, new and somehow fills a gap in the already rich literature on the regularity theory for nonlocal equations. Conversely, the contents of Chapter 3 are for the most part not original. Far from being an exhaustive summary of the known achievements in the subject, we intend this chapter as a useful collection of the regularity results that will be needed later in the thesis.

In Chapters 5 and 6 we are concerned with a minimization problem related to phase transition phenomena.

We study entire minimal configurations for a total energy functional obtained by coupling a standard Gibbs-type free energy with a nonlocal penalization term modelled upon a Gagliardo-type seminorm. The novelty of our work mostly resides in the introduction of this last term, thanks to which we are able to encompass the presence of long-range interactions between the particles constituting the medium. In particular, our model is general enough to allow for anisotropic effects (possibly changing at different scales of distances, too) and both finite- and infinite-range interactions.

More precisely, we consider a nonlocal Ginzburg-Landau-type energy of the form

$$\mathscr{E}_K(u) := \frac{1}{4} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} |u(x) - u(y)|^2 K(x, y)\, dxdy + \int_{\mathbb{R}^n} W(x, u(x))\, dx, \tag{28}$$

where $W = W(x, r) : \mathbb{R}^n \times \mathbb{R} \to [0, +\infty)$ is a smooth double-well potential, with zeroes at $r = \pm 1$. We remark that the minimizers of $\mathscr{E}_K$ in the full space $\mathbb{R}^n$ satisfy the nonlocal semilinear equation

$$L_K u = W_u(\cdot, u) \quad \text{in } \mathbb{R}^n, \tag{29}$$

with L_K as in (24). Note that when K is the kernel of the fractional Laplacian and W does not depend on the space variable x, equation (29) becomes

$$-(-\Delta)^s u = W'(u) \quad \text{in } \mathbb{R}^n, \tag{30}$$

which is often credited as a nonlocal analogue of the so-called *Allen-Cahn equation* - the classical, local one being just (30) with $s = 1$, formally.

The study of the solutions of the Allen-Cahn equation has been a deep field of research in the last three decades, both in the local and nonlocal case. Indeed, since the Ginzburg-Landau functional can be viewed as a prototype for the modelling of phase transitions within the Van der Walls-Cahn-Hilliard theory, solutions of the elliptic Allen-Cahn equation represent stationary configurations in this theory.

In the local case, it is well known by the pioneering works of De Giorgi, Modica and Mortola ([MM77, DG79, M87]) that a deep connection between the minimizers of Ginzburg-Landau functionals and minimal surfaces exists. It is probably this relation that prompted De Giorgi to make his famous conjecture on the symmetry of monotone entire solutions of the Allen-Cahn equation, which eventually paved the way for years of research in nonlinear analysis. See [BCN97, GG98, AC00, S09, dPKW11] for important contributions in this direction.

In the nonlocal scenario, there are interesting variations of the above mentioned problems which have attracted the attention of many mathematicians in recent years. An exhaustive report on the various achievements is beyond the scopes of this work and we instead refer the reader to the surveys [FraV13, BV15]. Nevertheless, we just recall here some of the contributions that are more closely related to our results.

The relationship between the solutions of the fractional Allen-Cahn equation (30) and minimal surfaces (both the classical ones and an appropriate nonlocal version of them) is studied in [SV12]. On the other hand, a suitable fractional version of De Giorgi conjecture may be stated as follows.

Let u be a bounded entire solution of (30), with $\partial_{x_n} u > 0$ in $\mathbb{R}^n$.
Is it true that u must be one-dimensional,
i.e. that there is $e \in S^{n-1}$ and $u_0 : \mathbb{R} \to \mathbb{R}$ such that $u(x) = u_0(e \cdot x)$ for any $x \in \mathbb{R}^n$,
at least when the dimension n is *low*? How low?

A positive answer to this question has been given in [SV09, CS11] for $n = 2$ and in [CC10, CC14] for $n = 3$ and $s \geqslant 1/2$. We also report the very recent [HR-OSV15], where the authors addressed the validity of such statement in the framework of equation (29), for a class of truncated kernels.

A far more basilar issue in the fractional setting is even the existence itself of one-dimensional solutions. In fact, due to the lack of a satisfactory nonlocal ODE theory, this problem is not trivial at all. In the case of the fractional Laplacian, it has been solved in [CS-M05], for $s = 1/2$, and in the papers [PSV13, CS14, CS15], for a general $s \in (0, 1)$. We also cite [AB98, AB98b], where similar results have been obtained for a class of operators driven by rather general integrable kernels.

In Chapter 6 we address precisely the existence of one-dimensional solutions for equation (29), under the assumptions that W is independent of x and K is translation-invariant, i.e.

$$K(x, y) = \bar{K}(x - y) \quad \text{for a.a. } x, y \in \mathbb{R}^n,$$

for some measurable $\bar{K} : \mathbb{R}^n \to [0, +\infty)$.

To obtain this result, we follow the lines of the arguments developed in [PSV13] and suitably adjust them in relation to the changes in our framework. Note that we do not adopt the viewpoint of, say, [CS15], as this relies on the so-called Caffarelli-Silvestre extension ([CS07]), while [PSV13] does not. This powerful tool enables the interpretation of equations driven by the fractional Laplacian as more common local equations in divergence form. Unfortunately, such extension theory is not available for nonlocal operators L_K which differ from the fractional Laplacian. In view of the generality allowed by our setting, we therefore need to undertake a more direct and intrinsically nonlocal approach.

On the other hand, in Chapter 5 we confront ourselves with a space-dependent functional. That is, we do not require K to be translation-invariant and we allow W to depend on x. Under these assumptions, there is no reason to expect the existence of one-dimensional solutions. Thus, we address a slightly different problem.

We consider the case of a periodic medium, which is modelled by supposing that K and W enjoy a periodicity property, namely that

$$K(x + k, y + k) = K(x, y) \quad \text{and} \quad W(x + k, r) = W(x, r), \tag{31}$$

for a.a. $x, y \in \mathbb{R}^n$, $r \in \mathbb{R}$ and any $k \in \mathbb{Z}^n$.

In the framework of equation (29), the solution $u : \mathbb{R}^n \to [-1, 1]$ represents a state parameter in a model of phase coexistence (the two "pure phases" being represented by -1 and 1). The periodicity condition in (31) takes into account a possible geometric (or crystalline) structure of the medium in which the phase transition takes place. The presence of a fractional exponent $s \in (0, 1)$ is motivated by models which try to take into account long-range particle interactions (as a matter of fact, these models may produce either a local or nonlocal tension effect, depending on the value of s, see [SV12, SV14]).

The level sets of the solution u have particular physical importance, since they correspond, at a large scale, to the interface between the two phases of the system. The question that we address in Chapter 5 is to find solutions of (29) whose intermediate level sets - say, between levels $-9/10$ and $9/10$ - lie in a given strip of universal size. The direction of this strip will be arbitrary and the size of the strip is bounded independently on the direction.

In addition to this geometric constraint on the level sets of the solution, we will also prescribe an energy condition. Though the associated energy functional (28) diverges (i.e. nontrivial solutions have infinite total energy in the whole of the space), it is possible to "localize" the nonlocal energy density in any fixed domain of interest and require that the solution has a minimal property with respect to any perturbation supported in this domain.

The existence of minimal solutions of phase transition equations whose level sets are confined in a strip goes back to [V04], where the local analogue of equation (29) was taken into account, and it is strictly related to the construction, performed in [CdlL01], of minimal surfaces which stay at a bounded distance from a plane (see also [H32, AB06]). Note that these types of results may be seen as the analogue in partial differential equations (or pseudo-differential equations) of the classical Aubry-Mather theory for dynamical systems, see [M90].

More specifically, we point out that the main result of [CdlL01] can be obtained as a limiting case of the one contained in [V04], by scaling the minimizers constructed there in a favourable way. Unfortunately, in the nonlocal framework of Chapter 5 we are not able to carry out efficiently this limiting procedure. This is due to a subtle modification in our proof that does not behave well under rescalings. However, we believe that an appropriate adaptation of our technique may lead to the result for minimal surfaces - either the classical or fractional ones, in the spirit of [SV12].

Nonlocal (or fractional) minimal surfaces have been introduced in [CRS10] as natural candidates for the limit of the interfaces of minimizers to phase transition models with long range interactions. We remark that, for a given $s \in (0,1)$, the boundary of a measurable set $E \subset \mathbb{R}^n$ is said to be s-minimal in a fixed open $\Omega \subseteq \mathbb{R}^n$ if

$$H_s[E](x) = 0 \quad \text{for any } x \in \partial E \cap \Omega.$$

Here H_s denotes the fractional mean curvature operator defined by

$$H_s[E](x) := \text{P.V.} \int_{\mathbb{R}^n} \frac{\chi_E(y) - \chi_{\mathcal{C}E}(y)}{|x-y|^{n+s}}\, dy,$$

and can be interpreted as the first variation of a suitable nonlocal perimeter. We refer to [V13, AV14, L15] or [CRS10] itself for introductions to the subject.

In Chapter 7, we focus our attention on the operator H_s and investigate its behaviour under smooth diffeomorphisms. In particular, we obtain a quantitative estimate of the difference between the s-minimal curvature of a set E and that of a perturbation $\Psi(E)$ in terms of the size of the diffeomorphism Ψ that connects them.

The results gathered in the second part are the content of the four papers [C15b, CV15, CP15, C15], which are partly written in collaboration with Tommaso Passalacqua and Enrico Valdinoci.

Chapter 3

Regularity results: estimates in Hölder spaces

3.1 Outline of the chapter

In this chapter we address the differentiability properties shared by the weak solutions of the linear nonlocal equation

$$-L_K u = f \quad \text{in } \Omega, \tag{3.1.1}$$

and of the associated Dirichlet problem

$$\begin{cases} -L_K u = f & \text{in } \Omega \\ u = g & \text{in } \mathbb{R}^n \setminus \Omega, \end{cases} \tag{3.1.2}$$

where L_K is the integral operator introduced in (24), Ω is a domain of $\mathbb{R}^n$ and f, g are measurable functions. Then, we use such results to obtain some informations on the behaviour of the solutions of the semilinear equation

$$L_K u = F(\cdot, u) \quad \text{in } \Omega, \tag{3.1.3}$$

for some Carathéodory function F.

In dependence on how Ω, f and g are chosen, a solution u may enjoy different regularity properties. We do not aim to present here an exhaustive treatise on the regularity theory for (3.1.1)-(3.1.2) and we instead refer the interested reader to the various contributions available in the literature on the subject (see e.g. [S06, S07, CS09, CS11, BFV14, K09, K11, DK12, R-OS14, R-O15, S14, AFV15]). In fact, we strictly focus on the statements that will be used in the prosecution of the thesis.

The chapter is divided into three sections.

In the opening Section 3.2, we specify the concepts of solutions that are taken into consideration. To do this, we introduce in detail the needed functional spaces.

The remaining two sections are devoted to the statements and proofs of the various regularity results. In Sections 3.3 and 3.4 we respectively deal with the case of a space-dependent and translation-invariant operator L_K. Accordingly, each section contains the preparatory material needed in Chapters 5 and 6.

The most general requirements on L_K, and thence on K, are those used in Section 3.3. There we require $K : \mathbb{R}^n \times \mathbb{R}^n \to [0, +\infty]$ to be measurable, symmetric, i.e.

$$K(x, y) = K(y, x) \quad \text{for a.a. } x, y \in \mathbb{R}^n, \tag{3.1.4}$$

and satisfy the condition

$$\frac{\lambda \chi_{B_{r_0}}(x-y)}{|x-y|^{n+2s}} \leqslant K(x,y) \leqslant \frac{\Lambda}{|x-y|^{n+2s}} \quad \text{for a.a. } x,y \in \mathbb{R}^n, \tag{3.1.5}$$

for some $\Lambda \geqslant \lambda > 0$, $r_0 > 0$ and $s \in (0,1)$.

On the other hand, in Section 3.4 we will restrict ourselves to translation-invariant operators and therefore to kernels of the form

$$K(x,y) = \bar{K}(x-y) \quad \text{for a.a. } x,y \in \mathbb{R}^n,$$

for some measurable $\bar{K}$. With a slight abuse of notation we will write simply K in place of $\bar{K}$. Consequently, we will consider a kernel $K : \mathbb{R}^n \to [0,+\infty]$ which satisfies

$$K(z) = K(-z) \quad \text{for a.a. } z \in \mathbb{R}^n, \tag{3.1.6}$$

and

$$\frac{\lambda \chi_{B_{r_0}}(z)}{|z|^{n+2s}} \leqslant K(z) \leqslant \frac{\Lambda}{|z|^{n+2s}} \quad \text{for a.a. } z \in \mathbb{R}^n. \tag{3.1.7}$$

Notice that conditions (3.1.5) and (3.1.7) prescribe a growth and ellipticity condition for K in terms of the kernel of the fractional Laplacian. While the growth requirement is asked at any scale, the ellipticity is ensured only in a neighbourhood of the origin. This indeed allows for a great generality of non-negative kernels, possibly non-homogeneous and truncated at infinity. However, for some technical purposes, in Section 3.4 we will sometimes need the stronger condition

$$\frac{\lambda}{|z|^{n+2s}} \leqslant K(z) \leqslant \frac{\Lambda}{|z|^{n+2s}} \quad \text{for a.a. } z \in \mathbb{R}^n, \tag{3.1.8}$$

according to which K is fully comparable with the kernel of the fractional Laplacian.

As a conclusive remark, we point out that a translation-invariant operator L_K, with K satisfying (3.1.6), may be written in other equivalent forms, such as

$$L_K u(x) = \frac{1}{2} \int_{\mathbb{R}^n} \left(u(x+z) + u(x-z) - 2u(x) \right) K(z)\, dz, \tag{3.1.9}$$

or

$$L_K u(x) = \int_{\mathbb{R}^n} \big(u(x+z) - u(x) - \chi_{B_{r_1}}(z) \nabla u(x) \cdot z \big) K(z)\, dz,$$

for any $r_1 > 0$. By doing this, we are able to represent L_K as a non-singular integral, a fact that simplifies many computations.

We are now ready to state the main definitions that will be needed in this chapter and in the subsequent ones.

3.2 Basic definitions

We begin by specifying the notions of solutions that will be adopted throughout the second part of the dissertation. To do this, we first need to introduce the less known functional spaces involved in our definitions. The kernel K is supposed here to satisfy the general hypotheses (3.1.4) and (3.1.5), when not differently stated.

Given any domain $\Omega \subseteq \mathbb{R}^n$, we consider the linear space

$$\mathbb{H}^K(\Omega) := \Big\{ u : \mathbb{R}^n \to \mathbb{R} \text{ measurable} : u|_\Omega \in L^2(\Omega) \text{ and } [u]_{\mathbb{H}^K(\Omega)} < +\infty \Big\},$$

where

$$[u]^2_{\mathbb{H}^K(\Omega)} := \frac{1}{2} \iint_{\mathscr{C}_\Omega} |u(x) - u(y)|^2 \, K(x,y) \, dxdy,$$

and

$$\mathscr{C}_\Omega := (\mathbb{R}^n \times \mathbb{R}^n) \setminus ((\mathbb{R}^n \setminus \Omega) \times (\mathbb{R}^n \setminus \Omega)) \subseteq \mathbb{R}^n \times \mathbb{R}^n. \tag{3.2.1}$$

We point out that

$$\|u\|_{\mathbb{H}^K(\Omega)} := \|u\|_{L^2(\Omega)} + [u]_{\mathbb{H}^K(\Omega)},$$

is a norm for the space $\mathbb{H}^K(\Omega)$, as K is positive near the origin, by (3.1.5). Moreover, when K fulfills the stronger condition

$$\frac{\lambda}{|x-y|^{n+2s}} \leqslant K(x,y) \leqslant \frac{\Lambda}{|x-y|^{n+2s}} \quad \text{for a.a. } x, y \in \mathbb{R}^n, \tag{3.2.2}$$

then $\mathbb{H}^K(\Omega)$ is the same as $\mathbb{H}^s(\Omega)$ - which is just $\mathbb{H}^k(\Omega)$, with $k(z) = |z|^{-n-2s}$ - considered in [SerV14], with equivalent norms. Note that $\mathbb{H}^K(\Omega)$ differs from the usual fractional Sobolev space $H^s(\Omega)$ in that the latter does not make any restrictions on the behaviour of its elements outside of Ω. It holds in fact $H^s(\mathbb{R}^n) = \mathbb{H}^s(\mathbb{R}^n) \subseteq \mathbb{H}^K(\mathbb{R}^n) \subset \mathbb{H}^K(\Omega) \subset H^s(\Omega)$. Furthermore, we set

$$\begin{aligned} \mathbb{H}^K_0(\Omega) &:= \left\{ u \in \mathbb{H}^K(\Omega) : u = 0 \text{ a.e. in } \mathbb{R}^n \setminus \Omega \right\} \\ &= \left\{ u \in \mathbb{H}^K(\mathbb{R}^n) : u = 0 \text{ a.e. in } \mathbb{R}^n \setminus \Omega \right\}. \end{aligned}$$

Remark 3.1. For a general kernel K satisfying (3.1.4) and (3.1.5), it actually holds

$$\mathbb{H}^K_0(\Omega) = \mathbb{H}^s_0(\Omega), \tag{3.2.3}$$

with equivalent norms, provided Ω is bounded. Here, $\mathbb{H}^s_0(\Omega)$ clearly denotes the subspace of $\mathbb{H}^s(\Omega)$ composed by the functions vanishing a.e. outside of Ω.

Notice that, if (3.2.2) is in force, then (3.2.3) is straightforward. Although not as obvious, the more general assumption (3.1.5) is still strong enough to imply (3.2.3). Indeed, while (3.1.5) ensures that K and the kernel of the fractional Laplacian are fully comparable only in a neighbourhood of the origin, both these two kernels are integrable at infinity. This and the fact that the functions in $\mathbb{H}^K_0(\Omega)$ and $\mathbb{H}^s_0(\Omega)$ are required to vanish outside of Ω (the fact that Ω has finite measure is of key importance, here) seem to hint at the validity of (3.2.3). Below is a rigorous justification of this quick insight.

First, observe that, by the right-hand inequality in (3.1.5), it clearly holds $\mathbb{H}^s_0(\Omega) \subseteq \mathbb{H}^K_0(\Omega)$, with the appropriate inequality for the respective norms. On the other hand, we claim that

$$[u]_{\mathbb{H}^s(\Omega)} \leqslant c \|u\|_{\mathbb{H}^K(\Omega)} \quad \text{for any } u \in \mathbb{H}^K_0(\Omega), \tag{3.2.4}$$

for some constant $c > 0$ depending only on n, s, λ, r_0 and $|\Omega|$. Note that, in view of (3.2.4), equivalence (3.2.3) would then follow. Thus, we only need to check (3.2.4). By using the left-hand side of (3.1.5), Young's inequality and the fact that $u = 0$ a.e. in $\mathbb{R}^n \setminus \Omega$, we compute[2]

$$\begin{aligned}[u]^2_{\mathbb{H}^s(\Omega)} &= \frac{1}{2}\int_{\mathbb{R}^n}\left(\int_{B_{r_0}(x)}\frac{|u(x)-u(y)|^2}{|x-y|^{n+2s}}\,dy+\int_{\mathbb{R}^n\setminus B_{r_0}(x)}\frac{|u(x)-u(y)|^2}{|x-y|^{n+2s}}\,dy\right)dx\\
&\leqslant \frac{1}{\lambda}[u]^2_{\mathbb{H}^K(\Omega)}+\int_\Omega\left(\int_{\mathbb{R}^n\setminus B_{r_0}(x)}\frac{2|u(x)|^2}{|x-y|^{n+2s}}\,dy+\int_{\Omega\setminus B_{r_0}(x)}\frac{2|u(y)|^2}{|x-y|^{n+2s}}\,dy\right)dx\\
&\leqslant \frac{1}{\lambda}[u]^2_{\mathbb{H}^K(\Omega)}+\frac{2}{r_0^{2s}}\left(n\alpha_n+\frac{|\Omega|}{r_0^n}\right)\|u\|^2_{L^2(\Omega)},\end{aligned}$$

which is (3.2.4).

As a consequence of Remark 3.1, we have that the map

$$\mathbb{H}^K_0(\Omega)\times\mathbb{H}^K_0(\Omega)\ni(u,v)\longmapsto\langle u,v\rangle_{L^2(\Omega)}+\mathcal{D}_K(u,v),$$

with

$$\mathcal{D}_K(u,v):=\frac{1}{2}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}(u(x)-u(y))\,(v(x)-v(y))\,K(x-y)\,dxdy, \tag{3.2.5}$$

is a Hilbert space inner product for $\mathbb{H}^K_0(\Omega)$, when Ω is bounded (see e.g. [SerV12, Lemma 7] or [FKV15, Lemma 2.3]). Moreover, if Ω also has continuous boundary and K is translation-invariant, then

$$\mathbb{H}^K_0(\Omega)=\overline{C^\infty_0(\Omega)}^{\|\cdot\|_{\mathbb{H}^K(\Omega)}}, \tag{3.2.6}$$

as shown in [FSV15]. We refer to [DNPV12, SerV12, SerV13, FKV15], to name a few, for additional informations on the above defined spaces and further generalizations.

Throughout the chapter we will almost always consider bounded solutions to (3.1.1). However, for some purposes it is useful to take into consideration a larger class of functions. To this aim, we introduce the weighted Lebesgue space

$$L^1_w(\mathbb{R}^n):=\Big\{u:\mathbb{R}^n\to\mathbb{R}\text{ measurable }:\|u\|_{L^1_w(\mathbb{R}^n)}<+\infty\Big\},$$

where

$$\|u\|_{L^1_w(\mathbb{R}^n)}:=\int_{\mathbb{R}^n}|u(x)|w(x)\,dx,$$

and the weight $w:\mathbb{R}^n\to[0,+\infty)$ is a measurable function. In what follows we always consider weights of the form

$$w_{x_0,\beta}(x)=\frac{1}{1+|x-x_0|^{n+\beta}}, \tag{3.2.7}$$

for $x_0\in\mathbb{R}^n$ and $\beta>0$. We denote the corresponding spaces just with $L^1_{x_0,\beta}(\mathbb{R}^n)$ and we adopt the same notation for their norms. Also, we simply write $L^1_\beta(\mathbb{R}^n)$ when x_0 is the origin. Notice that, in fact, the space $L^1_{x_0,\beta}(\mathbb{R}^n)$ does not depend on x_0 and different choices for the base point x_0 lead to equivalent norms. Lastly, we observe that, in consequence of the fact that $w_{x_0,\beta}\in L^1(\mathbb{R}^n)\cap L^\infty(\mathbb{R}^n)$, the space $L^1_\beta(\mathbb{R}^n)$ contains both $L^\infty(\mathbb{R}^n)$ and $L^1(\mathbb{R}^n)$.

[2]Throughout this and the following chapters, the symbol α_n is used to denote the volume of the unit ball of $\mathbb{R}^n$. That is,

$$\alpha_n:=|B_1|=\frac{\pi^{n/2}}{\Gamma((n+2)/2)}.$$

Accordingly, the $(n-1)$-dimensional Hausdorff measure of the sphere ∂B_1 is then given by $\mathcal{H}^{n-1}(\partial B_1)=n\alpha_n$.

With all this in hand, we may now head to the definitions of weak solutions of (3.1.1) and (3.1.2).

Let Ω be a bounded, Lipschitz domain of $\mathbb{R}^n$ and $f \in L^2(\Omega)$. We say that $u \in \mathbb{H}^K(\Omega)$ is a *weak solution of equation* (3.1.1) in Ω if

$$\mathcal{D}_K(u,\varphi) = \langle f, \varphi\rangle_{L^2(\Omega)} \quad \text{for any } \varphi \in \mathbb{H}^K_0(\Omega). \tag{3.2.8}$$

First, notice that the left-hand side of (3.2.8) is well-defined and finite, as can be seen by inspecting (3.2.5). Also, in view of (3.2.6), definition (3.2.8) may be relaxed by requiring it to hold for any $\varphi \in C_0^\infty(\Omega)$ only, without altering its meaning.

Similarly, $u \in \mathbb{H}^K(\Omega)$ is a *supersolution* of the equation (3.1.1) if

$$\mathcal{D}_K(u,\varphi) \geqslant \langle f, \varphi\rangle_{L^2(\mathbb{R}^n)} \quad \text{for any non-negative } \varphi \in \mathbb{H}^K_0(\Omega). \tag{3.2.9}$$

Analogously, one defines *subsolutions* of (3.1.1) by reverting the inequality in (3.2.9). It is almost immediate to check that a function u is a solution of (3.1.1) if and only if it is at the same time a super- and a subsolution.

On the other hand, given another function $g \in \mathbb{H}^K(\Omega)$, we say that $u \in \mathbb{H}^K(\Omega)$ is a *weak solution of the Dirichlet problem* (3.1.2) if $u - g \in \mathbb{H}^K_0(\Omega)$ and u weakly solves (3.1.1).

When Ω is not bounded, we may consider a generalized concept of weak solutions of (3.1.1). In this case, u is said to be a weak solution of (3.1.1) in Ω if, for any Lipschitz subdomain $\Omega' \subset\subset \Omega$, the function u belongs to $\mathbb{H}^K(\Omega')$ and weakly solves (3.1.1) in Ω'.

When the functions u, f and g have more regularity, we may of course strengthen the notion of solution under consideration. Indeed, when $u \in L^1_{2s}(\mathbb{R}^n) \cap C^{2s+\gamma}_{\mathrm{loc}}(\Omega)$, for some $\gamma > 0$, and f is, say, continuous in Ω, then u is a *pointwise solution* or, simply, a *solution* of (3.1.1) if the equation is satisfied at any point $x \in \Omega$. Similarly, if also $g \in C^{2s+\gamma}(\mathbb{R}^n \setminus \Omega)$, then u is a solution of (3.1.2) in Ω if (3.1.1) is satisfied in the pointwise sense in Ω and $u \equiv g$ outside of Ω.

It is immediate to see that $L_K u(x)$ is well-defined at any point $x \in \Omega$, when $u \in L^1_{2s}(\mathbb{R}^n) \cap C^{2s+\gamma}_{\mathrm{loc}}(\Omega)$. Also, it is not hard to check that if u is a weak solution of (3.1.1) and has such regularity, then the equation is also satisfied in the pointwise sense.

3.3 Space-dependent operators

Here we consider the case of space-dependent operators driven by kernels satisfying (3.1.4)-(3.1.5). For simplicity, we suppose $r_0 = 1$ in (3.1.5). In the main result of the section we show that, under these assumptions, the bounded solutions of (3.1.1) are Hölder continuous functions. The precise statement is as follows.

Theorem 3.2. *Let Ω be a bounded open set of $\mathbb{R}^n$, with $n \geqslant 2$, and $s_0 \in (0, 1/2)$ be a fixed parameter. Let $s \in [s_0, 1 - s_0]$ and K be a measurable kernel satisfying* (3.1.4) *and* (3.1.5)*, with $r_0 = 1$. If $f \in L^\infty(\Omega)$ and $u \in \mathbb{H}^K(\Omega) \cap L^\infty(\mathbb{R}^n)$ is a solution of* (3.1.1) *in Ω, then there exists an exponent $\alpha \in (0,1)$, only depending on n, s_0, λ and Λ, such that*

$$u \in C^{0,\alpha}_{\mathrm{loc}}(\Omega).$$

In particular, there exists a number $R_0 > 0$, depending only on n, s_0, λ and Λ, such that, for any point $x_0 \in \Omega$ and any radius $0 < R \leqslant R_0$ for which $B_R(x_0) \subset \Omega$, it holds

$$\operatorname*{osc}_{B_r(x_0)} u \leqslant 16 \left(\frac{r}{R}\right)^\alpha \Big[\|u\|_{L^\infty(\mathbb{R}^n)} + \|f\|_{L^\infty(B_R(x_0))} \Big], \tag{3.3.1}$$

for any $0 < r < R$.

Theorem 3.2 is an extension to nonlocal equations of the classical De Giorgi-Nash-Moser regularity theory. In recent years a great number of papers dealt with interior Hölder estimates for solutions of elliptic integro-differential equations, as for instance [S06, CS09, K09] and [K11]. However, since we have not been able to find a satisfactory reference for Theorem 3.2 in our exact setting, we provide here all the details of its proof.

Before advancing to the arguments that lead to Theorem 3.2, we highlight the fact that the regularity of the solutions of the semilinear equation (3.1.3) can be recovered from it.

Corollary 3.3. *Let u be a bounded open set of $\mathbb{R}^n$, with $n \geqslant 2$, and $s_0 \in (0, 1/2)$. Let $s \in [s_0, 1-s_0]$ and K be a measurable kernel satisfying (3.1.4) and (3.1.5), with $r_0 = 1$. Let $F : \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}$ be a Carathéodory function and $u \in \mathbb{H}^K(\Omega) \cap L^\infty(\mathbb{R}^n)$ be a solution of (3.1.3). Then, $u \in C^{0,\alpha}_{\mathrm{loc}}(\Omega)$, for some $\alpha \in (0,1)$. The exponent α only depends on n, s_0, λ and Λ, while the $C^{0,\alpha}$ norm of u on any $\Omega' \subset\subset \Omega$ may also depend on $\|u\|_{L^\infty(\mathbb{R}^n)}$, $\|F(\cdot, u)\|_{L^\infty(\Omega)}$ and* $\operatorname{dist}(\Omega', \partial\Omega)$.

The remaining part of the section is devoted to the proof of Theorem 3.2, which is based on the Moser's iteration technique and some arguments in [K09, K11].

We begin with a lemma dealing with non-negative supersolutions of (3.1.1).

Lemma 3.4. *Let $f \in L^\infty(B_1)$ and $u \in \mathbb{H}^K(B_1)$ be a non-negative supersolution of (3.1.1) in B_1. Suppose that*

$$u(x) \geqslant \|f\|_{L^\infty(B_1)} + \delta \quad \textit{for a.a. } x \in B_1, \tag{3.3.2}$$

for some $\delta > 0$. Then,

$$\left(\fint_{B_{1/2}} u(x)^{p_\star} \, dx \right)^{1/p_\star} \leqslant C_\star \left(\fint_{B_{1/2}} u(x)^{-p_\star} \, dx \right)^{-1/p_\star}, \tag{3.3.3}$$

for some constant $C_\star > 0$ and exponent $p_\star \in (0,1)$ which depend only on n, s_0, λ and Λ.

Proof. We plan to show that $\log u \in BMO(B_{1/2})$. To this aim, we claim that there exists a constant $c_1 > 0$, depending only on n, s_0, λ and Λ, such that

$$[\log u]_{H^s(B_r(z))} \leqslant c_1 r^{-s+n/2}, \tag{3.3.4}$$

holds true for any $z \in B_{1/2}$ and $r > 0$ for which $B_r(z) \subseteq B_{1/2}$.

In order to prove (3.3.4), we take a cut-off function $\eta \in C^\infty_c(\mathbb{R}^n)$ satisfying $0 \leqslant \eta \leqslant 1$ in $\mathbb{R}^n$, $\operatorname{supp}(\eta) = B_{3r/2}(z)$, $\eta = 1$ in $B_r(z)$ and $|\nabla \eta| \leqslant 4r^{-1}$ in $\mathbb{R}^n$. We test formulation (3.2.9) with $\varphi := \eta^2 u^{-1}$. Note that $\varphi \geqslant 0$ and $\varphi \in X_0(B_1)$ thanks to the definition of η and condition (3.3.2). Recalling (3.1.4), inequality (3.2.9) becomes

$$\begin{aligned}
&\int_{B_{3r/2}(z)} \frac{f(x)\eta^2(x)}{u(x)} \, dx \\
&\qquad \leqslant \int_{B_{2r}(z)} \int_{B_{2r}(z)} (u(x) - u(y)) \left(\frac{\eta^2(x)}{u(x)} - \frac{\eta^2(y)}{u(y)} \right) K(x,y) \, dx dy \\
&\qquad\quad + 2 \int_{B_{2r}(z)} \frac{\eta^2(y)}{u(y)} \left(\int_{\mathbb{R}^n \setminus B_{2r}(z)} (u(y) - u(x)) \, K(x,y) \, dx \right) dy \\
&\qquad =: I_1 + 2 I_2.
\end{aligned} \tag{3.3.5}$$

For any $x, y \in B_{2r}(z)$ we compute

$$\begin{aligned}
(u(x) - u(y)) \left(\frac{\eta^2(x)}{u(x)} - \frac{\eta^2(y)}{u(y)} \right) &= \eta^2(x) + \eta^2(y) - \frac{\eta^2(x) u(y)}{u(x)} - \frac{\eta^2(y) u(x)}{u(y)} \\
&= |\eta(x) - \eta(y)|^2 - \frac{|\eta(x) u(x) - \eta(y) u(y)|^2}{u(x) u(y)}.
\end{aligned}$$

Hence, using (3.1.5) together with the numerical inequality

$$(\log a - \log b)^2 \leqslant \frac{(a-b)^2}{ab},$$

that holds for any $a, b > 0$, we get

$$\begin{aligned} I_1 &= \int_{B_{2r}(z)} \int_{B_{2r}(z)} \left[|\eta(x) - \eta(y)|^2 - \frac{|\eta(x)u(x) - \eta(y)u(y)|^2}{u(x)u(y)} \right] K(x,y)\, dxdy \\ &\leqslant \frac{16\Lambda}{r^2} \int_{B_{2r}(z)} \int_{B_{2r}(z)} \frac{dxdy}{|x-y|^{n-2+2s}} - \lambda \int_{B_r(z)} \int_{B_r(z)} \frac{|u(x)-u(y)|^2}{u(x)u(y)} \frac{dxdy}{|x-y|^{n+2s}} \\ &\leqslant 2^{n+4} n \alpha_n^2 \Lambda r^{n-2} \int_0^{4r} \rho^{1-2s}\, d\rho - \lambda \int_{B_r(z)} \int_{B_r(z)} \frac{|\log u(x) - \log u(y)|^2}{|x-y|^{n+2s}}\, dxdy \\ &\leqslant \frac{2^{n+7} n \alpha_n^2 \Lambda}{s_0} r^{n-2s} - \lambda [\log u]^2_{H^s(B_r(z))}. \end{aligned} \tag{3.3.6}$$

On the other hand, by the non-negativity of u and again (3.1.5) we estimate

$$\begin{aligned} I_2 &= \int_{B_{3r/2}(z)} \frac{\eta^2(y)}{u(y)} \left(\int_{\mathbb{R}^n \setminus B_{2r}(z)} (u(y) - u(x))\, K(x,y)\, dx \right) dy \\ &\leqslant \Lambda \int_{B_{3r/2}(z)} \eta^2(y) \left(\int_{\mathbb{R}^n \setminus B_{2r}(z)} |x-y|^{-n-2s}\, dx \right) dy \\ &\leqslant \frac{2^{3n+1} n \alpha_n^2 \Lambda}{s_0} r^{n-2s}. \end{aligned} \tag{3.3.7}$$

Finally, using (3.3.2) we have

$$\int_{B_{3r/2}(z)} \frac{f(x)\eta^2(x)}{u(x)}\, dx \geqslant - \int_{B_{3r/2}(z)} \frac{|f(x)|}{u(x)}\, dx \geqslant - \frac{\|f\|_{L^\infty(B_1)} |B_{3r/2}|}{\|f\|_{L^\infty(B_1)} + \delta} \geqslant -2^n \alpha_n r^{n-2s},$$

since $r < 1$. Claim (3.3.4) then follows by combining this last equation with (3.3.5), (3.3.6) and (3.3.7).

We are now ready to show that $\log u \in BMO(B_{1/2})$. For a bounded $\Omega \subset \mathbb{R}^n$ and $v \in L^1(\Omega)$, write

$$(v)_\Omega := \frac{1}{|\Omega|} \int_\Omega v(x)\, dx.$$

Applying both Hölder's and fractional Poincaré's inequality, from (3.3.4) we obtain

$$\begin{aligned} \| \log u - (\log u)_{B_r(z)} \|_{L^1(B_r(z))} &\leqslant |B_r|^{1/2} \| \log u - (\log u)_{B_r(z)} \|_{L^2(B_r(z))} \\ &\leqslant c_2 r^{s+n/2} [\log u]_{H^s(B_r(z))} \\ &\leqslant c_3 r^n, \end{aligned}$$

for some $c_2, c_3 > 0$ which may depend on n, s_0, λ and Λ. Since the above inequality holds for any $B_r(z) \subseteq B_{1/2}$, we conclude that $\log u \in BMO(B_{1/2})$.

Estimate (3.3.3) then follows by the John-Nirenberg embedding in one of its equivalent forms (see, for instance, Theorem 6.25 of [GM12]). Observe that the exponent $p_\star$ given by such result is of the form of a dimensional constant divided by the $BMO(B_{1/2})$ semi-norm of $\log u$. This norm being bounded from above by c_3 and since we are free to make $p_\star$ smaller if necessary, it turns out that we can choose $p_\star \in (0,1)$ to depend only on n, s_0, λ and Λ. □

Next is the step of the proof in which the iterative argument really comes into play.

Lemma 3.5. *Let $f \in L^\infty(B_1)$ and $u \in \mathbb{H}^K(B_1)$ be a supersolution of (3.1.1) in B_1. Assume that u satisfies (3.3.2), for some $\delta > 0$. Then, for any $p_0 > 0$,*

$$\inf_{B_{1/4}} u \geqslant c_\sharp \left(\fint_{B_{1/2}} u(x)^{-p_0}\, dx \right)^{-1/p_0}, \tag{3.3.8}$$

for some constant $c_\sharp > 0$ which may depend on n, s_0, λ, Λ and p_0.

Proof. Fix $\theta \in (0,1)$. We claim that, for any $r \in (0, 1/2]$ and $p > 1$, it holds

$$\int_{B_{\theta r}} \int_{B_{\theta r}} \frac{\left| u(x)^{(-p+1)/2} - u(y)^{(-p+1)/2} \right|^2}{|x-y|^{n+2s}}\, dxdy \leqslant c_1 \frac{p^2}{(1-\theta)^2 r^{2s}} \int_{B_r} u(x)^{-p+1}\, dx, \tag{3.3.9}$$

for some constant $c_1 > 0$ depending on n, s_0, λ and Λ.

To prove (3.3.9), consider a cut-off $\eta \in C_c^\infty(\mathbb{R}^n)$ such that $0 \leqslant \eta \leqslant 1$ in $\mathbb{R}^n$, $\operatorname{supp}(\eta) = B_r$, $\eta = 1$ in $B_{\theta r}$ and $|\nabla \eta| \leqslant 2(1-\theta)^{-1} r^{-1}$ in $\mathbb{R}^n$, and plug $\varphi := \eta^{p+1} u^{-p}$ into (3.2.9). Inequality (3.3.9) then follows by arguing as in Lemma 3.5 of [K09] and noticing that, by (3.3.2),

$$\int_{B_r} \frac{f(x)\eta(x)^{p+1}}{u(x)^p}\, dx \geqslant -\int_{B_r} \frac{|f(x)| u(x)^{-p+1}}{u(x)}\, dx \geqslant -r^{-2s} \int_{B_r} u(x)^{-p+1}\, dx,$$

where we also used the fact that $r < 1$.

By using (3.3.9) in combination with the fractional Sobolev inequality, we then deduce

$$\left(\fint_{B_{\theta r}} u(x)^{\frac{n(-p+1)}{n-2s}}\, dx \right)^{(n-2s)/n} \leqslant c_2 \frac{p^2}{(1-\theta)^2 \theta^n} \fint_{B_r} u(x)^{-p+1}\, dx, \tag{3.3.10}$$

for some $c_2 \geqslant 1$ which depends only on n, s_0, λ and Λ.

We are now in position to run the iterative scheme, which is based on the fundamental estimate (3.3.10). For any $k \in \mathbb{N} \cup \{0\}$, define

$$r_k := \frac{1 + 2^{-k}}{4}, \quad p_k := \left(\frac{n}{n-2s} \right)^k p_0 \quad \text{and} \quad \Phi_k := \left(\fint_{B_{r_k}} u(x)^{-p_k}\, dx \right)^{1/p_k},$$

so that

$$\theta_k := \frac{r_{k+1}}{r_k} = \frac{1 + 2^{-k-1}}{1 + 2^{-k}} \in \left[\frac{3}{4}, 1 \right).$$

We apply (3.3.10) with $r = r_k$, $\theta = \theta_k$ and $p = 1 + p_k$, to get

$$\Phi_{k+1} \leqslant q_k \Phi_k, \tag{3.3.11}$$

for any $k \in \mathbb{N} \cup \{0\}$, where

$$q_k := \left[c_2 \frac{(1+p_k)^2}{(1-\theta_k)^2 \theta_k^n} \right]^{1/p_k}.$$

From (3.3.11) it then follows that

$$\Phi_k \leqslant \Phi_0 \prod_{j=0}^{k-1} q_j. \tag{3.3.12}$$

Now we observe that

$$1-\theta_k = \frac{2^{-k}-2^{-k-1}}{1+2^{-k}} = \frac{1}{2^{k+1}+2} \geqslant \frac{1}{2^{k+2}}.$$

Therefore, recalling that $\theta_k \geqslant 3/4$,

$$\frac{1}{(1-\theta_k)^2\theta_k^n} \leqslant 2^{2(k+2)}\left(\frac{4}{3}\right)^n \leqslant 2^{2k+n+4},$$

and hence

$$\log q_k \leqslant \frac{1}{p_k}\log\left[c_2(1+p_k)^2 2^{2k+n+4}\right] \leqslant \frac{1}{p_k}\log\left[c_3\left(\frac{2n}{n-2s}\right)^{2k}\right] \leqslant c_4\left(\frac{n-2s_0}{n}\right)^k k,$$

for some $c_3, c_4 > 0$ that may also depend on p_0. This implies that the product of the q_j's converges, as $k \to +\infty$. Thence, (3.3.8) follows from (3.3.12), since

$$\liminf_{k\to+\infty} \Phi_k \geqslant \lim_{k\to+\infty} |B_{r_k}|^{-1/p_k}\|u^{-1}\|_{L^{p_k}(B_{1/4})} = \sup_{B_{1/4}} u^{-1} = \left(\inf_{B_{1/4}} u\right)^{-1}. \qquad \square$$

By putting together Lemmata 3.4 and 3.5, we easily obtain the following *weak Harnack inequality*.

Corollary 3.6. *Let $r \in (0,1]$ and $f \in L^\infty(B_r)$. Assume that $u \in \mathbb{H}^K(B_r)\cap L^\infty(\mathbb{R}^n)$ is a non-negative supersolution of (3.1.1) in B_r. Then,*

$$\inf_{B_{r/4}} u + r^{2s}\|f\|_{L^\infty(B_r)} \geqslant c_\star \left(\fint_{B_{r/2}} u(x)^{p_\star}\right)^{1/p_\star}, \tag{3.3.13}$$

for some $c_\star \in (0,1)$ depending only on n, s_0, λ and Λ.

Proof. Assume for the moment $r = 1$. Let then $\delta > 0$ be a small parameter and define $u_\delta := u + \|f\|_{L^\infty(B_1)} + \delta$. Note that u_δ is still a non-negative supersolution of (3.1.1) in B_1 and that it satisfies (3.3.2). Thus, we are free to apply Lemmata 3.4 and 3.5 to u_δ and obtain that

$$\inf_{B_{1/4}} u + \|f\|_{L^\infty(B_1)} + \delta \geqslant \frac{c_\sharp}{C_\star}\left(\fint_{B_{1/2}} u(x)^{p_\star}\,dx\right)^{1/p_\star}.$$

Letting $\delta \to 0^+$ we obtain (3.3.13) when $r = 1$. For a general radius $r \leqslant 1$ the result follows by a simple scaling argument. $\square$

With the aid of Corollary 3.6, we can prove the following proposition, which will be the fundamental step in the conclusive inductive argument. In the literature, results of this kind are often known as *growth lemmata*.

Proposition 3.7. *There exist $\gamma \in (0, 2s_0)$ and $\eta \in (0,1)$, depending only on n, s_0, λ and Λ, such that for any $r \in (0,1]$, $f \in L^\infty(B_r)$ and $u \in \mathbb{H}^K(B_r)\cap L^\infty(\mathbb{R}^n)$ supersolution of (3.1.1) in B_r, for which*

$$u(x) \geqslant 0 \quad \textit{for a.a. } x \in B_{2r}, \tag{3.3.14}$$

$$\left|\left\{x \in B_{r/2} : u(x) \geqslant 1\right\}\right| \geqslant \frac{1}{2}|B_{r/2}|, \tag{3.3.15}$$

and

$$u(x) \geqslant -2\left(8\frac{|x|}{2r}\right)^\gamma + 2 \quad \textit{for a.a. } x \in \mathbb{R}^n \setminus B_{2r}, \tag{3.3.16}$$

hold true, then

$$\inf_{B_{r/4}} u + r^{2s}\|f\|_{L^\infty(B_r)} \geqslant \eta. \tag{3.3.17}$$

Proof. Write $u = u_+ - u_-$. Using (3.1.4) and (3.3.14), it is easy to see that u_+ is a supersolution of

$$\mathcal{D}_K(u_+, \cdot) = \tilde{f} \quad \text{in } B_r,$$

where

$$\tilde{f}(x) := f(x) - 2 \int_{\mathbb{R}^n \setminus B_{2r}} u_-(y) K(x,y)\, dy.$$

Applying Corollary 3.6 we get that

$$\inf_{B_{r/4}} u_+ + r^{2s} \|\tilde{f}\|_{L^\infty(B_r)} \geqslant c_\star \left(\fint_{B_{r/2}} u_+(x)^{p_\star} \right)^{1/p_\star}.$$

Using then hypotheses (3.3.14) and (3.3.15), this yields

$$\begin{aligned} \inf_{B_{r/4}} u + r^{2s} \|\tilde{f}\|_{L^\infty(B_r)} &\geqslant c_\star \left(\fint_{B_{r/2} \cap \{u \geqslant 1\}} u(x)^{p_\star} \right)^{1/p_\star} \\ &\geqslant c_\star \left(\frac{\left| \{ x \in B_{r/2} : u(x) \geqslant 1 \} \right|}{|B_{r/2}|} \right)^{1/p_\star} \\ &\geqslant c_\star 2^{-1/p_\star} =: 2\eta. \end{aligned} \tag{3.3.18}$$

Now we turn our attention to the L^∞ norm of $\tilde{f}$. First, we notice that (3.3.16) implies that

$$u_-(x) \leqslant 2 \left(8 \frac{|x|}{2r} \right)^\gamma - 2 \quad \text{for a.a. } x \in \mathbb{R}^n \setminus B_{2r},$$

as the right hand side of (3.3.16) is negative. Moreover, given $x \in B_r$ and $y \in \mathbb{R}^n \setminus B_{2r}$, it holds

$$|y - x| \geqslant |y| - |x| \geqslant |y| - \frac{|y|}{2} = \frac{|y|}{2}.$$

Consequently, recalling (3.1.5) we compute

$$\begin{aligned} \int_{\mathbb{R}^n \setminus B_{2r}} u_-(y) K(x,y)\, dy &\leqslant \Lambda \int_{\mathbb{R}^n \setminus B_{2r}} \frac{2 \left(8 \frac{|y|}{2r} \right)^\gamma - 2}{|x - y|^{n+2s}}\, dy \\ &\leqslant 2^{n+2s+1} \Lambda \left[\left(\frac{4}{r} \right)^\gamma \int_{\mathbb{R}^n \setminus B_{2r}} |y|^{\gamma - n - 2s}\, dy - \int_{\mathbb{R}^n \setminus B_{2r}} |y|^{-n-2s}\, dy \right] \\ &= 2^{n+1} n \alpha_n \Lambda \left[\frac{8^\gamma}{2s - \gamma} - \frac{1}{2s} \right] r^{-2s}, \end{aligned}$$

if $\gamma < 2s_0$. Notice that the term in brackets on the last line of the above formula converges to 0 as $\gamma \to 0^+$, uniformly in $s \geqslant s_0$. Therefore, we can find $\gamma > 0$, in dependence of n, s_0, λ and Λ, such that

$$\|\tilde{f}\|_{L^\infty(B_r)} \leqslant \|f\|_{L^\infty(B_r)} + r^{-2s} \eta.$$

Inequality (3.3.17) then follows by combining this with (3.3.18). □

We are now ready to move to the actual

Proof of Theorem 3.2. We focus on the proof of (3.3.1), as the Hölder continuity of u inside Ω would then easily follow. Furthermore, we may assume without loss of generality x_0 to be the origin.

Set

$$R_0 := \left(\frac{\eta}{4}\right)^{\frac{1}{2s_0}} < 1, \tag{3.3.19}$$

with η as in Proposition 3.7, and take $R \in (0, R_0]$. We claim that there exist a constant $\alpha \in (0,1)$, depending only on n, s, λ and Λ, a non-decreasing sequence $\{m_j\}$ and a non-increasing sequence $\{M_j\}$ of real numbers such that for any $j \in \mathbb{N} \cup \{0\}$

$$\begin{aligned} & m_j \leqslant u(x) \leqslant M_j \quad \text{for a.a. } x \in B_{8^{-j}R}, \\ & M_j - m_j = 8^{-j\alpha} L, \end{aligned} \tag{3.3.20}$$

with

$$L := 2\|u\|_{L^\infty(\mathbb{R}^n)} + \|f\|_{L^\infty(B_R)}. \tag{3.3.21}$$

We prove this by induction. Set $m_0 := -\|u\|_{L^\infty(\mathbb{R}^n)}$ and $M_0 := \|u\|_{L^\infty(\mathbb{R}^n)} + \|f\|_{L^\infty(B_R)}$. With this choice, property (3.3.20) clearly holds true for $j = 0$. Then, for a fixed $k \in \mathbb{N}$, we assume to have constructed the two sequences $\{m_j\}$ and $\{M_j\}$ up to $j = k-1$ in such a way that (3.3.20) is satisfied and show that we can also build m_k and M_k. For any $x \in \mathbb{R}^n$, define

$$v(x) := \frac{2 \cdot 8^{(k-1)\alpha}}{L} \left(u(x) - \frac{M_{k-1} + m_{k-1}}{2} \right),$$

with

$$\alpha := \min \left\{ \gamma, \frac{\log\left(\frac{4}{4-\eta}\right)}{\log 8} \right\}, \tag{3.3.22}$$

and γ, η as in Proposition 3.7. Since u is a solution of (3.1.1) in Ω, we deduce that v satisfies

$$-L_K v = \frac{2 \cdot 8^{(k-1)\alpha}}{L} f \quad \text{in } B_{8^{-(k-1)}R}. \tag{3.3.23}$$

Moreover,

$$|v(x)| \leqslant 1 \quad \text{for a.a. } x \in B_{8^{-(k-1)}R}. \tag{3.3.24}$$

Letting instead $x \in \mathbb{R}^n \setminus B_{8^{-(k-1)}R}$, there exists a unique $\ell \in \mathbb{N}$ for which

$$8^{-(k-\ell)} R \leqslant |x| < 8^{-(k-\ell-1)} R.$$

Writing $m_{-j} := m_0$ and $M_{-j} := M_0$ for every $j \in \mathbb{N}$, we compute

$$\begin{aligned} v(x) &\leqslant \frac{2 \cdot 8^{(k-1)\alpha}}{L} \left(M_{k-\ell-1} - m_{k-\ell-1} + m_{k-\ell-1} - \frac{M_{k-1} + m_{k-1}}{2} \right) \\ &\leqslant \frac{2 \cdot 8^{(k-1)\alpha}}{L} \left(M_{k-\ell-1} - m_{k-\ell-1} - \frac{M_{k-1} - m_{k-1}}{2} \right) \\ &\leqslant \frac{2 \cdot 8^{(k-1)\alpha}}{L} \left(8^{-(k-\ell-1)\alpha} L - \frac{8^{-(k-1)\alpha} L}{2} \right) \\ &= 2 \cdot 8^{\ell\alpha} - 1 \\ &\leqslant 2 \left(8 \frac{|x|}{8^{-(k-1)}R} \right)^{\alpha} - 1, \end{aligned} \tag{3.3.25}$$

Analogously, one checks that

$$v(x) \geqslant -2 \left(8 \frac{|x|}{8^{-(k-1)}R} \right)^{\alpha} + 1, \tag{3.3.26}$$

for a.a. $x \in \mathbb{R}^n \setminus B_{8^{-(k-1)}R}$.

We distinguish between the two mutually exclusive possibilities

(a) $\left|\left\{x \in B_{8^{-(k-1)}R/4} : v(x) \leqslant 0\right\}\right| \geqslant \frac{1}{2}|B_{8^{-(k-1)}R/4}|$, and

(b) $\left|\left\{x \in B_{8^{-(k-1)}R/4} : v(x) \leqslant 0\right\}\right| < \frac{1}{2}|B_{8^{-(k-1)}R/4}|$.

In case (a), set $\tilde{u} := 1 - v$. From (3.3.23) we deduce in particular that

$$-L_K \tilde{u} = -\frac{2 \cdot 8^{(k-1)\alpha}}{L} f \quad \text{in } B_{8^{-(k-1)}R/2}.$$

In view of (3.3.24) and (3.3.25) we apply Proposition 3.7 to $\tilde{u}$, with $r = 8^{-(k-1)}R/2$, and obtain that

$$\inf_{B_{8^{-(k-1)}R/8}} \tilde{u} + \left(\frac{8^{-(k-1)}R}{2}\right)^{2s} \left\| -\frac{2 \cdot 8^{(k-1)\alpha}}{L} f \right\|_{L^\infty(B_{8^{-(k-1)}R/2})} \geqslant \eta,$$

from which, by (3.3.21) and (3.3.19), it follows

$$\begin{aligned}
\sup_{B_{8^{-k}R}} v &\leqslant 1 - \eta + \left(\frac{8^{-(k-1)}R}{2}\right)^{2s} \left\| -\frac{2 \cdot 8^{(k-1)\alpha}}{L} f \right\|_{L^\infty(B_{8^{-(k-1)}R/2})} \\
&\leqslant 1 - \eta + 2 \cdot 8^{-(2s_0-\alpha)(k-1)} R_0^{2s_0} \frac{\|f\|_{L^\infty(B_R)}}{L} \\
&\leqslant 1 - \frac{\eta}{2}.
\end{aligned}$$

Note that we took advantage of the fact that $\alpha \leqslant \gamma < 2s_0$, by (3.3.22). If we translate this estimate back to u, applying (3.3.22) once again we finally get

$$\begin{aligned}
\sup_{B_{8^{-k}R}} u &\leqslant \left(1 - \frac{\eta}{2}\right) \frac{L}{2 \cdot 8^{(k-1)\alpha}} + \frac{M_{k-1} + m_{k-1}}{2} \\
&= \left(1 - \frac{\eta}{2}\right) \frac{M_{k-1} - m_{k-1}}{2} + \frac{M_{k-1} + m_{k-1}}{2} \\
&= m_{k-1} + \left(\frac{4-\eta}{4}\right) (M_{k-1} - m_{k-1}) \\
&\leqslant m_{k-1} + 8^{-k\alpha} L.
\end{aligned}$$

Accordingly, (3.3.20) is satisfied by setting $m_k := m_{k-1}$ and $M_k := m_{k-1} + 8^{-k\alpha}L$.

If on the other hand (b) holds we define $\tilde{u} := 1 + v$. With a completely analogous argument using (3.3.26) in place of (3.3.25), we end up estimating

$$\inf_{B_{8^{-k}R}} u \geqslant M_{k-1} - 8^{-k\alpha}L,$$

so that (3.3.20) again follows with $m_k := M_{k-1} - 8^{-k\alpha}L$ and $M_k := M_{k-1}$.

The proof of the theorem is therefore complete, as the bound in (3.3.1) is an immediate consequence of claim (3.3.20). □

3.4 Translation-invariant operators

This section is devoted to several regularity results for equations and Dirichlet problems involving operators with kernels that fulfill conditions (3.1.6)-(3.1.7). We complete this task in three steps: first, we restrict ourselves to kernels which are globally comparable to that of the fractional Laplacian, i.e. those for which (3.1.8) is true; in the following subsection we then remove this unnecessary hypothesis, thus extending the theory to general kernels satisfying the weaker (3.1.7); finally, we apply the results obtained to semilinear equations as (3.1.3).

3.4.1 Linear equations: positive kernels

In this subsection we enclose all the results that pertain to the linear setting given by (3.1.1)-(3.1.2), under the assumption that K satisfies (3.1.6) and (3.1.7). As a first step, we present an interior a priori estimate for the solutions of equation (3.1.1).

Proposition 3.8 ([DK12]). *Assume that K satisfies (3.1.6) and (3.1.8). Let $f \in L^\infty(B_1)$ and $u \in L^1_{2s}(\mathbb{R}^n) \cap C^2_{\rm loc}(B_1)$ be a solution of (3.1.1) in B_1. Then, $u \in C^\alpha(B_{1/2})$ for any $\alpha \in (0, \min\{2s, 1\})$ and it holds*

$$[u]_{C^\alpha(B_{1/2})} \leqslant C \left(\|f\|_{L^\infty(B_1)} + \|u\|_{L^1_{2s}(\mathbb{R}^n)} \right), \tag{3.4.1}$$

for some constant $C > 0$ which depends only on n, s, λ, Λ and α.

After this preliminary observation, we plan to establish global estimates for the solutions of the Dirichlet problem (3.1.2). For kernels which fulfill the homogeneity condition

$$K(z) = \frac{a(z/|z|)}{|z|^{n+2s}} \quad \text{for a.a. } z \in \mathbb{R}^n, \text{ with } \lambda \leqslant a(\zeta) \leqslant \Lambda \text{ for a.a. } \zeta \in S^{n-1}, \tag{3.4.2}$$

and, actually, more general homogeneous fully nonlinear operators, the optimal $C^s(\overline{\Omega})$ regularity has been established in [R-OS15]. In contrast, when K only satisfies (3.1.8), there is no hope for such boundary regularity, as discussed again in [R-OS15, Subsection 2.3]. In the next results we check that it still holds some $C^\alpha(\overline{\Omega})$ regularity, for $\alpha < s$.

In conformity with e.g. [CS09, CS11, R-OS15], we denote by $\mathscr{L}_0 = \mathscr{L}_0(s, \lambda, \Lambda)$ the class of operators $L = L_K$ of the form (3.1.9), whose kernels are measurable functions $K : \mathbb{R}^n \to [0, +\infty]$ which satisfy (3.1.6) and (3.1.8). The so-called *extremal Pucci operators* for the class $\mathscr{L}_0$ are defined by

$$M^+ u(x) = M^+_{\mathscr{L}_0} u(x) := \sup_{L \in \mathscr{L}_0} Lu(x) \quad \text{and} \quad M^- u(x) = M^-_{\mathscr{L}_0} u(x) := \inf_{L \in \mathscr{L}_0} Lu(x),$$

For $\beta \in (0, 2s)$ and $\nu \in S^{n-1}$, we consider the function

$$\psi^\beta_\nu(x) := (\nu \cdot x)^\beta_+ ,$$

defined for any $x \in \mathbb{R}^n$.

Proposition 3.9 ([R-OS15]). *In correspondence to any $\beta \in (0, 2s)$, there exists two real constants $\overline{C}(\beta)$ and $\underline{C}(\beta)$, which depend on n, s, λ and Λ, besides β, such that*

$$\begin{aligned} M^+ \psi^\beta_\nu(x) &= \overline{C}(\beta)(\nu \cdot x)^{\beta - 2s} \ \textit{in} \ \{\nu \cdot x > 0\}, \\ M^- \psi^\beta_\nu(x) &= \underline{C}(\beta)(\nu \cdot x)^{\beta - 2s} \ \textit{in} \ \{\nu \cdot x > 0\}, \end{aligned}$$

for every $\nu \in S^{n-1}$.

The constants $\overline{C}$, $\underline{C}$, viewed as functions of β, are continuous in $(0, 2s)$. Moreover, there exists two unique values $0 < \beta_1 < s < \beta_2 < 2s$, which also depend on n, s, λ and Λ, for which

$$\overline{C}(\beta_1) = 0 = \underline{C}(\beta_2),$$

and

$$\begin{aligned} \operatorname{sgn} \overline{C}(\beta) &= \operatorname{sgn}(\beta - \beta_1), \\ \operatorname{sgn} \underline{C}(\beta) &= \operatorname{sgn}(\beta - \beta_2), \end{aligned}$$

for any $\beta \in (0, 2s)$.

We observe that Proposition 3.9 is the merging of Proposition 2.7 and Corollary 2.8 in [R-OS15]. The fact that here the constants $\overline{C}$ and $\underline{C}$ do not depend on the direction ν is a consequence of the *isotropy* of the class $\mathscr{L}_0$. By this we mean that $\mathscr{L}_0$ is such that

$$L_K \in \mathscr{L}_0 \quad \text{if and only if} \quad L_{K_O} \in \mathscr{L}_0 \text{ for any } O \in SO(n),$$

where $K_O(z) := K(Oz)$. This implies that the Pucci operators M^+ and M^- are rotationally invariant.[3]

With the aid of the previous proposition, we are now ready to construct a barrier which will eventually prove the Hölder continuity of the solutions of (3.1.2) up to the boundary of Ω.

Lemma 3.10. *There exist three values $C_\star \geqslant 1$, $r \in (0,1)$ and $\beta \in (0,s)$, depending on n, s, λ, Λ, and a bounded, radial function $\varphi \in C^{0,\beta}(\mathbb{R}^n) \cap C^\infty(B_{1+r} \setminus \overline{B_1})$ such that*

$$\begin{cases} M^+\varphi \leqslant -1 & \text{in } B_{1+r} \setminus \overline{B_1} \\ \varphi = 0 & \text{in } B_1 \\ \varphi(x) \leqslant C_\star \left(|x|-1\right)^\beta & \text{for any } x \in \mathbb{R}^n \setminus B_1 \\ \varphi \geqslant 1 & \text{in } \mathbb{R}^n \setminus B_{1+r}. \end{cases} \tag{3.4.3}$$

Proof. Let $\beta_1 \in (0,s)$ be as given by Proposition 3.9. Let $\beta \in (0,\beta_1)$ and define

$$\varphi^{(\beta)}(x) := \operatorname{dist}(x, B_1)^\beta = (|x|-1)_+^\beta.$$

We claim that there exists two constants $\bar{c} > 0$ and $\bar{r} \in (0,1)$, depending on n, s, λ, Λ and β, such that

$$M^+\varphi^{(\beta)}(x) \leqslant -\bar{c}\left(|x|-1\right)^{\beta-2s} \text{ for any } x \in B_{1+\bar{r}} \setminus \overline{B_1}. \tag{3.4.4}$$

In order to verify this assertion, we reason as in the proof of Lemma 3.1 in [R-OS15]. We take $L = L_K \in \mathscr{L}_0$ and estimate $L\varphi^{(\beta)}(x_\rho)$, with $x_\rho = (0,\ldots,0,1+\rho)$ and $\rho \in (0,1)$ sufficiently small. To do this, we consider the function

$$\psi^\beta(x) := \psi^\beta_{e_n}(x - e_n) = (x_n - 1)_+^\beta\,.$$

It is easy to check that

$$\psi^\beta(x) \leqslant \varphi^{(\beta)}(x) \text{ for any } x \in \mathbb{R}^n,$$

and

$$\psi^\beta(0,\ldots,0,x_n) = \varphi^{(\beta)}(0,\ldots,0,x_n) \text{ for any } x_n \in \mathbb{R}. \tag{3.4.5}$$

By arguing as in the proof of [R-OS15, Lemma 3.1], we also obtain that

$$\left(\varphi^{(\beta)} - \psi^\beta\right)(x_\rho + z) \leqslant c_1 \begin{cases} \rho^{\beta-1}|z'|^2 & \text{if } z \in B_{\rho/2} \\ |z'|^{2\beta} & \text{if } z \in B_1 \setminus B_{\rho/2} \\ |z|^\beta & \text{if } z \in \mathbb{R}^n \setminus B_1, \end{cases}$$

[3] As noted in [CS09] the Pucci operators associated to the class $\mathscr{L}_0$ take the explicit forms

$$M^+u(x) = \frac{1}{2}\int_{\mathbb{R}^n} \frac{\Lambda\delta u(x,z)_+ - \lambda\delta u(x,z)_-}{|z|^{n+2s}}\,dz,$$
$$M^-u(x) = \frac{1}{2}\int_{\mathbb{R}^n} \frac{\lambda\delta u(x,z)_+ - \Lambda\delta u(x,z)_-}{|z|^{n+2s}}\,dz,$$

with $\delta u(x,z) := u(x+z) + u(x-z) - 2u(x)$. From this, it is also clear that M^+ and M^- are rotationally invariant.

for some constant $c_1 > 0$. Using this and (3.4.5), we estimate

$$\begin{aligned} L\big(\varphi^{(\beta)} - \psi^{\beta}\big)(x_\rho) &= \frac{1}{2}\int_{\mathbb{R}^n} \Big[\big(\varphi^{(\beta)} - \psi^{\beta}\big)(x_\rho + z) + \big(\varphi^{(\beta)} - \psi^{\beta}\big)(x_\rho - z)\Big] K(z)\, dz \\ &\leqslant c_1\Lambda\left(\int_{B_{\rho/2}} \frac{\rho^{\beta-1}|z'|^2}{|z|^{n+2s}}\, dz + \int_{B_1\setminus B_{\rho/2}} \frac{|z'|^{2\beta}}{|z|^{n+2s}}\, dz + \int_{\mathbb{R}^n\setminus B_1} \frac{|z|^{\beta}}{|z|^{n+2s}}\, dz\right) \\ &\leqslant \frac{c_2}{3}\left(\rho^{\beta+1-2s} + \rho^{2(\beta-s)} + 1\right) \\ &\leqslant c_2\rho^{2(\beta-s)}, \end{aligned}$$

for some $c_2 > 0$, since $\beta < \beta_1 < s$. Thus, recalling Proposition 3.9, we get

$$\begin{aligned} L\varphi^{(\beta)}(x_\rho) &= L\big(\varphi^{(\beta)} - \psi^{\beta}\big)(x_\rho) + L\psi^{\beta}(x_\rho) \leqslant c_2\rho^{2(\beta-s)} + M^+\psi^{\beta}(x_\rho) \\ &= c_2\rho^{2(\beta-s)} + \overline{C}(\beta)\rho^{\beta-2s} = \left(c_2\rho^{\beta} - \big|\overline{C}(\beta)\big|\right)\rho^{\beta-2s} \\ &\leqslant -\bar{c}\rho^{\beta-2s}, \end{aligned}$$

for some $\bar{c} > 0$, as $\overline{C}(\beta) < 0$, being $\beta < \beta_1$, and choosing $\rho < \bar{r}$, with $\bar{r} \in (0,1)$ small enough. Estimate (3.4.4) then follows by the independence of $\bar{c}, \bar{r}$ from $L \in \mathscr{L}_0$ and the rotational symmetry of M^+ and $\varphi^{(\beta)}$.

Furthermore, if we set

$$\widetilde{\varphi}^{(\beta)}(x) := \min\left\{\varphi^{(\beta)}(x), 1\right\} = \begin{cases} (|x|-1)_+^{\beta} & \text{if } x \in B_2 \\ 1 & \text{if } x \in \mathbb{R}^n \setminus B_2, \end{cases}$$

then it is not hard to check that

$$M^+\widetilde{\varphi}^{(\beta)}(x) \leqslant M^+\varphi^{(\beta)}(x) + c_3 \leqslant -\bar{c}\,(|x|-1)^{\beta-2s} + c_3 \text{ for any } x \in B_{1+\bar{r}} \setminus \overline{B_1},$$

for some $c_3 > 0$. Consequently, by taking a smaller $\bar{r} > 0$, if necessary, it follows that

$$M^+\widetilde{\varphi}^{(\beta)} \leqslant -1 \text{ in } B_{1+\bar{r}} \setminus \overline{B_1}.$$

The properties listed in (3.4.3) are then satisfied by $\varphi := C_\star\widetilde{\varphi}^{(\beta)}$, where $C_\star \geqslant 1$ is a constant chosen to have $\varphi \geqslant 1$ outside of $B_{1+\bar{r}}$. $\square$

Thanks to the supersolution provided by Lemma 3.10, we have

Corollary 3.11. *Assume that K satisfies (3.1.6) and (3.1.8). Let Ω be a bounded $C^{1,1}$ domain and $f \in L^\infty(\Omega)$. If $u \in \mathbb{H}^K(\Omega)$ is a weak solution of the problem (3.1.2), with $g = 0$, then*

$$|u(x)| \leqslant C_\sharp\left(\|f\|_{L^\infty(\Omega)} + \|u\|_{L^\infty(\mathbb{R}^n)}\right) \operatorname{dist}(x, \partial\Omega)^{\beta} \quad \textit{for a.a. } x \in \Omega, \tag{3.4.6}$$

for some $\beta \in (0, s)$ which depends only on n, s, λ, Λ and $C_\sharp > 0$ which may also depend on the $C^{1,1}$ norm of $\partial\Omega$.

Observe that the we do not need to require a priori the boundedness of u. Indeed, every weak solution of (3.1.2) is bounded and satisfies

$$\|u\|_{L^\infty(\Omega)} \leqslant C(\operatorname{diam}(\Omega))^{2s}\|f\|_{L^\infty(\Omega)},$$

with $C > 0$ depending on n, s and λ (see e.g. [R-O15, Corollary 5.2]). Note however that we do not make explicit use of this last estimate in (3.4.6), so that $C_\sharp$ does not depend on the size of Ω.

Proof of Corollary 3.11. Let $x_1 \in \partial\Omega$ and $B_\rho(x_0)$ be a ball that touches $\partial\Omega$ at x_1 from outside. Note that, due to the regularity of $\partial\Omega$, such ball exists and we may choose $\rho > 0$ in dependence of the $C^{1,1}$ norm of the boundary only. Let now φ be the function constructed in Lemma 3.10 and set

$$\tilde{\varphi}(x) := \left(\|f\|_{L^\infty(\Omega)} + \|u\|_{L^\infty(\mathbb{R}^n)}\right) \varphi\left(\frac{x - x_0}{\rho}\right),$$

for any $x \in \mathbb{R}^n$. Observe that

$$\begin{cases} M^+\tilde{\varphi} \leqslant -\rho^{-2s}\|f\|_{L^\infty(\Omega)} & \text{in } B_{\rho(1+r)}(x_0) \setminus \overline{B_\rho(x_0)} \\ \tilde{\varphi} = 0 & \text{in } B_\rho(x_0) \\ \tilde{\varphi}(x) \leqslant C_\star \dfrac{\|f\|_{L^\infty(\Omega)} + \|u\|_{L^\infty(\mathbb{R}^n)}}{\rho^\beta} (|x - x_0| - \rho)^\beta & \text{for any } x \in \mathbb{R}^n \setminus B_\rho(x_0) \\ \tilde{\varphi} \geqslant \|u\|_{L^\infty(\mathbb{R}^n)} & \text{in } \mathbb{R}^n \setminus B_{\rho(1+r)}(x_0), \end{cases} \tag{3.4.7}$$

where $r \in (0, 1)$ and $C_\star \geqslant 1$ are as in Lemma 3.10. By choosing $\rho \leqslant 1$, we obtain from (3.4.7) that

$$\begin{cases} L_K\tilde{\varphi} \leqslant L_K u & \text{in } \tilde{\Omega} := \Omega \cap B_{(1+r)\rho}(x_0) \\ u \leqslant \tilde{\varphi} & \text{in } \mathbb{R}^n \setminus \tilde{\Omega}. \end{cases}$$

Accordingly, by the comparison principle (see e.g. [R-O15, Corollary 4.2]) we get[4]

$$u(x) \leqslant \tilde{\varphi}(x) \leqslant C_\star \frac{\|f\|_{L^\infty(\Omega)} + \|u\|_{L^\infty(\mathbb{R}^n)}}{\rho^\beta} (|x - x_0| - \rho)^\beta, \tag{3.4.8}$$

for a.a. $x \in \tilde{\Omega}$.

Since this holds for any $x_1 \in \partial\Omega$, we deduce that

$$u(x) \leqslant C_\star \frac{\|f\|_{L^\infty(\Omega)} + \|u\|_{L^\infty(\mathbb{R}^n)}}{\rho^\beta} (\operatorname{dist}(x, \partial\Omega))^\beta, \tag{3.4.9}$$

for a.a. $x \in \Omega$ such that $\operatorname{dist}(x, \partial\Omega) < r\rho$. Actually, (3.4.9) holds for a.a. $x \in \Omega$.

By repeating the same argument with $-u$ in place of u, estimate (3.4.6) follows. □

We are now in position to prove the fundamental lemma that will lead to the desired global C^α estimates for the solutions of (3.1.2).

Lemma 3.12. *Assume that K satisfies (3.1.6) and (3.1.8). Let Ω be a bounded $C^{1,1}$ domain and $f \in L^\infty(\Omega)$. Let $u \in \mathbb{H}^K(\Omega)$ be a weak solution of the problem (3.1.2), with $g = 0$. Then, $u \in C^\beta_{\text{loc}}(\Omega)$, for some $\beta \in (0, s)$ depending only on n, s, λ and Λ. Moreover, given any $x_0 \in \Omega$ and setting $R := \operatorname{dist}(x_0, \partial\Omega)/2$, it holds*

$$[u]_{C^\beta(B_{R/2}(x_0))} \leqslant C\left(\|f\|_{L^\infty(\Omega)} + \|u\|_{L^\infty(\mathbb{R}^n)}\right), \tag{3.4.10}$$

for some constant $C > 0$ which depends only on n, s, λ, Λ and the $C^{1,1}$ norm of $\partial\Omega$.

[4] Note that such comparison principle involves *weak* super-/subsolutions. In our context, $\tilde{\varphi}$ is a pointwise supersolution, but may fail to belong to the energy space $\mathbb{H}^K(\tilde{\Omega})$, if β is too small. In this case, we would not be able to apply [R-O15, Corollary 4.2] directly.

To solve this issue, we argue as follows. First, we translate $B_\rho(x_0)$ away from the boundary of Ω and obtain a new ball $B_\rho(x_\varepsilon) \subset\subset \mathbb{R}^n \setminus \Omega$ such that $|x_\varepsilon - x_0| = \varepsilon$ and $\operatorname{dist}(B_\rho(x_\varepsilon), \Omega) = \varepsilon$, with $\varepsilon > 0$ small. Then, we label as $\tilde{\varphi}_\varepsilon$ the barrier associated to this new ball. The function $\tilde{\varphi}_\varepsilon$ is Lipschitz in a small neighbourhood of $\tilde{\Omega}_\varepsilon := \Omega \cap B_{(1+r)\rho}(x_\varepsilon)$ and therefore belongs to $\mathbb{H}^K(\tilde{\Omega}_\varepsilon)$. Accordingly, $\tilde{\varphi}_\varepsilon$ is a weak supersolution and we may apply the comparison principle to deduce that $u(x) \leqslant C_\star\rho^{-\beta}\left(\|f\|_{L^\infty(\Omega)} + \|u\|_{L^\infty(\mathbb{R}^n)}\right)(|x - x_\varepsilon| - \rho)^\beta$. Estimate (3.4.8) then follows by letting $\varepsilon \to 0^+$.

Proof. All along the proof, c will denote a positive constant depending on at most n, s, λ, Λ and the $C^{1,1}$ norm of $\partial\Omega$. The value of c may also change from line to line.

Note that $B_R(x_0) \subset B_{2R}(x_0) \subset \Omega$. Given $\varepsilon > 0$ small, we consider the standard mollifier η_ε and define $u_\varepsilon := u * \eta_\varepsilon$ and $f_\varepsilon := f * \eta_\varepsilon$ (where we suppose f to be extended to 0 outside of Ω). We clearly have $f_\varepsilon \in L^\infty(\mathbb{R}^n)$, $u_\varepsilon \in C^\infty(\mathbb{R}^n)$, with

$$\|f_\varepsilon\|_{L^\infty(\mathbb{R}^n)} \leqslant \|f\|_{L^\infty(\Omega)} \quad \text{and} \quad \|u_\varepsilon\|_{L^\infty(\mathbb{R}^n)} \leqslant \|u\|_{L^\infty(\mathbb{R}^n)}.$$

Furthermore, u_ε solves $-L_K u_\varepsilon = f_\varepsilon$ in $B_R(x_0)$.

Now we proceed to the actual proof of (3.4.10). In view of the interior estimate (3.4.1), we may suppose without loss of generality that $R \leqslant 1$. We set

$$\tilde{u}_\varepsilon(y) := u_\varepsilon(x_0 + Ry) \quad \text{for any } y \in \mathbb{R}^n.$$

A simple computation shows that $\tilde{u}_\varepsilon$ is a solution of

$$-L_{K_R}\tilde{u}_\varepsilon(y) = \tilde{f}_\varepsilon \quad \text{in } B_1,$$

where $K_R(z) = R^{n+2s}K(Rz)$ and $\tilde{f}_\varepsilon(y) := R^{2s} f_\varepsilon(x_0+Ry)$. Observe that K_R satisfies (3.1.6) and (3.1.8) with the same constants λ, Λ of K. Accordingly, we may use Lemma 3.8 to deduce that

$$[\tilde{u}_\varepsilon]_{C^\alpha(B_{1/2})} \leqslant c_1 \left(\|\tilde{f}_\varepsilon\|_{L^\infty(B_1)} + \|\tilde{u}_\varepsilon\|_{L^1_{2s}(\mathbb{R}^n)} \right), \tag{3.4.11}$$

for any $\alpha \in (0, \min\{2s, 1\})$ and with $c_1 > 0$ depending on n, s, λ, Λ and α.

On the one hand,

$$[\tilde{u}_\varepsilon]_{C^\alpha(B_{1/2})} = R^\alpha [u_\varepsilon]_{C^\alpha(B_{R/2}(x_0))}. \tag{3.4.12}$$

On the other hand, by (3.4.6) and the fact that u vanishes outside of Ω, we have that

$$\begin{aligned}
|u_\varepsilon(x)| &\leqslant \int_{B_\varepsilon(x)} |u(z)|\eta_\varepsilon(x-z)\,dz \\
&\leqslant C_\sharp \left(\|f\|_{L^\infty(\Omega)} + \|u\|_{L^\infty(\mathbb{R}^n)}\right) \int_{B_\varepsilon(x)} \chi_\Omega(z)\,\mathrm{dist}(z,\partial\Omega)^\beta \eta_\varepsilon(x-z)\,dz \\
&\leqslant C_\sharp \left(\|f\|_{L^\infty(\Omega)} + \|u\|_{L^\infty(\mathbb{R}^n)}\right) \left(\varepsilon + \mathrm{dist}(x,\partial\Omega)\right)^\beta.
\end{aligned}$$

Accordingly, if we take $\varepsilon \leqslant R$, we get

$$\begin{aligned}
|\tilde{u}_\varepsilon(y)| = |u_\varepsilon(x_0 + Ry)| &\leqslant C_\sharp \left(\|f\|_{L^\infty(\Omega)} + \|u\|_{L^\infty(\mathbb{R}^n)}\right) \left(\varepsilon + |Ry| + \mathrm{dist}(x_0,\partial\Omega)\right)^\beta \\
&\leqslant c \left(\|f\|_{L^\infty(\Omega)} + \|u\|_{L^\infty(\mathbb{R}^n)}\right) R^\beta \left(1 + |y|\right)^\beta,
\end{aligned}$$

for any $y \in \mathbb{R}^n$. Hence,

$$\int_{B_1} \frac{|\tilde{u}_\varepsilon(y)|}{1 + |y|^{n+2s}}\,dy \leqslant \alpha_n \|\tilde{u}_\varepsilon\|_{L^\infty(B_1)} \leqslant c \left(\|f\|_{L^\infty(\Omega)} + \|u\|_{L^\infty(\mathbb{R}^n)}\right) R^\beta,$$

while

$$\begin{aligned}
\int_{\mathbb{R}^n\setminus B_1} \frac{|\tilde{u}_\varepsilon(y)|}{1 + |y|^{n+2s}}\,dy &\leqslant c \left(\|f\|_{L^\infty(\Omega)} + \|u\|_{L^\infty(\mathbb{R}^n)}\right) R^\beta \int_{\mathbb{R}^n\setminus B_1} (1 + |y|)^{\beta-n-2s}\,dy \\
&\leqslant c \left(\|f\|_{L^\infty(\Omega)} + \|u\|_{L^\infty(\mathbb{R}^n)}\right) R^\beta,
\end{aligned}$$

since $\beta < 2s$. These last two estimates yield

$$\|\tilde{u}_\varepsilon\|_{L^1_{2s}(\mathbb{R}^n)} \leqslant c \left(\|f\|_{L^\infty(\Omega)} + \|u\|_{L^\infty(\mathbb{R}^n)}\right) R^\beta. \tag{3.4.13}$$

Moreover,

$$\|\tilde{f}_\varepsilon\|_{L^\infty(B_1)} \leqslant \|f\|_{L^\infty(\Omega)} R^{2s}. \tag{3.4.14}$$

By putting together (3.4.11), (3.4.12), (3.4.13) and (3.4.14), we finally obtain that

$$[u_\varepsilon]_{C^\alpha(B_{R/2}(x_0))} \leqslant c_2 \left(\|f\|_{L^\infty(\Omega)} + \|u\|_{L^\infty(\mathbb{R}^n)}\right) R^{\beta-\alpha},$$

for some $c_2 > 0$ depending on n, s, λ, Λ, α and the $C^{1,1}$ norm of $\partial\Omega$. By choosing $\alpha = \beta$ and letting $\varepsilon \to 0^+$, we are finally led to (3.4.10). □

Proposition 3.13. *Assume that the kernel K satisfies (3.1.6) and (3.1.8). Let $\Omega \subset \mathbb{R}^n$ be a bounded $C^{1,1}$ domain, $f \in L^\infty(\Omega)$ and $g \in C^{2s+\gamma}(\mathbb{R}^n \setminus \Omega)$, with $\gamma \in (0, 2-2s)$. If $u \in \mathbb{H}^K(\Omega)$ is a weak solution of the problem (3.1.2), then $u \in C^\alpha(\overline{\Omega})$, for some $\alpha \in (0, s)$ depending only on n, s, λ, Λ and γ, with*

$$\|u\|_{C^\alpha(\overline{\Omega})} \leqslant C \left(\|f\|_{L^\infty(\Omega)} + \|g\|_{C^{2s+\gamma}(\mathbb{R}^n\setminus\Omega)} + \|u\|_{L^\infty(\Omega)}\right), \tag{3.4.15}$$

for some constant $C > 0$ which depends only on n, s, λ, Λ, γ and Ω.

Proof. When $g \equiv 0$, formula (3.4.15) may be obtained as in the proof of [R-OS14, Proposition 1], by taking advantage of the estimate contained in Lemma 3.12. The general case then follows by arguing as in [R-O15, Remark 7.1]. □

Next we report a higher order interior regularity result.

Proposition 3.14 ([S14, R-O15])**.** *Assume that K satisfies (3.1.6) and (3.1.8). Let $f \in C^\alpha(B_1)$, for some $\alpha > 0$ such that $2s + \alpha$ is not an integer. Let $u \in \mathbb{H}^K(B_1) \cap C^\alpha(\mathbb{R}^n)$ be a bounded weak solution of (3.1.1) in B_1. Then, $u \in C^{2s+\alpha}(B_{1/2})$ and*

$$\|u\|_{C^{2s+\alpha}(B_{1/2})} \leqslant C \left(\|f\|_{C^\alpha(B_1)} + \|u\|_{C^\alpha(\mathbb{R}^n)}\right),$$

for some constant $C > 0$ which depends only on n, s, λ, Λ and α.

By combining this last result with Proposition 3.13, we obtain the following

Corollary 3.15. *Assume that the kernel K satisfies (3.1.6) and (3.1.8). Let $\Omega \subset \mathbb{R}^n$ be a bounded $C^{1,1}$ domain, $f \in C^\beta(\Omega)$ and $g \in C^{2s+\gamma}(\mathbb{R}^n \setminus \Omega)$, with $\beta \in (0,1)$ and $\gamma \in (0, 2-2s)$. If $u \in \mathbb{H}^K(\Omega)$ is a weak solution of (3.1.2), then $u \in C^{2s+\alpha}_{\mathrm{loc}}(\Omega)$, for some $\alpha \in (0, s)$ depending only on n, s, λ, Λ, β and γ. Also, for any domain $\Omega' \subset\subset \Omega$ it holds*

$$\|u\|_{C^{2s+\alpha}(\Omega')} \leqslant C \left(\|f\|_{C^\beta(\Omega)} + \|g\|_{C^{2s+\gamma}(\mathbb{R}^n\setminus\Omega)} + \|u\|_{L^\infty(\Omega)}\right),$$

for some constant $C > 0$ which depends only on n, s, λ, Λ, β, γ, Ω and Ω'.

In the next proposition we address the regularity of solutions in the whole space $\mathbb{R}^n$.

Proposition 3.16. *Assume that K satisfies (3.1.6) and (3.1.8). Let $u \in L^\infty(\mathbb{R}^n)$ be a weak solution of (3.1.1) in $\mathbb{R}^n$. Then,*

(i) if $f \in L^\infty(\mathbb{R}^n)$, then $u \in C^\alpha(\mathbb{R}^n)$ for any $\alpha \in (0, \min\{2s, 1\})$ and

$$\|u\|_{C^\alpha(\mathbb{R}^n)} \leqslant C \left(\|f\|_{L^\infty(\mathbb{R}^n)} + \|u\|_{L^\infty(\mathbb{R}^n)}\right),$$

for some constant $C > 0$ which depends only on n, s, λ, Λ and α;

(ii) if $f \in C^\alpha(\mathbb{R}^n)$, for some $\alpha \in (0,2)$ such that $2s+\alpha \neq 1,2,3$, then $u \in C^{2s+\alpha}(\mathbb{R}^n)$ and

$$\|u\|_{C^{2s+\alpha}(\mathbb{R}^n)} \leqslant C\left(\|f\|_{C^\alpha(\mathbb{R}^n)} + \|u\|_{L^\infty(\mathbb{R}^n)}\right),$$

for some constant $C > 0$ which depends only on n, s, λ, Λ and α.

Proof. Item (i) is an immediate consequence of Proposition 3.8 (up to an approximation argument).

On the other hand, to prove (ii) we first observe that $u \in C^\beta(\mathbb{R}^n)$, for any $\beta \in (0, \min\{2s,1\})$, in view of (i). Consider for the moment the case of $\alpha \in (0,1)$. If $s \in (\alpha/2, 1)$ we may take β to be larger than α. Consequently, both u and f belong to $C^\alpha(\mathbb{R}^n)$ and we are in position to use Proposition 3.14 and recover the $C^{2s+\alpha}$ regularity of u.

The case $s \in (0, \alpha/2]$ requires a more delicate argument, inspired by an iterative technique displayed in the proof of [PSV13, Lemma 6]. Let $k \geqslant 1$ be the only integer for which $s \in (\alpha/(2k+2), \alpha/(2k))]$. Applying Proposition 3.14 for k times, we get that $u \in C^{2ks+\beta}(\mathbb{R}^n)$ for any $\beta \in (0,2s)$, provided $2js+\beta \neq 1$ for each $j = 1,\dots,k$. Notice that we are allowed to use this result, since $\alpha \geqslant 2ks > 2js+\beta$ for any admissible β and any $j = 1,\dots,k-1$. But then, we can choose β in such a way that $2ks+\beta \geqslant \alpha$, as $(2k+2)s > \alpha$. Hence, $u \in C^\alpha(\mathbb{R}^n)$ and a further application of Proposition 3.14 leads to the thesis.

When $\alpha \in [1,2)$, we already know from the reasoning just displayed that $u \in C^{2s+\beta}(\mathbb{R}^n)$ for any $\beta \in (0,1)$. Then again, if $s \in ((\alpha-1)/2, 1)$, then $2s+\beta > \alpha$, for some β close enough to 1 and, consequently, we may use Proposition 3.14 to get that $u \in C^{2s+\alpha}(\mathbb{R}^n)$. Conversely, when $s \in (0, (\alpha-1)/2]$, we argue as before by splitting $(0,(\alpha-1)/2]$ into non-overlapping subintervals. Eventually, we obtain the thesis in this case too. □

We remark that the requirement $\alpha < 2$ in Proposition 3.16(ii) is only asked for simplicity of exposition. Indeed, one can obtain the result stated there for any $\alpha > 0$, in the spirit of Proposition 3.14. However, this formulation is general enough for our future purposes.

3.4.2 Linear equations: general kernels

Here, we extend some results of the previous subsection to operators driven by kernels K which only satisfy (3.1.7), instead of the stronger (3.1.8). To do this, we appropriately modify K far from the origin in order to obtain a new kernel $\widetilde{K}$ fulfilling (3.1.8). Then, the results will follow by studying the properties of the operator associated to the difference $\widetilde{K} - K$.

We define $K_{\text{ext}} : \mathbb{R}^n \to [0,+\infty)$ to be a radial function of class C^∞ satisfying

$$K_{\text{ext}}(z) = \begin{cases} 0 & \text{if } z \in B_{\frac{r_0}{2}} \\ \dfrac{\lambda}{|z|^{n+2s}} & \text{if } z \in \mathbb{R}^n \setminus B_{r_0}. \end{cases}$$

The function K_{ext} is clearly bounded. Also, it is not hard to check that $D^\alpha K_{\text{ext}} \in L^1(\mathbb{R}^n)$, for every multi-index $\alpha \in (\mathbb{N} \cup \{0\})^n$. We set

$$L_{K_{\text{ext}}} u(x) := \int_{\mathbb{R}^n} (u(x-z) - u(x))\, K_{\text{ext}}(z)\, dz.$$

Observe that $L_{K_{\text{ext}}} u$ is well-defined at a.a. $x \in \mathbb{R}^n$, provided $u \in L^\infty(\mathbb{R}^n)$. Furthermore,

$$L_{K_{\text{ext}}} u(x) = (u * K_{\text{ext}})(x) - \|K_{\text{ext}}\|_{L^1(\mathbb{R}^n)} u(x),$$

so that $L_{K_{\rm ext}}u$ essentially inherits the regularity properties of u. In particular,

$$\text{if } u\in L^\infty(\mathbb{R}^n), \text{ then } \begin{cases} L_{K_{\rm ext}}u\in L^\infty(\mathbb{R}^n), \text{ with} \\ \|L_{K_{\rm ext}}u\|_{L^\infty(\mathbb{R}^n)}\leqslant C_1\|u\|_{L^\infty(\mathbb{R}^n)}, \end{cases} \tag{3.4.16}$$

for some constant $C_1>0$ depending on $K_{\rm ext}$, and, given any open set $\Omega\subseteq\mathbb{R}^n$ and any $\alpha>0$,

$$\text{if } u\in L^\infty(\mathbb{R}^n)\cap C^\alpha(\Omega), \text{ then } \begin{cases} L_{K_{\rm ext}}u\in C^\alpha(\Omega), \text{ with} \\ \|L_{K_{\rm ext}}u\|_{C^\alpha(\Omega)}\leqslant C_2\|u\|_{C^\alpha(\Omega)}, \end{cases} \tag{3.4.17}$$

for some $C_2>0$ depending on $K_{\rm ext}$ and α.

Let now K be a kernel satisfying (3.1.6) and (3.1.7). We set $\widetilde{K}(z):=K(z)+K_{\rm ext}(z)$, for a.a. $z\in\mathbb{R}^n$. Notice that the new kernel $\widetilde{K}$ satisfies (3.1.6) and (3.1.8), with $\lambda+\Lambda$ in place of Λ. Also,

$$\text{if } u\in L^\infty(\mathbb{R}^n)\cap\mathbb{H}^K(\Omega), \text{ then } u\in\mathbb{H}^{\widetilde{K}}(\Omega), \tag{3.4.18}$$

for any bounded domain Ω.

By knowing all these facts, we are able to extend Proposition 3.13 to the case of general kernels satisfying (3.1.7) and obtain a global C^α regularity result for bounded solutions of the Dirichlet problem (3.1.2).

Proposition 3.17. *Assume that the kernel K satisfies (3.1.6) and (3.1.7). Let $\Omega\subset\mathbb{R}^n$ be a bounded $C^{1,1}$ domain, $f\in L^\infty(\Omega)$ and $g\in C^{2s+\gamma}(\mathbb{R}^n\setminus\Omega)$, with $\gamma\in(0,2-2s)$. If $u\in L^\infty(\mathbb{R}^n)\cap\mathbb{H}^K(\Omega)$ is a weak solution of the problem (3.1.2), then $u\in C^\alpha(\overline{\Omega})$, for some $\alpha\in(0,s)$ depending only on n, s, λ, Λ and γ, with*

$$\|u\|_{C^\alpha(\overline{\Omega})}\leqslant C\left(\|f\|_{L^\infty(\Omega)}+\|g\|_{C^{2s+\gamma}(\mathbb{R}^n\setminus\Omega)}+\|u\|_{L^\infty(\Omega)}\right),$$

for some constant $C>0$ which depends only on n, s, λ, Λ, r_0, γ and Ω.

Proof. By (3.4.18), we have that $u\in L^\infty(\mathbb{R}^n)\cap\mathbb{H}^{\widetilde{K}}(\Omega)$. Moreover, u is a weak solution of

$$\begin{cases} -L_{\widetilde{K}}u=f-L_{K_{\rm ext}}u & \text{in } \Omega \\ u=g & \text{in } \mathbb{R}^n\setminus\Omega. \end{cases}$$

Thanks to (3.4.16), the right-hand side $f-L_{K_{\rm ext}}u$ belongs to $L^\infty(\Omega)$, and the thesis then follows by an application of Proposition 3.13. □

Similarly, by using (3.4.17) and Corollary 3.15, we get

Proposition 3.18. *Assume that the kernel K satisfies (3.1.6) and (3.1.7). Let $\Omega\subset\mathbb{R}^n$ be a bounded $C^{1,1}$ domain, $f\in C^\beta(\Omega)$ and $g\in C^{2s+\gamma}(\mathbb{R}^n\setminus\Omega)$, with $\beta\in(0,1)$ and $\gamma\in(0,2-2s)$. If $u\in L^\infty(\mathbb{R}^n)\cap\mathbb{H}^K(\Omega)$ is a weak solution of (3.1.2), then $u\in C^{2s+\alpha}_{\rm loc}(\Omega)$, for some $\alpha\in(0,s)$ depending only on n, s, λ, Λ, β and γ. Also, for any domain $\Omega'\subset\subset\Omega$ it holds*

$$\|u\|_{C^{2s+\alpha}(\Omega')}\leqslant C\left(\|f\|_{C^\beta(\Omega)}+\|g\|_{C^{2s+\gamma}(\mathbb{R}^n\setminus\Omega)}+\|u\|_{L^\infty(\Omega)}\right),$$

for some constant $C>0$ which depends only on n, s, λ, Λ, r_0, β, γ, Ω and Ω'.

Finally, we extend Proposition 3.16 to obtain the following regularity result for entire solutions of (3.1.1).

Proposition 3.19. *Assume that K satisfies (3.1.6) and (3.1.7). Let $u\in L^\infty(\mathbb{R}^n)$ be a weak solution of (3.1.1) in $\mathbb{R}^n$. Then,* *(i)* *if* $f \in L^\infty(\mathbb{R}^n)$, *then* $u \in C^\alpha(\mathbb{R}^n)$ *for any* $\alpha \in (0, \min\{2s, 1\})$ *and*

$$\|u\|_{C^\alpha(\mathbb{R}^n)} \leqslant C\left(\|f\|_{L^\infty(\mathbb{R}^n)} + \|u\|_{L^\infty(\mathbb{R}^n)}\right),$$

for some constant $C > 0$ *which depends only on* n, s, λ, Λ, r_0 *and* α;

(ii) *if* $f \in C^\alpha(\mathbb{R}^n)$, *for some* $\alpha \in (0, 2)$ *such that* $2s + \alpha \neq 1, 2, 3$, *then* $u \in C^{2s+\alpha}(\mathbb{R}^n)$ *and*

$$\|u\|_{C^{2s+\alpha}(\mathbb{R}^n)} \leqslant C\left(\|f\|_{C^\alpha(\mathbb{R}^n)} + \|u\|_{L^\infty(\mathbb{R}^n)}\right),$$

for some constant $C > 0$ *which depends only on* n, s, λ, Λ, r_0 *and* α.

3.4.3 Semilinear equations

This conclusive subsection is devoted to a couple of results concerning semilinear equations. These propositions are modeled on the framework of equation (3.1.3), with F independent of x, and are the ones that will be more frequently exploited in the following chapters. We stress that K is asked here to satisfy (3.1.6) and (3.1.7) only.

First is a result for Dirichlet problems in smooth, bounded domains of $\mathbb{R}^n$.

Proposition 3.20. *Assume that the kernel* K *satisfies* (3.1.6) *and* (3.1.7). *Let* $\Omega \subset \mathbb{R}^n$ *be a bounded* $C^{1,1}$ *domain,* $F \in C^\beta_{\mathrm{loc}}(\mathbb{R})$ *and* $g \in C^{2s+\gamma}(\mathbb{R}^n \setminus \Omega)$, *with* $\beta \in (0, 1)$ *and* $\gamma \in (0, 2-2s)$. *If* $u \in L^\infty(\Omega) \cap \mathbb{H}^K(\Omega)$ *is a weak solution of*

$$\begin{cases} L_K u = F(u) & \text{in } \Omega \\ u = g & \text{in } \mathbb{R}^n \setminus \Omega, \end{cases}$$

then $u \in C^\alpha(\mathbb{R}^n) \cap C^{2s+\alpha}_{\mathrm{loc}}(\Omega)$, *for some* $\alpha \in (0, s)$ *depending only on* n, s, λ, Λ, β *and* γ.

Proof. Being F continuous and u bounded, it is clear that the composition $F(u)$ is also bounded. In view of this, we may apply Proposition 3.17 to deduce that $u \in C^{\alpha'}(\overline{\Omega})$, for some $\alpha' \in (0, s)$. Accordingly, u is Hölder continuous in the whole of $\mathbb{R}^n$. Furthermore, $F(u) \in C^{\beta\alpha'}(\Omega)$ and finally Proposition 3.18 implies that $u \in C^{2s+\alpha}_{\mathrm{loc}}(\Omega)$, with $\alpha \in (0, s)$. □

Next, we address the regularity of bounded solutions to semilinear equations in the full space $\mathbb{R}^n$.

Proposition 3.21. *Assume that* K *satisfies* (3.1.6) *and* (3.1.7). *Let* F *be of class* $C^{1,\beta}_{\mathrm{loc}}(\mathbb{R})$, *for some* $\beta > 0$, *and* $u \in L^\infty(\mathbb{R}^n)$ *be a weak solution of*

$$L_K u = F(u) \quad \text{in } \mathbb{R}^n.$$

Then, $u \in C^{1+2s+\alpha}(\mathbb{R}^n)$, *for some* $\alpha > 0$.

Proof. We observe that if we show that

$$u \in C^{1,\alpha}(\mathbb{R}^n) \text{ for some } \alpha \in (0, \beta], \tag{3.4.19}$$

then the proof would be over. Indeed, if u is this regular, then so is $F(u)$ and, hence, Proposition 3.19(ii) implies that $u \in C^{1+2s+\alpha}(\mathbb{R}^n)$.

Thus, we only have to prove (3.4.19). First, we remark that $F(u)$ is bounded. Thence, by Proposition 3.19(i) we deduce that u is of class $C^{\alpha'}(\mathbb{R}^n)$ for any $\alpha' \in (0, \min\{2s, 1\})$. Now we distinguish between the two cases $s \geqslant 1/2$ and $s < 1/2$.

When $s \in (1/2, 1)$, we have that $u \in C^{\alpha'}(\mathbb{R}^n)$ for any $\alpha' \in (0, 1)$. Consequently, $F(u) \in C^{\alpha'}(\mathbb{R}^n)$ and we may exploit Proposition 3.19(ii) to obtain that $u \in C^{2s+\alpha'}(\mathbb{R}^n)$ for any such α', provided $2s + \alpha' \neq 2$. Clearly, (3.4.19) follows.

The case of $s \in (0, 1/2]$ is slightly more involved. We deal with it by using an approach analogous to the one that we took in the second part of the proof of Proposition 3.19. Let $k \geqslant 1$ be the only integer for which $s \in (1/(2k+2), 1/(2k)]$. We already know that $u \in C^{\alpha'}(\mathbb{R}^n)$ for any $\alpha' \in (0, 2s)$. Thus, the composition $F(u)$ has the same regularity and we may apply Proposition 3.19(ii) to recover that $u \in C^{2s+\alpha'}(\mathbb{R}^n)$, provided $2s + \alpha' \neq 1$. By iterating this last step for k times, we get that $u \in C^{2ks+\alpha'}(\mathbb{R}^n)$ for any $\alpha' \in (0, 2s)$ such that $2js + \alpha' \neq 1$, for any $j = 1, \ldots, k$. But now $2ks + 2s > 1$ and thus (3.4.19) follows, as we may take α' as close to $2s$ (from below) as we desire. $\square$

Chapter 4

Regularity results: estimates in Sobolev and Nikol'skii spaces

4.1 Introduction

One of the cornerstones in the field of the regularity theory for weak solutions of second order linear elliptic differential equations is the existence of weak second derivatives. Indeed, let Ω be an open set of $\mathbb{R}^n$ and $u \in H^1(\Omega)$ a weak solution of

$$-\operatorname{div}(A(\cdot)\nabla u) = f \quad \text{in } \Omega, \tag{4.1.1}$$

where the $n \times n$ matrix $A = [a_{ij}]$ is uniformly elliptic, with entries $a_{ij} \in C^{0,1}_{\mathrm{loc}}(\Omega)$, and the right-hand term $f \in L^2(\Omega)$. Then, one gets that $u \in H^2_{\mathrm{loc}}(\Omega)$ and, for any domain $\Omega' \subset\subset \Omega$,

$$\|u\|_{H^2(\Omega')} \leqslant C \left(\|u\|_{L^2(\Omega)} + \|f\|_{L^2(\Omega)} \right),$$

for some constant $C > 0$ independent of u and f.

Such result is typically ascribed to Louis Nirenberg, who in [N55] obtained higher order Sobolev regularity for general linear elliptic equations. To do so, he introduced the by now classical *translation method*. In the setting of equation (4.1.1) the idea is basically to consider the difference quotients

$$D_i^h u(x) := \frac{u(x + he_i) - u(x)}{h},$$

for $i = 1, \dots, n$ and $h \neq 0$ suitably small in modulus, and use the equation itself to recover a uniform bound in h for the gradient of $D_i^h u$ in $L^2(\Omega')$. A compactness argument then shows that $u \in H^2_{\mathrm{loc}}(\Omega)$. Nice presentations of this technique are for instance contained in [E98] and [GM12].

After this, several generalizations were achieved. For example, the translation method has been successfully adapted to study nonlinear equations, too. Indeed, in [S77] and [D82] the authors deduce higher order regularity in both Sobolev and Besov classes for singular or degenerate operators of p-Laplacian type.

The object of this chapter is the attempt of a generalization of the above discussed higher differentiability to a nonlocal analogue of equation (4.1.1), modelled upon the fractional Laplacian.

Given any open set $\Omega \subset \mathbb{R}^n$, we consider a solution u of the linear equation (3.1.1), with $f \in L^2(\Omega)$. Here K is a measurable function which is comparable *in the small* to the kernel of the fractional Laplacian. A great generality of kernel is allowed, possibly not translation-invariant. However, if the kernel is not translation-invariant, we need to impose on K some sort of *joint* local $C^{0,s}$ regularity. We stress that this last hypothesis seems very natural to us. Indeed, while translation-invariant kernels correspond in the local framework to the constant coefficient case, asking K to be locally Hölder continuous is a legitimate counterpart to the Lipschitz regularity assumed on the matrix A in (4.1.1).

Here we show that a solution u of (3.1.1) has better weak *fractional* differentiability properties in the interior of Ω. By adapting the translation method to this nonlocal setting, we prove that

$$u \in N^{2s,2}_{\mathrm{loc}}(\Omega). \tag{4.1.2}$$

Notice that the symbol $N^{r,p}(\Omega)$, for $r > 0$ and $1 \leqslant p < +\infty$, denotes here the so-called *Nikol'skii space*.

Since both Nikol'skii and fractional Sobolev spaces are part of the wider class of Besov spaces, standard embedding results within this scale allow us to deduce from (4.1.2) that

$$u \in H^{2s-\varepsilon}_{\mathrm{loc}}(\Omega), \tag{4.1.3}$$

for any $\varepsilon > 0$.

We do not know whether or not (4.1.3) is the optimal interior regularity for solutions of (3.1.1) in the Sobolev class. While one would arguably expect u to belong to $H^{2s}_{\mathrm{loc}}(\Omega)$, there is no hope in general to extend such regularity up to the boundary, as discussed in Section 4.8. Finally, we point out that the exponent $2s - \varepsilon$ still provides Sobolev regularity for the gradient of u, when $s > 1/2$.

In the upcoming section we specify the framework in which the model is set. We give formal definitions of the notion of solution and of the class of kernels under consideration. Moreover, we introduce the various functional spaces that are necessary for these purposes. After such preliminary work, we are then in position to give the precise statements of our results.

4.2 Definitions and formal statements

Let $n \in \mathbb{N}$ and $s \in (0,1)$. The kernel $K : \mathbb{R}^n \times \mathbb{R}^n \to [0, +\infty]$ is assumed to be measurable and symmetric[5], that is

$$K(x,y) = K(y,x) \quad \text{for a.a. } x, y \in \mathbb{R}^n. \tag{4.2.1}$$

We also require K to satisfy

$$\lambda \leqslant |x-y|^{n+2s} K(x,y) \leqslant \Lambda \quad \text{for a.a. } x, y \in \mathbb{R}^n,\ |x-y| < 1, \tag{4.2.2a}$$

$$0 \leqslant |x-y|^{n+\beta} K(x,y) \leqslant M \quad \text{for a.a. } x, y \in \mathbb{R}^n,\ |x-y| \geqslant 1, \tag{4.2.2b}$$

for some constants $\Lambda \geqslant \lambda > 0$, $\beta, M > 0$, and

$$|x-y|^{n+2s} \left| K(x+z, y+z) - K(x,y) \right| \leqslant \Gamma |z|^s, \tag{4.2.3}$$

[5] We stress that the symmetry hypothesis does not really play much of a role here. Indeed, if one considers instead a non-symmetric kernel K, this can be written as the sum of its symmetric and anti-symmetric parts

$$K_{\mathrm{sym}}(x,y) := \frac{K(x,y) + K(x,y)}{2} \quad \text{and} \quad K_{\mathrm{asym}}(x,y) := \frac{K(x,y) - K(y,x)}{2}.$$

But then, it is easily shown that K_{asym} cancels out in (3.2.5), thus leading to an equation driven by the symmetric kernel K_{sym}. Hence, we may and do assume K symmetric from the outset.

In this regard, we refer the interested reader to [FKV15], where a class of integro-differential equations with non-symmetric kernels are studied.

for a.a. $x, y, z \in \mathbb{R}^n$, with $|x-y|, |z| < 1$, and for some $\Gamma > 0$.

Note that condition (4.2.2) is slightly more general than (3.1.5) in Chapter 3 (with $r_0 = 1$). In particular, (4.2.2b) allows for kernels having non-standard decay at infinity, with an order of homogeneity possibly greater than that near the origin.

On the other hand, (4.2.3) asserts that the map

$$(x, y) \longmapsto |x-y|^{n+2s} K(x, y),$$

is locally uniformly $C^{0,s}$ regular, jointly in the two variables x and y. Clearly, (4.2.3) is satisfied by translation-invariant kernels. But more general choices are possible, as for instance kernels of the type

$$K(x, y) = \frac{a(x, y)}{|x-y|^{n+2s}},$$

with $a \in C^{0,s}(\mathbb{R}^n \times \mathbb{R}^n)$. We also stress that (4.2.3) may be actually weakened by requiring it to hold only inside the set Ω where the equation will be valid.

By referring to Section 3.2 for the main definitions involved in the statements, we can now proceed to present the main contributions of the chapter. The first and principal result is given by

Theorem 4.1. *Let $s \in (0,1)$, $\beta > 0$ and $\Omega \subset \mathbb{R}^n$ be an open set. Assume that K satisfies assumptions (4.2.1), (4.2.2) and (4.2.3). Let $u \in \mathbb{H}^K(\Omega) \cap L^1_\beta(\mathbb{R}^n)$ be a solution of (3.1.1), with $f \in L^2(\Omega)$. Then, $u \in H^{2s-\varepsilon}_{\mathrm{loc}}(\Omega)$ for any small $\varepsilon > 0$ and, for any domain $\Omega' \subset\subset \Omega$,*

$$\|u\|_{H^{2s-\varepsilon}(\Omega')} \leqslant C \left(\|u\|_{L^2(\Omega)} + \|u\|_{L^1_\beta(\mathbb{R}^n)} + \|f\|_{L^2(\Omega)} \right), \tag{4.2.4}$$

for some constant $C > 0$ depending on n, s, β, λ, Λ, M, Γ, Ω, Ω' and ε.

The technique we adopt to prove Theorem 4.1 is basically the translation method of Nirenberg, suitably adjusted to cope with the difficulties arising in this fractional, nonlocal framework. However, this strategy does not immediately lead to an estimate in Sobolev spaces. In fact, it provides that the solution belongs to a slightly different functional space, which is well-studied in the literature and is often referred to as *Nikol'skii space*. We briefly introduce such class here below.

Let U be a domain of $\mathbb{R}^n$. Given $k \in \mathbb{N}$ and $z \in \mathbb{R}^n$, let

$$U_{kz} := \{x \in U : x + iz \in U \text{ for any } i = 1, \dots, k\}. \tag{4.2.5}$$

Observe that, by definition,

$$U_{kz} \subseteq U_{jz} \subseteq U \qquad \text{if } j, k \in \mathbb{N} \text{ and } j \leqslant k. \tag{4.2.6}$$

For any $z \in \mathbb{R}^n$ we also define $\tau_z u(x) := u(x+z)$ and

$$\Delta_z u(x) := \tau_z u(x) - u(x),$$

for any $x \in U_z$. Sometimes we will need to deal with increments along the *diagonal* for the kernel K, as previously done in (4.2.3). With a slight abuse of notation, we write

$$\tau_z K(x, y) := K(x+z, y+z) \quad \text{ and } \quad \Delta_z K(x, y) := \tau_z K(x, y) - K(x, y).$$

We also consider increments of higher orders. For any $k \in \mathbb{N}$ we set

$$\Delta^k_z u(x) := \Delta_z \Delta^{k-1}_z u(x) = \sum_{i=0}^{k} (-1)^{k-i} \binom{k}{i} \tau_{iz} u(x),$$

for any $x \in U_{kz}$, with the convention that $\Delta_z^0 u = u$. Of course, $\Delta_z^1 u = \Delta_z u$. Moreover, notice that by (4.2.6) all $\Delta_z^j u$, as $j = 0, 1, \ldots, k$, are well-defined in U_{kz}.

Given $s \in (0,2)$ and $1 \leqslant p < +\infty$, the Nikol'skii space $N^{s,p}(U)$ is defined as the space of functions $u \in L^p(U)$ such that

$$[u]_{N^{s,p}(U)} := \sup_{z \in \mathbb{R}^n \setminus \{0\}} |z|^{-s} \|\Delta_z^2 u\|_{L^p(U_{2z})} < +\infty. \tag{4.2.7}$$

The norm

$$\|u\|_{N^{s,p}(U)} := \|u\|_{L^p(U)} + [u]_{N^{s,p}(U)},$$

makes $N^{s,p}(U)$ a Banach space. We point out that the restriction to $s < 2$ is assumed here only to avoid unnecessary complications in the definition of the semi-norm (4.2.7). By the way, the above range for s is large enough for our scopes and, thus, there is no real need to deal with more general conditions. Nevertheless, such limitation will not be considered anymore in Section 4.3, where a deeper look at the space $N^{s,p}(U)$ will be given.

Now that the definition of Nikol'skii spaces has been recalled, we may finally head to our second main result.

Theorem 4.2. *Let $s \in (0,1)$, $\beta > 0$ and $\Omega \subset \mathbb{R}^n$ be an open set. Assume that K satisfies assumptions (4.2.1), (4.2.2) and (4.2.3). Let $u \in \mathbb{H}^K(\Omega) \cap L^1_\beta(\mathbb{R}^n)$ be a solution of (3.1.1), with $f \in L^2(\Omega)$. Then, $u \in N^{2s,2}_{\mathrm{loc}}(\Omega)$ and, for any domain $\Omega' \subset\subset \Omega$,*

$$\|u\|_{N^{2s,2}(\Omega')} \leqslant C \left(\|u\|_{L^2(\Omega)} + \|u\|_{L^1_\beta(\mathbb{R}^n)} + \|f\|_{L^2(\Omega)} \right), \tag{4.2.8}$$

for some constant $C > 0$ depending on n, s, β, λ, Λ, M, Γ, Ω and Ω'.

In light of this estimate, Theorem 4.1 follows more or less immediately. To see this, it is helpful to understand Sobolev and Nikol'skii spaces in the context of *Besov spaces*. For $s \in (0,2)$, $1 \leqslant p < +\infty$ and $1 \leqslant \lambda \leqslant +\infty$, the Besov space $B^{s,p}_\lambda(U)$ is the space of functions $u \in L^p(U)$ such that $[u]_{B^{s,p}_\lambda(U)} < +\infty$, where

$$[u]_{B^{s,p}_\lambda(U)} := \begin{cases} \left(\displaystyle\int_{\mathbb{R}^n} \left(|z|^{-s} \|\Delta_z^2 u\|_{L^p(U_{2z})} \right)^\lambda \frac{dz}{|z|^n} \right)^{1/\lambda} & \text{if } 1 \leqslant \lambda < +\infty, \\ \displaystyle\sup_{z \in \mathbb{R}^n \setminus \{0\}} |z|^{-s} \|\Delta_z^2 u\|_{L^p(U_{2z})} & \text{if } \lambda = +\infty. \end{cases}$$

Observe that, by definition, $B^{s,p}_\infty(U) = N^{s,p}(U)$, while the equivalence $B^{s,p}_p(U) = W^{s,p}(U)$ is also true, though less trivial. Then, since there exist continuous embeddings

$$B^{s,p}_\nu(U) \subset B^{r,p}_\lambda(U), \tag{4.2.9}$$

as $1 \leqslant \lambda \leqslant \nu \leqslant +\infty$ and $1 < r < s < +\infty$, it follows

$$N^{s,p}(U) \subset W^{r,p}(U).$$

Consequently, up to some minor details that will be discussed later in Section 4.7, Theorem 4.1 is a consequence of Theorem 4.2.

Of course, Theorem 4.2 and inclusion (4.2.9) yield estimates in many other Besov spaces for the solution u of (3.1.1). Basically, u lies in any $B^{2s-\varepsilon,2}_{\lambda,\mathrm{loc}}(\Omega)$, with $\varepsilon > 0$ and $1 \leqslant \lambda \leqslant +\infty$.

We point out here that throughout the chapter the same letter c is used to denote a positive constant which may change from line to line and depends on the various parameters involved.

The rest of the chapter is organized as follows.

In Section 4.3 we review some basic material on Sobolev and Nikol'skii spaces. To keep a leaner notation, we do not approach Besov spaces in their full generality and restrict in fact to the two classes to which we are interested. Despite every assertion of this section is classical and surely well-known to the experts, we choose to include here the few results that will be used afterwards, in order to make the work as self-contained as possible.

The subsequent two sections are devoted to some auxiliary results. Section 4.4 is concerned with a couple of technical lemmata that deal with a discrete integration by parts formula and an estimate for the defect of two translated balls. In Section 4.5, on the other hand, we prove a nonlocal version of the classical Caccioppoli inequality.

The main results are proved in Sections 4.6 and 4.7.

Finally, Section 4.8 contains some comments on the possible optimal global regularity for the Dirichlet problem (3.1.2) with homogeneous datum.

4.3 Preliminaries on Sobolev and Nikol'skii spaces

We collect here some general facts about fractional Sobolev spaces and Nikol'skii spaces. As said before, we avoid dealing with the wider class of Besov spaces in order not to burden the notation too much. For more complete and exhaustive presentations we refer the interested reader to the books by Triebel, [T83, T92, T06] and [T95].

We remark that the proofs displayed only make use of integration techniques, mostly inspired by [S90]. While some results can not be justified with such elementary arguments, we still provide specific references to the above mentioned books.

Let $U \subset \mathbb{R}^n$ be a bounded domain with C^∞ boundary[6]. Let $1 \leqslant p < +\infty$ and $s > 0$, with $s \notin \mathbb{N}$. Write $s = k + \sigma$, with $k \in \mathbb{N} \cup \{0\}$ and $\sigma \in (0,1)$. We recall that the fractional Sobolev space $W^{s,p}(U)$ is defined as the set of functions

$$W^{s,p}(U) := \Big\{ u \in W^{k,p}(U) : [D_\alpha u]_{W^{\sigma,p}(U)} < +\infty \text{ for any } |\alpha| = k \Big\},$$

where, for $v \in L^p(U)$,

$$[v]_{W^{\sigma,p}(U)} := \left(\int_U \int_U \frac{|v(x) - v(y)|^p}{|x-y|^{n+\sigma p}} \, dx dy \right)^{1/p}.$$

Clearly, α indicates a multi-index, i.e. $\alpha = (\alpha_1, \ldots, \alpha_n)$ with $\alpha_i \in \mathbb{N} \cup \{0\}$, and $|\alpha| = \alpha_1 + \cdots + \alpha_n$ is its modulus. Moreover, $W^{k,p}(\Omega)$, for $k \in \mathbb{N}$, denotes the standard Sobolev space and, when $k = 0$, we understand $W^{0,p}(U) = L^p(U)$. The space $W^{s,p}(U)$ equipped with the norm

$$\|u\|_{W^{s,p}(U)} := \|u\|_{W^{k,p}(U)} + \sum_{|\alpha|=k} [D^\alpha u]_{W^{\sigma,p}(U)},$$

is a Banach space.

[6]Most of the assertions contained in this section should be also true under less restrictive hypotheses on the boundary of the set. Of course, the definitions of the spaces require no assumptions at all on the boundary and other results are extended in the literature to Lipschitz sets. Unfortunately, we have not been able to find completely satisfactory references for Proposition 4.3.1, and its counterpart for Nikol'skii spaces, under such weaker assumptions. Anyway, the limitation to C^∞ domains will not have any influence on our applications.

Notice that, for $v \in L^p(U)$,

$$[v]_{W^{\sigma,p}(U)} = \left(\int_U \int_U \frac{|v(x) - v(y)|^p}{|x-y|^{n+\sigma p}} \, dxdy \right)^{1/p} = \left(\int_{\mathbb{R}^n} \left(\int_{U_z} \frac{|v(x+z) - v(x)|^p}{|z|^{n+\sigma p}} \, dx \right) dz \right)^{1/p} = \left(\int_{\mathbb{R}^n} \left(|z|^{-\sigma} \|\Delta_z v\|_{L^p(U_z)} \right)^p \frac{dz}{|z|^n} \right)^{1/p}.$$

In view of this fact, we have the following characterization for $W^{s,p}(U)$.

Proposition 4.3. *Let $1 \leqslant p < +\infty$ and $s > 0$. Let $k, l \in \mathbb{Z}$ be such that $0 \leqslant k < s$ and $l > s - k$. Then,*

$$\|u\|_{L^p(U)} + \sum_{|\alpha|=k} \left(\int_{\mathbb{R}^n} \left(|z|^{k-s} \|\Delta_z^l D^\alpha u\|_{L^p(U_{lz})} \right)^p \frac{dz}{|z|^n} \right)^{1/p}, \tag{4.3.1}$$

is a Banach space norm for $W^{s,p}(U)$, equivalent to $\|\cdot\|_{W^{s,p}(U)}$.

A reference for these equivalences is given by Theorem 4.4.2.1 at page 323 of [T95]. Note that the result is valid even if s is an integer.

Remark 4.4. In what follows, we will be mostly interested in norms with $k = 0$ and therefore $l > s$. In such cases, we stress that (4.3.1) may be replaced with the *restricted* norm

$$\|u\|_{L^p(U)} + \left(\int_{B_\delta} \left(|z|^{-s} \|\Delta_z^l u\|_{L^p(U_{lz})} \right)^p \frac{dz}{|z|^n} \right)^{1/p}, \tag{4.3.2}$$

for any $\delta > 0$, with no modifications to the space $W^{s,p}(U)$. Indeed, we have

$$\|\Delta_z^l u\|_{L^p(U_{lz})} \leqslant 2^l \|u\|_{L^p(U)},$$

so that

$$\left(\int_{\mathbb{R}^n \setminus B_\delta} \left(|z|^{-s} \|\Delta_z^l u\|_{L^p(U_{lz})} \right)^p \frac{dz}{|z|^n} \right)^{1/p} \leqslant 2^l \left(\frac{\mathcal{H}^{n-1}(\partial B_1)}{sp} \right)^{1/p} \delta^{-s} \|u\|_{L^p(U)}.$$

Consequently, the norms defined by (4.3.1) and (4.3.2) are equivalent.

The second class of fractional spaces which we are interested in are the Nikol'skii spaces. For $s = k + \sigma > 0$, with $k \in \mathbb{N} \cup \{0\}, \sigma \in (0,1]$, and $1 \leqslant p < +\infty$, define

$$N^{s,p}(U) := \left\{ u \in W^{k,p}(U) : [D^\alpha u]_{N^{\sigma,p}(U)} < +\infty \text{ for any } |\alpha| = k \right\},$$

where, for $v \in L^p(U)$,

$$[v]_{N^{\sigma,p}(U)} := \sup_{z \in \mathbb{R}^n \setminus \{0\}} |z|^{-\sigma} \|\Delta_z^2 v\|_{L^p(U_{2z})}.$$

It can be showed that $N^{s,p}(U)$ is a Banach space with respect to the norm

$$\|u\|_{N^{s,p}(U)} := \|u\|_{W^{k,p}(U)} + [u]_{N^{s,p}(U)}.$$

Notice that this definition of Nikol'skii space may seem to differ from that given in Section 4.2. In fact, this is not the case, as $N^{s,p}(U)$ can be equivalently endowed with any norm of the form

$$\|u\|_{L^p(U)} + \sum_{|\alpha|=k} \sup_{z \in \mathbb{R}^n \setminus \{0\}} |z|^{k-s} \|\Delta_z^l D^\alpha u\|_{L^p(U_{lz})}, \tag{4.3.3}$$

where $k, l \in \mathbb{Z}$ are such that $0 \leqslant k < s$ and $l > s - k$ (see again Theorem 4.4.2.1 of [T95]).

Remark 4.5. As for the Sobolev spaces, we will consider norms with $k = 0$ for the most of the time. We stress that in such cases (4.3.3) may be replaced with

$$\|u\|_{L^p(U)} + \sup_{0<|z|<\delta} |z|^{-s} \|\Delta_z^l u\|_{L^p(U_{lz})},$$

for any integer $l > s$ and any $\delta > 0$.

In the conclusive part of this section we study the mutual inclusion properties of $W^{s,p}(U)$ and $N^{s,p}(U)$. In order to do this, it will be useful to consider another family of equivalent norms. To this aim, for $l \in \mathbb{N}$ we introduce the so-called l-th *modulus of smoothness* of u

$$\omega_p^l(u;\eta) := \sup_{0<|z|<\eta} \|\Delta_z^l u\|_{L^p(U_{lz})},$$

defined for any $\eta > 0$. Then, we have

Proposition 4.6. *Let $s > 0$ and $1 \leqslant p < +\infty$. Let $l > s$ be an integer and $0 < \delta \leqslant +\infty$. Then,*

$$\|u\|_{L^p(U)} + \left(\int_0^\delta \left(\eta^{-s} \omega_p^l(u;\eta) \right)^p \frac{d\eta}{\eta} \right)^{1/p},$$

is a Banach space norm for $W^{s,p}(U)$, equivalent to $\|\cdot\|_{W^{s,p}(U)}$.

The same statement holds true for the norms

$$\|u\|_{L^p(U)} + \sup_{0<\eta<\delta} \eta^{-s} \omega_p^l(u;\eta),$$

and the space $N^{s,p}(U)$.

Proof. We only deal with the Sobolev space case, the Nikol'skii one being completely analogous and easier. Furthermore, we assume $\delta = 1$. Then, an argument similar to that presented in Remark 4.4 shows that the result can be extended to any δ.

For $u \in L^p(U)$ let

$$[u]^\flat_{W^{s,p}(U)} := \left(\int_{B_1} \left(|z|^{-s} \|\Delta_z^l u\|_{L^p(U_{lz})} \right)^p \frac{dz}{|z|^n} \right)^{1/p},$$

and

$$[u]^\sharp_{W^{s,p}(U)} := \left(\int_0^1 \left(\eta^{-s} \omega_p^l(u;\eta) \right)^p \frac{d\eta}{\eta} \right)^{1/p}.$$

We claim that there exists a constant $c \geqslant 1$ such that

$$c^{-1} [u]^\flat_{W^{s,p}(U)} \leqslant [u]^\sharp_{W^{s,p}(U)} \leqslant c \left(\|u\|_{L^p(U)} + [u]^\flat_{W^{s,p}(U)} \right), \tag{4.3.4}$$

for all $u \in L^p(U)$. In view of Proposition 4.3 and Remark 4.4, this concludes the proof.

To check the left hand inequality of (4.3.4) we first observe that

$$\|\Delta_z^l u\|_{L^p(U_{lz})} \leqslant \sup_{0<|y|<|z|} \|\Delta_y^l u\|_{L^p(U_{ly})} = \omega_p^l(u;|z|),$$

for any $z \in \mathbb{R}^n$. Then, using polar coordinates,

$$\begin{aligned}
[u]^\flat_{W^{s,p}(U)} &= \left(\int_{B_1} \left(|z|^{-s} \|\Delta_z^l u\|_{L^p(U_{lz})} \right)^p \frac{dz}{|z|^n} \right)^{1/p} \\
&\leqslant \left(\mathcal{H}^{n-1}(\partial B_1) \int_0^1 \left(\eta^{-s} \omega_p^l(u;\eta) \right)^p \frac{d\eta}{\eta} \right)^{1/p} \\
&= \mathcal{H}^{n-1}(\partial B_1)^{1/p} \, [u]^\sharp_{W^{s,p}(U)}.
\end{aligned}$$

Now we focus on the second inequality. In order to show its validity we need the following auxiliary result. For $x \in U$, $\eta > 0$ and $u \in L^p(U)$, let

$$
\begin{aligned}
V^l(x,\eta) &:= \{z \in B_\eta : x + \tau z \in U, \text{ for any } 0 \leqslant \tau \leqslant l\}, \\
M^l_\eta u(x) &:= \eta^{-n} \int_{V^l(x,\eta)} |\Delta^l_z u(x)|\, dz,
\end{aligned}
$$

and define

$$
\begin{aligned}
[u]^*_{W^{s,p}(U)} &:= \left(\int_0^1 \left(\eta^{-s} \|M^l_\eta u\|_{L^p(U)} \right)^p \frac{d\eta}{\eta} \right)^{1/p}, \\
\|u\|^*_{W^{s,p}(U)} &:= \|u\|_{L^p(U)} + [u]^*_{W^{s,p}(U)}.
\end{aligned} \tag{4.3.5}
$$

Then, by virtue of [T06, Theorem 1.118] we infer that

$$
[u]^\sharp_{W^{s,p}(U)} \leqslant c \|u\|^*_{W^{s,p}(U)}, \tag{4.3.6}
$$

for any $u \in L^p(U)$.

Applying the generalized Minkowski's inequality to the right-hand side of (4.3.5) and observing that

$$
\{(x,z) \in U \times \mathbb{R}^n : z \in V^l(x,\eta)\} \subseteq \{(x,z) \in U \times B_\eta : x \in U_{lz}\},
$$

we get

$$
\begin{aligned}
[u]^*_{W^{s,p}(U)} &= \left(\int_0^1 \eta^{-(s+n)p} \left(\int_U \left(\int_{V^l(x,\eta)} |\Delta^l_z u(x)|\, dz \right)^p dx \right) \frac{d\eta}{\eta} \right)^{1/p} \\
&\leqslant \left(\int_0^1 \eta^{-(s+n)p} \left(\int_{B_\eta} \|\Delta^l_z u\|_{L^p(U_{lz})}\, dz \right)^p \frac{d\eta}{\eta} \right)^{1/p}.
\end{aligned} \tag{4.3.7}
$$

Now, Jensen's inequality implies that

$$
\left(\int_{B_\eta} \|\Delta^l_z u\|_{L^p(U_{lz})}\, dz \right)^p \leqslant c\, \eta^{n(p-1)} \int_{B_\eta} \|\Delta^l_z u\|^p_{L^p(U_{lz})}\, dz,
$$

and hence (4.3.7) becomes

$$
[u]^*_{W^{s,p}(U)} \leqslant c \left(\int_0^1 \eta^{-n-1-sp} \left(\int_{B_\eta} \|\Delta^l_z u\|^p_{L^p(U_{lz})}\, dz \right) d\eta \right)^{1/p}.
$$

We finally switch to polar coordinates to compute

$$
\begin{aligned}
[u]^*_{W^{s,p}(U)} &\leqslant c \left(\int_0^1 \int_0^\eta \eta^{-n-1-sp} \left(\int_{\partial B_\rho} \|\Delta^l_z u\|^p_{L^p(U_{lz})}\, d\mathcal{H}^{n-1}(z) \right) d\rho\, d\eta \right)^{1/p} \\
&= c \left(\int_0^1 \left(\int_{\partial B_\rho} \|\Delta^l_z u\|^p_{L^p(U_{lz})}\, d\mathcal{H}^{n-1}(z) \right) \left(\int_\rho^1 \eta^{-n-1-sp}\, d\eta \right) d\rho \right)^{1/p} \\
&\leqslant c \left(\int_0^1 \left(\int_{\partial B_\rho} \|\Delta^l_z u\|^p_{L^p(U_{lz})}\, d\mathcal{H}^{n-1}(z) \right) \rho^{-n-sp}\, d\rho \right)^{1/p} \\
&= c[u]^\flat_{W^{s,p}(U)}.
\end{aligned}
$$

By combining this formula with (4.3.6), we obtain the right inequality of (4.3.4). Thus, the proof of the proposition is complete. □

We are now in position to prove the main results of this section, concerning the relation between Sobolev and Nikol'skii spaces. First, we have

Proposition 4.7. *Let $s > 0$ and $1 \leqslant p < +\infty$. Then,*

$$W^{s,p}(U) \subseteq N^{s,p}(U),$$

and there exists a constant $C > 0$, depending on n, s and p, such that

$$\|u\|_{N^{s,p}(U)} \leqslant C\|u\|_{W^{s,p}(U)},$$

for any $u \in L^p(U)$.

Proof. In view of Proposition 4.6 it is enough to prove that, if $l \in \mathbb{Z}$ is such that $l > s$, then

$$\sup_{\eta>0} \eta^{-s}\omega_p^l(u;\eta) \leqslant c \left(\int_0^{+\infty} \left(\eta^{-s}\omega_p^l(u;\eta) \right)^p \frac{d\eta}{\eta} \right)^{1/p}, \tag{4.3.8}$$

for some $c > 0$. But this is in turn an immediate consequence of the monotonicity of $\omega_p^l(u;\cdot)$. Indeed, $\omega_p^l(u;\eta) \geqslant \omega_p^l(u;t)$, for any $\eta \geqslant t$, and so

$$\left(\int_0^{+\infty} \left(\eta^{-s}\omega_p^l(u;\eta) \right)^p \frac{d\eta}{\eta} \right)^{1/p} \geqslant \left(\int_t^{+\infty} \left(\eta^{-s}\omega_p^l(u;t) \right)^p \frac{d\eta}{\eta} \right)^{1/p} = (sp)^{-1/p} t^{-s}\omega_p^l(u;t).$$

Inequality (4.3.8) is then obtained by taking the supremum as $t > 0$ on the right hand side. □

The following provides a partial converse to the above inclusion.

Proposition 4.8. *Let $s > r > 0$ and $1 \leqslant p < +\infty$. Then,*

$$N^{s,p}(U) \subseteq W^{r,p}(U),$$

and there exists a constant $C > 0$, depending on n, r, s and p, such that

$$\|u\|_{W^{r,p}(U)} \leqslant C\|u\|_{N^{s,p}(U)},$$

for any $u \in L^p(U)$.

Proof. The result follows by noticing that, for $l \in \mathbb{Z}$ with $l > s$,

$$\begin{aligned} \left(\int_0^1 \left(\eta^{-r}\omega_p^l(u;\eta) \right)^p \frac{d\eta}{\eta} \right)^{1/p} &= \left(\int_0^1 \eta^{(s-r)p} \left(\eta^{-s}\omega_p^l(u;\eta) \right)^p \frac{d\eta}{\eta} \right)^{1/p} \\ &\leqslant [(s-r)p]^{-1/p} \sup_{0<\eta<1} \eta^{-s}\omega_p^l(u;\eta), \end{aligned}$$

for any $u \in L^p(U)$, and recalling Proposition 4.6. □

4.4 Some auxiliary results

Before we can proceed to Sections 4.5 and 4.6, which contain the core argumentations leading to Theorem 4.2, we need to prove a couple of subsidiary result.

First, we prove the following discrete version of the standard integration by parts formula.

Lemma 4.9. *Let B_R be some ball of radius $R>0$ in $\mathbb{R}^n$. Assume that K satisfies assumptions (4.2.1) and (4.2.2). Let $u, v \in H^s(B_{8R})$, with v supported in B_{2R}. Then,*

$$\begin{aligned}
&\int_{B_{8R}}\int_{B_{8R}} (u(x)-u(y))\left(\Delta^2_{-z}v(x)-\Delta^2_{-z}v(y)\right)K(x,y)\,dxdy \\
&\quad= \int_{B_{6R}}\int_{B_{6R}} \left(\Delta^2_z u(x)-\Delta^2_z u(y)\right)(v(x)-v(y))\,K(x,y)\,dxdy \\
&\qquad+\sum_{i=1}^{2}(-1)^i\binom{2}{i}\int_{B_{6R}}\int_{B_{6R}} (\tau_{iz}u(x)-\tau_{iz}u(y))\,(v(x)-v(y))\,\Delta_{iz}K(x,y)\,dxdy \\
&\qquad-2\sum_{i=0}^{2}(-1)^i\binom{2}{i}\int_{B_{8R}}\int_{B_{8R}} (u(x)-u(y))\,\tau_{-iz}\chi_{\mathbb{R}^n\setminus B_{6R}}(x)\tau_{-iz}v(y) \\
&\qquad\qquad\times K(x,y)\,dxdy,
\end{aligned} \tag{4.4.1}$$

for any $z\in\mathbb{R}^n$ such that $|z|<R$.

Proof. We first expand the integral on the left hand side of (4.4.1), obtaining

$$\begin{aligned}
&\int_{B_{8R}}\int_{B_{8R}} (u(x)-u(y))\left(\Delta^2_{-z}v(x)-\Delta^2_{-z}v(y)\right)K(x,y)\,dxdy \\
&=\sum_{i=0}^{2}(-1)^i\binom{2}{i}\int_{B_{8R}}\int_{B_{8R}} (u(x)-u(y))\,(v(x-iz)-v(y-iz))\,K(x,y)\,dxdy.
\end{aligned} \tag{4.4.2}$$

Then, we write each term on the right hand side of (4.4.2) as[7]

$$\begin{aligned}
&\int_{B_{8R}}\int_{B_{8R}} (u(x)-u(y))\,(v(x-iz)-v(y-iz))\,K(x,y)\,dxdy \\
&\qquad= \int_{B_{6R}+iz}\int_{B_{6R}+iz} (u(x)-u(y))\,(v(x-iz)-v(y-iz))\,K(x,y)\,dxdy \\
&\qquad\quad-2\int_{B_{8R}}\int_{B_{8R}} (u(x)-u(y))\,\chi_{\mathbb{R}^n\setminus(B_{6R}+iz)}(x)v(y-iz)K(x,y)\,dxdy.
\end{aligned} \tag{4.4.3}$$

We apply the change of variables $\tilde{x}:=x-iz$, $\tilde{y}:=y-iz$ in the first integral, to get

$$\begin{aligned}
&\int_{B_{6R}+iz}\int_{B_{6R}+iz} (u(x)-u(y))\,(v(x-iz)-v(y-iz))\,K(x,y)\,dxdy \\
&\quad= \int_{B_{6R}}\int_{B_{6R}} (u(\tilde{x}+iz)-u(\tilde{y}+iz))\,(v(\tilde{x})-v(\tilde{y}))\,K(\tilde{x}+iz,\tilde{y}+iz)\,d\tilde{x}d\tilde{y}.
\end{aligned} \tag{4.4.4}$$

Writing then for $i=1,2$

$$K(\tilde{x}+iz,\tilde{y}+iz)=K(\tilde{x},\tilde{y})+\Delta_{iz}K(\tilde{x},\tilde{y}),$$

[7] The symbol $D+z$, where D is a set and z a vector of $\mathbb{R}^n$, identifies, as conventional, the set

$$\{y\in\mathbb{R}^n : y=x+z \text{ with } x\in D\}.$$

In the following formulae it is applied with D an Euclidean ball B_r. Also, it should not be confused with the notation $(B_r)_z$, which will be used later on in Section 4.6 and has to be understood in the sense of definition (4.2.5).

and relabeling the variables $\tilde{x}, \tilde{y}$ as x, y, formula (4.4.4) becomes

$$\begin{aligned}\int_{B_{6R}+iz}\int_{B_{6R}+iz} &(u(x)-u(y))\,(v(x-iz)-v(y-iz))\,K(x,y)\,dxdy\\ &= \int_{B_{6R}}\int_{B_{6R}} (u(x+iz)-u(y+iz))\,(v(x)-v(y))\,K(x,y)\,dxdy \qquad (4.4.5)\\ &\quad+ \int_{B_{6R}}\int_{B_{6R}} (u(x+iz)-u(y+iz))\,(v(x)-v(y))\,\Delta_{iz}K(x,y)\,dxdy.\end{aligned}$$

By using (4.4.3), (4.4.5) in (4.4.2) and noticing that $\tau_{-iz}\chi_{\mathbb{R}^n\setminus B_{6R}} = \chi_{\mathbb{R}^n\setminus(B_{6R}+iz)}$, we finally obtain (4.4.1). □

Then, we have the following result, in which we deduce an upper bound for the measure of the symmetric difference of two translated balls in terms of the modulus of the displacement vector. Despite the estimate is almost immediate, we include a proof of it for completeness.

We also refer to [S10] for a refined version of this result, holding for general bounded sets.

Lemma 4.10. *Let B_R be some ball of radius $R > 0$ in $\mathbb{R}^n$. Then, for any $z \in \mathbb{R}^n$,*

$$|B_R \Delta (B_R + z)| \leqslant C R^{n-1}|z|,$$

where $C > 0$ is a dimensional constant.

Proof. First, we observe that we may restrict ourselves to $|z| \leqslant R/2$, being the opposite case trivial. With the change of variables $y := x/R$, we scale

$$|B_R \Delta (B_R + z)| = 2\int_{B_R\setminus(B_R+z)} dx = 2R^n \int_{B_1\setminus(B_1+\hat{z})} dy,$$

where $\hat{z} = z/R$. Then, we easily check that

$$B_{1-|\hat{z}|} \subset B_1 + \hat{z},$$

to obtain

$$|B_R \Delta (B_R + z)| \leqslant 2R^n \int_{B_1\setminus B_{1-|\hat{z}|}} dy = \frac{2\mathcal{H}^{n-1}(\partial B_1)}{n} R^n \left[1 - (1-|\hat{z}|)^n\right].$$

The result then follows, since $1-(1-t)^n \leqslant nt$, for any $t \geqslant 0$. □

4.5 A Caccioppoli-type inequality

In this section we present an estimate for the H^s norm of a solution u of (3.1.1) reminiscent of the classical one by Caccioppoli. Results of this kind are by now well established also for nonlocal equations, for instance in [KMS15, DCKP15, BP14].

Proposition 4.11. *Let $s \in (0,1)$, $\beta > 0$ and $\Omega \subset \mathbb{R}^n$ be an open set. Fix a point $x_0 \in \Omega$ and let $r > 0$ be such that $B_r(x_0) \subset\subset \Omega$. Assume that K satisfies assumptions (4.2.1) and (4.2.2). Let $u \in \mathbb{H}^K(\Omega) \cap L^1_\beta(\mathbb{R}^n)$ be a solution of (3.1.1), with $f \in L^2(\Omega)$. Then,*

$$[u]_{H^s(B_r(x_0))} \leqslant C\left(\|u\|_{L^2(\Omega)} + \|u\|_{L^1_{x_0,\beta}(\mathbb{R}^n)} + \|f\|_{L^2(\Omega)}\right), \qquad (4.5.1)$$

for some constant $C > 0$ depending on n, s, β, λ, Λ, M, r and $\operatorname{dist}(B_r(x_0), \partial\Omega)$.

We stress that hypothesis (4.2.3) is not assumed here. Consequently, Proposition 4.11 holds for a general measurable K which only satisfies (4.2.2).

Proof of Proposition 4.11. Our argument follows the lines of those contained in the above mentioned papers. Anyway, we provide all the details for the reader's convenience.

First, observe that we may assume $r < 1/2$ for the beginning. The case of a general radius $r > 0$ will then follow by a covering argument. Take $R > 0$ in such a way that $r < R < 1/2$ and $B_R(x_0) \subset \Omega$. To simplify the notation, we write B_ρ instead of $B_\rho(x_0)$, for any $\rho > 0$.

Let $\eta \in C_0^\infty(\mathbb{R}^n)$ be a cut-off function such that

$$\begin{cases} \operatorname{supp} \eta \subset B_{(R+r)/2} & \\ 0 \leqslant \eta \leqslant 1 & \text{in } \mathbb{R}^n \\ \eta = 1 & \text{in } B_r \\ |\nabla \eta| \leqslant 4/(R-r) & \text{in } \mathbb{R}^n. \end{cases} \tag{4.5.2}$$

Testing (3.2.8) with $\varphi := \eta^2 u \in \mathbb{H}_0^K(\Omega)$ we get

$$\begin{aligned} &\int_{B_R} f(x)\eta^2(x)u(x)\,dx \\ &\quad = \frac{1}{2}\int_{B_R}\int_{B_R} (u(x)-u(y))\left(\eta^2(x)u(x)-\eta^2(y)u(y)\right)K(x,y)\,dxdy \\ &\quad\quad - \int_{\mathbb{R}^n\setminus B_R}\int_{B_R} (u(x)-u(y))\,\eta^2(y)u(y)K(x,y)\,dxdy \\ &\quad =: I - J. \end{aligned} \tag{4.5.3}$$

We estimate I. Notice that

$$\begin{aligned} &(u(x)-u(y))\left(\eta^2(x)u(x)-\eta^2(y)u(y)\right) \\ &\qquad = \eta^2(x)u^2(x) - \eta^2(x)u(x)u(y) - \eta^2(y)u(x)u(y) + \eta^2(y)u^2(y) \\ &\qquad = |\eta(x)u(x)-\eta(y)u(y)|^2 - |\eta(x)-\eta(y)|^2 u(x)u(y) \\ &\qquad \geqslant |\eta(x)u(x)-\eta(y)u(y)|^2 - |\eta(x)-\eta(y)|^2 |u(x)||u(y)|, \end{aligned}$$

and, therefore, using (4.2.2a),

$$\begin{aligned} I \geqslant\, &\frac{\lambda}{2}\int_{B_R}\int_{B_R} \frac{|\eta(x)u(x)-\eta(y)u(y)|^2}{|x-y|^{n+2s}}\,dxdy \\ &- \frac{\Lambda}{2}\int_{B_R}\int_{B_R} \frac{|\eta(x)-\eta(y)|^2|u(x)||u(y)|}{|x-y|^{n+2s}}\,dxdy. \end{aligned} \tag{4.5.4}$$

Applying (4.5.2) and Young's inequality, we deduce

$$\begin{aligned} \int_{B_R}\int_{B_R} \frac{|\eta(x)-\eta(y)|^2|u(x)||u(y)|}{|x-y|^{n+2s}}\,dxdy &\leqslant \frac{16}{(R-r)^2}\int_{B_R}\int_{B_R} \frac{|u(x)||u(y)|}{|x-y|^{n+2s-2}}\,dxdy \\ &\leqslant \frac{16}{(R-r)^2}\int_{B_R}\int_{B_R} \frac{|u(x)|^2}{|x-y|^{n+2s-2}}\,dxdy \\ &\leqslant c\|u\|^2_{L^2(B_R)}, \end{aligned}$$

which, together with (4.5.4), leads to

$$I \geqslant \frac{\lambda}{2}[\eta u]^2_{H^s(B_R)} - c\|u\|^2_{L^2(B_R)}. \tag{4.5.5}$$

We now deal with J. Let $x \in \mathbb{R}^n \setminus B_R$ and $y \in B_{(R+r)/2}$. Then,

$$|y - x_0| \leqslant \frac{R+r}{2} \leqslant \frac{R+r}{2R}|x - x_0|,$$

and so

$$|x - y| \geqslant |x - x_0| - |y - x_0| \geqslant \frac{R-r}{2R}|x - x_0| \geqslant \frac{R-r}{4}\left(1 + |x - x_0|\right),$$

since $R < 1$. In view of this and (4.2.2) we have

$$K(x,y) \leqslant \Lambda \frac{\chi_{[0,1)}(|x-y|)}{|x-y|^{n+2s}} + M \frac{\chi_{[1,+\infty)}(|x-y|)}{|x-y|^{n+\beta}} \leqslant \frac{c}{1 + |x - x_0|^{n+\beta}}. \tag{4.5.6}$$

Moreover, using (4.5.2) we write

$$|u(x) - u(y)||u(y)|\eta^2(y) \leqslant |u(x)||u(y)| + |u(y)|^2,$$

and hence by (4.5.6) and Young's inequality we get

$$\begin{aligned}
|J| &\leqslant c \int_{\mathbb{R}^n \setminus B_R} \left(\int_{B_{(R+r)/2}} \frac{|u(x) - u(y)||u(y)|\eta^2(y)}{1 + |x - x_0|^{n+\beta}} \, dy \right) dx \\
&\leqslant c \left[\int_{B_{(R+r)/2}} |u(y)|^2 \, dy + \left(\int_{\mathbb{R}^n \setminus B_R} \frac{|u(x)|}{1 + |x - x_0|^{n+\beta}} \, dx \right)^2 \right] \\
&\leqslant c \left(\|u\|^2_{L^2(B_R)} + \|u\|^2_{L^1_{x_0,\beta}(\mathbb{R}^n)} \right).
\end{aligned} \tag{4.5.7}$$

Finally, we easily compute

$$\left| \int_{B_R} f(x) u(x) \eta^2(x) \, dx \right| \leqslant \frac{1}{2} \left(\|u\|^2_{L^2(B_R)} + \|f\|^2_{L^2(\Omega)} \right). \tag{4.5.8}$$

Putting (4.5.3), (4.5.5), (4.5.7) and (4.5.8) together, we obtain

$$[u]_{H^s(B_r)} \leqslant [\eta u]_{H^s(B_R)} \leqslant c \left(\|u\|_{L^2(\Omega)} + \|u\|_{L^1_{x_0,\beta}(\mathbb{R}^n)} + \|f\|_{L^2(\Omega)} \right),$$

where the first inequality follows from (4.5.2). Thus, (4.5.1) is proved. □

4.6 Proof of Theorem 4.2

We are finally in position to proceed with the demonstration of our principal contribution.

Proof of Theorem 4.2. Let $x_0 \in \Omega$ and $R \in (0, 1/56)$ be such that $B_{56R}(x_0) \subset\subset \Omega$. In the following any ball B_r will always be assumed to be centered at x_0. Let $\eta \in C_0^\infty(\mathbb{R}^n)$ be a cut-off function satisfying

$$\begin{cases} \operatorname{supp} \eta \subset B_{2R} & \\ 0 \leqslant \eta \leqslant 1 & \text{in } \mathbb{R}^n \\ \eta = 1 & \text{in } B_R \\ |\nabla \eta| \leqslant 2/R & \text{in } \mathbb{R}^n. \end{cases} \tag{4.6.1}$$

Fix $z \in \mathbb{R}^n$, with $|z| < R$, and plug $\varphi := \Delta^2_{-z}\left(\eta^2 \Delta^2_z u\right) \in \mathbb{H}^K_0(\Omega)$ in formulation (3.2.8). Writing $U = \Delta^2_z u$, we have

$$\begin{aligned}\int_{B_{3R}} & f(x)\Delta^2_{-z}\left(\eta^2 U\right)(x)\,dx \\ &= \frac{1}{2}\int_{B_{8R}}\int_{B_{8R}} (u(x)-u(y))\left(\Delta^2_{-z}\left(\eta^2 U\right)(x) - \Delta^2_{-z}\left(\eta^2 U\right)(y)\right) K(x,y)\,dxdy \\ &\quad - \int_{\mathbb{R}^n\setminus B_{8R}}\int_{B_{8R}} (u(x)-u(y))\,\Delta^2_{-z}\left(\eta^2 U\right)(y) K(x,y)\,dydx \\ &=: I - J.\end{aligned} \tag{4.6.2}$$

We apply Lemma 4.9 to I with $v = \eta^2 U$, obtaining

$$\begin{aligned} I &= \frac{1}{2}\int_{B_{6R}}\int_{B_{6R}} (U(x)-U(y))\left(\eta^2(x)U(x) - \eta^2(y)U(y)\right) K(x,y)\,dxdy \\ &\quad + \frac{1}{2}\sum_{i=1}^{2}(-1)^i\binom{2}{i}\int_{B_{6R}}\int_{B_{6R}} (\tau_{iz}u(x) - \tau_{iz}u(y))\left(\left(\eta^2 U\right)(x) - \left(\eta^2 U\right)(y)\right) \\ &\qquad\qquad \times \Delta_{iz}K(x,y)\,dxdy \\ &\quad - \sum_{i=0}^{2}(-1)^i\binom{2}{i}\int_{B_{8R}}\int_{B_{8R}} (u(x)-u(y))\left(\tau_{-iz}\chi_{\mathbb{R}^n\setminus B_{6R}}(x)\tau_{-iz}\left(\eta^2 U\right)(y)\right) \\ &\qquad\qquad \times K(x,y)\,dxdy \\ &=: I_1 + I_2 - I_3. \end{aligned} \tag{4.6.3}$$

Arguing as we did to obtain (4.5.4) in Proposition 4.11, we recover

$$I_1 \geqslant \frac{\lambda}{2}[\eta\Delta^2_z u]^2_{H^s(B_{6R})} - c\|\Delta^2_z u\|^2_{L^2(B_{6R})}. \tag{4.6.4}$$

The term I_2 can be dealt with as follows. Applying (4.2.3) together with Young's inequality, we have

$$\begin{aligned}|I_2| &\leqslant \Gamma|z|^s\sum_{i=1}^{2}\int_{B_{6R}}\int_{B_{6R}} \frac{|\tau_{iz}u(x) - \tau_{iz}u(y)|\left|\left(\eta^2 U\right)(x) - \left(\eta^2 U\right)(y)\right|}{|x-y|^{n+2s}}\,dxdy \\ &\leqslant c|z|^s\left(\delta[u]^2_{H^s(B_{8R})} + \delta^{-1}[\eta^2\Delta^2_z u]^2_{H^s(B_{6R})}\right),\end{aligned}$$

with $\delta > 0$. Taking $\delta = \varepsilon^{-2}|z|^s$, for some small $\varepsilon > 0$, we get

$$|I_2| \leqslant c\left(\varepsilon^{-2}|z|^{2s}[u]^2_{H^s(B_{8R})} + \varepsilon^2[\eta^2\Delta^2_z u]^2_{H^s(B_{6R})}\right). \tag{4.6.5}$$

We now estimate I_3. By adding and subtracting the terms $\tau_{-2z}\chi_{\mathbb{R}^n\setminus B_{6R}}(x)\tau_{-z}(\eta^2 U)(y)$ and $\tau_{-z}\chi_{\mathbb{R}^n\setminus B_{6R}}(x)(\eta^2 U)(y)$, we see that

$$\begin{aligned} I_3 &= \sum_{i=0}^{1}\int_{B_{8R}}\int_{B_{8R}} (u(x)-u(y))\,\tau_{-(i+1)z}\chi_{\mathbb{R}^n\setminus B_{6R}}(x)\Delta_{-z}(\eta^2 U)(y-iz)K(x,y)\,dxdy \\ &\quad - \sum_{i=0}^{1}\int_{B_{8R}}\int_{B_{8R}} (u(x)-u(y))\,\Delta_{-z}\chi_{\mathbb{R}^n\setminus B_{6R}}(x-iz)\tau_{-iz}(\eta^2 U)(y)K(x,y)\,dxdy \\ &=: I_3^{(1)} - I_3^{(2)}.\end{aligned}$$

On the one hand, using (4.2.2a) and again the weighted Young's inequality,

$$\begin{aligned}\left|I_3^{(1)}\right| &\leqslant \Lambda \sum_{i=0}^{1} \int_{B_{3R}+iz} \left|\Delta_{-z}\left(\eta^2 U\right)(y-iz)\right| \left(\int_{B_{8R}\setminus(B_{6R}+(i+1)z)} \frac{|u(x)|+|u(y)|}{|x-y|^{n+2s}}\,dx\right) dy \\ &\leqslant c\left(\delta\|u\|^2_{L^2(B_{8R})} + \delta^{-1}\|\Delta_{-z}\left(\eta^2\Delta_z^2 u\right)\|^2_{L^2(B_{3R})}\right).\end{aligned}$$

On the other hand

$$\left|\Delta_{-z}\chi_{\mathbb{R}^n\setminus B_{6R}}(x-iz)\right| = \chi_{(B_{6R}+(i+1)z)\Delta(B_{6R}+iz)}(x),$$

and hence

$$\begin{aligned}\left|I_3^{(2)}\right| &\leqslant \Lambda \sum_{i=0}^{1} \int_{B_{2R}+iz} \left|\eta^2(y)U(y)\right| \left(\int_{(B_{6R}+(i+1)z)\Delta(B_{6R}+iz)} \frac{|u(x)|+|u(y)|}{|x-y|^{n+2s}}\,dx\right) dy \\ &\leqslant c\left(\gamma\left|(B_{6R}+z)\,\Delta B_{6R}\right|\|u\|^2_{L^2(B_{8R})} + \gamma^{-1}\|\Delta_z^2 u\|^2_{L^2(B_{3R})}\right),\end{aligned}$$

for any $\gamma > 0$. In view of Lemma 4.10 we have

$$|(B_{6R}+z)\,\Delta B_{6R}| \leqslant c|z|.$$

Therefore,

$$\left|I_3^{(2)}\right| \leqslant c\left(\gamma|z|\|u\|^2_{L^2(B_{8R})} + \gamma^{-1}\|\Delta_z^2 u\|^2_{L^2(B_{3R})}\right).$$

The choices $\delta = \varepsilon^{-2}|z|^{2s}$ and $\gamma = |z|^{2\sigma-1}$, for some

$$\sigma \geqslant \max\{s, 1/2\}, \tag{4.6.6}$$

then yield

$$|I_3| \leqslant c\left[\varepsilon^{-2}|z|^{2s}\|u\|^2_{L^2(B_{8R})} + \varepsilon^2|z|^{-2s}\|\Delta_{-z}\left(\eta^2\Delta_z^2 u\right)\|^2_{L^2(B_{3R})} + |z|^{1-2\sigma}\|\Delta_z^2 u\|^2_{L^2(B_{3R})}\right].$$

By combining (4.6.4) and (4.6.5) with the above inequality, recalling (4.6.3) and (4.6.6) we get

$$\begin{aligned}I \geqslant \frac{\lambda}{2}[\eta\Delta_z^2 u]^2_{H^s(B_{6R})} - c\Big[&\varepsilon^{-2}|z|^{2s}\|u\|^2_{H^s(B_{8R})} + |z|^{1-2\sigma}\|\Delta_z^2 u\|^2_{L^2(B_{6R})} \\ &+ \varepsilon^2\left([\eta^2\Delta_z^2 u]^2_{H^s(B_{6R})} + |z|^{-2s}\|\Delta_{-z}\left(\eta^2\Delta_z^2 u\right)\|^2_{L^2(B_{3R})}\right)\Big].\end{aligned} \tag{4.6.7}$$

Now, we turn our attention to J. Arguing as in (4.5.7), we use once again (4.2.2), (4.6.1) and Young's inequality to obtain

$$|J| \leqslant c\left[\delta\left(\|u\|^2_{L^2(B_{3R})} + \|u\|^2_{L^1_{x_0,\beta}(\mathbb{R}^n)}\right) + \delta^{-1}\|\Delta^2_{-z}\left(\eta^2\Delta_z^2 u\right)\|^2_{L^2(B_{3R})}\right],$$

for any $\delta > 0$. Setting again $\delta = \varepsilon^{-2}|z|^{2s}$, this becomes

$$|J| \leqslant c\left[\varepsilon^{-2}|z|^{2s}\left(\|u\|^2_{L^2(B_{3R})} + \|u\|^2_{L^1_{x_0,\beta}(\mathbb{R}^n)}\right) + \varepsilon^2|z|^{-2s}\|\Delta^2_{-z}\left(\eta^2\Delta_z u\right)\|^2_{L^2(B_{3R})}\right]. \tag{4.6.8}$$

Finally, we use Young's inequality as before to deduce

$$\left|\int_{B_{3R}} f(x)\Delta^2_{-z}\left(\eta^2 U\right)(x)\,dx\right| \leqslant c\left[\varepsilon^{-2}|z|^{2s}\|f\|^2_{L^2(\Omega)} + \varepsilon^2|z|^{-2s}\|\Delta^2_{-z}\left(\eta^2\Delta_z^2 u\right)\|^2_{L^2(B_{3R})}\right].$$

By combining this last estimation, (4.6.7), (4.6.8) with (4.6.2) and noticing that

$$\|\Delta^2_{-z}\left(\eta^2\Delta_z u\right)\|_{L^2(B_{3R})} \leqslant 2\|\Delta_{-z}\left(\eta^2\Delta_z u\right)\|_{L^2(B_{4R})},$$

we find

$$\begin{aligned}[\eta\Delta_z^2 u]_{H^s(B_{6R})} \leqslant c\Big[&\varepsilon\left([\eta^2\Delta_z^2 u]_{H^s(B_{6R})} + |z|^{-s}\|\Delta_{-z}\left(\eta^2\Delta_z^2 u\right)\|_{L^2(B_{4R})}\right)\\ &+|z|^{1/2-\sigma}\|\Delta_z^2 u\|_{L^2(B_{6R})}\\ &+\varepsilon^{-1}|z|^s\left(\|u\|_{H^s(B_{8R})}+\|u\|_{L^1_{x_0,\beta}(\mathbb{R}^n)}+\|f\|_{L^2(\Omega)}\right)\Big].\end{aligned} \tag{4.6.9}$$

In view of Proposition 4.7, we have[8]

$$\begin{aligned}\|\Delta_{-z}\left(\eta^2\Delta_z^2 u\right)\|_{L^2(B_{4R})} &\leqslant \|\Delta_{-z}\left(\eta^2\Delta_z^2 u\right)\|_{L^2((B_{5R})_{-z})}\\ &\leqslant |z|^s[\eta^2\Delta_z^2 u]_{N^{s,2}(B_{5R})}\\ &\leqslant c|z|^s\|\eta^2\Delta_z^2 u\|_{H^s(B_{5R})}.\end{aligned} \tag{4.6.10}$$

Moreover,

$$\begin{aligned}&\left|\left(\eta^2\Delta_z^2 u\right)(x) - \left(\eta^2\Delta_z^2 u\right)(y)\right|^2\\ &\qquad\leqslant 2\left(|\eta(x)|^2\left|\left(\eta\Delta_z^2 u\right)(x)-\left(\eta\Delta_z^2 u\right)(y)\right|^2 + \left|\left(\eta\Delta_z^2 u\right)(y)\right|^2|\eta(x)-\eta(y)|^2\right),\end{aligned}$$

and hence, recalling (4.6.1),

$$\begin{aligned}[\eta^2\Delta_z^2 u]^2_{H^s(B_{6R})} &\leqslant c\left[[\eta\Delta_z^2 u]^2_{H^s(B_{6R})} + \int_{B_{6R}}|\Delta_z^2 u(y)|^2\left[\int_{B_{6R}}|x-y|^{-n-2s+2}\,dx\right]dy\right]\\ &\leqslant c\left[[\eta\Delta_z^2 u]^2_{H^s(B_{6R})} + \|\Delta_z^2 u\|^2_{L^2(B_{6R})}\right].\end{aligned} \tag{4.6.11}$$

Consequently, if we choose ε suitably small, by (4.6.10), (4.6.11) and Proposition 4.11, estimate (4.6.9) becomes

$$\begin{aligned}[\eta\Delta_z^2 u]_{H^s(B_{6R})} \leqslant c\Big[&|z|^{1/2-\sigma}\|\Delta_z^2 u\|_{L^2(B_{6R})}\\ &+|z|^s\left(\|u\|_{L^2(\Omega)}+\|u\|_{L^1_{x_0,\beta}(\mathbb{R}^n)}+\|f\|_{L^2(\Omega)}\right)\Big],\end{aligned} \tag{4.6.12}$$

where we also employed (4.6.6). Applying again Proposition 4.7,

$$\|\Delta_w\left(\Delta_z^2 u\right)\|_{L^2((B_R)_w)} \leqslant |w|^s[\Delta_z^2 u]_{N^{s,2}(B_R)} \leqslant c|w|^s\|\Delta_z^2 u\|_{H^s(B_R)},$$

for any $w\in\mathbb{R}^n$. Taking $w=z$, from (4.2.6), (4.6.1), (4.6.6) and (4.6.12) we then get

$$\begin{aligned}\|\Delta_z^3 u\|_{L^2((B_R)_{3z})} &\leqslant \|\Delta_z^3 u\|_{L^2((B_R)_z)} \leqslant c|z|^s\|\Delta_z^2 u\|_{H^s(B_R)}\\ &\leqslant c|z|^s\left(\|\Delta_z^2 u\|_{L^2(B_R)} + [\eta\Delta_z^2 u]_{H^s(B_{6R})}\right)\\ &\leqslant c\Big[|z|^{1/2-\sigma+s}\|\Delta_z^2 u\|_{L^2(B_{6R})}\\ &\qquad+|z|^{2s}\left(\|u\|_{L^2(\Omega)}+\|u\|_{L^1_{x_0,\beta}(\mathbb{R}^n)}+\|f\|_{L^2(\Omega)}\right)\Big].\end{aligned} \tag{4.6.13}$$

[8] Here and in the remainder of the proof we freely swap between some of the equivalent norms of Nikol'skii spaces. In this regard, we recommend the reader to refer to Section 4.3 and, in particular, Remark 4.5.

Now we consider separately the two cases $s \in (0, 1/2]$ and $s \in (1/2, 1)$.

In the first situation, we set $\sigma = 1/2$. Notice that the choice is compatible with (4.6.6). By Proposition 4.7,

$$\|\Delta_z^2 u\|_{L^2(B_{6R})} \leqslant \|\Delta_z^2 u\|_{L^2((B_{7R})_z)} \leqslant |z|^s [u]_{N^{s,2}(B_{7R})} \leqslant c|z|^s \|u\|_{H^s(B_{7R})}. \tag{4.6.14}$$

Therefore, from (4.6.13)

$$\|\Delta_z^3 u\|_{L^2((B_R)_{3z})} \leqslant c|z|^{2s} \left([u]_{H^s(B_{56R})} + \|u\|_{L^2(\Omega)} + \|u\|_{L^1_{x_0,\beta}(\mathbb{R}^n)} + \|f\|_{L^2(\Omega)} \right), \tag{4.6.15}$$

and thus $u \in N^{2s,2}(B_R)$.

Now we address the more delicate case $s \in (1/2, 1)$. Here we choose $\sigma = s$ and first deduce from (4.6.13) and (4.6.14) that

$$\|\Delta_z^3 u\|_{L^2((B_R)_{3z})} \leqslant c|z|^{1/2+s} \left([u]_{H^s(B_{7R})} + \|u\|_{L^2(\Omega)} + \|u\|_{L^1_{x_0,\beta}(\mathbb{R}^n)} + \|f\|_{L^2(\Omega)} \right).$$

Note that such a σ is admissible for (4.6.6), since $s > 1/2$. Repeating the same argument with B_{8R} in place of B_R, we see that $u \in N^{1/2+s,2}(B_{8R})$ with

$$[u]_{N^{1/2+s,2}(B_{8R})} \leqslant c \left([u]_{H^s(B_{56R})} + \|u\|_{L^2(\Omega)} + \|u\|_{L^1_{x_0,\beta}(\mathbb{R}^n)} + \|f\|_{L^2(\Omega)} \right).$$

Consequently,

$$\begin{aligned} \|\Delta_z^2 u\|_{L^2(B_{6R})} &\leqslant \|\Delta_z^2 u\|_{L^2((B_{8R})_{2z})} \leqslant |z|^{1/2+s} [u]_{N^{1/2+s}(B_{8R})} \\ &\leqslant c|z|^{1/2+s} \left([u]_{H^s(B_{56R})} + \|u\|_{L^2(\Omega)} + \|u\|_{L^1_{x_0,\beta}(\mathbb{R}^n)} + \|f\|_{L^2(\Omega)} \right). \end{aligned}$$

Using this last estimate in combination with (4.6.13) and selecting $\sigma = 1$ there, again in agreement with (4.6.6), we conclude that $u \in N^{2s,2}(B_R)$ and (4.6.15) is true also for $s \in (1/2, 1)$.

Finally, we use Proposition 4.11 to control the Gagliardo semi-norm on the right hand side of (4.6.15) and recover

$$[u]_{N^{2s,2}(B_R)} \leqslant c \left(\|u\|_{L^2(\Omega)} + \|u\|_{L^1_{x_0,\beta}(\mathbb{R}^n)} + \|f\|_{L^2(\Omega)} \right). \tag{4.6.16}$$

Then, (4.2.8) follows for a general open $\Omega' \subset\subset \Omega$ by a standard covering argument.[9] □

We conclude this section with some brief comments on the technique just displayed.

To achieve the result we tested the equation with a function modelled on the double increment $\Delta_z^2 u$, which may seem a little unnatural and artificial. In fact, for $s \in (0, 1/2]$ the first order increment would have been sufficient. On the other hand, when $s > 1/2$ this strategy is no more conclusive, basically since it leads to $u \in N^{1/2+s,2}_{\mathrm{loc}}(\Omega)$ only. In order to take advantage of this intermediate regularity and then gain the extra $s - 1/2$ *derivatives*, we need the order of the increment to be at least 2.

[9]Note that the right hand side of (4.6.16) depends on the norm $\|\cdot\|_{L^1_{x_0,\beta}(\mathbb{R}^n)}$ which in turn varies with x_0. Consequently, while performing the covering argument one should take care that those norms depend on the centers of the covering balls. However, as noted in Section 4.2 such norms are all equivalent. The relative compactness of Ω' then allows the use of a finite number of balls, thus preventing the blow-up of the constant c.

4.7 Proof of Theorem 4.1

As previously discussed in Section 4.2, Theorem 4.1 essentially follows from Theorem 4.2, in light of the embedding of Proposition 4.8. The only detail left is that the results of Section 4.3 - specifically, Proposition 4.8 - are only proved for sets having smooth boundary.

But this is not a big drawback. As a matter of fact, we know that estimate (4.2.4) holds for any domain $\Omega' \subset\subset \Omega$, with $\partial\Omega' \in C^\infty$. Then, it can be further extended to any Ω', by noticing that it is always possible to find Ω'' with C^∞ boundary, such that $\Omega' \subset\subset \Omega'' \subset\subset \Omega$.

4.8 Towards the optimal regularity up to the boundary

In this conclusive section we briefly comment on the global Sobolev regularity for the Dirichlet problem (3.1.2).

For $x \in \mathbb{R}^n$, we define $u_s(x) := (x_n)_+^s$. The function u_s solves

$$\begin{cases} (-\Delta)^s u_s = 0 & \text{in } \mathbb{R}^n_+ := \mathbb{R}^{n-1} \times (0, +\infty) \\ u_s = 0 & \text{in } \mathbb{R}^n \setminus \mathbb{R}^n_+. \end{cases} \tag{4.8.1}$$

To see this, we write $u_s(x) = \mu_s(x_n)$, with $\mu_s(t) := t_+^s$ as $t \in \mathbb{R}$, and we compute for $x \in \mathbb{R}^n_+$

$$\begin{aligned} (-\Delta)^s u_s(x) &= \text{P.V.} \int_{\mathbb{R}^n} \frac{u_s(x) - u_s(y)}{|x-y|^{n+2s}}\, dy \\ &= \text{P.V.} \int_{\mathbb{R}} \frac{\mu_s(x_n) - \mu_s(y_n)}{|x_n - y_n|^{n+2s}} \left[\int_{\mathbb{R}^{n-1}} \left(1 + \frac{|x'-y'|^2}{|x_n - y_n|^2} \right)^{-\frac{n+2s}{2}} dy' \right] dy_n. \end{aligned}$$

Note that we use x' and y' to indicate the first $n-1$ components of x and y, respectively. Changing variables by setting $z' := |y_n - x_n|^{-1}(y' - x')$ in the inner integral, we get

$$(-\Delta)^s u_s(x) = \varpi_{n,s} (-\Delta)^s \mu_s(x_n),$$

where

$$\varpi_{n,s} := \int_{\mathbb{R}^{n-1}} \left(1 + |z'|^2\right)^{-\frac{n+2s}{2}} dz',$$

is a finite constant. Then, the equation in (4.8.1) follows from the fact that μ_s is s-harmonic in the half-line $(0, +\infty)$, as showed for instance in [CRSir10, R-OS14] or [BV15].

Of course, the function u_s is of class $C^{0,s}_{\text{loc}}(\mathbb{R}^n)$, but not $C^{0,\alpha}_{\text{loc}}(\mathbb{R}^n)$, with $\alpha > s$. On the other hand, the following proposition sheds some light on which could be the optimal Sobolev regularity of u_s, at least when $s \geqslant 1/2$.

Proposition 4.12. *Let $s \in [1/2, 1)$. Then, $u_s \notin H^{2s}_{\text{loc}}(\overline{\mathbb{R}^n_+})$.*

Proof. We focus on the case $s > 1/2$, as when $s = 1/2$ the computation is immediate.

Denoting with $B'_r(z')$ the $(n-1)$-dimensional open ball of radius r and center z' - with $B'_r := B'_r(0)$ as usual - and with Q the cylinder $B'_1 \times (0,1)$, we shall prove that

$$u_s \notin H^{2s}(Q). \tag{4.8.2}$$

First, setting

$$E := \int_0^1 \int_0^1 \frac{|\mu_s'(t) - \mu_s'(r)|^2}{|t-r|^{1+2(2s-1)}}\, dt dr,$$

we claim that

$$E \text{ is not finite.} \tag{4.8.3}$$

Assuming for the moment (4.8.3) to hold, we check that then (4.8.2) follows. While for $n = 1$ this is immediate, the case $n \geqslant 2$ requires some comments. Indeed,

$$\begin{aligned}\|u_s\|^2_{H^{2s}(Q)} &\geqslant \int_Q\int_Q \frac{|\nabla u_s(x) - \nabla u_s(y)|^2}{|x-y|^{n+2(2s-1)}}\,dxdy = \int_Q\int_Q \frac{|\mu_s'(x_n) - \mu_s'(y_n)|^2}{|x-y|^{n+2(2s-1)}}\,dxdy \\ &= \int_0^1\int_0^1 |\mu_s'(x_n) - \mu_s'(y_n)|^2 \left(\int_{B_1'}\int_{B_1'} \frac{dx'dy'}{(|x_n-y_n|^2 + |x'-y'|^2)^{\frac{n}{2}+2s-1}}\right) dx_n dy_n.\end{aligned}$$

For $\delta \in (0, 1/2)$ we consider the set

$$S(\delta) := \left\{(x', y') \in B_1' \times B_1' : |x'-y'| < \delta\right\} \subset \mathbb{R}^{n-1} \times \mathbb{R}^{n-1},$$

and we estimate its measure by computing

$$\begin{aligned}|S(\delta)| &= \int_{B_1'}\left(\int_{B_1' \cap B_\delta'(x')} dy'\right) dx' \geqslant \int_{B_{1-\delta}'}\left(\int_{B_\delta'(x')} dy'\right) dx' \\ &= |B_1'|^2(1-\delta)^{n-1}\delta^{n-1} \geqslant 2^{1-n}|B_1'|^2\delta^{n-1}.\end{aligned}$$

Noticing that on $S(|x_n - y_n|/4)$ it holds

$$|x_n - y_n|^2 + |x'-y'|^2 \leqslant \frac{17}{16}\,|x_n - y_n|^2,$$

and that $|x_n - y_n|/4 \leqslant 1/2$, we finally obtain

$$\begin{aligned}\|u_s\|^2_{H^{2s}(Q)} &\geqslant \left(\frac{16}{17}\right)^{\frac{n+2s}{2}} \int_0^1\int_0^1 \frac{|\mu_s'(x_n) - \mu_s'(y_n)|^2}{|x_n-y_n|^{n+2(2s-1)}} \left|S\left(\frac{|x_n-y_n|}{4}\right)\right| dx_n dy_n \\ &\geqslant \left(\frac{16}{17}\right)^{\frac{n+2s}{2}} 8^{1-n}|B_1'|^2\, E.\end{aligned}$$

Thus, (4.8.2) is valid.

To complete the proof of the proposition, we are only left to show that (4.8.3) is true. To do this, we first note that, for $t > 0$,

$$\begin{aligned}\mu_s'(t) &= st^{s-1}, \\ \mu_s''(t) &= s(s-1)t^{s-2} < 0.\end{aligned}$$

Accordingly, μ_s' is decreasing and for $0 < r < t < 1$ we have

$$\begin{aligned}|\mu_s'(t) - \mu_s'(r)| = \mu_s'(r) - \mu_s'(t) &= -\int_r^t \mu_s''(\tau)\,d\tau \\ &= s(1-s)\int_r^t \tau^{s-2}\,d\tau \geqslant s(1-s)t^{s-2}(t-r),\end{aligned}$$

so that

$$E \geqslant s^2(1-s)^2 \int_0^1 t^{2(s-2)}\left(\int_0^t (t-r)^{3-4s}dr\right) dt = \frac{s^2(1-s)}{4}\int_0^1 t^{-2s}dt.$$

Claim (4.8.3) then follows, since the integral on the right hand side of the above inequality does not converge. □

We remark that, for $s \in (0, 1/2)$, an almost identical argumentation leads to the conclusion that $u_s \notin H^{s+1/2}_{\mathrm{loc}}(\overline{\mathbb{R}^n_+})$.

Chapter 5

Plane-like minimizers in a periodic medium

5.1 Introduction and statement of the main result

The goal of this chapter is to construct solutions of a scalar, fractional Ginzburg-Landau (or Allen-Cahn) equation in a periodic medium, whose interface stays in a prescribed slab and whose energy is minimal among compact perturbations.

The simplest case that we have in mind is the nonlocal equation

$$(-\Delta)^s u(x) = Q(x)\,\big(u(x) - u^3(x)\big), \tag{5.1.1}$$

in which $s \in (0,1)$ is a fractional parameter and Q is a smooth function, bounded and bounded away from zero, and such that

$$Q(x+k) = Q(x) \ \text{ for every } \ k \in \mathbb{Z}^n. \tag{5.1.2}$$

Equations of this type naturally occur in other areas of applied mathematics, such as the Peierls-Nabarro model for crystal dislocations when $s = 1/2$, and for generalizations of this model when $s \in (0,1)$ (see e.g. [N97, DFV14]). Related problems also arise in models for diffusion of biological species (see e.g. [F12]).

As a matter of fact, we will consider here a more general equation than (5.1.1). Indeed, we will deal with operators that are more general than the fractional Laplacian, which can be also spatially heterogeneous and periodic, and also with more general forcing terms, which may possess different growths from the pure phases other than the classical quadratic growth.

Here are the details of the mathematical framework in which the work is set. For $n \geqslant 2$, we consider the formal energy functional

$$\mathscr{E}_K(u) := \frac{1}{4} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} |u(x) - u(y)|^2\, K(x,y)\, dx dy + \int_{\mathbb{R}^n} W(x, u(x))\, dx. \tag{5.1.3}$$

The kernel $K : \mathbb{R}^n \times \mathbb{R}^n \to [0,+\infty]$ is supposed to satisfy

$$K(x,y) = K(y,x) \quad \text{for a.a. } x, y \in \mathbb{R}^n, \tag{5.1.4}$$

and

$$\frac{\lambda\, \chi_{B_1}(x-y)}{|x-y|^{n+2s}} \leqslant K(x,y) \leqslant \frac{\Lambda}{|x-y|^{n+2s}} \quad \text{for a.a. } x, y \in \mathbb{R}^n, \tag{5.1.5}$$

for some $\Lambda \geqslant \lambda > 0$ and $s \in (0,1)$. Note that requirements (5.1.4) and (5.1.5) allow for a great variety of space-dependent, possibly truncated kernels. In particular, no regularity is asked on K.

The mapping W is a double-well potential, with zeros in -1 and 1. More specifically, we assume $W : \mathbb{R}^n \times \mathbb{R} \to [0,+\infty)$ to be a bounded measurable function for which

$$W(x, \pm 1) = 0 \quad \text{for a.a. } x \in \mathbb{R}^n, \tag{5.1.6}$$

and, for any $\theta \in [0,1)$,

$$\inf_{\substack{x \in \mathbb{R}^n \\ |r| \leqslant \theta}} W(x,r) \geqslant \gamma(\theta), \tag{5.1.7}$$

where γ is a non-increasing positive function of the interval $[0,1)$. Moreover, we require W to be differentiable in the second component, with partial derivative locally bounded in $r \in \mathbb{R}$, uniformly in $x \in \mathbb{R}^n$. Accordingly, we let

$$W(x,r),\ |W_r(x,r)| \leqslant W^* \quad \text{for a.a. } x \in \mathbb{R}^n \text{ and any } r \in [-1,1], \tag{5.1.8}$$

for some $W^* > 0$.

Since we are interested in modelling a periodic environment, we require both K and W to be periodic under integer translations. That is,

$$K(x+k, y+k) = K(x,y) \quad \text{for a.a. } x, y \in \mathbb{R}^n \text{ and any } k \in \mathbb{Z}^n, \tag{5.1.9}$$

and

$$W(x+k, r) = W(x,r) \quad \text{for a.a. } x \in \mathbb{R}^n \text{ and any } k \in \mathbb{Z}^n, \tag{5.1.10}$$

for any fixed $r \in \mathbb{R}$.

The assumptions listed above allow us to comprise a very general class of kernels and potentials.

As possible choices for K, we could indeed think of heterogeneous, isotropic kernels of the type

$$K(x,y) = \frac{a(x,y)}{|x-y|^{n+2s}},$$

for a measurable $a : \mathbb{R}^n \times \mathbb{R}^n \to [\lambda, \Lambda]$, or instead consider a translation invariant, but anisotropic K, as given by

$$K(x,y) = \frac{1}{\|x-y\|^{n+2s}},$$

with $\|\cdot\|$ a measurable norm in $\mathbb{R}^n$. Furthermore, one can combine both heterogeneity and anisotropy to obtain, for instance, kernels of the form

$$K(x,y) = \frac{1}{\langle A(x,y)(x-y), (x-y)\rangle^{\frac{n+2s}{2}}},$$

where A is a symmetric, uniformly elliptic $n \times n$ matrix with bounded entries.

Of course, the functions a and A should satisfy appropriate symmetry and periodicity conditions, in order that hypotheses (5.1.4) and (5.1.9) could be fulfilled by the resulting K's. Also, such functions may exhibit a degenerate behavior when x and y are far from each other (compare this with the left-hand side of (5.1.5)).

Important examples of admissible potentials W are given by

$$W(x,r) = Q(x)\left|1-r^2\right|^d \quad \text{or} \quad W(x,r) = Q(x)\left(1+\cos \pi r\right),$$

with $d > 1$ and Q a positive periodic function.[10] By taking $W(x,r) := Q(x)(1-r^2)^2$ and $K(x,y) := |x-y|^{-n-2s}$, one obtains that the critical points of the energy functional satisfy the model equation in (5.1.1) (up to normalization constants).

In the present work we look for minimizers of the functional $\mathscr{E}$ which connects the two pure phases -1 and 1, which are the zeroes of the potential W. In particular, given any vector $\omega \in \mathbb{R}^n \setminus \{0\}$, we address the existence of minimizers for which, roughly speaking, *most* of the transition between the pure states occurs in a strip orthogonal to ω and of universal width. Moreover, when ω is a rational vector, we want our minimizers to exhibit some kind of periodic behavior, consistent with that of the ambient space.

Note that we will often call a quantity *universal* if it depends at most on n, s, λ, Λ, W^* and on the function γ introduced in (5.1.7).

In order to formulate an exact statement, we introduce the following terminology. For a given $\omega \in \mathbb{Q}^n \setminus \{0\}$, we consider in $\mathbb{R}^n$ the relation $\sim_\omega$ defined by setting

$$x \sim_\omega y \quad \text{if and only if} \quad y - x = k \in \mathbb{Z}^n, \text{ with } \omega \cdot k = 0. \tag{5.1.11}$$

Notice that $\sim_\omega$ is an equivalence relation and that the associated quotient space

$$\widetilde{\mathbb{R}}^n_\omega := \mathbb{R}^n / \sim_\omega,$$

is topologically the Cartesian product of an $(n-1)$-dimensional torus and a line. We say that a function $u : \mathbb{R}^n \to \mathbb{R}$ is *periodic with respect to* $\sim_\omega$, or simply $\sim_\omega$*-periodic*, if u respects the equivalence relation $\sim_\omega$, i.e. if

$$u(x) = u(y) \quad \text{for any } x, y \in \mathbb{R}^n \text{ such that } x \sim_\omega y.$$

When no confusion may arise, we will indicate the relation $\sim_\omega$ just by $\sim$ and the resulting quotient space by $\widetilde{\mathbb{R}}^n$.

To specify the notion of minimizers that we take into consideration, we need to introduce an appropriate localized energy functional. Given a set $\Omega \subseteq \mathbb{R}^n$ and a function $u : \mathbb{R}^n \to \mathbb{R}$, we define the *total energy* $\mathscr{E}_K$ of u in Ω as

$$\mathscr{E}_K(u;\Omega) := \frac{1}{4} \iint_{\mathscr{C}_\Omega} |u(x) - u(y)|^2 K(x,y)\, dx dy + \int_\Omega W(x, u(x))\, dx, \tag{5.1.12}$$

with $\mathscr{C}_\Omega$ as in (3.2.1). Notice that when Ω is the whole space $\mathbb{R}^n$, then the energy (5.1.12) coincides with that anticipated in (5.1.3).

Sometimes, a more flexible notation for this functional will turn out to be useful. To this aim, recalling our symmetry assumption (5.1.4) on K, we will refer to $\mathscr{E}_K(u;\Omega)$ as the sum of the *kinetic part*[11]

$$\mathscr{K}(u;\Omega,\Omega) + 2\mathscr{K}(u;\Omega,\mathbb{R}^n \setminus \Omega),$$

[10] When comparing these assumptions with those usually found in the related literature on local functionals, see e.g. [CC95, CC06] or [V04], one realizes that the parameter d is asked there to range in the interval $(0,2]$. This is due essentially to the fact that our proofs do not rely on the density estimates established in those papers, but on the Hölder regularity results of Chapter 3.

If on the one hand this enables us to consider extremely flat potentials near the zeroes -1 and 1, which can be obtained by taking $d > 2$, on the other hand the Lipschitz continuity needed on W for the regularity results to apply imposes the bound $d > 1$. This is due to the fact that the regularity theory adopted is designed for solutions to integro-differential equations, instead of minimizers. We believe that if one was able to develop a nonlocal regularity theory in the spirit of [GG82], then the request $d > 1$ would become superfluous.

[11] We stress that the name *kinetic* does not hint at actual physical motivations. In fact, in the applications $\mathscr{K}$ is typically used to describe nonlocal interactions and elastic forces. However, we adopt this slight abuse of terminology in conformity with the classical jargon used for local Dirichlet energies in particle mechanics.

with

$$\mathscr{K}(u; U, V) := \frac{1}{4}\int_U \int_V |u(x) - u(y)|^2 K(x, y)\, dxdy,$$

for any $U, V \subseteq \mathbb{R}^n$, and the *potential part*

$$\mathscr{P}(u; \Omega) := \int_\Omega W(x, u(x))\, dx.$$

With this in hand, the notion of minimization inside a bounded set is described by the following

Definition 5.1. *Let Ω be a bounded subset of $\mathbb{R}^n$. A function u is said to be a* local minimizer *of $\mathscr{E}_K$ in Ω if $\mathscr{E}_K(u; \Omega) < +\infty$ and*

$$\mathscr{E}_K(u; \Omega) \leqslant \mathscr{E}_K(v; \Omega), \tag{5.1.13}$$

for any v which coincides with u in $\mathbb{R}^n \setminus \Omega$.

For simplicity, in Definition 5.1 and throughout the chapter we assume every set and every function to be measurable, even if it is not explicitly stated.

Remark 5.2. We point out that a minimizer u on Ω is also a minimizer on every subset of Ω. Though not obvious, this property is easily justified as follows.

Let $\Omega' \subset \Omega$ be measurable sets and v be a function coinciding with u outside Ω'. Recalling the notation introduced in (3.2.1), it is immediate to check that $\mathscr{C}_{\Omega'} \subset \mathscr{C}_\Omega$ and

$$\mathscr{C}_\Omega \setminus \mathscr{C}_{\Omega'} = \big((\Omega \setminus \Omega') \times (\Omega \setminus \Omega') \big) \cup \big((\Omega \setminus \Omega') \times (\mathbb{R}^n \setminus \Omega) \big) \cup \big((\mathbb{R}^n \setminus \Omega) \times (\Omega \setminus \Omega') \big).$$

Therefore, it follows that the integrands of the kinetic parts of $\mathscr{E}_K(u; \Omega)$ and $\mathscr{E}_K(v; \Omega)$ coincide on $\mathscr{C}_\Omega \setminus \mathscr{C}_{\Omega'}$. Since also the respective arguments of the potential terms are equal on $\Omega \setminus \Omega'$, by (5.1.13) we conclude that

$$\begin{aligned}
\mathscr{E}_K(u; \Omega') &= \mathscr{E}_K(u; \Omega) - \frac{1}{4}\iint_{\mathscr{C}_\Omega \setminus \mathscr{C}_{\Omega'}} |u(x) - u(y)|^2 K(x, y)\, dxdy - \mathscr{P}(u; \Omega \setminus \Omega') \\
&\leqslant \mathscr{E}_K(v; \Omega) - \frac{1}{4}\iint_{\mathscr{C}_\Omega \setminus \mathscr{C}_{\Omega'}} |v(x) - v(y)|^2 K(x, y)\, dxdy - \mathscr{P}(v; \Omega \setminus \Omega') \\
&= \mathscr{E}_K(v; \Omega').
\end{aligned}$$

Thus, u is a minimizer on Ω'.

Up to now we only discussed about local minimizers. Since we plan to construct functions which exhibit minimizing properties on the full space, we need to be precise on how we mean to extend Definition 5.1 to the whole of $\mathbb{R}^n$ (where the total energy functional may be divergent).

Definition 5.3. *A function u is said to be a* class A minimizer *of the functional $\mathscr{E}_K$ if it is a minimizer of $\mathscr{E}_K$ in Ω, for any bounded set $\Omega \subset \mathbb{R}^n$.*

Now that all the main ingredients have been introduced, we are ready to state formally the main result of the chapter.

Theorem 5.4. *Let $n \geqslant 2$ and $s \in (0, 1)$. Assume that the kernel K and the potential W satisfy* (5.1.4), (5.1.5), (5.1.9) *and* (5.1.6), (5.1.7), (5.1.8), (5.1.10), *respectively.*
For any fixed $\theta \in (0, 1)$, there is a constant $M_0 > 0$, depending only on θ and on universal

quantities, such that, given any $\omega \in \mathbb{R}^n \setminus \{0\}$, there exists a class A minimizer u_ω of the energy $\mathscr{E}_K$ for which the level set $\{|u_\omega| < \theta\}$ is contained in the strip

$$\left\{ x \in \mathbb{R}^n : \frac{\omega}{|\omega|} \cdot x \in [0, M_0] \right\}.$$

Moreover,

- *if $\omega \in \mathbb{Q}^n \setminus \{0\}$, then u_ω is periodic with respect to $\sim_\omega$, while*
- *if $\omega \in \mathbb{R}^n \setminus \mathbb{Q}^n$, then u_ω is the uniform limit on compact subsets of $\mathbb{R}^n$ of a sequence of periodic class A minimizers.*

We remark that Theorem 5.4 is new even in the model case in which $W(x,r) := Q(x)(1-r^2)^2$ and $K(x,y) := |x-y|^{-n-2s}$. In this case, Theorem 5.4 provides solutions of equation (5.1.1) (up to normalizing constants).

In the local case - which formally corresponds to taking $s = 1$ and can be effectively realized by replacing our kinetic term with the Dirichlet-type energy

$$\int \langle A(x) \nabla u(x), \nabla u(x) \rangle \, dx, \tag{5.1.14}$$

where A is a bounded, uniformly elliptic matrix - the result contained in Theorem 5.4 was proved by the second author in [V04]. After this, several generalizations were obtained, extending such result in many directions. See, for instance, [PV05, NV07, dlLV07, BV08] and [D13]. We also mention the pioneering work [CdlL01] of Caffarelli and de la Llave, where the two authors proved the existence of plane-like minimal surfaces with respect to periodic metrics of $\mathbb{R}^n$.

The proof of Theorem 5.4 makes use of a geometric and variational technique developed in [CdlL01] and [V04], suitably adapted in order to deal with nonlocal interactions. For a given rational direction $\omega \in \mathbb{Q}^n \setminus \{0\}$ and a fixed strip

$$\mathcal{S}_\omega^M := \{ x \in \mathbb{R}^n : \omega \cdot x \in [0, M] \},$$

with $M > 0$, one takes advantage of the identifications of the quotient space $\widetilde{\mathbb{R}}^n$ to gain the compactness needed to obtain a minimizer u_ω^M with respect to periodic perturbations supported inside $\mathcal{S}_\omega^M$. By construction, this minimizer is such that its interface $\{|u_\omega^M| < \theta\}$ is contained in the strip $\mathcal{S}_\omega^M$.

With the aid of some geometrical arguments, one then shows that u_ω^M becomes a class A minimizer for $\mathscr{E}_K$, provided $M/|\omega|$ is larger than some universal parameter M_0. The fact that the threshold M_0 is universal and that, in particular, it does not depend on the fixed direction ω is of key importance here and it allows, as a byproduct, to obtain the result for an irrational vector $\omega \in \mathbb{R}^n \setminus \mathbb{Q}^n$, by taking the limit of rational directions.

We remark that the nonlocal character of the energy $\mathscr{E}_K$ introduces several challenging difficulties into the above scheme.

First of all, the way the compactness is used to construct the minimizer u_ω^M is somehow not as straightforward as in the local case.

To have a glimpse of this difference, consider that in [V04] the candidate u_ω^M is by definition a minimizer with respect to $\sim$-periodic perturbations occurring in $\widetilde{\mathcal{S}_\omega^M}$. That is, one really considers the energy $\mathscr{E}$ driven by (5.1.14) as defined on the cylinder $\widetilde{\mathbb{R}}^n$ viewed as a manifold and obtain u_ω^M as the absolute minimizer of $\mathscr{E}$ within a particular class of functions defined on $\widetilde{\mathbb{R}}^n$. However, since the restriction of the local kinetic term (5.1.14) to a fundamental domain of $\widetilde{\mathbb{R}}^n$ only *sees* what happens inside that domain, it is clear that one is allowed in the local case to identify periodic perturbations and perturbations which are compactly supported inside $\widetilde{\mathbb{R}}^n$. As a result, u_ω^M is automatically a local minimizer for $\mathscr{E}$ in the strip $\mathcal{S}_\omega^M$.

As it is, this technique cannot work in the nonlocal setting. Indeed, let u be any $\sim$-periodic function and φ be compactly supported in a fixed fundamental region D of $\widetilde{\mathbb{R}}^n$: if we denote by $\tilde{\varphi}$ the $\sim$-periodic extension of $\varphi|_D$ to $\mathbb{R}^n$, then the two quantities $\mathscr{E}_K(u+\varphi; D)$ and $\mathscr{E}_K(u+\tilde{\varphi}; D)$, as defined in (5.1.12), are not equal in general.

In order to overcome this difficulty, we introduce an appropriate auxiliary functional $\mathscr{F}_\omega$ that is used to define the periodic minimizer u_ω^M. Then, it happens that u_ω^M is a local minimizer for the original energy $\mathscr{E}_K$, since $\mathscr{F}_\omega$ couples with $\mathscr{E}_K$ in a favorable way.

An additional difficulty comes from the different asymptotic properties of the energy in terms of the fractional parameter s. As a matter of fact, the threshold $s = 1/2$ distinguishes the local and nonlocal behavior of the functional at a large scale (see [SV12, SV14]) and it reflects into the finiteness or infiniteness of the energy of the one-dimensional transition layer. In our setting, this feature implies that not all the kernels K satisfying (5.1.5) can be dealt with at the same time. More precisely, when $s \leqslant 1/2$ the behaviour at infinity dictated by (5.1.5) causes infinite contributions coming from far. For this reason, at least at a first glance, it may seem necessary to restrict the class of admissible kernels by imposing some additional requirements on the decay of K at infinity. However, we will be able to remove this limitation by an appropriate limit procedure. Namely, we will first assume a fast decay property of the kernel to obtain the existence of a class A minimizer, but the estimates obtained will be independent of this additional assumption. Consequently, we will be able to extend the result to general kernels by treating them as limits of truncated ones.

Finally, we want to point out a possibly interesting difference between the proof displayed here and that of e.g. [CdlL01] and [V04]. In the existing literature, the technique that is typically adopted to show that u_ω^M is a class A minimizer relies on the so-called energy and density estimates.

These estimates respectively deal with the growth of the energy $\mathscr{E}$ of a local minimizer u inside *large* balls and the fractions of such balls occupied by a fixed level set of u. The latter, in particular, is a powerful tool first introduced by Caffarelli and Córdoba in [CC95] to study the uniform convergence of the level sets of a family of *scaled* minimizers.

Although such density estimates have been established in [SV14] in a nonlocal setting very close to ours, for some technical reasons we decided not to incorporate them into our argument (roughly speaking, the periodic setting is not immediately compatible with large balls in Euclidean spaces). In their place, we take advantage of the C^α bounds established in Section 3.3, along with a suitable version of the energy estimates.

Energy estimates for minimizers of nonlocal energies have been independently obtained in [CC14] and [SV14] (in different settings). Since this result was set in a slightly different framework than ours, we provide its proof in full details in Section 5.2.

The chapter is organized as follows.

Section 5.2 is devoted to an energy estimates. We stress that in this section both K and W are subjected to slightly more general requirements than those listed in the introduction (the statements of the results proved in there will contain the precise hypotheses needed for their proofs).

Section 5.3 is occupied by the main construction leading to the proof of Theorem 5.4. For the reader's ease, this section is in turn divided into seven short subsections. In each of these subsections, we will consider, respectively:

- the minimization arguments by compactness,
- the notion of minimal minimizer (i.e. the pointwise infimum of all the possible minimizers, which satisfy additional geometric and functional features),
- the doubling property (roughly, doubling the period does not change the minimal minimizer),
- the notion of minimization under compact perturbations,
- the Birkhoff property (namely, the level sets of the minimal minimizers are ordered by integer translations),
- the passage from constrained to unconstrained minimization (for large strips, we show that the constraint is irrelevant),
- the passage from rational to irrational slopes.

The argument displayed in Section 5.3 only works under an additional assumption on the decay rate of the kernel K at infinity. In the subsequent Section 5.4 we will show that this hypothesis can be in fact removed by a limit procedure. The proof of Theorem 5.4 will therefore be completed.

We conclude this chapter with the additional Section 5.5 which contains some auxiliary material and complements some technical steps in the proofs of our main results.

5.2 An energy estimate

We include here a result which addresses the growth of the energy $\mathscr{E}_K$ of local minimizers inside large balls. We point out that this estimate is set in a general framework. In particular, the periodicity of K and W encoded in (5.1.9) and (5.1.10) is not significant here. Writing

$$\Psi_s(R) := \begin{cases} R^{1-2s} & \text{if } s \in (0,1/2) \\ \log R & \text{if } s = 1/2 \\ 1 & \text{if } s \in (1/2,1), \end{cases} \tag{5.2.1}$$

we can state the following

Proposition 5.5. *Let $n \in \mathbb{N}$, $s \in (0,1)$, $x_0 \in \mathbb{R}^n$ and $R \geqslant 3$. Assume that K and W satisfy*[12] *(5.1.4), (5.1.5) and (5.1.6), (5.1.8), respectively. If $u : \mathbb{R}^n \to [-1,1]$ is a local minimizer of $\mathscr{E}_K$ in $B_{R+2}(x_0)$, then*

$$\mathscr{E}_K(u; B_R(x_0)) \leqslant C R^{n-1} \Psi_s(R), \tag{5.2.2}$$

for some constant $C > 0$ which depends on n, s, Λ and W^*.

The above proposition will play an important role later in Subsection 5.3.6, as it will imply that the interface region of a minimizer cannot be too wide.

Estimate (5.2.2) has first been proved in [CC14] and [SV14] for the fractional Laplacian. While in the first paper the authors use the harmonic extension of u to $\mathbb{R}^{n+1}_+$ to prove (5.2.2), in the latter work the result is obtained by explicitly computing the energy $\mathscr{E}$ of a suitable competitor of u. It turns out that this strategy is flexible enough to be adapted

[12] We observe that, at this level, only the boundedness of W encoded in (5.1.8) is relevant here. Thus, no assumption on the derivative W_r is necessary. See in particular the the proof of Proposition 5.5.

to our framework and the proof of Proposition 5.5 is actually an appropriate and careful modification of that of [SV14, Theorem 1.3].

Before heading to the proof of Proposition 5.5, we first need the following auxiliary result that will be also widely used in the following Section 5.3.

Lemma 5.6. *Let U, V be two measurable subsets of $\mathbb{R}^n$ and $u, v \in H^s_{\mathrm{loc}}(\mathbb{R}^n)$. Then,*

$$\mathscr{K}(\min\{u,v\};U,V) + \mathscr{K}(\max\{u,v\};U,V) \leqslant \mathscr{K}(u;U,V) + \mathscr{K}(v;U,V), \tag{5.2.3}$$

and

$$\mathscr{P}(\min\{u,v\};U) + \mathscr{P}(\max\{u,v\};U) = \mathscr{P}(u;U) + \mathscr{P}(v;V). \tag{5.2.4}$$

Proof. Since the derivation of identity (5.2.4) is quite straightforward, we focus on (5.2.3) only.

We write for simplicity $m := \min\{u,v\}$ and $M := \max\{u,v\}$. Observe that we may assume the right hand side of (5.2.3) to be finite, the result being otherwise obvious. In order to show (5.2.3), we actually prove the stronger pointwise relation

$$|m(x) - m(y)|^2 + |M(x) - M(y)|^2 \leqslant |u(x) - u(y)|^2 + |v(x) - v(y)|^2, \tag{5.2.5}$$

for a.a. $x, y \in \mathbb{R}^n$.

Let then x and y be two fixed points in $\mathbb{R}^n$. In order to check that (5.2.5) is true, we consider separately the two possibilities

i) $u(x) \leqslant v(x)$ and $u(y) \leqslant v(y)$, or $u(x) > v(x)$ and $u(y) > v(y)$;

ii) $u(x) \leqslant v(x)$ and $u(y) > v(y)$, or $u(x) > v(x)$ and $u(y) \leqslant v(y)$.

In the first situation it is immediate to see that (5.2.5) holds as an identity. Suppose then that point ii) occurs. If this is the case, we compute

$$\begin{aligned}
|m(x) - m(y)|^2 &+ |M(x) - M(y)|^2 \\
&= |u(x) - v(y)|^2 + |v(x) - u(y)|^2 \\
&= |u(x) - u(y)|^2 + |v(x) - v(y)|^2 + 2\,(u(x) - v(x))\,(u(y) - v(y)) \\
&\leqslant |u(x) - u(y)|^2 + |v(x) - v(y)|^2,
\end{aligned}$$

which is (5.2.5). The proof of the lemma is thus complete. □

Proof of Proposition 5.5. Without loss of generality, we assume x_0 to be the origin. In the course of the proof we will denote as c any positive constant which depends at most on n, s, Λ and W^*.

Let ψ be the radially symmetric function defined by

$$\psi(x) := 2\min\{(|x| - R - 1)_+, 1\} - 1 = \begin{cases} -1 & \text{if } x \in B_{R+1} \\ 2|x| - 2R - 1 & \text{if } x \in B_{R+2} \setminus B_{R+1} \\ 1 & \text{if } x \in \mathbb{R}^n \setminus B_{R+2}. \end{cases}$$

We claim that ψ satisfies (5.2.2) in B_{R+2}, that is

$$\mathscr{E}_K(\psi; B_{R+2}) \leqslant c R^{n-1} \Psi_s(R). \tag{5.2.6}$$

Indeed, let $x \in B_{R+2}$ and set $d(x) := \max\{R - |x|, 1\}$. It is easy to see that

$$|\psi(x) - \psi(y)| \leqslant 2 \begin{cases} d(x)^{-1}|x - y| & \text{if } |x - y| < d(x) \\ 1 & \text{if } |x - y| \geqslant d(x). \end{cases}$$

Consequently, applying (5.1.5) we compute

$$\int_{\mathbb{R}^n} |\psi(x)-\psi(y)|^2 K(x,y)\,dy \leqslant 4\omega_{n-1}\Lambda \left[d(x)^{-2} \int_0^{d(x)} \rho^{1-2s}\,d\rho + \int_{d(x)}^{+\infty} \rho^{-1-2s}\,d\rho \right]$$
$$\leqslant c d(x)^{-2s}.$$

Furthermore, using polar coordinates we get

$$\int_{B_{R+2}} d(x)^{-2s}\,dx = \int_{B_{R-1}} \frac{dx}{(R-|x|)^{2s}} + \int_{B_{R+2}\setminus B_{R-1}} dx \leqslant cR^{n-1}\Psi_s(R). \tag{5.2.7}$$

Hence,

$$\int_{B_{R+2}} \int_{\mathbb{R}^n} |\psi(x)-\psi(y)|^2 K(x,y)\,dxdy \leqslant cR^{n-1}\Psi_s(R).$$

Since by (5.1.8) and (5.1.6) we also have

$$\mathscr{P}(\psi, B_{R+2}) = \int_{B_{R+2}} W(x,\psi(x))\,dx \leqslant W^* \int_{B_{R+2}\setminus B_{R+1}} dx \leqslant cR^{n-1},$$

it is clear that estimate (5.2.6) follows.

Now, set $v := \min\{u,\psi\}$ and $w := \max\{u,\psi\}$. By the definition of ψ and the fact that $-1 \leqslant u \leqslant 1$, we observe that

$$u = v \quad \text{in } \mathbb{R}^n \setminus B_{R+2}, \tag{5.2.8}$$

and

$$u = w \quad \text{in } B_{R+1}. \tag{5.2.9}$$

By virtue of (5.2.9),

$$\mathscr{K}(u; B_R, B_R) = \mathscr{K}(w; B_R, B_R) \quad \text{and} \quad \mathscr{P}(u; B_R) = \mathscr{P}(w; B_R). \tag{5.2.10}$$

On the other hand, we claim that

$$\mathscr{K}(u; B_R, \mathbb{R}^n \setminus B_R) \leqslant \mathscr{K}(w; B_R, \mathbb{R}^n \setminus B_R) + cR^{n-1}\Psi_s(R). \tag{5.2.11}$$

Indeed, using (5.1.5), (5.2.9) and the fact that $|u|, |\psi| \leqslant 1$ a.e. in $\mathbb{R}^n$, we compute

$$\begin{aligned}
&\mathscr{K}(u; B_R, \mathbb{R}^n \setminus B_R) - \mathscr{K}(w; B_R, \mathbb{R}^n \setminus B_R) \\
&\qquad = \frac{1}{4} \int_{B_R} \left(\int_{\mathbb{R}^n\setminus B_{R+1}} \left[|u(x)-u(y)|^2 - |u(x)-w(y)|^2 \right] K(x,y)\,dy \right) dx \\
&\qquad \leqslant \Lambda \int_{B_R} \left(\int_{\mathbb{R}^n\setminus B_{R+1}} |x-y|^{-n-2s}\,dy \right) dx \leqslant c \int_{B_R} d(x)^{-2s}\,dx,
\end{aligned}$$

and claim (5.2.11) then follows from (5.2.7). Accordingly, by (5.2.11) and (5.2.10) we obtain that

$$\mathscr{E}_K(u; B_R) \leqslant \mathscr{E}_K(w; B_R) + cR^{n-1}\Psi_s(R). \tag{5.2.12}$$

We now take advantage of the minimality of u and (5.2.8) to deduce

$$\mathscr{E}_K(u; B_{R+2}) \leqslant \mathscr{E}_K(v; B_{R+2}).$$

Then, from this and Lemma 5.6 it follows immediately that

$$\mathscr{E}_K(w; B_R) \leqslant \mathscr{E}_K(w; B_{R+2}) \leqslant \mathscr{E}_K(\psi; B_{R+2}). \tag{5.2.13}$$

Note that the first inequality above is true as a consequence of the inclusion $\mathscr{C}_{B_R} \subset \mathscr{C}_{B_{R+2}}$ (see Remark 5.2). By applying in sequence (5.2.12), (5.2.13) and (5.2.6), we finally get (5.2.2). □

5.3 Proof of Theorem 5.4 for rapidly decaying kernels

The present section contains the proof of Theorem 5.4 under the additional assumption that K satisfies

$$K(x,y) \leqslant \frac{\Gamma}{|x-y|^{n+\beta}} \quad \text{for a.a. } x,y \in \mathbb{R}^n \text{ such that } |x-y| \geqslant \bar{R}, \text{ with } \beta > 1, \tag{5.3.1}$$

for some constants $\Gamma, \bar{R} > 0$. We stress that this hypothesis is merely technical and in fact it will be removed later in Section 5.4. However, we need the fast decay of the kernel K at infinity - ensured by the fact that $\beta > 1$ - in order to perform a delicate construction at some point (roughly speaking, the decay assumed in (5.3.1) is needed to ensure the existence of a competitor with finite energy in the large, but the geometric estimates will be independent of the quantities in (5.3.1) and this will allow us to perform a limit procedure). Hence, we assume (5.3.1) to hold in the whole section.

Notice that if $s > 1/2$, then (5.3.1) is automatically fulfilled in view of (5.1.5).

The argument leading to the proof of Theorem 5.4 is long and articulated. Therefore, we divide the section into several subsections which we hope will make the reading easier.

We first deal with the case of a rational direction ω. Under this assumption, we can take advantage of the equivalence relation $\sim_\omega$ defined in (5.1.11) to build the minimizer. This construction occupies Subsections 5.3.1-5.3.6.

Irrational directions - i.e. $\omega \in \mathbb{R}^n \setminus \mathbb{Q}^n$ - are then treated in Subsection 5.3.7 as limiting cases.

For simplicity of exposition, we restrict ourselves to consider $\theta = 9/10$. The general case is in no way different. Of course, the choice $9/10$ is made in order to represent a value of θ close to 1.

5.3.1 Minimization with respect to periodic perturbations

Let $\omega \in \mathbb{Q}^n \backslash \{0\}$ be fixed. Given a measurable function $u : \mathbb{R}^n \to \mathbb{R}$, we say that $u \in L^2_{\text{loc}}(\widetilde{\mathbb{R}}^n)$ if $u \in L^2_{\text{loc}}(\mathbb{R}^n)$ and u is periodic with respect to $\sim$. Given $A < B$, let

$$\mathcal{A}^{A,B}_\omega := \left\{ u \in L^2_{\text{loc}}(\widetilde{\mathbb{R}}^n) : u(x) \geqslant \frac{9}{10} \text{ if } \omega \cdot x \leqslant A \text{ and } u(x) \leqslant -\frac{9}{10} \text{ if } \omega \cdot x \geqslant B \right\},$$

be the set of admissible functions. We introduce the auxiliary functional

$$\begin{aligned} \mathscr{F}_\omega(u) &:= \mathscr{K}(u; \widetilde{\mathbb{R}}^n, \mathbb{R}^n) + \mathscr{P}(u; \widetilde{\mathbb{R}}^n) \\ &= \frac{1}{4} \int_{\widetilde{\mathbb{R}}^n} \int_{\mathbb{R}^n} |u(x) - u(y)|^2 K(x,y)\, dx dy + \int_{\widetilde{\mathbb{R}}^n} W(x, u(x))\, dx. \end{aligned} \tag{5.3.2}$$

Note that in the integrals above, $\widetilde{\mathbb{R}}^n$ stands for any fundamental domain of the relation $\sim$. In the following, we will often identify quotients with any of their respective fundamental domains.

The aim of this subsection is to prove the existence of an *absolute minimizer* of $\mathscr{F}_\omega$ within the class $\mathcal{A}^{A,B}_\omega$, that is a function $u \in \mathcal{A}^{A,B}_\omega$ such that $\mathscr{F}_\omega(u) \leqslant \mathscr{F}_\omega(v)$, for any $v \in \mathcal{A}^{A,B}_\omega$. Such minimizers are the building blocks of our construction, as will become clear in the sequel.

As a first step toward this goal, we show that $\mathscr{F}_\omega$ is not identically infinite on $\mathcal{A}^{A,B}_\omega$.

Lemma 5.7. *Let $\bar{u} \in \mathcal{A}_\omega^{A,B}$ be defined by setting $\bar{u}(x) := \bar{\mu}(\omega \cdot x)$, where $\bar{\mu}$ is the piecewise linear function given by*

$$\bar{\mu}(t) := \begin{cases} 1 & \text{if } t \leqslant A \\ 1 - \frac{2}{B-A}(t-A) & \text{if } A < t \leqslant B \\ -1 & \text{if } t > B. \end{cases}$$

Then, $\mathscr{F}_\omega(\bar{u}) < +\infty$.

Proof. Since $W(x, \cdot)$ vanishes at ± 1, for a.a. $x \in \mathbb{R}^n$, it is clear that the potential term of $\mathscr{F}_\omega$ evaluated at $\bar{u}$ is finite. Thus, we only need to estimate the kinetic term. To do this, by (5.1.5) and (5.3.1), it is in turn sufficient to show that

$$\int_{\widetilde{\mathbb{R}}^n} \left(\int_{B_{\bar{R}}(x)} \frac{|\bar{u}(x) - \bar{u}(y)|^2}{|x-y|^{n+2s}} \, dy + \int_{\mathbb{R}^n \setminus B_{\bar{R}}(x)} \frac{|\bar{u}(x) - \bar{u}(y)|^2}{|x-y|^{n+\beta}} \, dy \right) dx < +\infty. \tag{5.3.3}$$

Notice that, up to an affine transformation, we may take $\omega = e_n$. Moreover, we assume for simplicity that $A = 0$ and $B = 1$. In this setting, we have $\widetilde{\mathbb{R}}^n = [0,1]^{n-1} \times \mathbb{R}$ and, consequently, (5.3.3) is equivalent to

$$I := \int_{[0,1]^{n-1} \times \mathbb{R}} \left(\int_{B_{\bar{R}}(x)} \frac{|\bar{u}(x) - \bar{u}(y)|^2}{|x-y|^{n+2s}} \, dy \right) dx < +\infty, \tag{5.3.4}$$

and

$$J := \int_{[0,1]^{n-1} \times \mathbb{R}} \left(\int_{\mathbb{R}^n \setminus B_{\bar{R}}(x)} \frac{|\bar{u}(x) - \bar{u}(y)|^2}{|x-y|^{n+\beta}} \, dy \right) dx < +\infty. \tag{5.3.5}$$

By the definition of $\bar{u}$, it is clear that

$$I = \int_{[0,1]^{n-1} \times [-\bar{R}, \bar{R}+1]} \left(\int_{B_{\bar{R}}(x)} \frac{|\bar{u}(x) - \bar{u}(y)|^2}{|x-y|^{n+2s}} \, dy \right) dx.$$

Then, we take advantage of $\bar{u}$ being Lipschitz to compute, using polar coordinates,

$$I \leqslant 4 \int_{[0,1]^{n-1} \times [-\bar{R}, \bar{R}+1]} \left(\int_{B_{\bar{R}}(x)} \frac{dy}{|x-y|^{n+2s-2}} \right) dx = \frac{2n\alpha_n}{1-s} (2\bar{R}+1) \bar{R}^{2-2s},$$

which implies (5.3.4).

On the other hand, to prove (5.3.5) we first write $J = J_1 + J_2 + J_3$, where

$$J_1 := \int_{[0,1]^{n-1} \times [2,+\infty)} \left(\int_{\mathbb{R}^n \setminus B_{\bar{R}}(x)} \frac{|\bar{u}(x) - \bar{u}(y)|^2}{|x-y|^{n+\beta}} \, dy \right) dx,$$

$$J_2 := \int_{[0,1]^{n-1} \times (-\infty,-1]} \left(\int_{\mathbb{R}^n \setminus B_{\bar{R}}(x)} \frac{|\bar{u}(x) - \bar{u}(y)|^2}{|x-y|^{n+\beta}} \, dy \right) dx,$$

$$J_3 := \int_{[0,1]^{n-1} \times [-1,2]} \left(\int_{\mathbb{R}^n \setminus B_{\bar{R}}(x)} \frac{|\bar{u}(x) - \bar{u}(y)|^2}{|x-y|^{n+\beta}} \, dy \right) dx.$$

Using the definition of $\bar{u}$, we observe that

$$\begin{aligned} J_1 &\leqslant \int_{[0,1]^{n-1} \times [2,+\infty)} \left(\int_{\mathbb{R}^{n-1} \times (-\infty,1]} \frac{|-1-\bar{\mu}(y_n)|^2}{|x-y|^{n+\beta}} \, dy \right) dx \\ &\leqslant 4 \int_{[0,1]^{n-1} \times [2,+\infty)} \left(\int_{\mathbb{R}^{n-1} \times (-\infty,1]} \frac{dy}{|x-y|^{n+\beta}} \right) dx. \end{aligned}$$

Making the substitution $z' := (y' - x')/|x_n - y_n|$, we have

$$\begin{aligned}\int_{\mathbb{R}^{n-1}\times(-\infty,1]} \frac{dy}{|x-y|^{n+\beta}} &= \int_{-\infty}^{1} |x_n - y_n|^{-n-\beta} \left[\int_{\mathbb{R}^{n-1}} \left(1 + \frac{|x'-y'|^2}{|x_n-y_n|^2} \right)^{-\frac{n+\beta}{2}} dy' \right] dy_n \\ &= \int_{-\infty}^{1} |x_n - y_n|^{-1-\beta} \left[\int_{\mathbb{R}^{n-1}} \left(1 + |z'|^2\right)^{-\frac{n+\beta}{2}} dz' \right] dy_n \\ &= \frac{\Xi}{\beta} (x_n - 1)^{-\beta},\end{aligned}$$

where we denoted with Ξ the finite quantity

$$\int_{\mathbb{R}^{n-1}} \left(1 + |z'|^2\right)^{-\frac{n+\beta}{2}} dz'.$$

Accordingly,

$$J_1 \leqslant \frac{4\Xi}{\beta} \int_{2}^{+\infty} (x_n - 1)^{-\beta} dx_n = \frac{4\Xi}{(\beta-1)\beta},$$

since $\beta > 1$. Similarly, one checks that J_2 is finite too. The computation of J_3 is simpler. By taking advantage of the fact that $\bar{u}$ is a bounded function and switching to polar coordinates, we get

$$J_3 \leqslant 4 \int_{[0,1]^{n-1}\times[-1,2]} \left(\int_{\mathbb{R}^n \setminus B_{\bar{R}}(x)} \frac{dy}{|x-y|^{n+\beta}} \right) dx = \frac{12 n \alpha_n}{\beta} \bar{R}^{-\beta}.$$

Hence, (5.3.5) follows. □

We want to highlight how crucial condition (5.3.1) has been in the proof of the above lemma. Indeed, if the kernel K has a slower decay at infinity, the result is no longer true. Lemma 5.25 in Section 5.5 shows that, under this assumption, the functional $\mathscr{F}_\omega$ is nowhere finite on the whole class of admissible functions $\mathcal{A}_\omega^{A,B}$.

We also point out that this is the only part of the section in which we need the additional hypothesis (5.3.1) and future computations will involve neither β, nor $\bar{R}$, nor Γ.

With the aid of the finiteness result yielded by Lemma 5.7, we can now prove the existence of minimizers.

Proposition 5.8. *There exists an absolute minimizer of the functional $\mathscr{F}_\omega$ within the class $\mathcal{A}_\omega^{A,B}$.*

Proof. Our argumentation follows the lines of the standard Direct Method of the Calculus of Variations.

By Lemma 5.7 and the fact that $\mathscr{F}_\omega$ is non-negative, we know that

$$m := \inf\left\{ \mathscr{F}_\omega(u) : u \in \mathcal{A}_\omega^{A,B} \right\} \in [0, +\infty).$$

Let then $\{u_j\}_{j\in\mathbb{N}} \subseteq \mathcal{A}_\omega^{A,B}$ be a minimizing sequence. Observe that we may assume without loss of generality that

$$|u_j| \leqslant 1 \quad \text{a.e. in } \mathbb{R}^n, \tag{5.3.6}$$

as this restriction only makes the energy $\mathscr{F}_\omega$ decrease. Moreover, we fix an integer $k > \max\{-A, B\}$ and consider the Lipschitz domains

$$\Omega_k := \widetilde{\mathbb{R}}^n \cap \{x \in \mathbb{R}^n : |\omega \cdot x| \leqslant k\}.$$

By (5.3.6) and (5.1.5) we have

$$[u_j]^2_{H^s(\Omega_k)} \leqslant \int_{\Omega_k}\left(\int_{B_1(x)}\frac{|u_j(x)-u_j(y)|^2}{|x-y|^{n+2s}}\,dy\right)dx + 4\int_{\Omega_k}\left(\int_{\mathbb{R}^n\setminus B_1(x)}\frac{dy}{|x-y|^{n+2s}}\right)dx$$
$$\leqslant \frac{4}{\lambda}\mathscr{F}_\omega(u_j)+\frac{2n\alpha_n|\Omega_k|}{s},$$

so that $\{u_j\}$ is bounded in $H^s(\Omega_k)$, uniformly in j. By the compact embedding of $H^s(\Omega_k)$ into $L^2(\Omega_k)$ (see e.g. Theorem 7.1 of [DNPV12]), we then deduce that a subsequence of $\{u_j\}$ converges to some function u in $L^2(\Omega_k)$ and, thus, a.e. in Ω_k. Using a diagonal argument (on j and k), we may indeed find a subsequence $\{u_j^*\}$ of $\{u_j\}$ which converges to u a.e. in $\widetilde{\mathbb{R}}^n$. Furthermore, we may identify the u_j^*'s and u with their $\sim$-periodic extensions to $\mathbb{R}^n$ and thus obtain that such convergence is a.e. in the whole space $\mathbb{R}^n$. Accordingly, $u \in \mathcal{A}^{A,B}_\omega$ and an application of Fatou's lemma shows that $\mathscr{F}_\omega(u) = m$. This concludes the proof. □

5.3.2 The minimal minimizer

Denote by $\mathcal{M}^{A,B}_\omega$ the set composed by the absolute minimizers of $\mathscr{F}_\omega$ in $\mathcal{A}^{A,B}_\omega$, i.e.

$$\mathcal{M}^{A,B}_\omega := \left\{ u \in \mathcal{A}^{A,B}_\omega : \mathscr{F}_\omega(u) \leqslant \mathscr{F}_\omega(v) \text{ for any } v \in \mathcal{A}^{A,B}_\omega \right\}.$$

Clearly, $\mathcal{M}^{A,B}_\omega$ is not empty, as shown by Proposition 5.8. Here below we introduce a particular element of the class $\mathcal{M}^{A,B}_\omega$, that will turn out to be of central interest in the remainder of the chapter.

Definition 5.9. *We define the* minimal minimizer $u^{A,B}_\omega$ *as the infimum of* $\mathcal{M}^{A,B}_\omega$ *as a subset of the partially ordered set* $(\mathcal{A}^{A,B}_\omega, \leqslant)$*. More specifically,* $u^{A,B}_\omega$ *is the unique function of* $\mathcal{A}^{A,B}_\omega$ *for which*

$$u^{A,B}_\omega \leqslant u \text{ in } \mathbb{R}^n \text{ for every } u \in \mathcal{M}^{A,B}_\omega \tag{5.3.7}$$

and

$$\text{if } v \in \mathcal{A}^{A,B}_\omega \text{ is s.t. } v \leqslant u \text{ in } \mathbb{R}^n \text{ for every } u \in \mathcal{M}^{A,B}_\omega, \text{ then } v \leqslant u^{A,B}_\omega \text{ in } \mathbb{R}^n. \tag{5.3.8}$$

Of course, the existence of the minimal minimizer is far from being established. Aim of the subsection is to prove that such function is in fact well-defined and that it belongs to $\mathcal{M}^{A,B}_\omega$ itself.

In order to construct $u^{A,B}_\omega$ we first need to show that the set $\mathcal{M}^{A,B}_\omega$ is closed with respect to the operation of taking the minimum between two of its elements. To do this, we actually prove a stronger fact, which will be needed, in its full generality, only later in Subsection 5.3.5.

Lemma 5.10. *Let $A \leqslant A'$ and $B \leqslant B'$, with $A < B$ and $A' < B'$. If $u \in \mathcal{M}^{A,B}_\omega$ and $v \in \mathcal{M}^{A',B'}_\omega$, then $\min\{u,v\} \in \mathcal{M}^{A,B}_\omega$.*

Proof. First, notice that $\min\{u,v\} \in \mathcal{A}^{A,B}_\omega$ and $\max\{u,v\} \in \mathcal{A}^{A',B'}_\omega$. Moreover, employing Lemma 5.6 we deduce

$$\mathscr{F}_\omega(\min\{u,v\}) + \mathscr{F}_\omega(\max\{u,v\}) \leqslant \mathscr{F}_\omega(u) + \mathscr{F}_\omega(v).$$

Taking advantage of this inequality, together with the fact that $v \in \mathcal{M}^{A',B'}_\omega$, we get

$$\mathscr{F}_\omega(\min\{u,v\}) + \mathscr{F}_\omega(\max\{u,v\}) \leqslant \mathscr{F}_\omega(u) + \mathscr{F}_\omega(\max\{u,v\}),$$

which in turn implies that

$$\mathscr{F}_\omega(\min\{u,v\}) \leqslant \mathscr{F}_\omega(u).$$

Consequently, $\min\{u,v\} \in \mathcal{M}_\omega^{A,B}$. □

By choosing $A = A'$ and $B = B'$, we obtain the desired

Corollary 5.11. *Let $u, v \in \mathcal{M}_\omega^{A,B}$. Then, $\min\{u,v\} \in \mathcal{M}_\omega^{A,B}$.*

Now that we know that the minimum between two - and, consequently, any finite number of - minimizers is still a minimizer, we can show that also the infimum over a *countable* family of elements of $\mathcal{M}_\omega^{A,B}$ belongs to $\mathcal{M}_\omega^{A,B}$.

Lemma 5.12. *Let $\{u_j\}_{j\in\mathbb{N}}$ be a sequence of elements of $\mathcal{M}_\omega^{A,B}$. Then, $\inf\limits_{j\in\mathbb{N}} u_j \in \mathcal{M}_\omega^{A,B}$.*

Proof. Write $u_* := \inf\limits_{j\in\mathbb{N}} u_j$. We define inductively the auxiliary sequence

$$v_j := \begin{cases} u_1 & \text{if } j = 1 \\ \min\{v_{j-1}, u_j\} & \text{if } j \geqslant 2. \end{cases}$$

By Corollary 5.11, we know that $\{v_j\} \subseteq \mathcal{M}_\omega^{A,B}$. Moreover, v_j converges to u_* a.e. in $\mathbb{R}^n$. An application of Fatou's lemma then yields that $u_* \in \mathcal{A}_\omega^{A,B}$ and

$$\mathscr{F}_\omega(u_*) \leqslant \lim_{j\to+\infty} \mathscr{F}_\omega(v_j) = \mathscr{F}_\omega(v_k),$$

for any $k \in \mathbb{N}$. Therefore, $u_* \in \mathcal{M}_\omega^{A,B}$. □

Finally, we are in position to prove the main result of the present subsection.

Proposition 5.13. *The minimal minimizer $u_\omega^{A,B}$, as given by Definition 5.9, exists and belongs to $\mathcal{M}_\omega^{A,B}$.*

Proof. The set $\mathcal{M}_\omega^{A,B}$ is separable with respect to convergence a.e., i.e. there exists a sequence $\{u_j\}_{j\in\mathbb{N}} \subseteq \mathcal{M}_\omega^{A,B}$ such that for any $u \in \mathcal{M}_\omega^{A,B}$ we may pick a subsequence $\{u_{j_k}\}$ which converges to u a.e. in $\mathbb{R}^n$. A rigorous proof of this fact can be found in Proposition B.2 of Appendix B. Set

$$u_\omega^{A,B} := \inf_{j\in\mathbb{N}} u_j.$$

By Lemma 5.12, we already know that $u_\omega^{A,B} \in \mathcal{M}_\omega^{A,B}$. We claim that $u_\omega^{A,B}$ is the minimal minimizer, i.e. that satisfies the properties (5.3.7) and (5.3.8) listed in Definition 5.9.

Take $u \in \mathcal{M}_\omega^{A,B}$ and let $\{u_{j_k}\}$ be a subsequence of $\{u_j\}$ converging to u a.e. in $\mathbb{R}^n$. By definition, $u_\omega^{A,B} \leqslant u_{j_k}$ in $\mathbb{R}^n$, for any $k \in \mathbb{N}$. Hence, taking the limit as $k \to +\infty$, condition (5.3.7) follows.

Now we turn our attention to (5.3.8) and we assume that there exists $v \in \mathcal{A}_\omega^{A,B}$ such that $v \leqslant u$, for any $u \in \mathcal{M}_\omega^{M}$. Then, in particular, we have $v \leqslant u_j$, for any $j \in \mathbb{N}$ which implies $v \leqslant u_\omega^{A,B}$. Thus, (5.3.8) follows and the proof of the proposition is complete. □

5.3.3 The doubling property

An important feature of the minimal minimizer is the so-called *doubling property* (or *no-symmetry-breaking property*). Namely, we prove in this subsection that $u_\omega^{A,B}$ is still the minimal minimizer with respect to functions having periodicity multiple of $\sim$. In order to formulate precisely this result, we need a few more notation.

Let $z_1, \dots, z_{n-1} \in \mathbb{Z}^n$ denote some vectors spanning the $(n-1)$-dimensional lattice induced by $\sim$. Thus, any $k \in \mathbb{Z}^n$ such that $\omega \cdot k = 0$ may be written as

$$k = \sum_{i=1}^{n-1} \mu_i z_i,$$

for some $\mu_1, \dots, \mu_{n-1} \in \mathbb{Z}$. For a fixed $m \in \mathbb{N}^{n-1}$, we introduce the equivalence relation $\sim_m$, defined by setting

$$x \sim_m y \quad \text{if and only if} \quad x - y = \sum_{i=1}^{n-1} \mu_i m_i z_i, \text{ for some } \mu_1, \dots, \mu_{n-1} \in \mathbb{Z}.$$

Also, set $\widetilde{\mathbb{R}}^n_m := \mathbb{R}^n / \sim_m$ and denote by $L^2_{\text{loc}}(\widetilde{\mathbb{R}}^n_m)$ the space of $\sim_m$-periodic functions which belong to $L^2_{\text{loc}}(\mathbb{R}^n)$. Note that $\widetilde{\mathbb{R}}^n_m$ contains exactly $m_1 \cdot \ldots \cdot m_{n-1}$ copies of $\widetilde{\mathbb{R}}^n$. Indeed, the relation $\sim_m$ is weaker than $\sim$ and $L^2_{\text{loc}}(\widetilde{\mathbb{R}}^n) \subseteq L^2_{\text{loc}}(\widetilde{\mathbb{R}}^n_m)$. We consider the space of admissible functions

$$\mathcal{A}^{A,B}_{\omega,m} := \left\{ u \in L^2_{\text{loc}}(\widetilde{\mathbb{R}}^n_m) : u(x) \geqslant \frac{9}{10} \text{ if } \omega \cdot x \leqslant A \text{ and } u(x) \leqslant -\frac{9}{10} \text{ if } \omega \cdot x \geqslant B \right\},$$

related to this new equivalence relation, together with the set of absolute minimizers

$$\mathcal{M}^{A,B}_{\omega,m} := \left\{ u \in \mathcal{A}^{A,B}_{\omega,m} : \mathscr{F}_{\omega,m}(u) \leqslant \mathscr{F}_{\omega,m}(v) \text{ for any } v \in \mathcal{A}^{A,B}_{\omega,m} \right\},$$

of the functional

$$\begin{aligned} \mathscr{F}_{\omega,m}(u) &:= \mathscr{K}(u; \widetilde{\mathbb{R}}^n_m, \mathbb{R}^n) + \mathscr{P}(u; \widetilde{\mathbb{R}}^n_m) \\ &= \frac{1}{4} \int_{\widetilde{\mathbb{R}}^n_m} \int_{\mathbb{R}^n} |u(x) - u(y)|^2 K(x,y)\, dx dy + \int_{\widetilde{\mathbb{R}}^n_m} W(x, u(x))\, dx. \end{aligned}$$

We indicate with $u^{A,B}_{\omega,m}$ the minimal minimizer of the class $\mathcal{M}^{A,B}_{\omega,m}$. Of course, its existence is granted by the same arguments of Subsection 5.3.2.

Finally, given a function $u : \mathbb{R}^n \to \mathbb{R}$ and a vector $z \in \mathbb{R}^n$, we denote the translation of u in the direction z as

$$\tau_z u(x) := u(x - z) \quad \text{for any } x \in \mathbb{R}^n. \tag{5.3.9}$$

After this preliminary work, we can now prove that the minimal minimizer in a class of larger period coincides with the one in a class of smaller period:

Proposition 5.14. *For any $m \in \mathbb{N}^{n-1}$, it holds $u^{A,B}_{\omega,m} = u^{A,B}_\omega$.*

Proof. For simplicity of exposition we restrict ourselves to the case in which $m_1 = 2$ and $m_i = 1$, for every $i = 2, \dots, n-1$. The approach in the general case would be analogous, but much heavier in notation.

We begin by showing that $u_{\omega,m}^{A,B} \leqslant u_\omega^{A,B}$. Notice that the inequality follows if we prove that $u_\omega^{A,B} \in \mathcal{M}_{\omega,m}^{A,B}$. To see this, we consider the translation $\tau_{z_1} u_{\omega,m}^{A,B}$ of $u_{\omega,m}^{A,B}$ in the *doubled* direction z_1. Clearly, $\tau_{z_1} u_{\omega,m}^{A,B} \in \mathcal{M}_{\omega,m}^{A,B}$. Then, we define

$$\hat{u}_{\omega,m}^{A,B} := \min \left\{ u_{\omega,m}^{A,B}, \tau_{z_1} u_{\omega,m}^{A,B} \right\}.$$

Observe that $\hat{u}_{\omega,m}^{A,B}$ is $\sim$-periodic and hence belongs to $\mathcal{A}_\omega^{A,B}$. Then,

$$\mathscr{F}_{\omega,m}(u_\omega^{A,B}) = 2\mathscr{F}_\omega(u_\omega^{A,B}) \leqslant 2\mathscr{F}_\omega(\hat{u}_{\omega,m}^{A,B}) = \mathscr{F}_{\omega,m}(\hat{u}_{\omega,m}^{A,B}) \leqslant \mathscr{F}_{\omega,m}(u_{\omega,m}^{A,B}),$$

where the last inequality follows by Lemma 5.6, arguing as in the proof of Lemma 5.10. Accordingly, we deduce that $u_\omega^{A,B} \in \mathcal{M}_{\omega,m}^{A,B}$ and so $u_{\omega,m}^{A,B} \leqslant u_\omega^{A,B}$, since $u_{\omega,m}^{A,B}$ is the minimal minimizer of $\mathcal{M}_{\omega,m}^{A,B}$.

On the other hand, being $\hat{u}_{\omega,m}^{A,B} \in \mathcal{M}_{\omega,m}^{A,B}$ and $u_\omega^{A,B} \in \mathcal{A}_{\omega,m}^{A,B}$, we have

$$\mathscr{F}_\omega(\hat{u}_{\omega,m}^{A,B}) = \frac{1}{2}\mathscr{F}_{\omega,m}(\hat{u}_{\omega,m}^{A,B}) \leqslant \frac{1}{2}\mathscr{F}_{\omega,m}(u_\omega^{A,B}) = \mathscr{F}_\omega(u_\omega^{A,B}),$$

which implies that $\hat{u}_{\omega,m}^{A,B} \in \mathcal{M}_\omega^{A,B}$. Consequently, $u_\omega^{A,B} \leqslant \hat{u}_{\omega,m}^{A,B} \leqslant u_{\omega,m}^{A,B}$, and the proposition is therefore proved. □

5.3.4 Minimization with respect to compact perturbations

In the previous subsections we have been concerned with functionals of the type $\mathscr{F}_{\omega,m}$. We proved that absolute minimizers for such functionals exist in particular classes of $\sim_m$-periodic functions. Since our ultimate goal is the construction of class A minimizers for the energy $\mathscr{E}_K$, we now need to show that the elements of $\mathcal{M}_\omega^{A,B}$ are also minimizers of $\mathscr{E}_K$ with respect to compact perturbations occurring within the strip

$$\mathcal{S}_\omega^{A,B} := \{x \in \mathbb{R}^n : \omega \cdot x \in [A,B]\}. \tag{5.3.10}$$

In what follows, it will also be useful to introduce the quotient

$$\widetilde{\mathcal{S}}_{\omega,m}^{A,B} := \mathcal{S}_\omega^{A,B} / \sim_m . \tag{5.3.11}$$

The first result of the subsection addresses a general relationship intervening between the two functionals $\mathscr{E}_K$ and $\mathscr{F}_{\omega,m}$.

Lemma 5.15. *Let $u \in \mathcal{A}_{\omega,m}^{A,B}$ be a bounded function with finite $\mathscr{F}_{\omega,m}$ energy. Given an open set Ω compactly contained in $\widetilde{\mathcal{S}}_{\omega,m}^{A,B}$,*[13] *let v be another bounded function such that $u = v$ outside Ω and set $\varphi := v - u$. Denoting with $\tilde{v}$ and $\tilde{\varphi}$ the $\sim_m$-periodic extensions to $\mathbb{R}^n$ of $v|_{\widetilde{\mathbb{R}}^n_m}$ and $\varphi|_{\widetilde{\mathbb{R}}^n_m}$, respectively, it then holds*

$$\begin{aligned}\mathscr{E}_K(v;\widetilde{\mathbb{R}}^n_m) - \mathscr{E}_K(u;\widetilde{\mathbb{R}}^n_m) = {} & \mathscr{F}_{\omega,m}(\tilde{v}) - \mathscr{F}_{\omega,m}(u) \\ & + \frac{1}{2}\int_{\widetilde{\mathbb{R}}^n_m}\int_{\mathbb{R}^n\setminus\widetilde{\mathbb{R}}^n_m} \tilde{\varphi}(x)\tilde{\varphi}(y)K(x,y)\,dxdy.\end{aligned} \tag{5.3.12}$$

In particular, if $u \in \mathcal{M}_{\omega,m}^{A,B}$, then

$$\mathscr{E}_K(v;\widetilde{\mathbb{R}}^n_m) - \mathscr{E}_K(u;\widetilde{\mathbb{R}}^n_m) \geqslant \frac{1}{2}\int_{\widetilde{\mathbb{R}}^n_m}\int_{\mathbb{R}^n\setminus\widetilde{\mathbb{R}}^n_m} \tilde{\varphi}(x)\tilde{\varphi}(y)K(x,y)\,dxdy. \tag{5.3.13}$$

[13] We stress that here Ω is meant to be compactly contained in a fundamental domain of $\widetilde{\mathcal{S}}_{\omega,m}^{A,B}$, and not only in the quotient set itself. The difference is that we do not allow Ω to touch the *lateral* boundary of the domain.

Note that the integral written on the right-hand sides of (5.3.12) and (5.3.13) is finite, since φ is compactly supported on $\widetilde{\mathcal{S}}^{A,B}_{\omega,m}$ and bounded. For a justification of this fact, see Lemma 5.26 in Section 5.5.

Proof of Lemma 5.15. For simplicity, we restrict ourselves to consider $m = (1, \dots, 1)$, the general case being completely analogous. Moreover, it is enough to prove formula (5.3.12), as (5.3.13) then easily follows by noticing that $\tilde{v} \in \mathcal{A}^{A,B}_{\omega,m}$.

Recalling definition (5.1.3), we first inspect the term $\mathscr{K}(v; \widetilde{\mathbb{R}}^n, \mathbb{R}^n \setminus \widetilde{\mathbb{R}}^n)$. To this aim, let $x \in \widetilde{\mathbb{R}}^n$ and $y \in \mathbb{R}^n \setminus \widetilde{\mathbb{R}}^n$. We compute

$$\begin{aligned}
|v(x) - v(y)|^2 &= |u(x) + \varphi(x) - u(y)|^2 \\
&= |u(x) + \tilde{\varphi}(x) - u(y) - \tilde{\varphi}(y)|^2 + 2\tilde{\varphi}(y)\left(u(x) + \tilde{\varphi}(x) - u(y)\right) - \tilde{\varphi}(y)^2 \\
&= |\tilde{v}(x) - \tilde{v}(y)|^2 + |u(x) - u(y)|^2 - |u(x) - u(y) - \tilde{\varphi}(y)|^2 + 2\tilde{\varphi}(x)\tilde{\varphi}(y),
\end{aligned}$$

and thus

$$\begin{aligned}
\mathscr{K}(v; \widetilde{\mathbb{R}}^n, \mathbb{R}^n \setminus \widetilde{\mathbb{R}}^n) = \mathscr{K}(\tilde{v}, \widetilde{\mathbb{R}}^n, \mathbb{R}^n \setminus \widetilde{\mathbb{R}}^n) &+ \mathscr{K}(u; \widetilde{\mathbb{R}}^n, \mathbb{R}^n \setminus \widetilde{\mathbb{R}}^n) \\
&- \frac{1}{4} \int_{\widetilde{\mathbb{R}}^n} \left(\int_{\mathbb{R}^n \setminus \widetilde{\mathbb{R}}^n} |u(x) - u(y) - \tilde{\varphi}(y)|^2 K(x,y)\, dy \right) dx \\
&+ \frac{1}{2} \int_{\widetilde{\mathbb{R}}^n} \int_{\mathbb{R}^n \setminus \widetilde{\mathbb{R}}^n} \tilde{\varphi}(x)\tilde{\varphi}(y) K(x,y)\, dx dy.
\end{aligned} \tag{5.3.14}$$

Notice now that

$$\mathbb{R}^n \setminus \widetilde{\mathbb{R}}^n = \bigcup_{\substack{k \in \mathbb{Z}^n \setminus \{0\} \\ \omega \cdot k = 0}} \left(\widetilde{\mathbb{R}}^n + k \right),$$

so that we may write the integral on the second line of (5.3.14) as

$$\sum_{\substack{k \in \mathbb{Z}^n \setminus \{0\} \\ \omega \cdot k = 0}} \int_{\widetilde{\mathbb{R}}^n} \left(\int_{\widetilde{\mathbb{R}}^n + k} |u(x) - u(y) - \tilde{\varphi}(y)|^2 K(x,y)\, dy \right) dx.$$

By changing variables as $w := x - k$, $z := y - k$, recalling (5.1.9) and taking advantage of the periodicity of u and $\tilde{\varphi}$, we find that

$$\begin{aligned}
\int_{\widetilde{\mathbb{R}}^n} &\left(\int_{\widetilde{\mathbb{R}}^n + k} |u(x) - u(y) - \tilde{\varphi}(y)|^2 K(x,y)\, dy \right) dx \\
&= \int_{\widetilde{\mathbb{R}}^n - k} \left(\int_{\widetilde{\mathbb{R}}^n} |u(w) - u(z) - \tilde{\varphi}(z)|^2 K(w,z)\, dz \right) dw \\
&= \int_{\widetilde{\mathbb{R}}^n - k} \left(\int_{\widetilde{\mathbb{R}}^n} |v(w) - v(z)|^2 K(w,z)\, dz \right) dw.
\end{aligned}$$

By summing up on k this identity, (5.3.14) becomes

$$\begin{aligned}
\mathscr{K}(v; \widetilde{\mathbb{R}}^n, \mathbb{R}^n \setminus \widetilde{\mathbb{R}}^n) = \mathscr{K}(\tilde{v}, \widetilde{\mathbb{R}}^n, \mathbb{R}^n \setminus \widetilde{\mathbb{R}}^n) + \mathscr{K}(u; \widetilde{\mathbb{R}}^n, \mathbb{R}^n \setminus \widetilde{\mathbb{R}}^n) - \mathscr{K}(v; \mathbb{R}^n \setminus \widetilde{\mathbb{R}}^n, \widetilde{\mathbb{R}}^n) \\
+ \frac{1}{2} \int_{\widetilde{\mathbb{R}}^n} \int_{\mathbb{R}^n \setminus \widetilde{\mathbb{R}}^n} \tilde{\varphi}(x)\tilde{\varphi}(y) K(x,y)\, dx dy.
\end{aligned}$$

The thesis then follows by noticing that

$$\mathscr{K}(v; \widetilde{\mathbb{R}}^n, \widetilde{\mathbb{R}}^n) = \mathscr{K}(\tilde{v}; \widetilde{\mathbb{R}}^n, \widetilde{\mathbb{R}}^n) \quad \text{and} \quad \mathscr{P}(v; \widetilde{\mathbb{R}}^n) = \mathscr{P}(\tilde{v}; \widetilde{\mathbb{R}}^n),$$

and recalling the definitions of $\mathscr{E}_K$ and $\mathscr{F}_\omega$. □

With this in hand, we may state the following proposition, where we prove that the absolute minimizers of $\mathscr{F}_{\omega,m}$ in the class $\mathcal{A}^{A,B}_{\omega,m}$ also minimizes $\mathscr{E}_K$ with respect to compact perturbations occurring inside $\widetilde{\mathcal{S}}^{A,B}_{\omega,m}$

Proposition 5.16. *Let $u \in \mathcal{M}^{A,B}_{\omega,m}$. Then, u is a local minimizer of $\mathscr{E}_K$ in every open set Ω compactly contained in $\widetilde{\mathcal{S}}^{A,B}_{\omega,m}$, that is*

$$\mathscr{E}_K(u;\Omega) \leqslant \mathscr{E}_K(v;\Omega), \tag{5.3.15}$$

for any v which coincides with u outside Ω.

Proof. First of all, we assume without loss of generality that $\mathscr{E}_K(v;\Omega) < +\infty$ and $|v| \leqslant 1$ a.e. in $\mathbb{R}^n$. Set $\varphi := v - u$ and observe that φ is supported on Ω. We will show that inequality (5.3.15) holds on the larger region $\widetilde{\mathbb{R}}^n_m$, in place of Ω, i.e.

$$\mathscr{E}_K(u;\widetilde{\mathbb{R}}^n_m) \leqslant \mathscr{E}_K(v;\widetilde{\mathbb{R}}^n_m). \tag{5.3.16}$$

This will imply (5.3.15), in light of Remark 5.2.

To prove (5.3.16), we notice that if φ is either non-negative or non-positive, then (5.3.16) follows as a direct consequence of inequality (5.3.13). On the other hand, if φ is sign-changing, we consider the minimum and the maximum between u and $u+\varphi$. Recalling Lemma 5.6 it is immediate to see that

$$\mathscr{E}_K(\min\{u,u+\varphi\};\widetilde{\mathbb{R}}^n_m) + \mathscr{E}_K(\max\{u,u+\varphi\};\widetilde{\mathbb{R}}^n_m) \leqslant \mathscr{E}_K(u;\widetilde{\mathbb{R}}^n_m) + \mathscr{E}_K(u+\varphi;\widetilde{\mathbb{R}}^n_m).$$

Moreover, since it holds

$$\min\{u,u+\varphi\} = u - \varphi_- \quad \text{and} \quad \max\{u,u+\varphi\} = u + \varphi_+,$$

we may apply (5.3.13) and get

$$\begin{aligned} 2\,\mathscr{E}_K(u;\widetilde{\mathbb{R}}^n_m) &\leqslant \mathscr{E}_K(u-\varphi_-;\widetilde{\mathbb{R}}^n_m) + \mathscr{E}_K(u+\varphi_+;\widetilde{\mathbb{R}}^n_m) \\ &= \mathscr{E}_K(\min\{u,u+\varphi\};\widetilde{\mathbb{R}}^n_m) + \mathscr{E}_K(\max\{u,u+\varphi\};\widetilde{\mathbb{R}}^n_m) \\ &\leqslant \mathscr{E}_K(u;\widetilde{\mathbb{R}}^n_m) + \mathscr{E}_K(u+\varphi;\widetilde{\mathbb{R}}^n_m). \end{aligned}$$

This leads to (5.3.16). □

From this proposition and the results of Subsection 5.3.3, we immediately deduce the following

Corollary 5.17. *The minimal minimizer $u^{A,B}_\omega$ is a local minimizer of $\mathscr{E}_K$ in every bounded open set Ω compactly contained in the strip $\mathcal{S}^{A,B}_\omega$.*

Proof. Given Ω, we take $m \in \mathbb{N}^{n-1}$ large enough in order to have $\Omega \subset\subset \widetilde{\mathcal{S}}^{A,B}_{\omega,m}$. In view of Proposition 5.14, $u^{A,B}_\omega$ is the minimal minimizer with respect to $\mathcal{M}^{A,B}_{\omega,m}$. But then, by Proposition 5.16, $u^{A,B}_\omega$ is a local minimizer of $\mathscr{E}_K$ in Ω. □

5.3.5 The Birkhoff property

In this subsection we introduce an interesting geometric feature shared by the level sets of the minimal minimizer: the *Birkhoff property* (also known in the literature as "non-self-intersection property"). Namely, the level sets of the minimal minimizers are ordered under translations.

In order to give a formal definition of this property, the following notation will be useful.

Similarly to what we did in (5.3.9) for functions, we consider the translation of a set $E \subseteq \mathbb{R}^n$ with respect to a vector $z \in \mathbb{R}^n$

$$\tau_z E := E + z = \{x + z : x \in E\}. \tag{5.3.17}$$

Notice that, with this notation, the translation of a sublevel set then is given by

$$\tau_z\{u < \theta\} = \{\tau_z u < \theta\}, \tag{5.3.18}$$

and analogously for the superlevel sets.

Definition 5.18. *Let E be a subset of $\mathbb{R}^n$. We say that E has the* Birkhoff property *with respect to a vector $\varpi \in \mathbb{R}^n$ if:*

- *$\tau_k E \subseteq E$, for any $k \in \mathbb{Z}^n$ such that $\varpi \cdot k \leqslant 0$, and*
- *$\tau_k E \supseteq E$, for any $k \in \mathbb{Z}^n$ such that $\varpi \cdot k \geqslant 0$.*

Before exploring the connection between the minimal minimizer and the Birkhoff property, we present a proposition which addresses Birkhoff sets from an abstract point of view and displays a rigidity feature of those of such sets that have *fat* interior.

Proposition 5.19. *Let $E \subseteq \mathbb{R}^n$ be a set satisfying the Birkhoff property with respect to a vector $\varpi \in \mathbb{R}^n \setminus \{0\}$. If E contains a ball of radius $\sqrt{n}$, then it also contains a half-space which includes the center of the ball, has delimiting hyperplane orthogonal to ϖ and is such that ϖ points outside of it.*

Proof. Let $B_{\sqrt{n}}(x_0)$ be the ball of radius $\sqrt{n}$ and center x_0 contained in E. By the Birkhoff property, it holds

$$\bigcup_{\substack{k\in\mathbb{Z}^n\\ \varpi\cdot k\leqslant 0}} \tau_k B_{\sqrt{n}}(x_0) \subseteq \bigcup_{\substack{k\in\mathbb{Z}^n\\ \varpi\cdot k\leqslant 0}} \tau_k E \subseteq E.$$

The thesis now follows by observing that the set on the left-hand side above contains the half-space $\{\varpi \cdot (x - x_0) < \varepsilon\}$, for some $\varepsilon > 0$. □

Now we show that the level sets of the minimal minimizer are Birkhoff sets. Recalling the relation between translations and level sets established in (5.3.18), we have

Proposition 5.20. *Let $\theta \in \mathbb{R}$. Then, the superlevel set $\left\{u_\omega^{A,B} > \theta\right\}$ has the Birkhoff property with respect to ω. Explicitly,*

- *$\left\{\tau_k u_\omega^{A,B} > \theta\right\} \subseteq \left\{u_\omega^{A,B} > \theta\right\}$, for any $k \in \mathbb{Z}^n$ such that $\omega \cdot k \leqslant 0$, and*
- *$\left\{\tau_k u_\omega^{A,B} > \theta\right\} \supseteq \left\{u_\omega^{A,B} > \theta\right\}$, for any $k \in \mathbb{Z}^n$ such that $\omega \cdot k \geqslant 0$.*

Analogously, the sublevel set $\{u_\omega^{A,B} < \theta\}$ has the Birkhoff property with respect to $-\omega$. The same statements still hold if we replace strict level sets with broad ones.

Proof. Let $v := \min\{u_\omega^{A,B}, \tau_k u_\omega^{A,B}\}$ and observe that $\tau_k u_\omega^{A,B}$ is the minimal minimizer with respect to the strip $\tau_k \mathcal{S}_\omega^{A,B} = \mathcal{S}_\omega^{A+\omega\cdot k, B+\omega\cdot k}$. If $\omega \cdot k \leqslant 0$ then by Lemma 5.10 it follows that $v \in \mathcal{M}_\omega^{A+\omega\cdot k, B+\omega\cdot k}$. Thus, $\tau_k u_\omega^{A,B} \leqslant v \leqslant u_\omega^{A,B}$ and hence

$$\left\{\tau_k u_\omega^{A,B} > \theta\right\} \subseteq \left\{u_\omega^{A,B} > \theta\right\}.$$

On the other hand, if $\omega \cdot k \geqslant 0$ then $v \in \mathcal{M}_\omega^{A,B}$ and therefore

$$\left\{ u_\omega^{A,B} < \theta \right\} \subseteq \left\{ \tau_k u_\omega^{A,B} < \theta \right\}.$$

The conclusion for the sublevel set $\{u_\omega^{A,B} \leqslant \theta\}$ follows observing that a set $E \subseteq \mathbb{R}^n$ is Birkhoff with respect to a vector $\varpi \in \mathbb{R}^n$ if and only if $\mathbb{R}^n \setminus E$ is Birkhoff with respect to $-\varpi$.

Finally, by writing

$$\left\{ u_\omega^{A,B} < \theta \right\} = \bigcup_{k \in \mathbb{N}} \left\{ u_\omega^{A,B} \leqslant \theta - 1/k \right\},$$

and noticing that the union of a family of sets that are Birkhoff with respect to a mutual vector is itself Birkhoff with respect to the same vector, we deduce that $\{u_\omega^{A,B} < \theta\}$ has the Birkhoff property with respect to $-\omega$. In a similar way one checks that the superlevel set $\{u_\omega^{A,B} \geqslant \theta\}$ is Birkhoff with respect to ω. □

5.3.6 Unconstrained and class A minimization

From now on we mainly restrict our attention to strips of the form

$$\mathcal{S}_\omega^M := \mathcal{S}_\omega^{0,M} = \{ x \in \mathbb{R}^n : \omega \cdot x \in [0, M] \}.$$

We simply write $\mathcal{A}_\omega^M$ for the space $\mathcal{A}_\omega^{0,M}$ of admissible functions, $\mathcal{M}_\omega^M$ for the absolute minimizers and u_ω^M for the minimal minimizer. We also assume $M > 10|\omega|$, in order to avoid degeneracies caused by too narrow strips.

The main purpose of this subsection is to show that the minimal minimizer u_ω^M becomes unconstrained for large, universal values of $M/|\omega|$. By *unconstrained* we mean that u_ω^M no longer *feels* the boundary data prescribed outside the strip $\mathcal{S}_\omega^M$ and gains additional minimizing properties in the whole space $\mathbb{R}^n$. Of course, we will be more precise on this later in Proposition 5.23.

We begin by adapting the results of Sections 3.3 and 5.2 to the minimal minimizer u_ω^M. Recall that u_ω^M is a local minimizer for $\mathscr{E}_K$ inside the strip $\mathcal{S}_\omega^M$, thanks to Corollary 5.17. In particular, $u_{\omega,m}^{A,B}$ is a bounded weak solution of the Euler-Lagrange equation

$$L_K u_{\omega,m}^{A,B} = W'(u_{\omega,m}^{A,B}) \quad \text{in } \mathcal{S}_\omega^M,$$

with L_K as in (24). In view of Corollary 3.3, we then deduce that there exist universal quantities $\alpha \in (0,1)$ and $C_1 \geqslant 1$ for which

$$\|u_\omega^M\|_{C^{0,\alpha}(S)} \leqslant C_1, \tag{5.3.19}$$

for any open set $S \subset\subset \mathcal{S}_\omega^M$ such that $\operatorname{dist}\left(S, \partial \mathcal{S}_\omega^M\right) \geqslant 1$.

On the other hand, Proposition 5.5 tells that, given $x_0 \in \mathcal{S}_\omega^M$ and $R \geqslant 3$ in such a way that $B_{R+2}(x_0) \subset\subset \mathcal{S}_\omega^M$, it holds

$$\mathscr{E}_K(u_\omega^M; B_R(x_0)) \leqslant C_2 R^{n-1} \Psi_s(R), \tag{5.3.20}$$

for a universal constant $C_2 > 0$. Recall that $\Psi_s(R)$ was defined in (5.2.1).

Now that (5.3.19) and (5.3.20) are established, we may proceed to the core proposition of the present subsection.

Proposition 5.21. *There exists a universal $M_0 > 0$ such that if $M \geqslant M_0|\omega|$, then the superlevel set $\{u_\omega^M > -9/10\}$ is at least at distance 1 from the upper constraint $\{\omega \cdot x = M\}$ delimiting $\mathcal{S}_\omega^M$.*

Proof. In the course of this proof we will often indicate balls and cubes without any explicit mention of their center. Thus, B will be for instance used to denote a ball not necessarily centered at the origin, in contrast with the notation adopted in the rest of the chapter.

We claim that

$$\begin{aligned}&\text{there exists a universal constant } M_0 \geqslant 8n \text{ such that, for any } M \geqslant M_0|\omega|,\\ &\text{we can find a ball } B_{\sqrt{n}}(\bar{z}) \subset\subset \mathcal{S}_\omega^M, \text{ for some } \bar{z} \in \mathcal{S}_\omega^M, \text{ on which}\\ &\text{either } u_\omega^M \geqslant 9/10 \text{ or } u_\omega^M \leqslant -9/10.\end{aligned} \tag{5.3.21}$$

Let $M \geqslant 8n|\omega|$ be given and suppose that for any ball $\tilde{B}$ of radius $\sqrt{n}$ compactly contained in $\mathcal{S}_\omega^M$, there exists a point $\tilde{x} \in \tilde{B}$ such that $|u_\omega^M(\tilde{x})| < 9/10$. If we show that $M/|\omega|$ is less or equal to a universal value M_0, claim (5.3.21) would then be true.

Let $k \geqslant 2$ be the only integer for which

$$k \leqslant \frac{M}{4n|\omega|} < k+1. \tag{5.3.22}$$

Take a point $x_0 \in \mathcal{S}_\omega^M$ lying on the hyperplane $\{\omega \cdot x = M/2\}$ and consider the ball $B = B_{nk}(x_0)$. By (5.3.22), we have that $B \subset\subset \mathcal{S}_\omega^M$, with

$$\operatorname{dist}\left(B, \partial\mathcal{S}_\omega^M\right) = \frac{M}{2|\omega|} - nk \geqslant nk \geqslant 4. \tag{5.3.23}$$

Consequently, we may apply the bound in (5.3.19) to deduce that

$$\|u_\omega^M\|_{C^{0,\alpha}(B)} \leqslant C_1. \tag{5.3.24}$$

Let now Q be a cube of sides $2\sqrt{n}k$, centered at x_0. Of course, $Q \subset B$. It is easy to see that Q may be partitioned (up to a negligible set) into a collection $\{Q_j\}_{j=1}^{k^n}$ of cubes with sides of length $2\sqrt{n}$, parallel to those of Q. Moreover, we denote with $B_j \subset Q_j$ the ball of radius $\sqrt{n}$ having the same center of Q_j. See Figure 5.1.

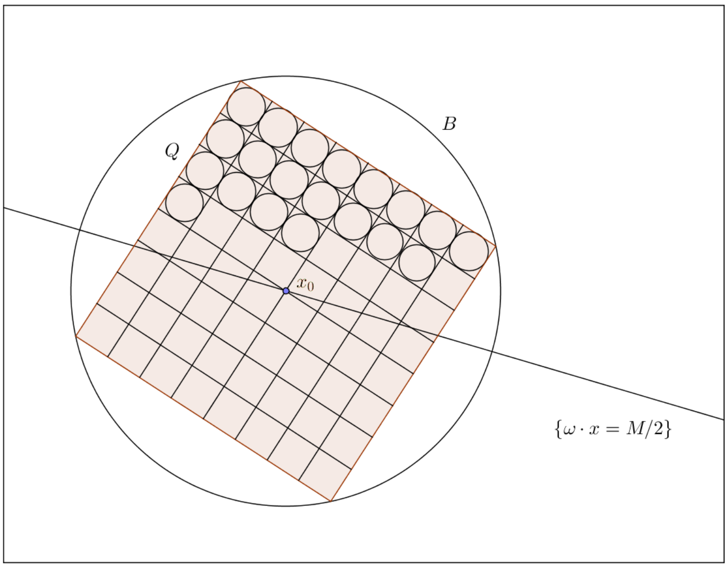

Figure 5.1: The partition of the cube Q into the subcubes Q_j's and the concentric balls B_j's.

In view of our starting assumption, for any $j = 1, \dots, k^n$ there exists a point $\tilde{x}_j \in B_j$ at which $|u_\omega^M(\tilde{x}_j)| < 9/10$. We claim that

$$|u_\omega^M| < 99/100 \quad \text{in } B_{r_0}(\tilde{x}_j), \tag{5.3.25}$$

for some universal radius $r_0 \in (0,1)$. Indeed, setting $r_0 := (9/(100C_1))^{1/\alpha}$, by (5.3.24) we get

$$|u_\omega^M(x)| \leqslant |u_\omega^M(\tilde{x}_j)| + C_1|x-\tilde{x}_j|^\alpha < \frac{9}{10} + C_1 r_0^\alpha = \frac{99}{100},$$

for any $x \in B_{r_0}(\tilde{x}_j)$. Hence, (5.3.25) is established. Furthermore, since $\tilde{x}_j \in B_j \subset Q_j$, we have

$$|B_{r_0}(\tilde{x}_j) \cap Q_j| \geqslant \frac{1}{2^n}\,|B_{r_0}(\tilde{x}_j)| = \frac{\alpha_n}{2^n} r_0^n. \tag{5.3.26}$$

By combining (5.3.25) and (5.3.26), recalling (5.1.7) we compute

$$\begin{aligned}
\mathscr{P}\left(u_\omega^M; B\right) &\geqslant \mathscr{P}\left(u_\omega^M; Q\right) = \sum_{j=1}^{k^n} \mathscr{P}\left(u_\omega^M; Q_j\right) \\
&\geqslant \sum_{j=1}^{k^n} \mathscr{P}\left(u_\omega^M; B_{r_0}(\tilde{x}_j) \cap Q_j\right) = \sum_{j=1}^{k^n} \int_{B_{r_0}(\tilde{x}_j)\cap Q_j} W\left(x, u_\omega^M(x)\right)\, dx \\
&\geqslant \gamma\left(\frac{99}{100}\right) \sum_{j=1}^{k^n} |B_{r_0}(\tilde{x}_j) \cap Q_j| \geqslant \frac{\alpha_n}{2^n} r_0^n \gamma\left(\frac{99}{100}\right) k^n \\
&=: C_3 k^n,
\end{aligned}$$

with $C_3 > 0$ universal. On the other hand, (5.3.20) implies that

$$\mathscr{P}(u_\omega^M; B) \leqslant \mathscr{E}_K(u_\omega^M; B) \leqslant C_2\,(nk)^{n-1}\,\Psi_s\,(nk) \leqslant C_4 k^{n-1}\Psi_s(k),$$

for some universal $C_4 > 0$. Note that the energy estimate (5.3.20) may be applied to the ball B thanks to (5.3.23). Comparing the last two inequalities and recalling (5.2.1), we find out that k cannot be greater than a universal constant. By (5.3.22), the same holds true for the quotient $M/|\omega|$ and hence (5.3.21) follows.

Now, we want to rule out the possibility of u_ω^M being greater or equal to $9/10$ on $B_{\sqrt{n}}(\bar{z})$, thus showing that $u_\omega^M \leqslant -9/10$ in $B_{\sqrt{n}}(\bar{z})$. By contradiction, assume that

$$u_\omega^M \geqslant 9/10 \quad \text{in } B_{\sqrt{n}}(\bar{z}). \tag{5.3.27}$$

In view of Proposition 5.20 the set $\left\{u_\omega^M \geqslant 9/10\right\}$ has the Birkhoff property with respect to ω. Hence, thanks to (5.3.27) and Proposition 5.19, this superlevel set contains the half-space $\Pi_- := \{\omega\cdot(x-\bar{z}) < 0\}$. Since $B_{\sqrt{n}}(\bar{z}) \subset \mathcal{S}_\omega^M$, we then deduce that the distance of $\partial\Pi_-$ from the lower constraint $\{\omega\cdot x = 0\}$ is at least 1. Accordingly, if we assume without loss of generality that $\omega_1 > 0$, then the translation $\tau_{-e_1} u_\omega^M$ belongs to $\mathcal{A}_\omega^M$ (recall definition (5.3.9)). But then, the periodicity assumptions (5.1.9)-(5.1.10) imply that $\mathscr{F}_\omega(\tau_{-e_1}u_\omega^M) = \mathscr{F}_\omega(u_\omega^M)$ and thus $\tau_{-e_1}u_\omega^M \in \mathcal{M}_\omega^M$. Being u_ω^M the minimal minimizer, we conclude that

$$u_\omega^M(x+e_1) = \tau_{-e_1}u_\omega^M(x) \geqslant u_\omega^M(x) \quad \text{for a.a. } x \in \mathbb{R}^n.$$

By iterating this inequality we then find that

$$u_\omega^M(x+\ell e_1) \geqslant u_\omega^M(x) \geqslant \frac{9}{10} \quad \text{for a.a. } x \in \Pi_- \text{ and any } \ell \in \mathbb{N},$$

i.e., $u_\omega^M \geqslant 9/10$ a.e. in $\mathbb{R}^n$, in contradiction with the fact that, by construction, $u_\omega^M \leqslant -9/10$ in $\{\omega\cdot x \geqslant M\}$.

As a result, $u_\omega^M \leqslant -9/10$ on the ball $B_{\sqrt{n}}(\bar{z})$. The proof then finishes by applying once again Propositions 5.20 and 5.19 to the sublevel set $\left\{u_\omega^M \leqslant -9/10\right\}$. $\square$

Corollary 5.22. *If $M \geqslant M_0|\omega|$, then $u_\omega^M = u_\omega^{M+a}$, for any $a \geqslant 0$.*

Proof. Fix $M \geqslant M_0|\omega|$ and $a \in [0,1]$. By applying Proposition 5.21 to the minimal minimizer u_ω^{M+a}, we find that $u_\omega^{M+a} \leqslant -9/10$ a.e. in the half-space $\{\omega \cdot x \geqslant M\}$. Hence, $u_\omega^{M+a} \in \mathcal{A}_\omega^M$ and $\mathscr{F}_\omega(u_\omega^M) \leqslant \mathscr{F}_\omega(u_\omega^{M+a})$, by the minimization properties of u_ω^M. On the other hand, clearly $u_\omega^M \in \mathcal{A}_\omega^{M+a}$, so that we also have $\mathscr{F}_\omega(u_\omega^{M+a}) \leqslant \mathscr{F}_\omega(u_\omega^M)$. Thus, both u_ω^M and u_ω^{M+a} belong to $\mathcal{M}_\omega^M \cap \mathcal{M}_\omega^{M+a}$ and, consequently, they define the same function.

By iteration, the arguments extends to any $a \geqslant 0$. □

This corollary essentially tells that when $M/|\omega|$ is greater than the universal constant M_0 found in Proposition 5.21, then the upper constraint $\{\omega \cdot x = M\}$ becomes immaterial for the minimal minimizer u_ω^M, which starts attaining values below the threshold $-9/10$ well before touching that constraint.

The next result shows that a similar behavior also occurs with the lower constraint $\{\omega \cdot x = 0\}$, thus proving that the minimal minimizer is unconstrained. Recalling the notation introduced right above Lemma 5.10, we state the following

Proposition 5.23. *If $M \geqslant M_0|\omega|$, then u_ω^M is unconstrained, that is $u_\omega^M \in \mathcal{M}_\omega^{-a,M+a}$, for any $a \geqslant 0$.*

Proof. Let $k \in \mathbb{Z}^n$ be such that $\omega \cdot k \geqslant a$. Given $v \in \mathcal{A}_\omega^{-a,M+a}$, we consider its translation $\tau_k v \in \mathcal{A}_\omega^{M+a+\omega\cdot k}$. By Corollary 5.22, it then holds $\mathscr{F}_\omega(u_\omega^M) \leqslant \mathscr{F}_\omega(\tau_k v)$. The thesis then follows, as $\mathscr{F}_\omega(v) = \mathscr{F}_\omega(\tau_k v)$ by (5.1.9)-(5.1.10). □

To conclude the subsection, we combine the previous proposition with the results of Subsection 5.3.4 and obtain that u_ω^M is indeed a class A minimizer.

Theorem 5.24. *If $M \geqslant M_0|\omega|$, then u_ω^M is a class A minimizer of the functional $\mathscr{E}_K$.*

Proof. Let Ω be any given bounded subset of $\mathbb{R}^n$. Take $a \geqslant 0$ and $m \in \mathbb{Z}^{n-1}$ large enough to have Ω compactly contained in the quotient $\widetilde{\mathcal{S}}_{\omega,m}^{-a,M+a}$ (recall notation (5.3.11)). By virtue of Proposition 5.14 we know that $u_\omega^{-a,M+a}$ is the minimal minimizer of the class $\mathcal{M}_{\omega,m}^{-a,M+a}$. On the other hand, Proposition 5.23 yields $\mathscr{F}_\omega(u_\omega^M) = \mathscr{F}_\omega(u_\omega^{-a,M+a})$. Recalling the terminology introduced in Subsection 5.3.3, we then have

$$\mathscr{F}_{\omega,m}(u_\omega^M) = c_m \mathscr{F}_\omega(u_\omega^M) = c_m \mathscr{F}_\omega(u_\omega^{-a,M+a}) = \mathscr{F}_{\omega,m}(u_\omega^{-a,M+a}),$$

with $c_m = \prod_{i=1}^{n-1} m_i$. Hence, $u_\omega^M \in \mathcal{M}_{\omega,m}^{-a,M+a}$ and Proposition 5.16 implies that u_ω^M is a local minimizer of $\mathscr{E}_K$ in Ω. □

5.3.7 The case of irrational directions

Here we finish the proof of Theorem 5.4 for kernels satisfying hypothesis (5.3.1), by extending the results obtained in the previous subsections to irrational vectors ω. This task is accomplished by means of an approximation argument, whose most technical steps are inspired by [BV08, Section 7].

Fix $\omega \in \mathbb{R}^n \setminus \mathbb{Q}^n$ and consider a sequence $\{\omega_j\}_{j\in\mathbb{N}} \subset \mathbb{Q}^n \setminus \{0\}$ converging to ω. Denote with u_j the class A minimizer corresponding to ω_j, given by our construction. We recall that $u_j \in H^s_{\mathrm{loc}}(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)$, with $|u_j| \leqslant 1$ in $\mathbb{R}^n$, and that

$$\left\{x \in \mathbb{R}^n : |u_j(x)| \leqslant \frac{9}{10}\right\} \subseteq \left\{x \in \mathbb{R}^n : \frac{\omega_j}{|\omega_j|} \cdot x \in [0, M_0]\right\}, \tag{5.3.28}$$

for any $j \in \mathbb{N}$. Moreover, by Corollary 3.3, the u_j's are uniformly bounded in $C^{0,\alpha}(\mathbb{R}^n)$, for some universal $\alpha \in (0,1)$. Hence, by Arzelà-Ascoli theorem there exists a subsequence of $\{u_j\}$ - which, without loss of generality, we will assume to be $\{u_j\}$ itself - converging to some continuous function u, uniformly on compact subsets of $\mathbb{R}^n$.

Of course, $|u| \leqslant 1$ in $\mathbb{R}^n$. Also, (5.3.28) passes to the limit, so that the same inclusion holds with u and ω replacing u_j and ω_j. In order to finish the proof of Theorem 5.4 we therefore only need to show that u is a class A minimizer of $\mathscr{E}_K$. To do this, let $R \geqslant 1$ be a fixed number: we claim that u is a local minimizer of $\mathscr{E}_K$ in B_R, that is $\mathscr{E}_K(u; B_R) < +\infty$ and

$$\mathscr{E}_K(u; B_R) \leqslant \mathscr{E}_K(u + \varphi; B_R) \quad \text{for any } \varphi \text{ supported inside } B_R. \tag{5.3.29}$$

Observe that, going back to Remark 5.2, this implies that u is a class A minimizer.

To see that (5.3.29) is true, we first apply Proposition 5.5 to u_j and obtain that

$$\mathscr{E}_K(u_j; B_{R+1}) \leqslant C_R, \tag{5.3.30}$$

for some constant $C_R > 0$ independent of j. Furthermore, by Fatou's lemma, we know that

$$\mathscr{E}_K(u; B_{R+\tau}) \leqslant \liminf_{j \to +\infty} \mathscr{E}_K(u_j; B_{R+\tau}), \tag{5.3.31}$$

for any $\tau \in [0,1]$, and thus, in particular,

$$\mathscr{E}_K(u; B_R) \leqslant \mathscr{E}_K(u; B_{R+1}) \leqslant C_R < +\infty. \tag{5.3.32}$$

Recall that $\mathscr{E}_K(u; \cdot)$ is monotone non-decreasing with respect to set inclusion.

Now, we deal with the limit on the right-hand side of (5.3.31).

Let $\{\varepsilon_j\}_{j\in\mathbb{N}}$ be the sequence of positive real numbers given by

$$\varepsilon_j := \|u_j - u\|_{L^\infty(B_{R+1})}. \tag{5.3.33}$$

Clearly, ε_j converges to 0 and we may also assume $\varepsilon_j \leqslant 1/2$ for any j. Take $\eta_j \in C_c^\infty(\mathbb{R}^n)$ to be a cut-off function satisfying $0 \leqslant \eta_j \leqslant 1$ in $\mathbb{R}^n$, $\eta_j = 1$ in B_R, $\operatorname{supp}(\eta_j) \subseteq B_{R+\varepsilon_j}$ and $|\nabla \eta_j| \leqslant 2/\varepsilon_j$ in $\mathbb{R}^n$. Let φ be as in (5.3.29) and suppose without loss of generality that $\varphi \in L^\infty(\mathbb{R}^n)$. We are also allowed to assume $\mathscr{E}_K(u + \varphi; B_R) < +\infty$, formula (5.3.29) being trivially satisfied otherwise. As a consequence of this, (5.3.32), (5.1.5) and the boundedness of u and φ, we have that $\varphi \in H^s(B_{R+1})$. We define $v := u + \varphi$ and

$$v_j := \eta_j u + (1 - \eta_j) u_j + \varphi \quad \text{in } \mathbb{R}^n.$$

Notice that $v_j = v$ in B_R and $v_j = u_j$ in $\mathbb{R}^n \setminus B_{R+\varepsilon_j}$. Accordingly, v_j is an admissible competitor for u_j in $B_{R+\varepsilon_j}$ and thus

$$\mathscr{E}_K(u_j; B_{R+\varepsilon_j}) \leqslant \mathscr{E}_K(v_j; B_{R+\varepsilon_j}), \tag{5.3.34}$$

in view of the minimizing property of u_j. Furthermore, v_j converges to v uniformly on compact subsets of $\mathbb{R}^n$ and, in particular,

$$\|v_j - v\|_{L^\infty(B_{R+1})} \leqslant \|u_j - u\|_{L^\infty(B_{R+1})} = \varepsilon_j.$$

Fix a number $\delta \in (0,1)$ and take j big enough to have $\varepsilon_j < \delta/2$. We address the right-hand side of (5.3.34). Concerning its kinetic part, we decompose the domain of integration $\mathscr{C}_{B_{R+\varepsilon_j}}$ as

$$\mathscr{C}_{B_{R+\varepsilon_j}} = D_\delta \cup E_{j,\delta} \cup F_{j,\delta}, \tag{5.3.35}$$

where, up to sets of measure zero,

$$\begin{aligned}
D_\delta &:= (B_R \times B_R) \cup (B_R \times (B_{R+\delta} \setminus B_R)) \cup ((B_{R+\delta} \setminus B_R) \times B_R)\,,\\
E_{j,\delta} &:= \Big(\mathscr{C}_{B_{R+\varepsilon_j}} \cap (B_{R+\delta} \times B_{R+\delta})\Big) \setminus D_\delta,\\
F_{j,\delta} &:= \mathscr{C}_{B_{R+\varepsilon_j}} \setminus (B_{R+\delta} \times B_{R+\delta})\,.
\end{aligned}$$

See Figure 5.2. Also set

$$F_\delta := \mathscr{C}_{B_R} \setminus (B_{R+\delta} \times B_{R+\delta})\,,$$

and observe that, analogously to (5.3.35), it holds

$$\mathscr{C}_{B_R} = D_\delta \cup F_\delta. \tag{5.3.36}$$

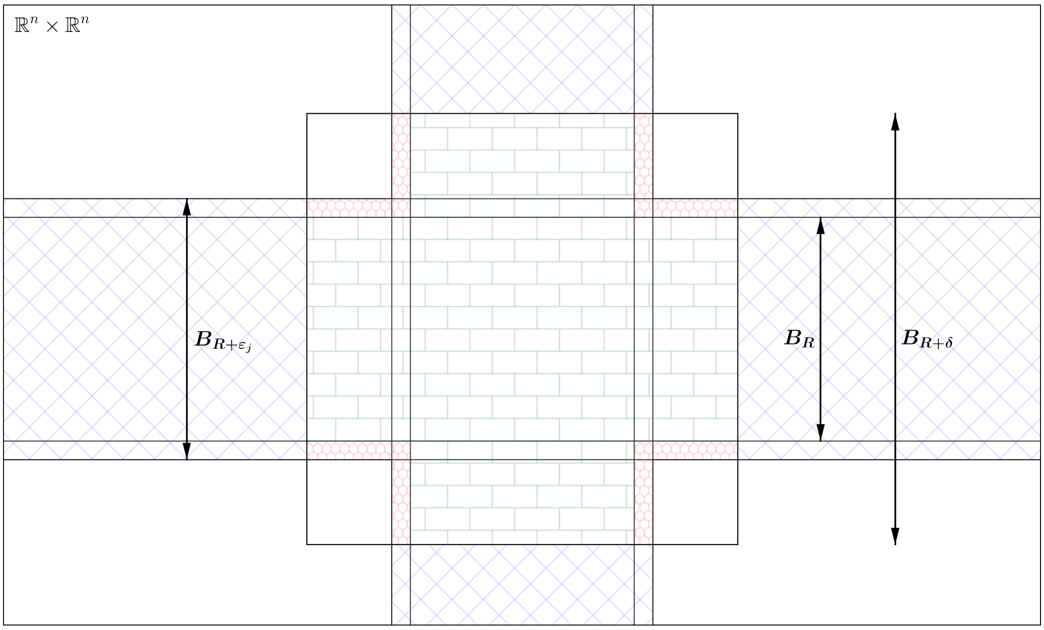

Figure 5.2: The decomposition of the region $\mathscr{C}_{B_{R+\varepsilon_j}}$ as given by (5.3.35). The set D_δ is rendered in the 'brick' texture, $E_{j,\delta}$ in the 'honeycomb' one and the 'diagonal crosshatch' is used to denote $F_{j,\delta}$.

First, we deal with the tail term of $\mathscr{E}_K$, which corresponds to $F_{j,\delta}$. Note that $F_{j,\delta}$ may be written as the union of $B_{R+\varepsilon_j} \times (\mathbb{R}^n \setminus B_{R+\delta})$ and $(\mathbb{R}^n \setminus B_{R+\delta}) \times B_{R+\varepsilon_j}$. By (5.1.4), it is clearly enough to study what happens inside the first set of this union. Given $x \in B_{R+\varepsilon_j}$ and $y \in \mathbb{R}^n \setminus B_{R+\delta}$, we have

$$|v_j(x) - v_j(y)| = |v_j(x) - u_j(y)| \leqslant 3 + |\varphi(x)|.$$

Moreover, $|x| \leqslant R + \varepsilon_j \leqslant [(R + \delta/2)/(R + \delta)]|y|$ and thus

$$|x - y| \geqslant |y| - |x| \geqslant \frac{\delta}{2(R+\delta)}|y|.$$

Using (5.1.5), for any $x \in B_{R+1}$ and $y \in \mathbb{R}^n \setminus B_{R+\delta}$ we get

$$|v_j(x) - v_j(y)|^2\, K(x, y)\chi_{B_{R+\varepsilon_j}}(x) \leqslant C\, \frac{1 + |\varphi(x)|^2}{|y|^{n+2s}} \in L^1\left(B_{R+1} \times (\mathbb{R}^n \setminus B_{R+\delta})\right),$$

for some constant $C > 0$ independent of j. Recalling that v_j converges pointwise to v in $\mathbb{R}^n$, by the Dominated Convergence Theorem we conclude that

$$\lim_{j\to+\infty} \iint_{F_{j,\delta}} |v_j(x) - v_j(y)|^2 K(x,y)\,dxdy = \iint_{F_\delta} |v(x) - v(y)|^2 K(x,y)\,dxdy. \tag{5.3.37}$$

Now, we focus on $E_{j,\delta}$. By the triangle inequality, for any $x, y \in B_{R+1}$ we write

$$\begin{aligned} |v_j(x) - v_j(y)| &\leqslant |\eta_j(x) - \eta_j(y)|\,|u(x) - u_j(x)| + |\eta_j(y)|\,|u(x) - u(y)| \\ &\quad + |1 - \eta_j(y)|\,|u_j(x) - u_j(y)| + |\varphi(x) - \varphi(y)| \\ &\leqslant \varepsilon_j\,|\eta_j(x) - \eta_j(y)| + |u(x) - u(y)| + |u_j(x) - u_j(y)| + |\varphi(x) - \varphi(y)|\,, \end{aligned}$$

where we also used (5.3.33) and that $|\eta_j| \leqslant 1$. Hence, taking advantage of (5.1.5) and the regularity of η_j,

$$\begin{aligned} &\left[\iint_{E_{j,\delta}} |v_j(x) - v_j(y)|^2 K(x,y)\,dxdy\right]^{\frac{1}{2}} \\ &\quad\leqslant \left[4\Lambda \iint_{E_{j,\delta}} \frac{dxdy}{|x-y|^{n-2+2s}}\right]^{\frac{1}{2}} + \left[\iint_{E_{j,\delta}} |u(x) - u(y)|^2 K(x,y)\,dxdy\right]^{\frac{1}{2}} \\ &\qquad + \left[\iint_{E_{j,\delta}} |u_j(x) - u_j(y)|^2 K(x,y)\,dxdy\right]^{\frac{1}{2}} + \left[\iint_{E_{j,\delta}} |\varphi(x) - \varphi(y)|^2 K(x,y)\,dxdy\right]^{\frac{1}{2}}. \end{aligned}$$

Note that the arguments of the first, second and fourth integrals on the right-hand side above are integrable on the set $B_{R+1} \times B_{R+1}$, which contains $E_{j,\delta}$. Thus, by the absolute continuity of the Lebesgue measure in $\mathbb{R}^n \times \mathbb{R}^n$, it follows that those integrals go to zero, as $j \to +\infty$ (observe in this regard that $|E_{j,\delta}| \to 0$). Moreover, in view of (5.3.30), we conclude that

$$\iint_{E_{j,\delta}} |v_j(x) - v_j(y)|^2 K(x,y)\,dxdy \leqslant \iint_{E_{j,\delta}} |u_j(x) - u_j(y)|^2 K(x,y)\,dxdy + \rho_j, \tag{5.3.38}$$

for some sequence $\{\rho_j\}$ of positive real numbers such that

$$\lim_{j\to+\infty} \rho_j = 0. \tag{5.3.39}$$

We are left with the term involving D_δ. We recall that $v_j = v$ in B_R, so that

$$\int_{B_R}\int_{B_R} |v_j(x) - v_j(y)|^2 K(x,y)\,dxdy = \int_{B_R}\int_{B_R} |v(x) - v(y)|^2 K(x,y)\,dxdy. \tag{5.3.40}$$

Therefore, we just need to examine the complement $D_\delta \setminus (B_R \times B_R)$ and thus, by symmetry, the region $B_R \times (B_{R+\delta} \setminus B_R)$ only. Letting $x \in B_R$ and $y \in B_{R+\delta} \setminus B_R$, by (5.3.33) we have

$$\begin{aligned} |v_j(x) - v_j(y)| = |v(x) - v_j(y)| &\leqslant |v(x) - v(y)| + |1 - \eta_j(y)|\,|u(y) - u_j(y)| \\ &= |v(x) - v(y)| + |\eta_j(x) - \eta_j(y)|\,|u(y) - u_j(y)| \\ &\leqslant |v(x) - v(y)| + \varepsilon_j\,|\eta_j(x) - \eta_j(y)|\,. \end{aligned}$$

Then, by the definition of η_j and (5.1.5) we get

$$\begin{aligned}
&\left[\int_{B_R}\int_{B_{R+\delta}\setminus B_R} |v_j(x)-v_j(y)|^2\, K(x,y)\,dxdy\right]^{\frac{1}{2}} \\
&\qquad \leqslant \left[\int_{B_R}\int_{B_{R+\delta}\setminus B_R} |v(x)-v(y)|^2\, K(x,y)\,dxdy\right]^{\frac{1}{2}} + \left[\int_{B_R}\int_{B_{R+\delta}\setminus B_R} \frac{4\Lambda\,dxdy}{|x-y|^{n-2+2s}}\right]^{\frac{1}{2}} \\
&\qquad \leqslant \left[\int_{B_R}\int_{B_{R+\delta}\setminus B_R} |v(x)-v(y)|^2\, K(x,y)\,dxdy\right]^{\frac{1}{2}} + C\,|B_{R+\delta}\setminus B_R|^{\frac{1}{2}},
\end{aligned}$$

for some constant $C>0$ independent of j and δ. Recalling (5.3.40), we may thence conclude that there exists a function $r:(0,1)\to(0,+\infty)$ for which

$$\lim_{\delta\to 0^+} r(\delta)=0, \tag{5.3.41}$$

and

$$\iint_{D_\delta} |v_j(x)-v_j(y)|^2\, K(x,y)\,dxdy \leqslant \iint_{D_\delta} |v(x)-v(y)|^2\, K(x,y)\,dxdy + r(\delta), \tag{5.3.42}$$

for any j big enough.

Observe now that for the potential term of $\mathscr{E}_K$ we may simply estimate

$$\mathscr{P}(v_j;B_{R+\varepsilon_j}) \leqslant \mathscr{P}(v;B_R) + W^*\left|B_{R+\varepsilon_j}\setminus B_R\right|.$$

Taking advantage of decomposition (5.3.35) on both sides of (5.3.34) and using inequalities (5.3.38), (5.3.42), we write

$$\begin{aligned}
&\frac{1}{4}\iint_{D_\delta\cup E_{j,\delta}\cup F_{j,\delta}} |u_j(x)-u_j(y)|^2\, K(x,y)\,dxdy + \mathscr{P}(u_j;B_{R+\varepsilon_j}) \\
&\quad = \mathscr{E}_K(u_j;B_{R+\varepsilon_j}) \leqslant \mathscr{E}_K(v_j;B_{R+\varepsilon_j}) \\
&\quad \leqslant \frac{1}{4}\iint_{D_\delta} |v(x)-v(y)|^2\, K(x,y)\,dxdy + \frac{1}{4}\iint_{E_{j,\delta}} |u_j(x)-u_j(y)|^2\, K(x,y)\,dxdy \\
&\qquad + \frac{1}{4}\iint_{F_{j,\delta}} |v_j(x)-v_j(y)|^2\, K(x,y)\,dxdy + \mathscr{P}(v;B_R) + W^*\left|B_{R+\varepsilon_j}\setminus B_R\right| + r(\delta) + \rho_j,
\end{aligned}$$

which in turn simplifies to

$$\begin{aligned}
&\frac{1}{4}\iint_{D_\delta\cup F_{j,\delta}} |u_j(x)-u_j(y)|^2\, K(x,y)\,dxdy + \mathscr{P}(u_j;B_{R+\varepsilon_j}) \\
&\qquad\qquad \leqslant \frac{1}{4}\iint_{D_\delta} |v(x)-v(y)|^2\, K(x,y)\,dxdy + \frac{1}{4}\iint_{F_{j,\delta}} |v_j(x)-v_j(y)|^2\, K(x,y)\,dxdy \\
&\qquad\qquad\quad + \mathscr{P}(v;B_R) + W^*\left|B_{R+\varepsilon_j}\setminus B_R\right| + r(\delta) + \rho_j.
\end{aligned}$$

If we exploit the fact that $\mathscr{C}_{B_R}\subset D_\delta\cup F_{j,\delta}$ and recall (5.3.36), (5.3.37), (5.3.39), by taking the limit in j in the previous formula we find

$$\limsup_{j\to+\infty} \mathscr{E}_K(u_j;B_R) \leqslant \mathscr{E}_K(v;B_R) + r(\delta).$$

Putting together this last inequality with (5.3.31), we finally obtain

$$\mathscr{E}_K(u;B_R) \leqslant \mathscr{E}_K(v;B_R) + r(\delta).$$

Then, (5.3.29) follows from the arbitrariness of δ and (5.3.41). We conclude that u is a class A minimizer of $\mathscr{E}_K$.

5.4 Proof of Theorem 5.4 for general kernels

In this section we complete the proof of Theorem 5.4, by extending the results of Section 5.3 to kernels which do not necessarily satisfy condition (5.3.1). This can be done in consequence of the fact that none of the estimates established there involve any of the parameters appearing in (5.3.1). This enables us to perform a limit argument analogous to that of Subsection 5.3.7.

Let K be a kernel satisfying (5.1.4), (5.1.5) and (5.1.9) only. Given any monotone increasing sequence $\{R_j\}_{j\in\mathbb{N}} \subset [2, +\infty)$ which diverges to $+\infty$, we set

$$K_j(x,y) := K(x,y)\chi_{[0,R_j]}(|x-y|) \quad \text{for any } x,y \in \mathbb{R}^n.$$

Notice that the new truncated kernel K_j still satisfies hypotheses (5.1.4), (5.1.5) and (5.1.9). Moreover, K_j clearly fulfills the additional requirement (5.3.1) with $\bar{R} = R_j$.

For a fixed direction $\omega \in \mathbb{R}^n \setminus \{0\}$, let u_j be the plane-like class A minimizer for $\mathscr{E}_{K_j}$ directed along ω. The existence of such minimizers is a consequence of Section 5.3, as K_j verifies (5.3.1). It holds

$$\left\{ x \in \mathbb{R}^n : |u_j(x)| \leqslant \frac{9}{10} \right\} \subseteq \left\{ x \in \mathbb{R}^n : \frac{\omega}{|\omega|} \cdot x \in [0, M_0] \right\}, \tag{5.4.1}$$

for a universal value $M_0 > 0$. Furthermore, $|u_j| \leqslant 1$ in $\mathbb{R}^n$ and, in view of Corollary 3.3, $\|u_j\|_{C^{0,\alpha}(\mathbb{R}^n)} \leqslant C$, for some $\alpha \in (0,1]$ and $C > 0$. We highlight the fact that we can choose M_0, α and C to be independent of j, since each K_j satisfies (5.1.5) with the same structural constants. Accordingly, by Arzelà-Ascoli theorem $\{u_j\}$ converges, up to a subsequence, to a continuous function u, uniformly on compact subset of $\mathbb{R}^n$.

Observe that u satisfies (5.4.1). Also, if ω is rational then each u_j is $\sim$-periodic and, consequently, so is u. To prove that u is a class A minimizer, fix $R \geqslant 1$ and consider a perturbation φ, with $\operatorname{supp}(\varphi) \subset\subset B_R$. We know that

$$\mathscr{E}_{K_j}(u_j; B_R) \leqslant \mathscr{E}_{K_j}(u_j + \varphi; B_R) \quad \text{for any } j \in \mathbb{N}.$$

On the one hand, a simple application of Fatou's lemma implies that

$$\mathscr{E}_K(u; B_R) \leqslant \liminf_{j \to +\infty} \mathscr{E}_{K_j}(u_j; B_R).$$

On the other hand, following the strategy presented in Subsection 5.3.7 it is not hard to see that we also have

$$\limsup_{j \to +\infty} \mathscr{E}_{K_j}(u_j; B_R) \leqslant \mathscr{E}_K(u + \varphi; B_R).$$

It follows that u is a class A minimizer of $\mathscr{E}_K$ and the proof of Theorem 5.4 is therefore complete.

5.5 Some auxiliary results

In this last section we enclose a couple of lemmata which cover some technical aspects that we faced throughout the chapter.

We begin with an observation on the necessity of hypothesis (5.3.1) for the validity of the computations of Section 5.3. We refer to Subsection 5.3.1, in particular, for the notation employed in the statement.

Lemma 5.25. *Assume that K is a measurable kernel satisfying*

$$K(x,y) \geqslant \frac{\gamma}{|x-y|^{n+\beta}} \quad \textit{for a.a. } x,y \in \mathbb{R}^n \textit{ such that } |x-y| \geqslant \bar{R}, \textit{ with } \beta \in (0,1],$$

for some $\gamma, \bar{R} > 0$. Then, given any two real numbers $A < B$, it holds

$$\int_{\{\omega\cdot x \leqslant A\}} \int_{\widetilde{\mathbb{R}}^n \cap \{\omega \cdot x \geqslant B\}} |u(x)-u(y)|^2 K(x,y)\, dxdy = +\infty, \tag{5.5.1}$$

for any $u \in \mathcal{A}_\omega^{A,B}$. Consequently, $\mathscr{F}_\omega \equiv +\infty$ on $\mathcal{A}_\omega^{A,B}$.

Proof. Of course, we may take $\omega = e_n$, $A = 0$ and $B = 1$. Then,

$$\{\omega \cdot x \leqslant A\} = \mathbb{R}^{n-1} \times (-\infty, 0] \quad \text{and} \quad \widetilde{\mathbb{R}}^n \cap \{\omega \cdot x \geqslant B\} = [0,1]^{n-1} \times [1, +\infty).$$

Under these conditions, the left hand side of (5.5.1) is controlled from below by

$$I := \gamma \int_{[0,1]^{n-1}\times[1,+\infty)} \left(\int_{(\mathbb{R}^{n-1}\times(-\infty,0])\setminus B_{\bar{R}}(x)} \frac{|u(x)-u(y)|^2}{|x-y|^{n+\beta}}\, dy \right) dx.$$

Since $u \in \mathcal{A}_{e_n}^{0,1}$, it follows that for any $x, y \in \mathbb{R}^n$ such that $x_n \geqslant 1$ and $y_n \leqslant 0$,

$$|u(x)-u(y)| = u(y) - u(x) \geqslant \frac{9}{10} - \left(-\frac{9}{10}\right) = \frac{9}{5} \geqslant 1.$$

Hence,

$$I \geqslant \gamma \int_{[0,1]^{n-1}\times[\bar{R}+1,+\infty)} \left(\int_{\mathbb{R}^{n-1}\times(-\infty,0]} \frac{dy}{|x-y|^{n+\beta}} \right) dx.$$

Arguing as in the proof of Lemma (5.7), it is easy to check that

$$\int_{\mathbb{R}^{n-1}\times(-\infty,0]} \frac{dy}{|x-y|^{n+\beta}} = c x_n^{-\beta},$$

for some constant $c > 0$ independent of x. Accordingly,

$$I \geqslant c\gamma \int_{\bar{R}+1}^{+\infty} x_n^{-\beta} dx_n = +\infty,$$

since $\beta \leqslant 1$. The thesis then follows. $\square$

Next is a lemma that ensures the finiteness of the integral appearing on the right-hand side of (5.3.12), in Subsection 5.3.4.

Lemma 5.26. *Let $\varphi \in L^\infty(\mathbb{R}^n)$ have support compactly contained in $\widetilde{\mathcal{S}}_{\omega,m}^{A,B}$, in the sense of footnote 13 at page 132. Denote with $\tilde{\varphi}$ the $\sim_m$-periodic extension to $\mathbb{R}^n$ of $\varphi|_{\widetilde{\mathbb{R}}^n_m}$. Then, the integral*

$$\int_{\widetilde{\mathbb{R}}^n_m} \int_{\mathbb{R}^n\setminus\widetilde{\mathbb{R}}^n_m} \frac{|\tilde{\varphi}(x)||\tilde{\varphi}(y)|}{|x-y|^{n+2s}}\, dxdy, \tag{5.5.2}$$

is finite.

Proof. Assume for simplicity that $\omega = e_n$ and $m = (1, \dots, 1)$. With this choices, we identify $\widetilde{\mathbb{R}}^n$ with its fundamental region $Q'_{1/2} \times \mathbb{R}$.

We split the domain of integration of (5.5.2) as

$$\widetilde{\mathbb{R}}^n \times \left(\mathbb{R}^n \setminus \widetilde{\mathbb{R}}^n\right) = \left(\widetilde{\mathbb{R}}^n \times \mathcal{D}_1\right) \cup \left(\widetilde{\mathbb{R}}^n \times \mathcal{D}_2\right),$$

with

$$\mathcal{D}_1 := \left(Q'_{\sqrt{n-1}} \setminus Q'_{1/2}\right) \times \mathbb{R} \quad \text{and} \quad \mathcal{D}_2 := \left(\mathbb{R}^{n-1} \setminus Q'_{\sqrt{n-1}}\right) \times \mathbb{R}.$$

We first deal with the integral involving the region $\mathcal{D}_1$. In view of the hypothesis on the support of φ, we have

$$\operatorname{dist}\left(\overline{\operatorname{supp}(\varphi)}, \mathcal{D}_1\right) \geqslant \delta,$$

for some $\delta > 0$. Therefore, we estimate

$$\begin{aligned} \int_{\widetilde{\mathbb{R}}^n} \int_{\mathcal{D}_1} \frac{|\tilde{\varphi}(x)||\tilde{\varphi}(y)|}{|x-y|^{n+2s}} \, dx dy &\leqslant \|\varphi\|^2_{L^\infty(\mathbb{R}^n)} \int_{\operatorname{supp}(\varphi)} \int_{\mathcal{D}_1 \cap \{x_n \in [A,B]\}} \frac{dx dy}{|x-y|^{n+2s}} \\ &\leqslant \|\varphi\|^2_{L^\infty(\mathbb{R}^n)} \delta^{-n-2s} \left[2\sqrt{n-1}\right]^{n-1} (B-A)^2, \end{aligned}$$

where we also used the fact that $\operatorname{supp}(\varphi)$ is contained in the strip $\mathbb{R}^{n-1} \times [A, B]$.

On the other hand, if $x \in \widetilde{\mathbb{R}}^n$ and $y \in \mathcal{D}_2$, then $|x'| \leqslant \sqrt{n-1}/2$ and $|y'| \geqslant \sqrt{n-1}$. Hence,

$$|x-y| \geqslant |x'-y'| \geqslant |y'| - |x'| \geqslant \frac{|y'|}{2},$$

and thus

$$\begin{aligned} \int_{\widetilde{\mathbb{R}}^n} \int_{\mathcal{D}_2} \frac{|\tilde{\varphi}(x)||\tilde{\varphi}(y)|}{|x-y|^{n+2s}} \, dx dy &\leqslant 2^{n+2s} \|\varphi\|^2_{L^\infty(\mathbb{R}^n)} (B-A)^2 \int_{\mathbb{R}^{n-1} \setminus B'_{\sqrt{n-1}}} \frac{dy'}{|y'|^{n+2s}} \\ &\leqslant c_n \|\varphi\|^2_{L^\infty(\mathbb{R}^n)} (B-A)^2, \end{aligned}$$

for some dimensional constant $c_n > 0$. This concludes the proof. $\square$

Chapter 6

One-dimensional minimizers for translation-invariant functionals

6.1 Description of the setting and main results

In this chapter we establish the existence of one-dimensional solutions of the nonlocal Allen-Cahn-type equation

$$L_K u = W'(u), \tag{6.1.1}$$

in the full space $\mathbb{R}^n$. Observe that L_K is the integral operator introduced in (24), with K translation-invariant, and W is a double-well potential similar to the one we considered in Chapter 5, but independent of the space variable x. The detailed formulation of the setting is as follows.

Given a domain $\Omega \subseteq \mathbb{R}^n$, for some integer $n \geqslant 1$, we consider the energy functional

$$\mathscr{E}_K(u, \Omega) := \mathscr{K}_K(u, \Omega) + \mathscr{P}(u, \Omega), \tag{6.1.2}$$

where the nonlocal interaction term $\mathscr{K}_K$ and the potential term $\mathscr{P}$ are respectively defined as

$$\mathscr{K}_K(u, \Omega) := \frac{1}{4} \iint_{\mathscr{C}_\Omega} |u(x) - u(y)|^2 K(x - y)\, dxdy,$$

with $\mathscr{C}_\Omega$ as in (3.2.1), and

$$\mathscr{P}(u, \Omega) := \int_\Omega W(u(x))\, dx.$$

Here, $K : \mathbb{R}^n \to [0, +\infty]$ is required to fulfill the symmetry condition

$$K(z) = K(-z) \quad \text{for a.a. } z \in \mathbb{R}^n, \tag{6.1.3}$$

along with various growth and ellipticity assumptions. We highlight the fact that no regularity is assumed on K.

The main ellipticity hypothesis on K will be

$$\frac{\lambda \chi_{B_{r_0}(z)}}{|z|^{n+2s}} \leqslant K(z) \leqslant \frac{\Lambda}{|z|^{n+2s}} \quad \text{for a.a. } z \in \mathbb{R}^n, \tag{6.1.4}$$

for some $\Lambda \geqslant \lambda > 0$, $r_0 > 0$ and $s \in (0, 1)$. Notice that condition (6.1.4) is very general and allows for a great variety of translation invariant kernels only locally comparable to that of the fractional Laplacian. For instance, under (6.1.4) we encompass truncated kernels of the form

$$K(z) = \chi_{B_{r_0}(z)} \frac{a(z)}{|z|^{n+2s}}, \tag{6.1.5}$$

with a bounded and positive, which have been considered in [KKL14, HR-OSV15]. Kernels satisfying (6.1.4), and even broader similar requirements, are by now widely studied. See e.g. [K09, K11, DCKP14, DCKP15].

For some purposes, we will need the kernel K to satisfy the stronger condition

$$\frac{\lambda}{|z|^{n+2s}} \leqslant K(z) \leqslant \frac{\Lambda}{|z|^{n+2s}} \quad \text{for a.a. } z \in \mathbb{R}^n. \tag{6.1.6}$$

Assumption (6.1.6) differs from (6.1.4) in that K is here required to control the kernel of the fractional Laplacian at all scales and not only in a neighbourhood of the origin. Such hypothesis is more frequently adopted in the literature. To name a few, see [CS09, CS11, S14, KMS15, KMS15b].

Finally, to obtain some additional specific results we will restrict ourselves to homogeneous kernels. That is, we will ask K to be in the form

$$K(z) = \frac{a(z/|z|)}{|z|^{n+2s}} \quad \text{for a.a. } z \in \mathbb{R}^n, \text{ with } \lambda \leqslant a(\zeta) \leqslant \Lambda \text{ for a.a. } \zeta \in S^{n-1}. \tag{6.1.7}$$

Note that, in dimension $n = 1$, this and the symmetry condition (6.1.3) force K to be the kernel of the fractional Laplacian, up to a multiplicative constant, i.e.

$$K(z) = \lambda_\star |z|^{-1-2s} \quad \text{for a.a. } z \in \mathbb{R}, \tag{6.1.8}$$

for some $\lambda_\star \in [\lambda, \Lambda]$. We remark that this condition and other generalizations in the same spirit are also often considered in the literature. The interest in (6.1.7) is motivated, for example, by its relationship with stable Lévy processes in probability theory. On the analysis side, they often lead to slightly sharper results, especially in regularity theory. We refer the interested reader to the works [FV14, R-OS14b, R-OS15, R-OV15].

On the other hand, the term $\mathscr{P}$ is driven by a smooth, even double-well potential W with wells at ± 1. More precisely, $W : \mathbb{R} \to [0, +\infty)$ is a function of class $C^{2,\beta}_{\mathrm{loc}}(\mathbb{R})$, for some $\beta > 0$, such that

$$W(r) > 0 \quad \text{for any } r \in (-1, 1), \tag{6.1.9}$$
$$W(\pm 1) = W'(\pm 1) = 0, \tag{6.1.10}$$
$$W''(\pm 1) > 0, \tag{6.1.11}$$

and

$$W(r) = W(-r) \quad \text{for any } r \in [-1, 1]. \tag{6.1.12}$$

A typical example for W is represented by the choice

$$W(r) = \frac{(1 - r^2)^2}{4}.$$

In this chapter we focus on the study of the minimizers for the nonlocal energy functional (6.1.2). Note that such minimizers are particular solutions of the Euler-Lagrange equation (6.1.1). The exact notion of minimizers of (6.1.2) in $\mathbb{R}^n$ that we will adopt is that of *class A minimizers*, which has been introduced by Definition 5.3, in Chapter 5.

We now proceed to the statements of the main result contained in the chapter.

Our first contribution focuses on the construction of class A minimizers for $\mathscr{E}_K$ in one dimension. More precisely, we prove the existence and essential uniqueness of a monotone class A minimizer in the class

$$\mathcal{X} := \left\{ f \in L^1_{\mathrm{loc}}(\mathbb{R}) \, : \, \lim_{x \to \pm\infty} f(x) = \pm 1 \right\}, \tag{6.1.13}$$

of admissible functions. Furthermore, we establish some sharp estimates for the behaviour of such minimizer at infinity and the growth of its energy $\mathscr{E}_K$ when evaluated on large intervals. To do this, we introduce the quantities

$$\mathscr{G}_*(u) := \liminf_{R\to+\infty} \frac{\mathscr{E}_K(u,[-R,R])}{\Psi_s(R)}, \qquad \mathscr{G}^*(u) := \limsup_{R\to+\infty} \frac{\mathscr{E}_K(u,[-R,R])}{\Psi_s(R)}, \tag{6.1.14}$$

and

$$\mathscr{G}(u) := \lim_{R\to+\infty} \frac{\mathscr{E}_K(u,[-R,R])}{\Psi_s(R)},$$

provided this last limit exists, where Ψ_s was defined in (5.2.1). The term Ψ_s is a scaling factor that will play an important role by compensating the possible blow up of the energy $\mathscr{E}_K$ at infinity.

The precise statement of the first result is as follows.

Theorem 6.1. *Let $n = 1$ and $s \in (0,1)$. Assume that K and W respectively satisfy conditions (6.1.3), (6.1.4) and (6.1.9), (6.1.10), (6.1.11), (6.1.12).*
There exists an odd, strictly increasing class A minimizer $u_0 \in \mathcal{X}$ for $\mathscr{E}_K$. The minimizer u_0 is of class $C^{1+2s+\alpha}(\mathbb{R})$, for some $\alpha > 0$,[14] and is the unique (up to translations) non-decreasing solution of the Euler-Lagrange equation

$$L_K u = W'(u) \quad \text{in } \mathbb{R}, \tag{6.1.15}$$

in the class $\mathcal{X}$.
Moreover, there exists a constant $C \geqslant 1$ such that the following estimates hold:

$$|u_0(x) - \operatorname{sgn}(x)| \leqslant \frac{C}{|x|^{2s}} \quad \text{and} \quad |u_0'(x)| \leqslant \frac{C}{|x|^{1+2s}}, \tag{6.1.16}$$

for any large $|x|$,

$$\int_{-R}^{R} \int_{\mathbb{R}\setminus[-R,R]} |u_0(x) - u_0(y)|^2 K(x-y)\, dxdy \leqslant C R^{1-2s}, \tag{6.1.17}$$

for any large $R > 0$, and

$$\mathscr{G}^*(u_0) < +\infty. \tag{6.1.18}$$

If in addition K satisfies (6.1.6), then we also have

$$\int_{-R}^{R} \int_{\mathbb{R}\setminus[-R,R]} |u_0(x) - u_0(y)|^2 K(x-y)\, dxdy \geqslant \frac{1}{C} R^{1-2s}, \tag{6.1.19}$$

for any large $R > 0$, and

$$\mathscr{G}_*(u_0) > 0. \tag{6.1.20}$$

Finally, if $s = 1/2$ and K satisfies (6.1.7) - in its form (6.1.8) -, then

$$\mathscr{G}(u_0) = \lim_{R\to+\infty} \frac{\mathscr{E}_K(u_0,[-R,R])}{\log R} \quad \text{exists and is finite}, \tag{6.1.21}$$

and it holds

$$\mathscr{G}(u_0) = \frac{\lambda_\star}{2} \left(\lim_{x\to+\infty} u_0(x) - \lim_{x\to-\infty} u_0(x) \right)^2 = 2\lambda_\star. \tag{6.1.22}$$

[14] Given a non-integer $\gamma > 0$ and a set $\Omega \subseteq \mathbb{R}^n$, in this chapter we indicate with $C^\gamma(\Omega)$ the space composed by functions of $C^{\lfloor\gamma\rfloor}(\Omega)$ whose partial derivatives of order $\lfloor\gamma\rfloor$ are globally Hölder continuous in Ω, with exponent $\gamma - \lfloor\gamma\rfloor$. Although no ambiguity should derive from this choice, we will always prefer the more common notation $C^{\lfloor\gamma\rfloor,\gamma-\lfloor\gamma\rfloor}$ whenever the value of $\lfloor\gamma\rfloor$ is known.

Remark 6.2. Observe that the oddness of u_0 is a consequence of the parity assumption (6.1.12). We stress that, apart from this, such hypothesis on the potential W is only used at a technical point in Section 6.4, in order to successfully perform a limiting procedure. We strongly believe that an appropriate adaptation of the arguments contained in [PSV13, Sections 3 and 4] may lead to the construction of non-symmetric class A minimizers, in the absence of (6.1.12).

Remark 6.3. Note that, when (6.1.4) is in force with $s > 1/2$, the existence and finiteness of $\mathscr{G}(u_0)$ can be easily deduced. Indeed, in such case,

$$\mathscr{G}(u_0) = \lim_{R \to +\infty} \mathscr{E}_K(u_0, [-R, R]) = \mathscr{E}_K(u_0, \mathbb{R}),$$

since the limit exists in view of the monotonicity of the energy (recall Remark 5.2 in Chapter 5). Moreover, we also know that $\mathscr{G}(u_0)$ is finite, thanks to (6.1.18). It is also immediate to check that $\mathscr{G}(u_0) > 0$, as, otherwise, u_0 would be constant.

Remark 6.4. When $s = 1/2$, a careful analysis of the proof of (6.1.21), provided by Proposition 6.26 in Section 6.4, shows that such conclusion still holds if hypothesis (6.1.8) on K is replaced by the requirement that the limit function

$$K_\infty(z) := \lim_{R \to +\infty} R^2 K(Rz) \text{ exists for a.a. } z > 1 \text{ and is measurable.} \tag{6.1.23}$$

We stress that condition (6.1.23) is really weaker than (6.1.8). Indeed, (6.1.23) is satisfied for instance by any kernel of the form

$$K(z) = \frac{\lambda_\star + \sigma(|z|)}{|z|^2},$$

with $\lambda_\star > 0$ and $\sigma : [0, +\infty) \to [0, +\infty)$ measurable, bounded and admitting limit at infinity.

If K only satisfies (6.1.23), besides (6.1.3) and (6.1.4), then (6.1.22) clearly can not be valid as it is. Nevertheless, by following the proof of Proposition 6.26, it is not hard to see that in such case

$$\mathscr{G}(u_0) = 2 \int_1^{+\infty} K_\infty(z)\, dz. \tag{6.1.24}$$

Note that $K_\infty \in L^1((1, +\infty))$ as a consequence of (6.1.23) and (6.1.6). Thus, the right-hand side above is finite.

Furthermore, we point out that (6.1.21) is trivially satisfied by any truncated kernels, such as for instance those of the form (6.1.5). In this case, $K_\infty \equiv 0$ and therefore $\mathscr{G}(u_0) = 0$, in view of (6.1.24). Besides being interesting on its own, this fact reveals in particular that condition (6.1.4) is not strong enough for (6.1.20) to hold, at least for the case $s = 1/2$.

We believe that an interesting related problem would be to understand whether conclusion (6.1.21) holds for a larger class of kernels or even for any general K satisfying (6.1.3) and (6.1.4)/(6.1.6).

Now that we have established the existence of class A minimizers on the real line, we can address the problem of how this construction translates to the n-dimensional setting, with $n \geqslant 2$. In particular, we shall prove the existence of a *one-dimensional* class A minimizer, that is a class A minimizer for $\mathscr{E}_K$ in $\mathbb{R}^n$ that depends only on one single variable, say x_n.

To do this, given a kernel $K : \mathbb{R}^n \to [0, +\infty]$ satisfying (6.1.3) and (6.1.4), let $k : \mathbb{R} \to [0, +\infty]$ be the kernel defined by[15]

$$k(t) := \frac{1}{\varpi} \int_{\mathbb{R}^{n-1}} K\left(z', \frac{t}{\varpi}\right) dz', \tag{6.1.25}$$

where

$$\varpi := \left[\int_{\mathbb{R}^{n-1}} \left(1 + |y'|^2\right)^{-\frac{n+2s}{2}} dy'\right]^{-\frac{1}{2s}}. \tag{6.1.26}$$

Note that the quantity ϖ is well-defined and positive (see e.g. (6.2.49) for a proof of this fact). The kernel k is a measurable function which clearly fulfills the symmetry requirement (6.1.3), as K does. Furthermore, it is also easy to see that k satisfies (6.1.4). Indeed, by applying the change of variables $y' := \varpi z'/t$, we compute

$$\begin{aligned} k(t) &= \frac{|t|^{n-1}}{\varpi^n} \int_{\mathbb{R}^{n-1}} K\left(\frac{t}{\varpi} y', \frac{t}{\varpi}\right) dy' \geqslant \frac{|t|^{n-1}}{\varpi^n} \int_{B'_{\sqrt{\frac{r_0^2 \varpi^2}{t^2} - 1}}} K\left(\frac{t}{\varpi} y', \frac{t}{\varpi}\right) dy' \\ &\geqslant \frac{\lambda \varpi^{2s}}{|t|^{1+2s}} \int_{B'_{\sqrt{\frac{r_0^2 \varpi^2}{t^2} - 1}}} \left(1 + |y'|^2\right)^{-\frac{n+2s}{2}} dy' \geqslant \frac{\lambda \varpi^{2s}}{|t|^{1+2s}} \int_{B'_1} \left(1 + |y'|^2\right)^{-\frac{n+2s}{2}} dy' \\ &= \frac{\tilde{\lambda}}{|t|^{1+2s}}, \end{aligned}$$

for some $\tilde{\lambda} > 0$, provided $t < \tilde{r}_0 := \varpi r_0/\sqrt{2}$. Similarly one checks that the right-hand side inequality in (6.1.4) holds true too. Then, we consider the minimizer u_0 for the energy $\mathscr{E}_k$ given by Theorem 6.1. We extend it to n-dimensions by setting

$$u^*(x) := u_0(\varpi x_n) \quad \text{for any } x \in \mathbb{R}^n. \tag{6.1.27}$$

In the next result we show that u^* is a class A minimizer for $\mathscr{E}_K$ and deduce some interesting facts on the asymptotics of the energy $\mathscr{E}_K(u^*, B_R)$, for $R > 0$ big.

Theorem 6.5. *Let $n \geqslant 2$ and $s \in (0, 1)$. Assume that K and W respectively satisfy conditions (6.1.3), (6.1.4) and (6.1.9), (6.1.10), (6.1.11), (6.1.12).*
Then, the function u^* defined in (6.1.27) is a class A minimizer for $\mathscr{E}_K$.
Furthermore, the following statements holds true.

- *If $s \in (0, 1/2)$, then there exists a constant $C \geqslant 1$ such that*

$$\int_{B_R} \int_{\mathbb{R}^n \setminus B_R} |u^*(x) - u^*(y)|^2 K(x - y)\, dx dy \leqslant C R^{n-2s}, \tag{6.1.28}$$

for any large $R > 0$. Also, if K satisfies (6.1.6), then

$$\int_{B_R} \int_{\mathbb{R}^n \setminus B_R} |u^*(x) - u^*(y)|^2 K(x - y)\, dx dy \geqslant \frac{1}{C} R^{n-2s}, \tag{6.1.29}$$

for any large $R > 0$.

[15] We reserve the primed notations x', y', z' for variables in $\mathbb{R}^{n-1}$ or, equivalently, in the hyperplane $\mathbb{R}^{n-1} \times \{0\}$ of $\mathbb{R}^n$. Similarly, we often denote with $B'_r(x'_0)$ the open $(n-1)$-dimensional ball with radius r and center x'_0. As for n-dimensional balls, B'_r stands for the ball centered at the origin.

- *If $s = 1/2$, then*[16]

$$\liminf_{R\to+\infty} \frac{\mathscr{E}_K(u^*, B_R)}{R^{n-1}\log R} = \frac{\alpha_{n-1}}{\varpi}\,\mathscr{G}_*(u_0), \qquad \limsup_{R\to+\infty} \frac{\mathscr{E}_K(u^*, B_R)}{R^{n-1}\log R} = \frac{\alpha_{n-1}}{\varpi}\,\mathscr{G}^*(u_0), \tag{6.1.30}$$

and

$$\lim_{R\to+\infty} \frac{1}{R^{n-1}\log R} \int_{B_R}\int_{\mathbb{R}^n\setminus B_R} |u^*(x) - u^*(y)|^2\, K(x-y)\, dxdy = 0. \tag{6.1.31}$$

Also, if K satisfies (6.1.7), then the inferior and superior limits in (6.1.30) are equal and it actually holds

$$\lim_{R\to+\infty} \frac{\mathscr{E}_K(u^*, B_R)}{R^{n-1}\log R} = \frac{\alpha_{n-1}}{\varpi}\,\mathscr{G}(u_0) = \frac{2\lambda_\star\alpha_{n-1}}{\varpi}, \tag{6.1.32}$$

with

$$\lambda_\star := \varpi^{2s}\int_{\mathbb{R}^{n-1}} K\left(y', 1\right)\, dy'. \tag{6.1.33}$$

- *If $s > 1/2$, then*

$$\lim_{R\to+\infty} \frac{\mathscr{E}_K(u^*, B_R)}{R^{n-1}} = \frac{\alpha_{n-1}}{\varpi}\,\mathscr{E}_K(u_0, \mathbb{R}), \tag{6.1.34}$$

and

$$\lim_{R\to+\infty} \frac{1}{R^{n-1}} \int_{B_R}\int_{\mathbb{R}^n\setminus B_R} |u^*(x) - u^*(y)|^2\, K(x-y)\, dxdy = 0. \tag{6.1.35}$$

Note that Theorem 6.5 is the generalization of [PSV13, Theorem 3] to our setting. To prove it, we also extend the techniques of [PSV13, Section 5] to rather general integral operators driven by possibly non-homogeneous and truncated kernels (and correct some minor flaws).

The verification of the fact that u^* is a class A minimizer is based on the following argument. By Theorem 6.1, we know that the function u_0 is a class A minimizer for $\mathscr{E}_k$ and a solution of

$$L_k u_0 = W'(u_0) \quad \text{in } \mathbb{R}.$$

A simple computation (see (6.5.4) in Section 6.5) then shows that u^* is a solution of

$$L_K u^* = W'(u^*) \quad \text{in } \mathbb{R}^n.$$

To obtain that u^* is actually a class A minimizer for $\mathscr{E}_K$, we rely on a general result that connects class A minimizers and monotone solutions with prescribed limits at infinity in one fixed direction.

Theorem 6.6. *Let $n \geqslant 1$ and $s \in (0,1)$. Assume that K and W respectively satisfy conditions (6.1.3), (6.1.4) and (6.1.10). Let $u : \mathbb{R}^n \to (-1,1)$ be a function of class $C^{1+2s+\gamma}(\mathbb{R}^n)$, for some $\gamma > 0$. Suppose that u is a solution of*

$$L_K u = W'(u) \quad \textit{in } \mathbb{R}^n, \tag{6.1.36}$$

which satisfies

$$\partial_{x_n} u(x) > 0 \quad \textit{for any } x \in \mathbb{R}^n, \tag{6.1.37}$$

and

$$\lim_{x_n\to\pm\infty} u(x', x_n) = \pm 1 \quad \textit{for any } x' \in \mathbb{R}^{n-1}. \tag{6.1.38}$$

Then, u is a class A minimizer for $\mathscr{E}_K$.

[16] Recall that, as prescribed in footnote 2 at page 78, the symbol α_{n-1} indicates the measure of the unit ball B'_1 of $\mathbb{R}^{n-1}$.

We observe that both Theorems 6.5 and 6.6 are of course still valid if we replace the direction e_n with a generic direction $e \in S^{n-1}$. This can be seen for instance by applying an appropriate rotation in the base space $\mathbb{R}^n$.

Remark 6.7. Hypothesis (6.1.37) may be relaxed to a weak monotonicity assumption. That is, we can replace it with

$$\partial_{x_n} u(x) \geqslant 0 \quad \text{for any } x \in \mathbb{R}^n, \tag{6.1.39}$$

without altering the validity of Theorem 6.6. Indeed, it can be shown that if u satisfies (6.1.36), (6.1.38) and (6.1.39), then u in fact satisfies (6.1.37). See Lemma 6.14 in Subsection 6.2.2 for a proof of this fact.

The remainder of the chapter is organized as follows.

In Section 6.2 we collect a few preliminary results regarding auxiliary barriers, equations, minimizers and some integral manipulations. In particular, we point the attention of the reader to Subsection 6.2.2, where we obtain a strong comparison principle for semilinear equations driven by the operator L_K, for a rather general class of non-negative kernels K.

The conclusive three sections are devoted to the proofs of the main results. In Section 6.3 we show the validity of Theorem 6.6. The subsequent Section 6.4 contains the arguments leading to the proof of Theorem 6.1, while the verification of Theorem 6.5 occupies the final Section 6.5.

6.2 Auxiliary results

In this section we include a few preliminary lemmata that will be employed throughout the remainder of the chapter to prove the main theorems.

6.2.1 Barriers and applications

Here we construct a couple of useful auxiliary functions that will be needed later on. We begin by introducing the following barrier.

Lemma 6.8. *Let $n \geqslant 1$, $s \in (0,1)$ and assume that K satisfies (6.1.3) and (6.1.4). Given any $\tau > 0$ there exists a constant $C \geqslant 1$, which may depend on n, s, Λ and τ, such that for any $R \geqslant C$ we can construct a symmetric radially non-decreasing function*

$$w \in C^{1,1}\left(\mathbb{R}^n, \left[-1 + C^{-1} R^{-2s}, 1\right]\right), \tag{6.2.1}$$

with

$$w = 1 \quad \text{in } \mathbb{R}^n \setminus B_R, \tag{6.2.2}$$

which satisfies

$$|L_K w(x)| \leqslant \tau \left(1 + w(x)\right), \tag{6.2.3}$$

and

$$\frac{1}{C}\left(R + 1 - |x|\right)^{-2s} \leqslant 1 + w(x) \leqslant C \left(R + 1 - |x|\right)^{-2s}, \tag{6.2.4}$$

for any $x \in B_R$.

Barriers like the one considered in Lemma 6.8 have been first constructed in [SV14, PSV13] for the fractional Laplacian. In order to prove Lemma 6.8, we first need the following

Lemma 6.9. *Let $n \in \mathbb{N}$, $s \in (0,1)$ and assume that K satisfies (6.1.3) and (6.1.4). Then, given $x \in \mathbb{R}^n$, $\rho > 0$ and $\psi \in L^\infty(\mathbb{R}^n) \cap C^{1,1}(B_\rho(x))$, it holds*

$$|L_K\psi(x)| \leqslant C\left(\|\psi\|_{L^\infty(\mathbb{R}^n)}\rho^{-2s} + \|\nabla^2\psi\|_{L^\infty(B_\rho(x))}\rho^{2(1-s)}\right), \tag{6.2.5}$$

for some constant $C > 0$ which depends only on n, s and Λ.

Proof. We have

$$\begin{aligned}|L_K\psi(x)| &\leqslant \int_{\mathbb{R}^n\setminus B_\rho(x)} |\psi(x)-\psi(y)|K(x-y)\,dy \\ &\quad + \int_{B_\rho(x)} |\psi(x)-\psi(y)+\nabla\psi(x)\cdot(y-x)|K(x-y)\,dy \\ &\quad + \left|\text{P.V.}\int_{B_\rho(x)} \nabla\psi(x)\cdot(y-x)K(x-y)\,dy\right| \\ &=: I_1 + I_2 + I_3.\end{aligned}$$

First, taking advantage of (6.1.3), it is easy to see that $I_3 = 0$. On the other hand, by (6.1.4),

$$I_1 \leqslant 2\Lambda\|\psi\|_{L^\infty(\mathbb{R}^n)} \int_{\mathbb{R}^n\setminus B_\rho(x)} |x-y|^{-n-2s}\,dy = \frac{\alpha_{n-1}\Lambda\|\psi\|_{L^\infty(\mathbb{R}^n)}\rho^{-2s}}{s}.$$

Finally, the regularity of ψ and again (6.1.4) imply that

$$I_2 \leqslant \Lambda\|\nabla^2\psi\|_{L^\infty(B_\rho(x))} \int_{B_\rho(x)} |x-y|^{2-n-2s}\,dy = \frac{\alpha_{n-1}\Lambda\|\nabla^2\psi\|_{L^\infty(B_\rho(x))}\rho^{2(1-s)}}{2(1-s)}.$$

Formula (6.2.5) then plainly follows. □

Proof of Lemma 6.8. Fix a value

$$r_1 \geqslant 2^{3/s}, \tag{6.2.6}$$

and let $r \geqslant r_1$. Then, set $\ell(t) := (r-t)^{-2s}$, for any $0 \leqslant t < r$, and define

$$\gamma_r := \left[\ell(r-1) - \ell(r/2) - \ell'(r/2)\,(r/2-1)\right]^{-1}.$$

Note that

$$\begin{aligned}\ell(r-1) - \ell(r/2) - \ell'(r/2)(r/2-1) &= 1 - 2^{2s}\left(1+2s-4sr^{-1}\right)r^{-2s} \\ &\geqslant 1 - 12{r_1}^{-2s} \\ &\geqslant 1/2,\end{aligned}$$

for any $r \geqslant r_1$. Thus, γ_r is well-defined and

$$1 < \gamma_r \leqslant 2. \tag{6.2.7}$$

Consider the function $h : [0,+\infty) \to [0,1]$ defined by

$$h(t) := \begin{cases} 0 & \text{if } t \in [0, r/2) \\ \gamma_r\left(\ell(t) - \ell(r/2) - \ell'(r/2)(t-r/2)\right) & \text{if } t \in [r/2, r-1) \\ 1 & \text{if } t \geqslant r-1. \end{cases}$$

We have

$$h(r/2) = 0, \quad h'(r/2) = 0 \quad \text{and} \quad h(r-1) = 1,$$

so that $h \in C^{0,1}([0,+\infty))$. Furthermore, recalling (6.2.7), for $t \in (r/2, r)$ it holds

$$\begin{aligned} |h'(t)| &= \gamma_r |\ell'(t) - \ell'(r/2)| = 2s\gamma_r \left[(r-t)^{-2s-1} - (r/2)^{-2s-1}\right] \leqslant 4(r-t)^{-2s-1} \\ |h''(t)| &= \gamma_r |\ell''(t)| = 2s(2s+1)\gamma_r (r-t)^{-2s-2} \leqslant 12(r-t)^{-2s-2}. \end{aligned} \tag{6.2.8}$$

We want to modify h between $r-2$ and $r-1$ in order to obtain a new function g of class $C^{1,1}$. To do this, let $\eta \in C^\infty([0,+\infty))$ be a cut-off function with $0 \leqslant \eta \leqslant 1$, $\eta = 1$ in $[0, r-7/4]$, $\eta = 0$ in $[r-5/4, +\infty)$, $-4 \leqslant \eta' \leqslant 0$ and $|\eta''| \leqslant 32$. We then set

$$g(t) := \eta(t)h(t) + 1 - \eta(t) \quad \text{for any } t \geqslant 0.$$

Of course, $g \in C^{1,1}([0,+\infty))$, $0 \leqslant g \leqslant 1$ and g coincides with h outside of $(r-2, r-1)$. On the other hand, by (6.2.8), for $t \in (r-2, r-1)$ we have

$$\begin{aligned} |g'(t)| &\leqslant |h'(t)|\chi_{[r-2,r-5/4]}(t) + 4(1-h(t)) \\ &\leqslant 4(r-t)^{-2s-1}\chi_{[r-2,r-5/4]}(t) + 4 \\ &\leqslant 8, \end{aligned}$$

and

$$\begin{aligned} |g''(t)| &\leqslant |h''(t)|\chi_{[r-2,r-5/4]}(t) + 8|h'(t)|\chi_{[r-2,r-5/4]}(t) + 32(1-h(t)) \\ &\leqslant 12(r-t)^{-2s-2}\chi_{[r-2,r-5/4]}(t) + 32(r-t)^{-2s-1}\chi_{[r-2,r-5/4]}(t) + 32 \\ &\leqslant 76. \end{aligned}$$

By combining these last two estimates with (6.2.8) we conclude that there exists an absolute constant $c_1 > 0$ such that

$$|g'(t)| \leqslant c_1 \min\{(r-t)^{-2s-1}, 1\} \quad \text{and} \quad |g''(t)| \leqslant c_1 \min\{(r-t)^{-2s-2}, 1\}, \tag{6.2.9}$$

for a.a. $t \in [0, r]$. Moreover, we claim that

$$\min\left\{(r-t)^{-2s}, 1\right\} \leqslant g(t) + 16r^{-2s} \leqslant 20 \min\left\{(r-t)^{-2s}, 1\right\} \quad \text{for any } t \in [0, r]. \tag{6.2.10}$$

Since the right-hand inequality of (6.2.10) follows almost directly from the definition of g, we may focus on the left estimation. The bound clearly holds when $t \geqslant r-1$, as $g = 1$ there. If $t \in [0, r-1)$, using (6.2.7) we have

$$\begin{aligned} g(t) \geqslant h(t) &\geqslant \ell(t) - \ell(r/2) - \ell'(r/2)\,(t - r/2) \\ &= (r-t)^{-2s} - (r/2)^{-2s} - 2s(r/2)^{-2s-1}\,(t-r/2) \\ &\geqslant (r-t)^{-2s} - 2^{2s}\,(1+2s)\,r^{-2s} \\ &\geqslant (r-t)^{-2s} - 16r^{-2s}. \end{aligned}$$

In any case, (6.2.10) is established.

Let now $v(x) := g(|x|)$, for any $x \in \mathbb{R}^n$. By the properties of g, we recover that $v \in C^{1,1}(\mathbb{R}^n)$ is radially symmetric, radially non-decreasing and satisfies $v = 0$ in $B_{r/2}$, $v = 1$ in $\mathbb{R}^n \setminus B_r$. Moreover, we infer from (6.2.10) that, for $x \in B_r$, it holds

$$\min\left\{(r-|x|)^{-2s}, 1\right\} \leqslant v(x) + 16r^{-2s} \leqslant 20 \min\left\{(r-|x|)^{-2s}, 1\right\}. \tag{6.2.11}$$

We claim that for any $x \in B_r$

$$\|\nabla^2 v\|_{L^\infty(B_{\max\{(r-|x|)/2,1\}}(x))} \leqslant c_2 \max\left\{\frac{r-|x|}{2},1\right\}^{-2s-2}, \tag{6.2.12}$$

for some dimensional constant $c_2 > 0$. To prove (6.2.12), we first consider $y \in B_r \setminus \overline{B_{r/2}}$ and compute, using (6.2.9),

$$\begin{aligned}
\left|\nabla^2 v(y)\right| &\leqslant n^2\left(|g''(|y|)| + 2\frac{|g'(|y|)|}{|y|}\right)\\
&\leqslant n^2 c_1\left(\min\left\{\left(\frac{r-|y|}{2}\right)^{-2s-2},1\right\} + \frac{2}{|y|}\min\left\{\left(\frac{r-|y|}{2}\right)^{-2s-1},1\right\}\right)\\
&\leqslant 2n^2 c_1 \min\{(r-|y|)^{-2s-2},1\},
\end{aligned} \tag{6.2.13}$$

where to obtain the last inequality we took advantage of the fact that $|y| \geqslant r/2 \geqslant r-|y|$ and $r \geqslant r_1 \geqslant 4$, by (6.2.6). Naturally, (6.2.13) extends also to the case of $y \in B_{r/2}$, since $\nabla^2 v$ vanishes identically there. Fix now $x \in B_r$ and take any $y \in B_{(r-|x|)/2}(x)$. Clearly, $y \in B_r$. Also,

$$|y| \leqslant |y-x| + |x| \leqslant \frac{r-|x|}{2} + |x| = \frac{r+|x|}{2},$$

and thus

$$r - |y| \geqslant r - \frac{r+|x|}{2} = \frac{r-|x|}{2}.$$

By this and (6.2.13), it follows that

$$\left|\nabla^2 v(y)\right| \leqslant 2n^2 c_1 \min\left\{\left(\frac{r-|x|}{2}\right)^{-2s-2},1\right\} \quad \text{for any } y \in B_{\frac{r-|x|}{2}}(x). \tag{6.2.14}$$

Finally, when $(r-|x|)/2 \leqslant 1$ we use again (6.2.13) to deduce that (6.2.14) holds also for $y \in B_1(x) \cap B_r$. Then, (6.2.12) follows.

With this in hand, we can deduce an estimate for $L_{K_\sigma} v$ in B_r, where K_σ is the scaled kernel defined by

$$K_\sigma(z) := \sigma^{n+2s} K(\sigma z), \quad \text{for a.a. } z \in \mathbb{R}^n,$$

with $\sigma > 0$. Observe that K_σ satisfies conditions (6.1.3) and (6.1.4) with the same λ, Λ of K (and with r_0/σ in place of r_0). We apply Lemma 6.9 with $\rho = \max\{(r-|x|)/2, 1\}$. Using (6.2.12) and (6.2.11), we obtain that for any $x \in B_r$

$$\begin{aligned}
|L_{K_\sigma} v(x)| &\leqslant c_3\Bigg[\|v\|_{L^\infty(\mathbb{R}^n)} \max\left\{\frac{r-|x|}{2},1\right\}^{-2s}\\
&\qquad + \|\nabla^2 v\|_{L^\infty(B_{\max\{(r-|x|)/2,1\}}(x))} \max\left\{\frac{r-|x|}{2},1\right\}^{2(1-s)}\Bigg]\\
&\leqslant c_4 \min\left\{(r-|x|)^{-2s},1\right\}\\
&\leqslant c_4\left(v(x) + 16 r^{-2s}\right).
\end{aligned} \tag{6.2.15}$$

for some constants $c_3, c_4 > 0$ which may depend on n, s and Λ.

We are now able to construct the function w. We take $R \geqslant R_0$, with

$$R_0 := \left(\frac{c_4}{\tau}\right)^{\frac{1}{2s}} r_1, \tag{6.2.16}$$

and set

$$r := \frac{r_1}{R_0} R. \tag{6.2.17}$$

Notice that $r \geqslant r_1$. We then define

$$w(x) := (2-\beta) v\left(\frac{r}{R} x\right) + \beta - 1,$$

where $\beta := 32 r^{-2s}$. Clearly, $\beta \in (0,1)$, since $r \geqslant r_1 \geqslant 3^{1/s}$, by (6.2.6).

The function w thus obtained inherits all the qualitative properties of v. That is, w is of class $C^{1,1}(\mathbb{R}^n)$, is radially symmetric and radially non-decreasing. Moreover, $w = \beta - 1$ in $B_{R/2}$ and $w = 1$ in $\mathbb{R}^n \setminus B_R$. Now we check that w satisfies properties (6.2.3) and (6.2.4).

By changing variables appropriately, applying (6.2.15) and recalling definitions (6.2.16)-(6.2.17), we get

$$\begin{aligned}
|L_K w(x)| &= (2-\beta)\left(\frac{R}{r}\right)^{-2s} \left| L_{K_{R/r}} v\left(\frac{r}{R}x\right)\right| \\
&\leqslant c_4 (2-\beta)\left(\frac{R}{r}\right)^{-2s} \left(v\left(\frac{r}{R}x\right) + 16 r^{-2s}\right) \\
&\leqslant c_4 \left(\frac{R}{r}\right)^{-2s} \left((2-\beta) v\left(\frac{r}{R}x\right) + 32 r^{-2s}\right) \\
&= \tau\left(1 + w(x)\right),
\end{aligned}$$

for any $x \in B_R$. Thus, (6.2.3) is established. The validity of the inequalities in (6.2.4) basically relies on (6.2.11). Namely, being β positive and taking advantage of the upper estimate of (6.2.11), along with (6.2.16) and (6.2.17), we have, for any $x \in B_R$,

$$\begin{aligned}
1 + w(x) &\leqslant 2\left[v\left(\frac{r}{R}x\right) + 16 r^{-2s}\right] \\
&\leqslant 40 \min\left\{ \frac{c_4}{\tau} \left(R - |x|\right)^{-2s}, 1\right\} \\
&\leqslant c_5 \left(R + 1 - |x|\right)^{-2s},
\end{aligned}$$

for some constant $c_5 > 0$ which depends on n, s, Λ and τ. The left-hand inequality of (6.2.4) follows similarly. Indeed, we first note that $2 - \beta \geqslant 1$, since $\beta \leqslant 1$. Hence, by this, (6.2.11), (6.2.16) and (6.2.17), we get

$$1 + w(x) \geqslant v\left(\frac{r}{R}x\right) + 16 r^{-2s} \geqslant \min\left\{ \frac{c_4}{\tau}\left(R - |x|\right)^{-2s}, 1\right\} \geqslant c_6 \left(R + 1 - |x|\right)^{-2s},$$

for some $c_6 > 0$ which depends on the same parameters as c_5. The proof of the lemma is therefore complete, as we may take $C := \max\left\{1, R_0, \frac{1}{32}\left(r_1/R_0\right)^{2s}, c_5, 1/c_6\right\}$. □

In the next result, we obtain another useful barrier in a one-dimensional setting.

Lemma 6.10. *Let $n = 1$, $s \in (0,1)$ and assume that K satisfies (6.1.3) and (6.1.4). Let $\eta \in C^2(\mathbb{R})$ be a positive function such that*

$$\eta(x) = \frac{1}{|x|^{1+2s}} \quad \textit{for any } x \in \mathbb{R} \setminus (-1,1). \tag{6.2.18}$$

Then,

$$L_K \eta \leqslant \Gamma \eta \quad \textit{in } \mathbb{R} \setminus (-4,4), \tag{6.2.19}$$

for some constant $\Gamma \geqslant 1$ depending only on s, Λ and $\|\eta\|_{C^2([-1,1])}$.

Proof. First of all, observe that $\|\eta\|_{C^2(\mathbb{R})} \leqslant C_1\|\eta\|_{C^2([-1,1])}$, for some constant $C_1 \geqslant 1$ depending only on s. Also notice that, by symmetry, we may restrict ourselves to consider $x > 0$.

For $x > 4$, we write

$$\begin{aligned} L_K\eta(x) &= \int_{\mathbb{R}} \left[\eta(y) - \eta(x) - \chi_{(-1,1)}(y-x)\eta'(x)(y-x)\right] K(y-x)\,dy \\ &= I_1(x) + I_2(x) + I_3(x), \end{aligned}$$

where

$$\begin{aligned} I_1(x) &:= \int_{-1}^{1} [\eta(y) - \eta(x)]\, K(y-x)\,dy, \\ I_2(x) &:= \int_{\{|x-y|\geqslant 1\}\setminus(-1,1)} [\eta(y) - \eta(x)]\, K(y-x)\,dy, \\ I_3(x) &:= \int_{\{|x-y|<1\}\setminus(-1,1)} \left[\eta(y) - \eta(x) - \eta'(x)(y-x)\right] K(y-x)\,dy. \end{aligned}$$

The first term is easy to handle. We simply take advantage of (6.1.4), together with the boundedness of η, to get

$$|I_1(x)| \leqslant 2\Lambda\|\eta\|_{L^(\mathbb{R})} \int_{-1}^{1} \frac{dy}{(x-y)^{1+2s}} \leqslant \frac{4\Lambda C_1\|\eta\|_{C^1([-1,1])}}{(x-1)^{1+2s}} \leqslant \frac{2^5\Lambda C_1\|\eta\|_{C^1([-1,1])}}{x^{1+2s}}, \qquad (6.2.20)$$

where the last inequality is due to the fact that $x - 1 > x/2$, as $x > 4$.

To deal with the second integral, first note that since y is such that $|x-y| \geqslant 1$ and $|y| \geqslant 1$, it follows that

$$y \in (-\infty, -1) \cup (1, x-1) \cup (x+1, +\infty). \qquad (6.2.21)$$

Also, since both $\eta(x)$ and $\eta(y)$ are in the form (6.2.18), then the integrand of $I_2(x)$ is non-negative if and only if $|y| \leqslant x$. By comparing this with (6.2.21), we actually estimate

$$\begin{aligned} I_2(x) &\leqslant \Lambda \int_{(-x,-1)\cup(1,x-1)} \left[\frac{1}{|y|^{1+2s}} - \frac{1}{x^{1+2s}}\right] \frac{dy}{(x-y)^{1+2s}} \\ &= \Lambda\Big(I_2^1(x) + I_2^2(x) + I_2^3(x)\Big), \end{aligned}$$

where

$$\begin{aligned} I_{2,1}(x) &:= \int_{\left\{1\leqslant|y|<\frac{x}{2}\right\}} \left[\frac{1}{|y|^{1+2s}} - \frac{1}{x^{1+2s}}\right] \frac{dy}{(x-y)^{1+2s}}, \\ I_{2,2}(x) &:= \int_{\left\{\frac{x}{2}\leqslant|y|<x-1\right\}} \left[\frac{1}{|y|^{1+2s}} - \frac{1}{x^{1+2s}}\right] \frac{dy}{(x-y)^{1+2s}}, \\ I_{2,3}(x) &:= \int_{-x}^{-x+1} \left[\frac{1}{|y|^{1+2s}} - \frac{1}{x^{1+2s}}\right] \frac{dy}{(x-y)^{1+2s}}. \end{aligned}$$

First,

$$I_{2,1}(x) \leqslant \frac{2}{(x/2)^{1+2s}} \int_1^{\frac{x}{2}} \frac{dy}{y^{1+2s}} \leqslant \frac{2^{1+2s}}{s}\,\frac{1}{x^{1+2s}}.$$

Then,

$$I_{2,2}(x) \leqslant \frac{2^{2+2s}}{x^{1+2s}} \int_{\frac{x}{2}}^{x-1} \frac{dy}{(x-y)^{1+2s}} \leqslant \frac{2^{1+2s}}{s}\,\frac{1}{x^{1+2s}}.$$

Finally,

$$I_{2,3}(x) \leqslant \frac{1}{(x-1)^{1+2s}(2x-1)^{1+2s}} \int_{-x}^{-x+1} dy \leqslant \frac{2^{1+2s}}{x^{2+4s}}.$$

By putting together the last three inequalities, we obtain

$$I_2(x) \leqslant \frac{C_2}{x^{1+2s}}, \tag{6.2.22}$$

for some constant $C_2 \geqslant 1$ which only depends on s and Λ.

In order to obtain an estimate for I_3, we first write the Taylor expansion of its integrand around x. We get

$$\eta(y) - \eta(x) - \eta'(x)(y-x) = \frac{\eta''(z_y)}{2}(y-x)^2,$$

for some z_x lying on the segment joining x and y. Since $3 < x-1 \leqslant z_x \leqslant x+1$, we have

$$\eta''(z_y) = \frac{2(1+2s)(1+s)}{z_y^{3+2s}} \leqslant \frac{12}{(x-1)^{3+2s}} \leqslant \frac{3 \cdot 2^{5+2s}}{x^{3+2s}}.$$

Accordingly,

$$|I_3(x)| \leqslant \frac{2^{6+2s}\Lambda}{x^{3+2s}} \int_{x-1}^{x+1} |y-x|^{1-2s}\, dy = \frac{2^{6+2s}\Lambda}{1-s} \frac{1}{x^{3+2s}}. \tag{6.2.23}$$

The combination of (6.2.20), (6.2.22) and (6.2.23) yields the desired (6.2.19). □

With the aid of the previous function, one can prove the following bound from above for the decay at infinity of a subsolution of the linear equation

$$L_K u = \delta u, \quad \text{with } \delta > 0,$$

set on the real line, away from the origin.

Lemma 6.11. *Let $n = 1$, $s \in (0,1)$ and assume that K satisfies (6.1.3) and (6.1.4). Let R_0, δ be given positive constants. Let $v \in C^{2s+\gamma}(\mathbb{R})$, for some $\gamma > 0$, be a bounded function satisfying*

$$L_K v \geqslant \delta v \quad \text{in } \mathbb{R} \setminus [-R_0, R_0]. \tag{6.2.24}$$

Then,

$$v(x) \leqslant \frac{C}{|x|^{1+2s}} \quad \text{for any } x \in \mathbb{R}, \tag{6.2.25}$$

for some constant $C > 0$ possibly depending on s, λ, Λ, R_0, δ and $\|v\|_{L^\infty(\mathbb{R})}$.

Proof. Of course, we may suppose that $\|v\|_{L^\infty(\mathbb{R})} > 0$ since, otherwise, the claim is trivially satisfied.

Let η and Γ be as in Lemma 6.10, with $\|\eta\|_{C^2(\mathbb{R})} \leqslant 50$. Set

$$a := \left(\frac{\delta}{2\Gamma}\right)^{\frac{1}{2s}}, \tag{6.2.26}$$

and

$$\zeta(x) := \eta(ax) \quad \text{for any } x \in \mathbb{R}.$$

Note that

$$L_K \zeta(x) = a^{2s} L_{K_a} \eta(ax),$$

where $K_a(z) := a^{-1-2s}K(az)$ is another kernel satisfying (6.1.3) and (6.1.4), possibly with a different r_0. Consequently, by Lemma 6.10 and (6.2.26),

$$L_K\zeta(x) \leqslant a^{2s}\Gamma\eta(ax) = \frac{\delta}{2}\zeta(x), \tag{6.2.27}$$

for any $|x| > R_1 := \max\{R_0, 4/a\}$.

We now set

$$\bar{C} := \frac{4\|v\|_{L^\infty(\mathbb{R})}}{\min\limits_{[-aR_1,aR_1]} \eta} = \frac{4\|v\|_{L^\infty(\mathbb{R})}}{\min\limits_{[-R_1,R_1]} \zeta}, \tag{6.2.28}$$

and claim that

$$v(x) \leqslant \bar{C}\zeta(x) \quad \text{for any } x \in \mathbb{R}. \tag{6.2.29}$$

To prove (6.2.29), let $b \in [0, +\infty)$ and define

$$v_b(x) := \bar{C}\zeta(x) + b - v(x) \quad \text{for any } x \in \mathbb{R}.$$

Set

$$\mathcal{B} := \Big\{b \in [0, +\infty) : v_b(x) \geqslant 0 \text{ for any } x \in \mathbb{R}\Big\},$$

and

$$b_0 := \inf \mathcal{B}.$$

First, observe that if $b > \|v\|_{L^\infty(\mathbb{R})}$, then $b \in \mathcal{B}$. Hence, $b_0 \in [0, +\infty)$. We now claim that

$$b_0 = 0. \tag{6.2.30}$$

To check (6.2.30), we argue by contradiction and suppose that

$$b_0 > 0. \tag{6.2.31}$$

Notice that then

$$v_{b_0}(x) \geqslant 0 \quad \text{for any } x \in \mathbb{R}, \tag{6.2.32}$$

and there exists a sequence of points $\{x_k\}_{k\in\mathbb{N}} \subset \mathbb{R}$ such that

$$v_{b_0-\frac{1}{k}}(x_k) < 0 \quad \text{for any } k \in \mathbb{N}.$$

Accordingly,

$$v_{b_0}(x_k) = v_{b_0-\frac{1}{k}}(x_k) + \frac{1}{k} < \frac{1}{k}, \tag{6.2.33}$$

and

$$\zeta(x_k) = \frac{v_{b_0}(x_k) + v(x_k) - b_0}{\bar{C}} < \frac{\|v\|_{L^\infty(\mathbb{R})} + \frac{1}{k}}{\bar{C}} \leqslant \frac{2\|v\|_{L^\infty(\mathbb{R})}}{\bar{C}} = \frac{\min_{[-R_1,R_1]} \zeta}{2},$$

if k is sufficiently large. Thus, $|x_k| > R_1$. Then, recalling (6.2.27) and (6.2.24),

$$L_K v_{b_0}(x_k) = \bar{C} L_K \zeta(x_k) - L_K v(x_k) \leqslant \frac{\bar{C}\delta}{2}\zeta(x_k) - \delta v(x_k) < \delta\left(v_{b_0}(x_k) - b_0\right). \tag{6.2.34}$$

Set now $\tilde{v}_k(x) := v_{b_0}(x + x_k)$. Note that, by (6.2.32), (6.2.33) and (6.2.34),

$$\tilde{v}_k(x) \geqslant 0 \quad \text{for any } x \in \mathbb{R}, \tag{6.2.35}$$

$$0 \leqslant \tilde{v}_k(0) < \frac{1}{k}, \tag{6.2.36}$$

$$L_K\tilde{v}_k(0) < \delta\left(\tilde{v}_k(0) - b_0\right), \tag{6.2.37}$$

and
$$\|\tilde{v}_k\|_{L^\infty(\mathbb{R})} = \|v_{b_0}\|_{L^\infty(\mathbb{R})} \leqslant \bar{C}\|\zeta\|_{L^\infty(\mathbb{R})} + b_0 + \|v\|_{L^\infty(\mathbb{R})}.$$
Consequently, by Arzelà-Ascoli theorem, up to a subsequence, $\tilde{v}_k \to \tilde{v}_\infty$ uniformly on compact sets. In view of (6.2.35) and (6.2.36),
$$\tilde{v}_\infty(0) = 0 \leqslant \tilde{v}_\infty(x) \quad \text{for any } x \in \mathbb{R}. \tag{6.2.38}$$
Moreover, by (6.1.4) and the regularity of v and ζ,
$$[\tilde{v}_k(x) + \tilde{v}_k(-x) - 2\tilde{v}_k(0)]\, K(x) \leqslant c_1 \left[|x|^{\gamma-1}\chi_{(-1,1)}(x) + |x|^{-1-2s}\chi_{\mathbb{R}\setminus(-1,1)}\right] \in L^1(\mathbb{R}),$$
for some constant $c_1 > 0$ independent of k. Therefore, we may apply the Dominated Convergence Theorem to deduce that
$$\lim_{k\to+\infty} L_K\tilde{v}_k(0) = \frac{1}{2}\int_{\mathbb{R}} [\tilde{v}_\infty(x) + \tilde{v}_\infty(-x) - 2\tilde{v}_\infty(0)]\, K(x)\, dx \geqslant 0,$$
where the last inequality follows from (6.2.38). By comparing this with (6.2.37) and recalling (6.2.38), we get
$$-\delta b_0 = \delta\left(\tilde{v}_\infty(0) - b_0\right) = \delta \lim_{k\to+\infty} \left(\tilde{v}_k(0) - b_0\right) \geqslant \lim_{k\to+\infty} L_K\tilde{v}_k(0) \geqslant 0,$$
which contradicts (6.2.31), since $\delta > 0$.

Consequently, $b_0 = 0$. This means that
$$0 \leqslant v_b(x) = v_0(x) + b \quad \text{for any } x \in \mathbb{R},$$
for any $b > 0$, that is
$$v_0(x) \geqslant 0 \quad \text{for any } x \in \mathbb{R}.$$
This leads to (6.2.25). □

6.2.2 A strong comparison principle

This subsection focuses on the derivation of a strong comparison principle for semilinear equations. We will heavily rely on such result throughout both Sections 6.3 and 6.4.

Proposition 6.12. *Let $n \geqslant 1$, $s \in (0,1)$ and assume that K satisfies (6.1.3) and (6.1.4). Let $f_1, f_2 : \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}$ be two continuous functions. Let Ω be a domain of $\mathbb{R}^n$ and $v, w \in L^\infty(\mathbb{R}^n) \cap C^{2s+\gamma}(\Omega)$, for some $\gamma > 0$, be such that*
$$\begin{cases} L_K v \leqslant f_1(\cdot, v) & \text{in } \Omega \\ L_K w \geqslant f_2(\cdot, w) & \text{in } \Omega \\ v \geqslant w & \text{in } \mathbb{R}^n. \end{cases}$$
Suppose furthermore that
$$f_1(x, w(x)) \leqslant f_2(x, w(x)) \quad \text{for any } x \in \Omega. \tag{6.2.39}$$
If there exists a point $x_0 \in \Omega$ at which $v(x_0) = w(x_0)$, then $v \equiv w$ in the whole Ω.

In the technical hypothesis (6.2.39) the two right-hand sides f_1 and f_2 are required to be appropriately ordered on the range of the subsolution w. The conclusion of the proposition is still true if (6.2.39) is asked to hold on the range of v, instead. Of course, (6.2.39) is clearly satisfied when f_1 and f_2 are the same function.

Proof of Proposition 6.12. Let $\varphi := v - w$ and set

$$\mathcal{Z}_\varphi := \Big\{ x \in \Omega : \varphi(x) = 0 \Big\}.$$

By assumption, we know that $\mathcal{Z}_\varphi$ is non-empty, as $x_0 \in \mathcal{Z}_\varphi$. Moreover, $\mathcal{Z}_\varphi$ is closed, thanks to the continuity of φ in Ω. We now claim that $\mathcal{Z}_\varphi$ is also open. Indeed, let $\bar{x} \in \mathcal{Z}_\varphi$. Clearly, $\varphi \geqslant 0$ in $\mathbb{R}^n$, $\varphi(\bar{x}) = 0$ and

$$L_K \varphi(\bar{x}) \leqslant f_1(\bar{x}, v(\bar{x})) - f_2(\bar{x}, w(\bar{x})) = f_1(\bar{x}, w(\bar{x})) - f_2(\bar{x}, w(\bar{x})) \leqslant 0,$$

in view of (6.2.39). Accordingly,

$$\begin{aligned} 0 \geqslant L_K \varphi(\bar{x}) &= \frac{1}{2} \int_{\mathbb{R}^n} \left(\varphi(\bar{x} + z) + \varphi(\bar{x} - z) - 2\varphi(\bar{x}) \right) K(z)\, dz \\ &= \frac{1}{2} \int_{\mathbb{R}^n} \left(\varphi(\bar{x} + z) + \varphi(\bar{x} - z) \right) K(z)\, dz \\ &\geqslant 0. \end{aligned}$$

Since, by condition (6.1.4), the kernel K is positive in B_{r_0}, we deduce that $\varphi = 0$ a.a. in $B_{r_0}(\bar{x})$. That is, $\Omega \cap B_{r_0} \subseteq \mathcal{Z}_\varphi$. Hence, $\mathcal{Z}_\varphi$ is open and, by the connectedness of Ω, we get that $\mathcal{Z}_\varphi = \Omega$. This concludes the proof. $\square$

Remark 6.13. By inspecting the proof just displayed, we see that the only hypothesis that we really used on K to deduce the strong comparison principle is its positivity in a small neighbourhood of the origin. This requirement is of course implied by assumption (6.1.4). But much more different kernels may also enjoy it, such as for instance integrable ones.

As a first application of Proposition 6.12, we can now justify the assertion contained in Remark 6.7.

Lemma 6.14. *Let $n \geqslant 1$ and $s \in (0, 1)$. Assume that K satisfies (6.1.3) and (6.1.4). Let $u \in C^{1+2s+\gamma}(\mathbb{R}^n)$, for some $\gamma > 0$, be a solution of (6.1.36) which satisfies (6.1.38) and (6.1.39). Then, u also satisfies (6.1.37).*

Proof. In view of the regularity of u, we may differentiate (6.1.36) in direction e_n and find that $\partial_{x_n} u$ solves the equation

$$L_K \partial_{x_n} u = W''(u) \partial_{x_n} u \quad \text{in } \mathbb{R}^n. \tag{6.2.40}$$

Suppose now by contradiction that there exists $x_0 \in \mathbb{R}^n$ at which $\partial_{x_n} u(x_0) = 0$. If this is the case, then by Proposition 6.12 we deduce that $\partial_{x_n} u = 0$ in the whole of $\mathbb{R}^n$, which contradicts hypothesis (6.1.38). Note that we can apply such proposition since the function identically equal to 0 is another solution of (6.2.40) and $\partial_{x_n} u \geqslant 0$, according to (6.1.39). We therefore conclude that (6.1.37) holds true. $\square$

6.2.3 Existence and stability results

In this subsection we gather a couple of lemmata concerning the existence of local minimizers for $\mathscr{E}_K$ in a given domain (recall Definition 5.1 in Chapter 5) and the stability of semilinear equations like (6.1.1) under locally uniform limits.

We begin with the existence result.

Lemma 6.15. *Let $n \geqslant 1$ and $s \in (0,1)$. Assume that K and W satisfy (6.1.3), (6.1.4) and (6.1.10), respectively. Let $\Omega \subset \mathbb{R}^n$ be a bounded Lipschitz domain and $w_0 : \mathbb{R}^n \to [-1,1]$ be a measurable function. Suppose that there exists another measurable function w which coincides with w_0 in $\mathbb{R}^n \setminus \Omega$ and such that*

$$\mathscr{E}_K(w,\Omega) < +\infty.$$

Then, there exists a local minimizer $u_* : \mathbb{R}^n \to [-1,1]$ for $\mathscr{E}_K$ in Ω which coincides with w_0 in $\mathbb{R}^n \setminus \Omega$.

Proof. Consider a minimizing sequence $\{u_j\}_{j\in\mathbb{N}}$, that is

$$\begin{cases} u_j = w_0 \quad \text{in } \mathbb{R}^n \setminus \Omega \\ \mathscr{E}_K(u_j,\Omega) \leqslant \mathscr{E}_K(w,\Omega) \\ \lim\limits_{j\to+\infty} \mathscr{E}_K(u_j,\Omega) = \inf\left\{\mathscr{E}_K(v,\Omega) : \, v = w_0 \text{ in } \mathbb{R}^n \setminus \Omega\right\} =: \mu. \end{cases}$$

Furthermore, by (6.1.10) we may assume without loss of generality that

$$|u_j| \leqslant 1 \quad \text{in } \mathbb{R}^n,$$

for any $j \in \mathbb{N}$. In view of this and (6.1.4), we compute

$$\begin{aligned} [u_j]^2_{H^s(\Omega)} &\leqslant \int_\Omega \left(\frac{1}{\lambda} \int_{\Omega\cap B_{r_0}(x)} |u_j(x)-u_j(y)|^2 \, K(x-y)\, dy + \int_{\Omega\setminus B_{r_0}(x)} \frac{4\|u_j\|^2_{L^\infty(R^n)}}{|x-y|^{n+2s}}\, dy \right) dx \\ &\leqslant \frac{4}{\lambda}\, \mathscr{E}_K(u_j,\Omega) + \frac{4|\Omega|^2}{r_0^{n+2s}} \|u_j\|_{L^\infty(\mathbb{R}^n)} \\ &\leqslant c\left(\mathscr{E}_K(w,\Omega)+1\right), \end{aligned}$$

for some constant $c > 0$ independent of j. Hence, $\{u_j\}$ is bounded in $H^s(\Omega)$ and then, using e.g. [DNPV12, Theorem 7.1], we deduce that $\{u_j\}$ converges, up to a subsequence, to some u_* in $L^2(\Omega)$ and, thus, a.e. in Ω. Fatou's Lemma then yields that

$$\mathscr{E}_K(u_*,\Omega) \leqslant \liminf_{j\to+\infty} \mathscr{E}_K(u_j,\Omega) = \mu.$$

This concludes the proof. □

Secondly, we have the stability lemma.

Lemma 6.16. *Let $n \geqslant 1$, $s \in (0,1)$ and assume that K satisfies (6.1.3) and (6.1.4). Let $\Omega \subset \mathbb{R}^n$ be a bounded Lipschitz domain and $\{v_j\}_{j\in\mathbb{N}} \subset \mathbb{H}^K(\Omega) \cap L^\infty(\mathbb{R}^n)$ a sequence of functions. Assume that v_j is a weak solution of*

$$L_K v_j = W'(v_j) \quad \text{in } \Omega, \tag{6.2.41}$$

and that there exists a constant $C > 0$ such that

$$[v_j]_{\mathbb{H}^K(\Omega)} + \|v_j\|_{L^\infty(\mathbb{R}^n)} \leqslant C, \tag{6.2.42}$$

for any $j \in \mathbb{N}$. Suppose furthermore that v_j converges to a function v uniformly on compact subsets of $\mathbb{R}^n$. Then, $v \in \mathbb{H}^K(\Omega) \cap L^\infty(\mathbb{R}^n)$ and is a weak solution of

$$L_K v = W'(v) \quad \text{in } \Omega. \tag{6.2.43}$$

Proof. First of all, it is clear that v belongs to $L^\infty(\mathbb{R}^n)$, as $v_j \to v$ locally uniformly in $\mathbb{R}^n$ and (6.2.42) holds. It is immediate to check that $v \in \mathbb{H}^K(\Omega)$, since, by (6.2.42) and Fatou's lemma,

$$\begin{aligned}
[v]^2_{\mathbb{H}^K(\Omega)} &= \frac{1}{2} \iint_{\mathbb{R}^{2n}\setminus(\mathbb{R}^n\setminus\Omega)^2} |v(x) - v(y)|^2 \, K(x-y) \, dx dy \\
&\leqslant \frac{1}{2} \liminf_{j\to+\infty} \iint_{\mathbb{R}^{2n}\setminus(\mathbb{R}^n\setminus\Omega)^2} |v_j(x) - v_j(y)|^2 \, K(x-y) \, dx dy \\
&= \liminf_{j\to+\infty} [v_j]^2_{\mathbb{H}^K(\Omega)} \\
&\leqslant C^2.
\end{aligned}$$

Now we show that v is a weak solution of (6.2.43). Fix $\varphi \in C_0^\infty(\Omega)$. Since v_j is a weak solution of (6.2.41), we have that

$$\begin{aligned}
\int_\Omega W'(v_j(x))\varphi(x)\, dx &= -\frac{1}{2} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} (v_j(x) - v_j(y)) \, (\varphi(x) - \varphi(y)) \, K(x-y) \, dx dy \\
&= \int_\Omega v_j(x) L_K \varphi(x) \, dx.
\end{aligned} \tag{6.2.44}$$

Notice now that $L_K\varphi \in L^\infty(\Omega) \subset L^1(\Omega)$. Indeed, by (6.1.6) we have

$$\begin{aligned}
|L_K\varphi(x)| &= \frac{1}{2} \left| \int_{\mathbb{R}^n} [\varphi(x+z) + \varphi(x-z) - 2\varphi(x)] \, K(z) \, dz \right| \\
&\leqslant \frac{\Lambda}{2} \left[4\|\varphi\|_{L^\infty(\mathbb{R}^n)} \int_{B_1} \frac{dz}{|z|^{n+2s}} + \|\nabla^2\varphi\|_{L^\infty(\mathbb{R}^n)} \int_{\mathbb{R}^n\setminus B_1} \frac{dz}{|z|^{n-2+2s}} \right] \\
&\leqslant \Lambda n \alpha_n \|\varphi\|_{C^2(\mathbb{R}^n)} \left[\frac{1}{s} + \frac{1}{1-s} \right],
\end{aligned}$$

for any $x \in \Omega$. Hence, by the Dominated Convergence Theorem and the continuity of W', we may take the limit as $j \to +\infty$ in (6.2.44) and deduce that

$$\int_\Omega v(x) L_K\varphi(x)\, dx = \int_\Omega W'(v(x))\varphi(x)\, dx.$$

Since we have already showed that $v \in \mathbb{H}^K(\Omega)$, it easily follows that v is a weak solution of (6.2.43). □

6.2.4 Some integral computations

We conclude the section with a couple of results aimed at establishing an upper bound for the quantity

$$J_{\alpha,n}(\rho,\sigma) := \int_{B_\rho} \int_{\mathbb{R}^n\setminus B_\sigma} \left(1 + |x-y|^2\right)^{-\alpha} dx dy, \quad \text{for } \rho,\sigma > 0 \text{ and } \alpha > \frac{n}{2}. \tag{6.2.45}$$

This will play an important role later in Section 6.5, to perform some computations needed for the proof of Theorem 6.5.

First, we have the following

Lemma 6.17. *Let $n \geqslant 1$, $\alpha \in (n/2, +\infty)$ and $\rho > \sigma > 0$. Then, given any $\delta \in (0,1)$ satisfying*

$$\frac{n}{2\alpha} < \delta < \frac{n+1}{2\alpha}, \tag{6.2.46}$$

it holds

$$\int_{B_\rho}\int_{B_\rho\setminus B_\sigma}(1+|x-y|^2)^{-\alpha}\,dxdy \leqslant C_1\left(\rho^n-\sigma^n\right), \tag{6.2.47}$$

and

$$\int_{B_\rho}\int_{\mathbb{R}^n\setminus B_\rho}(1+|x-y|^2)^{-\alpha}\,dxdy \leqslant C_2\rho^{2n-2\delta\alpha}, \tag{6.2.48}$$

for some constants $C_1>0$, which depends only on n and α, and $C_2>0$, which may also depend on δ.

Proof. All along the proof, c will denote any positive constant depending on n and α, whose value may change from line to line.

We begin by establishing (6.2.47). Changing variables appropriately we compute

$$\begin{aligned}\int_{B_\rho}\int_{B_\rho\setminus B_\sigma}(1+|x-y|^2)^{-\alpha}\,dxdy &\leqslant \int_{B_\rho\setminus B_\sigma}\left(\int_{\mathbb{R}^n}(1+|z|^2)^{-\alpha}\,dz\right)dx\\ &= |B_\rho\setminus B_\sigma|\int_{\mathbb{R}^n}(1+|z|^2)^{-\alpha}\,dz\\ &= c_1\left(\rho^n-\sigma^n\right),\end{aligned}$$

for some constant $c_1>0$ depending on n and α. This is true since

$$\begin{aligned}\int_{\mathbb{R}^n}(1+|z|^2)^{-\alpha}\,dz &= c\int_0^{+\infty}\left(1+r^2\right)^{-\alpha}r^{n-1}\,dr\\ &\leqslant c\left(\int_0^1 r^{n-1}\,dr+\int_1^{+\infty}r^{n-2\alpha-1}\,dr\right)\\ &= c\left(\frac{1}{n}-\frac{1}{n-2\alpha}\right)\\ &< +\infty,\end{aligned} \tag{6.2.49}$$

as $n-2\alpha<0$. Therefore, (6.2.47) is proved.

We now address (6.2.48). Consider any real number $0<\delta<1$. From now on, c is allowed to depend on δ too. Applying Young's inequality with weight δ, we get

$$1+|x-y|^2=(1-\delta)\left(\frac{1}{(1-\delta)^{1-\delta}}\right)^{1/(1-\delta)}+\delta\left(\frac{|x-y|^{2\delta}}{\delta^\delta}\right)^{1/\delta}\geqslant\frac{|x-y|^{2\delta}}{(1-\delta)^{1-\delta}\delta^\delta}.$$

We estimate

$$\begin{aligned}\int_{B_\rho}\int_{\mathbb{R}^n\setminus B_\rho}(1+|x-y|^2)^{-\alpha}\,dxdy &\leqslant c\int_{B_\rho}\left(\int_{\mathbb{R}^n\setminus B_{\rho-|y|}(y)}|x-y|^{-2\delta\alpha}\,dx\right)dy\\ &= c\int_{B_\rho}\left(\int_{\rho-|y|}^{+\infty}r^{n-1-2\delta\alpha}\,dr\right)dy.\end{aligned} \tag{6.2.50}$$

Now, we require δ to satisfy (6.2.46). Under this restriction, $n-2\delta\alpha<0$ and thus (6.2.50) becomes

$$\int_{B_\rho}\int_{\mathbb{R}^n\setminus B_\rho}(1+|x-y|^2)^{-\alpha}\,dxdy\leqslant c\int_{B_\rho}(\rho-|y|)^{n-2\delta\alpha}\,dy\leqslant c\rho^{n-1}\int_0^\rho(\rho-r)^{n-2\delta\alpha}\,dr.$$

But then, (6.2.46) also implies that $n-2\delta\alpha+1>0$, so that

$$\int_{B_\rho}\int_{B_{2\rho}\setminus B_\rho}(1+|x-y|^2)^{-\alpha}\,dxdy\leqslant c\rho^{2n-2\delta\alpha},$$

which is (6.2.48). □

From Lemma 6.17 we immediately get the desired estimate for J.

Corollary 6.18. *Let $n \geqslant 1$, $\alpha \in (n/2, +\infty)$ and $\rho, \sigma > 0$. Then, given any $\delta \in (0,1)$ satisfying (6.2.46),*

$$J_{\alpha,n}(\rho,\sigma) \leqslant C\Big(\rho^{2n-2\delta\alpha} + \max\{\rho^n - \sigma^n, 0\}\Big),$$

for some constant $C > 0$ which depends on n, α and δ.

6.3 Proof of Theorem 6.6

In this section we present a proof of Theorem 6.6. We stress that the argument displayed is an adaptation of that of [PSV13, Theorem 1], in accordance with the changes in our setting.

Step 1. Arguing by contradiction, we suppose that u is not a class A minimizer for $\mathscr{E}_K$. Recalling Definition 5.3 in Chapter 5, there exists a bounded domain $\Omega \subset \mathbb{R}^n$ in which u is not a local minimizer. According to Remark 5.2, we may further assume that $\Omega = B_R$, for some $R > 0$. Thus, there exists a function φ supported in B_R such that

$$\mathscr{E}_K(u + \varphi, B_R) < \mathscr{E}_K(u, B_R).$$

Note that this implies in particular that $\mathscr{E}_K(u + \varphi, B_R)$ is finite. Hence, we may apply Lemma 6.15 with $w = u + \varphi$ and find a minimizer u_* for $\mathscr{E}_K(\cdot, B_R)$ among all functions v such that $v = u$ outside of B_R. Observe that Lemma 6.15 also tells us that

$$|u_*| \leqslant 1 \quad \text{in } \mathbb{R}^n.$$

Since we assumed by contradiction that u is not a minimizer then there exists a point $x_0 \in \mathbb{R}^n$ at which

$$u_*(x_0) \neq u(x_0).$$

We suppose in fact that

$$u_*(x_0) > u(x_0). \tag{6.3.1}$$

A specular argument can be provided in case the opposite inequality holds. By the minimizing property of u_* we have that u_* is a weak solution of

$$L_K u_* = W'(u_*) \quad \text{in } B_R. \tag{6.3.2}$$

Therefore, we may apply Proposition 3.20 of Chapter 3 to conclude that u_* is continuous in the whole of $\mathbb{R}^n$. Also, observe that, by the same proposition, u_* is of class $C^{2s+\alpha}$ in the interior of B_R and thus (6.3.2) holds in the pointwise sense.

Step 2. Now we can prove that

$$|u_*| < 1, \tag{6.3.3}$$

using the assumptions on the potential W.

Indeed, suppose that there exists $\bar{x} \in \mathbb{R}^n$ at which, e.g., $u_*(\bar{x}) = -1$. Since $|u| < 1$ and u_* coincides with u outside B_R we conclude that $\bar{x} \in B_R$. Hence, by also recalling (6.3.2) and (6.1.10), we are in position to apply Proposition 6.12 (with $v = u_*$ and $w = -1$) to deduce that $u_* \equiv -1$ in B_R. But this and the continuity of u_* up to the boundary of B_R contradict the assumption that $u_* \equiv u$ outside B_R, as $|u| < 1$. Then (6.3.3) holds true.

Step 3. We claim that there exists $\bar{k} \in \mathbb{R}$ such that

$$\text{if } k \geqslant \bar{k}, \text{ then } u(x + ke_n) \geqslant u_*(x) \quad \text{for any } x \in \mathbb{R}^n. \tag{6.3.4}$$

Again we argue by contradiction and suppose that there exist two sequences $k_j > 0$ and $x^{(j)} \in \mathbb{R}^n$ such that $k_j \to +\infty$ as $j \to +\infty$ and

$$u(x^{(j)} + k_j e_n) < u_*(x^{(j)}). \tag{6.3.5}$$

Since u is monotone in the e_n direction by assumption (6.1.37) and $k_j \geqslant 0$, it follows that

$$u(x^{(j)}) < u_*(x^{(j)}),$$

and therefore $x^{(j)} \in B_R$. Hence, up to a subsequence, $x^{(j)}$ converges to some $x_* \in \overline{B_R}$. But now, taking advantage of assumption (6.1.38), inequality (6.3.5) and the continuity of u_* in $\overline{B_R}$, we find

$$1 = \lim_{j\to+\infty} u(x^{(j)} + k_j e_n) \leqslant \lim_{j\to+\infty} u_*(x^{(j)}) = u_*(x_*).$$

But this is in contradiction with (6.3.3) and so (6.3.4) is proved.

Step 4. Now we can take $\hat{k}$ as the least possible value of $\bar{k}$ for which (6.3.4) holds. Thus, there exist two sequences $\eta_j > 0$ and $y^{(j)} \in \mathbb{R}^n$ for which

$$u(y^{(j)} + (\hat{k} - \eta_j)e_n) \leqslant u_*(y^{(j)}), \tag{6.3.6}$$

and $\eta_j \to 0^+$ as $j \to +\infty$. Now, by (6.3.1) and (6.3.4) we have that

$$u(x_0) < u_*(x_0) \leqslant u(x_0 + \hat{k}e_n),$$

so that

$$\hat{k} > 0, \tag{6.3.7}$$

by the monotonicity of u.

We claim that there exists $J \in \mathbb{N}$ such that

$$y^{(j)} \in B_R \quad \text{for any } j \geqslant J. \tag{6.3.8}$$

By contradiction, if $y^{(j)} \in \mathbb{R}^n \setminus B_R$ for infinitely many j's, by (6.3.6) and the fact that $u_* \equiv u$ outside of B_R, we would have that

$$u(y^{(j)} + (\hat{k} - \eta_j)e_n) \leqslant u_*(y^{(j)}) = u(y^{(j)}).$$

But then

$$\hat{k} - \eta_j \leqslant 0,$$

by the monotonicity of u, and thus, by letting j go to $+\infty$, we would get $\hat{k} \leqslant 0$. But this is contradicts (6.3.7) and hence (6.3.8) holds true.

Step 5. In view of the previous deduction, we can assume that

$$\lim_{j\to+\infty} y^{(j)} = y_*,$$

for some y_* in the closure of B_R. Taking the limit as $j \to +\infty$ in (6.3.6) and recalling (6.3.4), we then get

$$u(y_* + \hat{k}e_n) = u_*(y_*). \tag{6.3.9}$$

But using once again the strict monotonicity of u and recalling (6.3.7), we are led to

$$u(y_*) < u_*(y_*).$$

Consequently, $y_* \in B_R$, as u and u_* coincide outside of B_R.

Define now $v(x) := u(x + \hat{k}e_n)$, for any $x \in \mathbb{R}^n$. By (6.1.36), (6.3.2) and (6.3.4), we know that

$$\begin{cases} L_K v = W'(v) & \text{in } \mathbb{R}^n \\ L_K u_* = W'(u_*) & \text{in } B_R \\ v \geqslant u_* & \text{in } \mathbb{R}^n. \end{cases}$$

Also, by (6.3.9), we have that $v(y_*) = u_*(y_*)$. Thence, by applying Proposition 6.12 (with $w = u_*$ and $\Omega = B_R$) we obtain that $v \equiv u_*$ in the whole B_R.

The strict monotonicity of u, (6.3.7) and the continuity of u and u_* up to the boundary of B_R imply in turn that

$$u(x) < u(x + \hat{k}e_n) = u_*(x) \quad \text{for any } x \in \overline{B_R},$$

contradicting the fact that u coincides with u_* outside B_R. Thus, the proof is complete.

6.4 Proof of Theorem 6.1

Here we show the existence of a class A minimizer in dimension $n = 1$, thus proving Theorem 6.1. To do so, we first deal with a constraint minimization problem on intervals, in Subsection 6.4.1. Then, in Subsection 6.4.2, we obtain the existence of local minimizers on the whole real line $\mathbb{R}$. Finally, the conclusive Subsections 6.4.3, 6.4.4 and 6.4.5 are devoted to the study of the various estimates involved in the statement of Theorem 6.1.

6.4.1 Minimizers on intervals

In the first proposition of the subsection we deal with the existence of local minimizers on *large* real intervals and prove some key estimates for their energies.

Lemma 6.19. *For any $M > 3$, there exists a local minimizer $v_{[-M,M]} : \mathbb{R} \to [-1,1]$ for $\mathscr{E}_K$ in $[-M,M]$, such that $v_{[-M,M]}(x) = -1$ for any $x \leqslant -M$ and $v_{[-M,M]}(x) = 1$ for any $x \geqslant M$. Moreover, $v_{[-M,M]}$ is odd, non-decreasing and is the unique solution of the Dirichlet problem*

$$\begin{cases} L_K u = W'(u) & \text{in } (-M,M) \\ u = -1 & \text{in } (-\infty,-M] \\ u = 1 & \text{in } [M,+\infty). \end{cases} \tag{6.4.1}$$

Finally, there exists a constant $C \geqslant 1$, depending only on s, Λ and W, for which

$$\mathscr{E}_K(v_{[-M,M]}, J) \leqslant C\Psi_s(|J|), \tag{6.4.2}$$

where J is either $[-M,M]$ or any subinterval of $[-M,M]$ such that $|J| > 6$ and $\operatorname{dist}(J, \mathbb{R} \setminus (-M,M)) > 2$.

Recall that the quantity Ψ_s was defined in (5.2.1).

Proof of Lemma 6.19. Consider the piecewise linear function $h : \mathbb{R} \to [-1,1]$ defined by

$$h(x) := \begin{cases} -1 & \text{if } x \leqslant -1 \\ x & \text{if } -1 < x \leqslant 1 \\ 1 & \text{if } x > 1. \end{cases}$$

By arguing as in [PSV13, Lemma 2] and taking advantage of the right-hand inequality in (6.1.4), it is easy to check that

$$\mathscr{E}_K(h, [-M,M]) \leqslant c\Psi_s(M) < +\infty, \tag{6.4.3}$$

for some constant $c > 0$ only depending on s, Λ and W. The existence of a local minimizer $v_{[-M,M]} : \mathbb{R} \to [-1,1]$ then derives from Lemma 6.15. Note that (6.4.3) also establishes (6.4.2) for $J = [-M,M]$. Estimate (6.4.2) for a general interval $J \subset [-M,M]]$ with $|J| > 6$ and $\operatorname{dist}(J, \mathbb{R} \setminus (-M,M)) > 2$ follows from Proposition 5.5 in Chapter 5,[17] by observing that $v_{[-M,M]}$ is also a local minimizer in any subinterval of $[-M,M]$ (recall Remark 5.2 in Chapter 5).

Then, we address the monotonicity of $v_{[-M,M]}$. First, note that $v_{[-M,M]}$ is a weak solution of (6.4.1) and, therefore, by Proposition 3.20, $v_{[-M,M]} \in C^{\alpha}(\mathbb{R}) \cap C^{2s+\alpha}_{\mathrm{loc}}((-M,M))$, for some $\alpha > 0$. In particular, $v_{[-M,M]}$ is a pointwise solution of (6.4.1). Now, we claim that

$$|v_{[-M,M]}(x)| < 1 \quad \text{for any } x \in (-M,M). \tag{6.4.4}$$

Indeed, if (6.4.4) does not hold, then there is $x_0 \in (-M,M)$ at which, say, $v_{[-M,M]}(x_0) = -1$. But then, in view of Proposition 6.12, we deduce that $v_{[-M,M]} = -1$ in the whole of $(-M,M)$, which clearly contradicts the continuity of $v_{[-M,M]}$ at M. Thus, (6.4.4) is valid.

Given $\tau \geqslant 0$, set $u(x) := v_{[-M,M]}(x)$ and $u_\tau(x) := v_{[-M,M]}(x-\tau)$, for any $x \in \mathbb{R}$. Note that

$$u_\tau(x) = u(x) \quad \text{for any } x \in (-\infty, -M] \cup [M+\tau, +\infty). \tag{6.4.5}$$

We define

$$\hat{\tau}_0 := \inf \Big\{ \tau_0 > 0 : u_\tau \leqslant u \text{ in } \mathbb{R}, \text{ for any } \tau \geqslant \tau_0 \Big\}.$$

By construction, it holds $\hat{\tau}_0 \in [0, 2M]$. Observe that if we show that

$$\hat{\tau}_0 = 0, \tag{6.4.6}$$

then the monotonicity of $v_{[-M,M]}$ would follow. To prove (6.4.6), we argue by contradiction and in fact suppose that $\hat{\tau}_0 \in (0, 2M]$. As a result,

$$u_{\hat{\tau}_0} \leqslant u \quad \text{in } \mathbb{R}, \tag{6.4.7}$$

and there exist two sequences $\varepsilon_j > 0$ and $x_j \in \mathbb{R}$ such that $\varepsilon_j \to 0$ as $j \to +\infty$ and

$$u_{\hat{\tau}_0 - \varepsilon_j}(x_j) > u(x_j), \tag{6.4.8}$$

for any $j \in \mathbb{N}$. Moreover, by (6.4.5), we have that $x_j \in (-M, M + \hat{\tau}_0 - \varepsilon_j)$, so that x_j converges to some $x_0 \in [-M, M + \hat{\tau}_0]$, up to subsequences. Using (6.4.7) and (6.4.8), it then follows that

$$u_{\hat{\tau}_0}(x_0) = u(x_0), \tag{6.4.9}$$

while by (6.4.4) we further deduce that $\hat{\tau}_0 < 2M$ and $x_0 \in (-M + \hat{\tau}_0, M)$. By virtue of (6.4.1), (6.4.7) and (6.4.9), we may now apply Proposition 6.12 and obtain that $u_{\hat{\tau}_0}(x) = u(x)$, for any $x \in (-M + \hat{\tau}_0, M)$. By (6.4.4) and the continuity of $v_{[-M,M]}$, we are then led to

$$1 > v_{[-M,M]}(M - \hat{\tau}_0) = u_{\hat{\tau}_0}(M) = u(M) = v_{[-M,M]}(M) = 1,$$

which is a contradiction. Accordingly, (6.4.6) is true and therefore $v_{[-M,M]}$ is non-decreasing.

Now, we show that $v_{[-M,M]}$ is the unique solution of the Dirichlet problem (6.4.1). Let w be a solution of (6.4.1). By Proposition 3.20, we know that $w \in C^{\alpha}(\mathbb{R}) \cap C^{2s+\alpha}_{\mathrm{loc}}((-M,M))$,

[17] Note that Proposition 5.5 was proved under assumption (5.1.5) on K, which is the analogous of requirement (6.1.4) here, with $r_0 = 1$. However, the proof of that result only exploits the right-hand inequality of (5.1.5) and thus it is valid in the framework of this chapter too.

for some $\alpha > 0$. Furthermore, by arguing as in the proof of (6.4.4), we get that $|w(x)| < 1$, for any $x \in (-M, M)$. We claim that

$$w \leqslant v_{[-M,M]} \quad \text{in } \mathbb{R}. \tag{6.4.10}$$

To prove it, we take any $\tau \geqslant 0$ and set $w_\tau(x) := w(x-\tau)$, for any $x \in \mathbb{R}$. Note that $w_\tau(x) = v_{[-M,M]}(x)$, for any $x \in (-\infty, -M] \cup [M+\tau, +\infty)$. Set then

$$\bar{\tau}_0 := \inf \left\{ \tau_0 > 0 : w_\tau \leqslant v_{[-M,M]} \text{ in } \mathbb{R}, \text{ for any } \tau \geqslant \tau_0 \right\} \in [0, 2M).$$

Clearly, (6.4.10) would follow if we prove that $\bar{\tau}_0 = 0$. We thus argue by contradiction and suppose that $\bar{\tau}_0 > 0$. Then, it is not hard to show that $w_{\bar{\tau}} \leqslant v_{[-M,M]}$ in $\mathbb{R}$ and that there exists a point $x_0 \in (-M + \hat{\tau}_0, M)$ at which $w_{\bar{\tau}_0}(x_0) = v_{[-M,M]}(x_0)$. But then, by Proposition (6.12) we deduce that $w_{\bar{\tau}} = v_{[-M,M]}$ in the whole interval $[-M + \hat{\tau}_0, M]$, which is a contradiction, since $\hat{\tau}_0 > 0$. Accordingly, (6.4.10) is valid. With a completely analogous argument we obtain that the converse inequality is also true and, therefore, that $w = v_{[-M,M]}$.

Finally, we are left to prove that $v_{[-M,M]}$ is an odd function. To do this, we define

$$z(x) := -v_{[-M,M]}(-x) \quad \text{for any } x \in \mathbb{R}.$$

Clearly, we have that $z(x) = -1$ for any $x \leqslant -M$ and $z(x) = 1$ for any $x \geqslant M$. Moreover,

$$L_K z(x) = -L_K v_{[-M,M]}(-x) = -W'(v_{[-M,M]}(-x)) = -W'(-z(x)),$$

for any $x \in (-M, M)$. By taking advantage of (6.1.12), we have that W' is odd in $[-1, 1]$ and we conclude that z is a solution of (6.4.1). Hence, $z = v_{[-M,M]}$, by uniqueness, and $v_{[-M,M]}$ is odd. □

6.4.2 Minimizers on the real line

We now use the results obtained in the previous subsection to deduce the existence of a class A minimizer for $\mathscr{E}_K$ in $\mathbb{R}$.

Recalling definitions (6.1.13) and (6.1.14), we introduce the set of monotone minimizers

$$\mathcal{M} := \left\{ u \in \mathcal{X} : u \text{ is a non-decreasing class A minimizer for } \mathscr{E}_K \right\}.$$

In the next proposition we show that the class $\mathcal{M}$ defined above contains at least one element.

Proposition 6.20. *The set $\mathcal{M}$ is not empty. In particular, there exists an odd class A minimizer $u_0 : \mathbb{R} \to (-1, 1)$ for $\mathscr{E}_K$, which is $C^{1+2s+\alpha}(\mathbb{R}^n)$ regular, for some $\alpha > 0$, and satisfies*

$$u_0(0) = 0, \tag{6.4.11}$$

$$u_0'(x) > 0 \quad \textit{for any } x \in \mathbb{R}, \tag{6.4.12}$$

$$\lim_{x \to \pm\infty} u_0(x) = \pm 1, \tag{6.4.13}$$

and (6.1.18).

Proof. Let $M > 5$ and consider the local minimizer $v_{[-M,M]} : \mathbb{R} \to [-1,1]$ given by Lemma 6.19. Recall that $v_{[-M,M]}$ is an odd, non-decreasing function such that $v_{[-M,M]}(x) = -1$ if $x \leqslant -M$ and $v_{[-M,M]}(x) = 1$ if $x \geqslant M$. Moreover,

$$\frac{1}{2}\left[v_{[-M,M]}\right]^2_{\mathbb{H}^K(J)} \leqslant \mathscr{E}_K(v_{[-M,M]}, J) \leqslant C_1 \Psi_s(|J|), \tag{6.4.14}$$

where either $J = [-M,M]$ or J is any subinterval of $[-M,M]$, with $|J| > 6$ and $\operatorname{dist}(J, \mathbb{R} \setminus [-M,M]) > 2$. Note that $C_1 \geqslant 1$ is a constant depending only on s, Λ and W. Also, $v_{[-M,M]}$ is a solution of

$$L_K v_{[-M,M]} = W'(v_{[-M,M]}) \quad \text{in } (-M,M), \tag{6.4.15}$$

and thus by Proposition 3.20 we deduce that $v_{[-M,M]} \in C^\alpha(\mathbb{R})$, for some $\alpha \in (0,1)$, with Hölder norm bounded independently[18] of M.

In view of this and Arzelà-Ascoli theorem, we may assume that $v_{[-M,M]}$ converges to a continuous function u_0, uniformly on compacts subsets of $\mathbb{R}$, as $M \to +\infty$. By the oddness $v_{[-M,M]}$, we have that $v_{[-M,M]}(0) = 0$, for any M. Accordingly, u_0 satisfies (6.4.11). Also, u_0 is odd, non-decreasing and weakly satisfies

$$L_K u_0 = W'(u_0) \quad \text{in } \mathbb{R}, \tag{6.4.16}$$

in view of (6.4.14), (6.4.15) and Lemma 6.16. By Proposition 3.21, it then follows that $u_0 \in C^{1+2s+\alpha}(\mathbb{R})$, for some $\alpha > 0$.

Now we prove that $u_0 \in \mathcal{M}$, thus concluding the proof of the proposition. In order to do this, we first show that (6.1.18) holds true. To check it, we fix $R > 4$ and address the energy of $v_{[-M,M]}$ inside the interval $[-R,R]$. By taking M suitably large in dependence of R if necessary, by (6.4.14) we have that

$$\mathscr{E}_K(v_{[-M,M]}, [-R,R]) \leqslant C \Psi_s(R),$$

for some constant $C > 0$ independent of M and R. The finiteness condition (6.1.18) then follows by letting R go to $+\infty$ in the above inequality, thanks to Fatou's lemma.

Next, we check that (6.4.13) holds true. In view of the monotonicity of u_0 and (6.4.11), we know that there exist two numbers $-1 \leqslant a_- \leqslant 0 \leqslant a_+ \leqslant 1$ such that

$$\lim_{x \to \pm\infty} u_0(x) = a_\pm.$$

We prove here that $a_+ = 1$, while a completely analogous argument shows that $a_- = -1$ holds too. Suppose by contradiction that $a_+ < 1$ and notice that $u_0(x) \in [0, a_+]$ for any $x \geqslant 0$. Set

$$\kappa := \inf_{x \geqslant 0} W(u_0(x)) = \inf_{r \in [0,a_+]} W(r).$$

By taking advantage of (6.1.9) in combination with the fact that $a_+ < 1$, we deduce that $\kappa > 0$. Consequently,

$$\mathscr{G}^*(u_0) \geqslant \limsup_{R \to +\infty} \frac{1}{\Psi_s(R)} \int_0^R W(u_0(x))\, dx \geqslant \kappa \lim_{R \to +\infty} \frac{R}{\Psi_s(R)} = +\infty,$$

in contradiction with (6.1.18). Thence, (6.4.13) is valid. In particular, $u_0 \in \mathcal{X}$.

[18] A careful inspection of the proof of Proposition 3.13 - on which Propositions 3.20 is based - shows that the Hölder norm of the solution of the Dirichlet problem (3.1.2) is bounded by a constant that does not depend on Ω as a whole, but only on the $C^{1,1}$ norm of its boundary. In particular, when $n = 1$ the constant is independent on the reference interval. As a result, we can conclude that the $C^\alpha(\mathbb{R})$ norm of $v_{[-M,M]}$ is independent of M.

Finally, the monotonicity of u_0, (6.4.16), (6.4.13) and Lemma 6.14 imply that u_0 satisfies (6.4.12). By virtue of this, (6.4.16) and (6.4.13), the function u_0 fulfills the hypotheses of Theorem 6.6. Therefore, it follows that u_0 is a class A minimizer. By this and again (6.4.12), we conclude that $u_0 \in \mathcal{M}$. The proof of the proposition is thus complete. □

Next, we address the problem of assessing how *big* the set $\mathcal{M}$ is. In Proposition 6.20, we have established that $\mathcal{M}$ contains at least one element u_0. Clearly, it also contains the translations $u_0(\cdot - k)$, for any $k \in \mathbb{R}$. We are thence led to study the subclasses

$$\mathcal{M}_{x_0} := \Big\{ u \in \mathcal{M} \, : \, x_0 = \sup \{ x \in \mathbb{R} : u(x) < 0 \} \Big\},$$

for any fixed $x_0 \in \mathbb{R}$. Of course, we have that

$$\mathcal{M} = \bigcup_{x_0 \in \mathbb{R}} \mathcal{M}_{x_0} \quad \text{and} \quad \mathcal{M}_{x_0} \cap \mathcal{M}_{x_1} = \varnothing, \text{ if } x_0 \neq x_1.$$

Also,

$$u \in \mathcal{M}_{x_0} \quad \text{if and only if} \quad u(\cdot + x_0) \in \mathcal{M}_0, \tag{6.4.17}$$

for any $x_0 \in \mathbb{R}$. It turns out that each of these subclasses is a singleton, as shown by the following

Proposition 6.21. *For any fixed $x_0 \in \mathbb{R}$, the class $\mathcal{M}_{x_0}$ consists of one single element u_{x_0}. More specifically, $u_{x_0} : \mathbb{R} \to (-1, 1)$ is a class A minimizer for $\mathscr{E}_K$, which is $C^{1+2s+\alpha}$ regular, for some $\alpha > 0$, and satisfies (6.4.12), (6.4.13), (6.1.18) and $u_{x_0}(x_0) = 0$.*

Proof. In light of (6.4.17), it is enough to prove the statement for the point $x_0 = 0$. Note that the function u_0 constructed in Proposition 6.20 belongs to $\mathcal{M}_0$. Let $u \in \mathcal{M}_0$. Observe that u is a weak solution of (6.4.16) and, hence, by Proposition 3.21, that $u \in C^{1+2s+\alpha}(\mathbb{R})$, for some $\alpha > 0$. Also, $|u| \leqslant 1$ in $\mathbb{R}$, since u satisfies (6.4.13) and it is non-decreasing. If we show that $u = u_0$, then the proof would be over.

First, we notice that there exists a small value $\varepsilon_0 > 0$ such that, for any $\varepsilon \in (0, \varepsilon_0)$, we can find a $\bar{k}_\varepsilon \in \mathbb{R}$ for which

$$\text{if } k \leqslant \bar{k}_\varepsilon, \text{ then } u(x-k) + \varepsilon > u_0(x) \text{ for any } x \in \mathbb{R}. \tag{6.4.18}$$

This is true as a consequence of both u_0 and u having values in $[-1, 1]$ and satisfying (6.4.13). Then, we start sliding the graph of $u + \varepsilon$ to the right until it first touches that of u_0. That is, we take $\hat{k}_\varepsilon$ as the largest possible value of $\bar{k}_\varepsilon$ for which (6.4.18) holds true, and find a point $x_\varepsilon \in \mathbb{R}$ at which

$$u(x_\varepsilon - \hat{k}_\varepsilon) + \varepsilon = u_0(x_\varepsilon).$$

Again, this is possible in view of the continuity and the behaviour at $\pm\infty$ of u and u_0. Set now $u_\varepsilon(x) := u(x - \hat{k}_\varepsilon) + \varepsilon$ and observe that, by definition of $\hat{k}_\varepsilon$, it holds

$$\begin{cases} u_\varepsilon(x) \geqslant u_0(x) & \text{for any } x \in \mathbb{R} \\ u_\varepsilon(x_\varepsilon) = u_0(x_\varepsilon). \end{cases} \tag{6.4.19}$$

Now we claim that

$$x_\varepsilon \text{ is bounded as } \varepsilon \to 0^+. \tag{6.4.20}$$

By contradiction, suppose that there is a sequence of values $\varepsilon_j > 0$ for which $\varepsilon_j \to 0^+$ and, say, $x_{\varepsilon_j} \to +\infty$, as $j \to +\infty$. By (6.1.11), we can pick a small value $c > 0$ such that W' is monotone non-decreasing in $[1 - c, 1]$. Fix a real number $M > 0$ large enough to have $u_0(M) > 1 - c/2$. Notice that $x_{\varepsilon_j} > M$ and $\varepsilon_j < c/2$, provided j is sufficiently large. By this and the monotonicity of u_0, we have

$$u_0(x) \geqslant u_0(x) - \varepsilon_j \geqslant u_0(M) - \frac{c}{2} > 1 - c,$$

for any $x \in (M, +\infty)$. Hence, recalling the monotonicity of W' in $[1 - c, 1]$, we obtain

$$W'(u_0(x) - \varepsilon_j) \leqslant W'(u_0(x)) \quad \text{for any } x \in (M, +\infty). \tag{6.4.21}$$

Observe now that, since both u and u_0 satisfy (6.4.16),

$$\begin{cases} L_K u_{\varepsilon_j} = W'(u_{\varepsilon_j} - \varepsilon_j) & \text{in } \mathbb{R} \\ L_K u_0 = W'(u_0) & \text{in } \mathbb{R}. \end{cases}$$

Consequently, by this and (6.4.19), we are able to use Proposition 6.12 - with $\Omega = (M, +\infty)$ and $f_1(r) = W'(r - \varepsilon_j)$, $f_2(r) = W'(r)$ - to deduce that

$$u_{\varepsilon_j}(x) = u_0(x) \quad \text{for any } x > M, \tag{6.4.22}$$

provided j is large enough. Notice that the validity of condition (6.2.39) there is ensured by (6.4.21). But (6.4.22) is contradictory, as can be seen for instance by letting $x \to +\infty$. A symmetrical argument shows that we reach a contradiction also if $x_{\varepsilon_j} \to -\infty$. Thus, (6.4.20) follows. As a result of (6.4.20), we have that, up to a subsequence,

$$\lim_{\varepsilon \to 0^+} x_\varepsilon = x_0, \tag{6.4.23}$$

for some $x_0 \in \mathbb{R}$.

Then, we claim that

$$\hat{k}_\varepsilon \text{ is bounded as } \varepsilon \to 0^+. \tag{6.4.24}$$

Again, we argue by contradiction and suppose that $\hat{k}_{\varepsilon_j} \to \pm\infty$ on an infinitesimal sequence $\varepsilon_j > 0$. Applying the identity on the second line of (6.4.19) and (6.4.23), we obtain

$$\mp 1 = \lim_{j \to +\infty} \left[u(x_{\varepsilon_j} - \hat{k}_{\varepsilon_j}) + \varepsilon_j \right] = \lim_{j \to +\infty} u_{\varepsilon_j}(x_{\varepsilon_j}) = \lim_{j \to +\infty} u_0(x_{\varepsilon_j}) = u_0(x_0),$$

which is not the case, since u_0 has values in $(-1, 1)$. Thence, (6.4.24) holds and, up to a subsequence,

$$\lim_{\varepsilon \to 0^+} \hat{k}_\varepsilon = \hat{k}_0, \tag{6.4.25}$$

for some $\hat{k}_0 \in \mathbb{R}$.

By virtue of (6.4.23) and (6.4.25), we may finally let $\varepsilon \to 0^+$ in (6.4.19), to find that

$$\begin{cases} L_K u(x - \hat{k}_0) = W'(u(x - \hat{k}_0)) & \text{for any } x \in \mathbb{R} \\ L_K u_0(x) = W'(u_0(x)) & \text{for any } x \in \mathbb{R} \\ u(x - \hat{k}_0) \geqslant u_0(x) & \text{for any } x \in \mathbb{R} \\ u(x_0 - \hat{k}_0) = u_0(x_0). \end{cases}$$

By applying once again Proposition 6.12, we infer that $u(x - \hat{k}_0) = u_0(x)$ for any $x \in \mathbb{R}$. Then, as $u, u_0 \in \mathcal{M}_0$, we conclude that $u(0) = 0 = u_0(0)$. Recalling (6.4.12), it follows that $\hat{k}_0 = 0$ and hence $u = u_0$. The proposition is thus proved. □

6.4.3 Further estimates: general kernels

Up to now, we have established the existence - and essential uniqueness - of the minimizer u_0 in the class $\mathcal{X}$. Moreover, we already know by construction that u_0 is strictly increasing and that (6.1.18) holds true.

In this subsection we show that estimates (6.1.16) and (6.1.17) are also valid. These results are the content of the following two propositions.

Proposition 6.22. *The function u_0 constructed in Proposition 6.20 satisfies the decay estimates* (6.1.16).

Proof. We begin by addressing the validity of the first estimate in (6.1.16). Obviously, we may restrict ourselves to prove only that there exists $R_1, C_1 > 0$ such that

$$u_0(x) \leqslant -1 + \frac{C_1}{|x|^{2s}}, \tag{6.4.26}$$

if $x \leqslant -R_1$.

To do this, first observe that, by (6.1.11),

$$W'(t) \geqslant W'(r) + c(t-r) \quad \text{for any } r \leqslant t \text{ such that } r, t \in [-1, -1+c], \tag{6.4.27}$$

for some $c \in (0, 1/2)$. Take now $\tau = c$ in Lemma 6.8 and for any $R \geqslant C$ consider the barrier w constructed there. By (6.2.1), (6.2.2) and (6.4.13), there exists $k_0 \in \mathbb{R}$ such that

$$\text{for any } k \in (-\infty, k_0), \text{ it holds } u_0(x) < w(x-k) \text{ for any } x \in \mathbb{R}. \tag{6.4.28}$$

Now, let $\bar{k}_0$ be the largest k_0 for which (6.4.28) is true. Clearly,

$$u_0(x) \leqslant w(x - \bar{k}_0) \quad \text{for any } x \in \mathbb{R}. \tag{6.4.29}$$

Also, it is not hard to check that there exists

$$\bar{x} \in (\bar{k}_0 - R, \bar{k}_0 + R), \tag{6.4.30}$$

at which

$$u_0(\bar{x}) = w(\bar{x} - \bar{k}_0). \tag{6.4.31}$$

We claim that

$$u_0(\bar{x}) \geqslant -1 + c. \tag{6.4.32}$$

To prove it, we argue by contradiction and suppose indeed that

$$u_0(\bar{x}) \in (-1, -1+c). \tag{6.4.33}$$

Define

$$\Omega := \left\{ x \in (\bar{k}_0 - R, \bar{k}_0 + R) : u_0(x) < -1 + c \right\},$$

and note that, by (6.4.33) and the continuity and monotonicity of u_0, we have that Ω is an open domain with

$$(\bar{k}_0 - R, \bar{x}] \subset \Omega. \tag{6.4.34}$$

Setting now $\bar{w}(x) := w(x - \bar{k}_0)$, by (6.2.3), (6.4.16), (6.4.29) and (6.4.31), we know that

$$\begin{cases} L_K \bar{w} \leqslant c(1 + \bar{w}) & \text{in } (\bar{k}_0 - R, \bar{k}_0 + R) \\ L_K u_0 = W'(u_0) & \text{in } \mathbb{R} \\ \bar{w} \geqslant u_0 & \text{in } \mathbb{R} \\ \bar{w}(\bar{x}) = u_0(\bar{x}). \end{cases}$$

Furthermore, notice that, by taking $t = u_0(x)$ and $r = -1$ in (6.4.27) and recalling (6.1.10),

$$W'(u_0(x)) \geqslant c(1 + u_0(x)) \quad \text{for any } x \in \Omega.$$

In view of this last consideration, we are then in position to apply Proposition 6.12 and obtain that $u_0(x) = \bar{w}(x)$, for any $x \in \Omega$. But then, by (6.4.34), the continuity of $u_0, \bar{w}$ and (6.2.2),

$$1 > u_0(\bar{k}_0 - R) = \bar{w}(\bar{k}_0 - R) = w(-R) = 1,$$

which is a contradiction. Consequently, (6.4.32) holds true.

In view of (6.2.4), (6.4.31), (6.4.30) and (6.4.32) we now get

$$C(R + 1 - |\bar{x} - \bar{k}_0|)^{-2s} \geqslant 1 + w(\bar{x} - \bar{k}_0) = 1 + u_0(\bar{x}) \geqslant c,$$

so that

$$|\bar{x} - \bar{k}_0| \geqslant R - c', \tag{6.4.35}$$

for some $c' > 0$. Moreover,

$$\bar{x} \geqslant \bar{k}_0. \tag{6.4.36}$$

To check (6.4.36), we argue by contradiction and suppose that $\bar{x} < \bar{k}_0$. Set $\hat{k} := 2\bar{x} - \bar{k}_0$ and notice then that $\hat{k} < \bar{k}_0$. Accordingly, by (6.4.28) and (6.4.31) we deduce that

$$w(\hat{k}_0 - \bar{x}) = w(\bar{x} - \hat{k}) > u_0(\bar{x}) = w(\bar{x} - \hat{k}_0),$$

in contradiction with the parity of w. Thus, (6.4.36) is true.

In consequence of (6.4.30), (6.4.35) and (6.4.36), we see that

$$\bar{x} - \bar{k}_0 \in [R - c', R]. \tag{6.4.37}$$

Let $\kappa > 0$ be chosen in such a way that $u_0(-\kappa) = -1 + c$. By the monotonicity of u_0, we clearly have $-\kappa \leqslant \bar{x}$ and

$$u_0(x - \kappa) \leqslant u_0(x + \bar{x}) \quad \text{for any } x \in \mathbb{R}. \tag{6.4.38}$$

Take now any $y \in [R/2, R]$. By (6.4.37) and taking a larger R if necessary, we have that $\bar{x} - y - \bar{k}_0 \in [-R/2, R/2]$. Consequently, by (6.2.4),

$$1 + w(\bar{x} - y - \bar{k}_0) \leqslant C(R + 1 - |\bar{x} - y - \bar{k}_0|)^{-2s} \leqslant C\left(\frac{R}{2}\right)^{-2s} \leqslant 4Cy^{-2s}.$$

By combining this with (6.4.29) and (6.4.38), we then get

$$u_0(-\kappa - y) \leqslant u_0(\bar{x} - y) \leqslant w(\bar{x} - y - \bar{k}_0) \leqslant -1 + 4Cy^{-2s} \quad \text{for any } y \in \left[\frac{R}{2}, R\right].$$

Since κ is a positive constant and R may be chosen arbitrarily large, it is almost immediate to check that this implies (6.4.26). Accordingly, the first estimate in (6.1.16) is established.

Now, we head to the proof of the second estimate of (6.1.16). We first remark that, since $u_0 \in C^{1+2s+\alpha}(\mathbb{R})$, for some $\alpha > 0$, we may differentiate equation (6.4.16) and deduce that u_0' solves

$$L_K u_0' = W''(u_0)u_0' \quad \text{in } \mathbb{R}. \tag{6.4.39}$$

Observe now that, in view of (6.1.11) and the fact that u_0 satisfies (6.4.13), we can take $R_0 > 0$ big enough to have $W''(u_0(x)) \geqslant \delta$ for any $x \in \mathbb{R}$ such that $|x| > R_0$ and for some constant $\delta > 0$. By virtue of this, (6.4.39) and (6.4.12), we then obtain that

$$L_K u_0' \geqslant \delta u_0' \quad \text{in } \mathbb{R} \setminus [-R_0, R_0].$$

The thesis now follows from Lemma 6.11. $\square$

Proposition 6.23. *The upper tail energy estimate* (6.1.17) *holds true.*

Proof. All along the proof, we denote with c any positive constant, whose value may change from line to line.

First we notice that, by the second estimate in (6.1.16),

$$\|u_0'\|_{L^\infty\left(\left[t-\frac{|t|}{2},t+\frac{|t|}{2}\right]\right)} \leqslant \frac{c}{|t|^{1+2s}}, \tag{6.4.40}$$

for $|t|$ sufficiently large. Moreover, given any $\rho > 0$, by the fact that $|u_0| \leqslant 1$, we compute

$$\begin{aligned} \int_{\mathbb{R}} \frac{|u_0(x)-u_0(t)|^2}{|x-t|^{1+2s}}\,dx &\leqslant \int_0^{t+\rho} \frac{2\|u_0'\|^2_{L^\infty([t-\rho,t+\rho])}}{|x-t|^{-1+2s}}\,dx + \int_{t+\rho}^{+\infty} \frac{8\,dx}{|x-t|^{1+2s}} \\ &\leqslant c\left(\|u_0'\|^2_{L^\infty([t-\rho,t+\rho])}\rho^2+1\right)\rho^{-2s}. \end{aligned} \tag{6.4.41}$$

We claim that

$$\beta(t) := \frac{1}{4}\int_{\mathbb{R}} |u_0(x)-u_0(t)|^2\,K(x-t)\,dx + W(u_0(t)) \leqslant \frac{c}{1+|t|^{2s}}, \tag{6.4.42}$$

for any $t \in \mathbb{R}$. We actually prove the stronger

$$\frac{1}{4}\int_{\mathbb{R}} \frac{|u_0(x)-u_0(t)|^2}{|x-t|^{1+2s}}\,dx + W(u_0(t)) \leqslant \frac{c}{1+|t|^{2s}}, \tag{6.4.43}$$

for any $t \in \mathbb{R}$. Observe that (6.4.43) implies (6.4.42), thanks to the right-hand inequality of (6.1.4).

To prove (6.4.43), we first plug $\rho = |t|/2$ into (6.4.41). In view of (6.4.40) we get

$$\int_{\mathbb{R}} \frac{|u_0(x)-u_0(t)|^2}{|x-t|^{1+2s}}\,dx \leqslant \frac{c}{|t|^{2s}}, \tag{6.4.44}$$

provided $|t|$ is large enough. Also, $u_0' \in L^\infty(\mathbb{R})$ and thus, by choosing e.g. $\rho = 1$ in (6.4.41),

$$\int_{\mathbb{R}} \frac{|u_0(x)-u_0(t)|^2}{|x-t|^{1+2s}}\,dx \leqslant c, \tag{6.4.45}$$

for any $t \in \mathbb{R}$. On the other hand, W is of class C^2 and satisfies (6.1.10). Hence, recalling the first estimate of (6.1.16) we obtain

$$\begin{aligned} W(u_0(t)) = W(u_0(t)) - W(1) &= \int_1^{u_0(t)} W'(\tau)\,d\tau = \int_{u_0(t)}^1 [W'(1)-W'(\tau)]\,d\tau \\ &\leqslant \|W''\|_{L^\infty([-1,1])}\int_{u_0(t)}^1 (1-\tau)\,d\tau = \frac{\|W''\|_{L^\infty([-1,1])}}{2}(1-u_0(t))^2 \\ &\leqslant \frac{c}{|t|^{4s}}, \end{aligned}$$

if t is close enough to 1. Similarly, one prove that the same is true when t approaches -1. By this and the boundedness of W we get that

$$W(u_0(t)) \leqslant \frac{c}{1+|t|^{4s}}, \tag{6.4.46}$$

for any $t \in \mathbb{R}$. The combination of (6.4.44), (6.4.45) and (6.4.46) leads to (6.4.43).

With the aid of the previous computations, we may now head to the actual proof of (6.1.17). We have

$$\begin{aligned}\int_{-R}^{R}\int_{\mathbb{R}\setminus[-R,R]} &|u_0(x)-u_0(y)|^2\,K(x-y)\,dxdy\\ &\leqslant \int_{-\frac{R}{2}}^{\frac{R}{2}}\left(\int_R^{+\infty}\frac{8\Lambda\,dx}{|x-y|^{1+2s}}\right)dy + 4\int_{\{\frac{R}{2}<|y|\leqslant R\}}\beta(y)\,dy\\ &\leqslant c\left[\int_{-\frac{R}{2}}^{\frac{R}{2}}(R-y)^{-2s}\,dy+\int_{\frac{R}{2}}^{R}\frac{dy}{1+y^{2s}}\right]\\ &\leqslant cR^{1-2s}.\end{aligned}$$

This finishes the proof of the proposition. □

Notice that we did not really need inequality (6.4.46) to prove Proposition 6.23. However, we included such estimate for the potential term, as it will turn out to be helpful later in Section 6.5.

6.4.4 Further estimates: positive kernels

Here we tackle (6.1.19) and (6.1.20). Since both of them are estimates from below, to prove them we assume the more restrictive condition (6.1.6) on K. Thus, (6.1.6) will be implicitly required throughout the subsection.

Proposition 6.24. *The lower tail energy estimate* (6.1.19) *holds true.*

Proof. Let $R>0$ be large enough to have

$$u_0(x)\geqslant\frac{1}{2}\text{ for any }x\geqslant R\qquad\text{and}\qquad u_0(y)\leqslant-\frac{1}{2}\text{ for any }y\leqslant-\frac{R}{4}.$$

For such values of R, using (6.1.6) we compute

$$\begin{aligned}\int_{-\frac{R}{2}}^{-\frac{R}{4}}\int_R^{+\infty}|u_0(x)-u_0(y)|^2\,K(x-y)\,dxdy &\geqslant \lambda\int_{-\frac{R}{2}}^{-\frac{R}{4}}\left(\int_R^{+\infty}\frac{dx}{|x-y|^{1+2s}}\right)dy\\ &=\frac{\lambda}{2s}\int_{-\frac{R}{2}}^{-\frac{R}{4}}(R-y)^{-2s}dy\\ &\geqslant\frac{\lambda}{2^{3+2s}s}R^{1-2s}.\end{aligned}\tag{6.4.47}$$

Formula (6.1.17) then immediately follows. □

We conclude this subsection with a lemma that gives a sharp lower bound for the total energy $\mathscr{E}_K(u_0,[-R,R])$, when $s=1/2$.

Lemma 6.25. *Let $s=1/2$. There exists a constant $c>0$ such that*

$$\mathscr{E}_K(u_0,[-R,R])\geqslant c\log R,\tag{6.4.48}$$

for any R large enough.

Proof. Choose $k_0 > 1$ in a way that

$$u_0(x) \geqslant \frac{1}{2} \text{ for any } x \geqslant k_0 \qquad \text{and} \qquad u_0(y) \leqslant -\frac{1}{2} \text{ for any } y \leqslant -k_0. \tag{6.4.49}$$

Let $\ell > k \geqslant k_0$ and define

$$I_{k,\ell} := \int_{-\ell}^{-k} \int_{k}^{\ell} |u_0(x) - u_0(y)|^2 K(x-y)\, dxdy.$$

By (6.4.49) and (6.1.6) we compute

$$I_{k,\ell} \geqslant \lambda \int_{-\ell}^{-k} \left(\int_{k}^{\ell} \frac{dx}{(x-y)^2} \right) dy = \lambda \int_{-\ell}^{-k} \left(\frac{1}{k-y} - \frac{1}{\ell - y} \right) dy = \lambda \log \frac{(k+\ell)^2}{4k\ell}.$$

If we set $\ell = 10k$, the above inequality becomes

$$I_{k,10k} \geqslant \lambda \log \frac{121k^2}{40k^2} > \lambda. \tag{6.4.50}$$

Take now any R satisfying

$$R > 100k_0^2, \tag{6.4.51}$$

and let $M > 0$ be the largest integer for which $10^M k_0 \leqslant R$. Notice that then

$$10^{M+1} k_0 > R,$$

which, along with (6.4.51), implies

$$M > \log_{10} \frac{R}{k_0} - 1 = \frac{\log \frac{R}{k_0} - \log 10}{\log 10} \geqslant \frac{\log R}{2 \log 10}.$$

By this and (6.4.50), we conclude that

$$\int_{-R}^{R} \int_{-R}^{R} |u_0(x) - u_0(y)|^2 K(x-y)\, dxdy \geqslant I_{k_0, 10^M k_0} \geqslant \sum_{j=1}^{M} I_{10^{j-1}k_0, 10^j k_0} \geqslant \lambda M \geqslant \frac{\lambda \log R}{2 \log 10},$$

which gives (6.4.48). □

Notice that we can now conclude that (6.1.20) is true. Indeed, when $s > 1/2$ this is obvious (see Remark 6.3). On the other hand, if $s < 1/2$ this fact immediately follows from (6.1.19), while for $s = 1/2$ it is a consequence of Lemma 6.25.

6.4.5 Further estimates: homogeneous kernels

Finally, we address the validity of (6.1.22). To this aim, we suppose $s = 1/2$. Unfortunately, we are able to prove such result only for homogeneous kernels, that is - since $n = 1$ - only for those kernels which are multiples of the kernel of the fractional Laplacian.

Proposition 6.26. *Let $s = 1/2$ and suppose that K is in the form (6.1.8). Then, (6.1.22) holds true.*

Proof. First of all, we remark that, in view of the right-hand inequality in (6.1.17), we already know that

$$\lim_{R \to +\infty} \frac{1}{\log R} \int_{-R}^{R} \int_{\mathbb{R} \setminus [-R,R]} |u_0(x) - u_0(y)|^2 K(x-y)\, dxdy = 0.$$

Hence,

$$\lim_{R\to+\infty}\frac{\mathscr{E}_K(u_0,[-R,R])}{\log R}=\lim_{R\to+\infty}\frac{\displaystyle\int_{-R}^{R}\beta(x)\,dx}{\log R},$$

with β as in (6.4.42).

To compute this limit, we use L'Hôpital's rule. Observe that we are allowed to use such method, since, by Lemma 6.25, the numerator of the quotient written above diverges, as $R\to+\infty$. By the Fundamental Theorem of Calculus, we have

$$\lim_{R\to+\infty}\frac{\mathscr{E}_K(u_0,[-R,R])}{\log R}=\lim_{R\to+\infty}\frac{\dfrac{d}{dR}\displaystyle\int_{-R}^{R}\beta(x)\,dx}{\dfrac{d}{dR}\log R}=\lim_{R\to+\infty}R\Big(\beta(R)+\beta(-R)\Big). \tag{6.4.52}$$

Now, we show that

$$\lim_{R\to+\infty}R\,\beta(\pm R)=\frac{\lambda_\star}{4}\left(\lim_{x\to+\infty}u_0(x)-\lim_{x\to-\infty}u_0(x)\right)^2=\lambda_\star. \tag{6.4.53}$$

Notice that (6.4.52) and (6.4.53) immediately lead to (6.1.22).

We only deal with the limit of $R\beta(R)$ in (6.4.53), the term with the minus sign being completely analogous. We claim that

$$\lim_{R\to+\infty}R\,W(u_0(R))=0, \tag{6.4.54}$$

and

$$\lim_{R\to+\infty}R\int_{-1}^{+\infty}|u_0(R)-u_0(y)|^2\,K(R-y)\,dy=0. \tag{6.4.55}$$

Observe that (6.4.54) immediately follows from estimate (6.4.46). On the other hand, to prove (6.4.55), we fix $k_0>0$ large enough to have, by (6.1.16),

$$|u_0'(t)|\leqslant\frac{c_3}{t^2}\quad\text{for any } t\geqslant k_0,$$

for some $c_3>0$. Then,

$$|u_0(R)-u_0(y)|^2\leqslant\left|\int_y^R|u_0'(t)|\,dt\right|^2\leqslant c_3^2\left|\int_y^R\frac{dt}{t^2}\right|^2=c_3^2\left|\frac{1}{y}-\frac{1}{R}\right|^2=c_3^2\frac{(R-y)^2}{R^2y^2},$$

for any $y\geqslant k_0$, so that, by the right-hand inequality in (6.1.4),

$$\int_{k_0}^{+\infty}|u_0(R)-u_0(y)|^2\,K(R-y)\,dy\leqslant\frac{c_3^2\,\Lambda}{R^2}\int_{k_0}^{+\infty}\frac{dy}{y^2}=\frac{c_3^2\,\Lambda}{k_0R^2}. \tag{6.4.56}$$

Also, since $|u_0|\leqslant 1$, by choosing $R>2k_0$ we get

$$\int_{-1}^{k_0}|u_0(R)-u_0(y)|^2\,K(R-y)\,dy\leqslant 4\Lambda\int_{-1}^{k_0}\frac{dy}{(R-y)^2}\leqslant\frac{8\Lambda(1+k_0)}{R^2}. \tag{6.4.57}$$

Estimates (6.4.56) and (6.4.57) combined yield (6.4.55).

In view of (6.4.54) and (6.4.55), we end up with

$$\lim_{R\to+\infty}R\,\beta(R)=\frac{1}{4}\lim_{R\to+\infty}R\int_{-\infty}^{-1}|u_0(R)-u_0(y)|^2\,K(R-y)\,dy.$$

By changing variables as $y = R(1-z)$, this becomes

$$\lim_{R\to+\infty} R\,\beta(R) = \frac{1}{4}\lim_{R\to+\infty} R^2 \int_{1+\frac{1}{R}}^{+\infty} |u_0(R) - u_0(R(1-z))|^2\, K(Rz)\, dz. \tag{6.4.58}$$

Note that so far we never used that K is in the form (6.1.8), but only the growth assumption in (6.1.4). We do it now. By taking advantage of (6.1.8), formula (6.4.58) reduces to

$$\lim_{R\to+\infty} R\,\beta(R) = \frac{\lambda_\star}{4}\lim_{R\to+\infty}\int_1^{+\infty} \phi_R(z)\, dz,$$

where

$$\phi_R(z) := \frac{|u_0(R) - u_0(R(1-z))|^2}{z^2}\chi_{(1+\frac{1}{R},+\infty)}(z) \quad \text{for a.a. } z \in (1,+\infty).$$

Observe that

$$|\phi_R(z)| \leqslant \frac{4}{z^2} \in L^1((1,+\infty)),$$

and

$$\lim_{R\to+\infty}\phi_R(z) = \frac{\left|\lim\limits_{x\to+\infty} u_0(x) - \lim\limits_{x\to-\infty} u_0(x)\right|^2}{z^2} = \frac{4}{z^2},$$

for any $z > 1$. Thus, by the Dominated Convergence Theorem,

$$\lim_{R\to+\infty} R\,\beta(R) = \frac{\lambda_\star}{4}\int_1^{+\infty}\frac{4}{z^2}\, dz = \lambda_\star,$$

which concludes the proof of the proposition. □

Thanks to the various results displayed in the last subsections, the proof of Theorem 6.1 is now complete.

6.5 Proof of Theorem 6.5

In this conclusive section, we finally address the proof of Theorem 6.5. Our argument essentially follows the lines of that displayed in [PSV13, Section 5]. We stress that, aside from the obvious modifications due to the different framework in which this chapter is set, we also correct some small mistakes present in [PSV13].

Recalling definition (6.1.27), we have to prove that u^* is a class A minimizer for $\mathscr{E}_K$ and that it satisfies assertions (6.1.28)-(6.1.35).

First of all, recall that u_0 and, consequently, u^* are of class $C^{1+2s+\alpha}$, for some $\alpha > 0$. Then, notice that

$$\partial_{x_n} u^*(x) = \varpi u_0'(\varpi x_n) > 0 \quad \text{for any } x \in \mathbb{R}^n, \tag{6.5.1}$$

and

$$\lim_{x_n\to\pm\infty} u^*(x', x_n) = \lim_{x_n\to\pm\infty} u_0(\varpi x_n) = \pm 1 \quad \text{for any } x' \in \mathbb{R}^{n-1}. \tag{6.5.2}$$

Thus, by (6.5.1), (6.5.2) and Theorem 6.6, we are only left to show that u^* solves

$$L_K u^* = W'(u^*) \quad \text{in } \mathbb{R}^n, \tag{6.5.3}$$

to prove that u^* is a class A minimizer for $\mathscr{E}_K$. This is indeed quite straightforward. By substituting $t := \varpi z_n$, we compute

$$\begin{aligned} L_K u^*(x) &= \frac{1}{2}\int_{\mathbb{R}^n} \left(u^*(x+z) + u^*(x-z) - 2u^*(x)\right) K(z)\, dz \\ &= \frac{1}{2}\int_{\mathbb{R}} \left(u_0(\varpi x_n + t) + u_0(\varpi x_n - t) - 2u_0(\varpi x_n)\right) k(t)\, dt \\ &= L_k u_0(\varpi x_n), \end{aligned} \tag{6.5.4}$$

for any $x \in \mathbb{R}^n$. Recall that the kernel k was defined in (6.1.25). Therefore, since u_0 is a solution of

$$L_k u_0 = W'(u_0) \quad \text{in } \mathbb{R},$$

by (6.5.4) we obtain

$$L_K u^*(x) = L_k u_0(\varpi x_n) = W'(u_0(\varpi x_n)) = W'(u^*(x)) \quad \text{for any } x \in \mathbb{R}^n,$$

which is (6.5.3).

Thus, we are left to prove formulae (6.1.28)-(6.1.35). In the remainder of the section, we will frequently denote with c any positive constant, whose value may change from line to line. Also, the radius R will be always implicitly assumed large.

Set

$$I_{n,s}(R) := \int_{B_R}\int_{\mathbb{R}^n \setminus B_R} \frac{|u^*(x) - u^*(y)|^2}{|x-y|^{n+2s}}\, dxdy.$$

First, we claim that

$$\text{if } s \in [1/2, 1), \text{ then } \lim_{R \to +\infty} \frac{I_{n,s}(R)}{R^{n-1}\Psi_s(R)} = 0. \tag{6.5.5}$$

and

$$\text{if } s \in (0, 1/2), \text{ then } I_{n,s}(R) \leqslant cR^{n-2s}. \tag{6.5.6}$$

Recall that Ψ_s was defined in (5.2.1). Note that, thanks to the right-hand inequality in (6.1.4), claim (6.5.5) would then imply formulae (6.1.31) and (6.1.35), while (6.5.6) would yield (6.1.28).

To prove (6.5.5) and (6.5.6), we write $I_{n,s}(R) = S_{n,s}(R) + T_{n,s}(R)$, where

$$\begin{aligned} S_{n,s}(R) &:= \int_{B_R}\int_{(\mathbb{R}^n \setminus B_R)\cap\{|x_n| \leqslant R\}} \frac{|u^*(x) - u^*(y)|^2}{|x-y|^{n+2s}}\, dxdy, \\ T_{n,s}(R) &:= \int_{B_R}\int_{\{|x_n| > R\}} \frac{|u^*(x) - u^*(y)|^2}{|x-y|^{n+2s}}\, dxdy. \end{aligned}$$

First, we deal with the term $T_{n,s}(R)$. We compute

$$\begin{aligned} T_{n,s}(R) &= \int_{-R}^{R}\int_{\{|x_n|>R\}}\int_{B'_{\sqrt{R^2 - |y_n|^2}}}\int_{\mathbb{R}^{n-1}} \frac{|u^*(x) - u^*(y)|^2}{|x-y|^{n+2s}}\, dx'dy'dx_ndy_n \\ &= \int_{-R}^{R}\int_{\{|x_n|>R\}} \frac{|u_0(\varpi x_n) - u_0(\varpi y_n)|^2}{|x_n - y_n|^{n+2s}} \int_{B'_{\sqrt{R^2 - |y_n|^2}}} H(y', |x_n - y_n|)\, dy'dx_ndy_n, \end{aligned}$$

where for $\rho > 0$ we define

$$H(y', \rho) := \int_{\mathbb{R}^{n-1}} \frac{dx'}{\left(1 + \frac{|x'-y'|^2}{\rho^2}\right)^{\frac{n+2s}{2}}}.$$

If we change variables by setting $z' = (x'-y')/\rho$, we get

$$H(y',\rho) = \rho^{n-1}\int_{\mathbb{R}^{n-1}} \frac{dz'}{(1+|z'|^2)^{\frac{n+2s}{2}}} = \frac{\rho^{n-1}}{\varpi^{2s}},$$

by recalling definition (6.1.26). Thus, we get

$$\begin{aligned} T_{n,s}(R) &= \frac{\alpha_{n-1}}{\varpi^{2s}} \int_{-R}^{R}\int_{\{|x_n|>R\}} \frac{|u_0(\varpi x_n)-u_0(\varpi y_n)|^2}{|x_n-y_n|^{1+2s}} \left[R^2-|y_n|^2\right]^{\frac{n-1}{2}} dx_n dy_n \\ &\leqslant \frac{\alpha_{n-1}}{\varpi^{2s}} R^{n-1} \int_{-R}^{R}\int_{\{|x_n|>R\}} \frac{|u_0(\varpi x_n)-u_0(\varpi y_n)|^2}{|x_n-y_n|^{1+2s}} dx_n dy_n, \end{aligned} \tag{6.5.7}$$

recalling (6.1.26). By exploiting (6.1.17), we then get

$$T_{n,s}(R) \leqslant cR^{n-2s}. \tag{6.5.8}$$

Now, we address the term $S_{n,s}(R)$. We compute

$$\begin{aligned} S_{n,s}(R) &= \int_{-R}^{R}\int_{-R}^{R}\int_{B'_{\sqrt{R^2-|y_n|^2}}}\int_{\mathbb{R}^{n-1}\setminus B'_{\sqrt{R^2-|x_n|^2}}} \frac{|u^*(x)-u^*(y)|^2}{|x-y|^{n+2s}} dx'dy'dx_n dy_n \\ &= \int_{-R}^{R}\int_{-R}^{R} \frac{|u_0(\varpi x_n)-u_0(\varpi y_n)|^2}{|x_n-y_n|^{2-n+2s}} \tilde{J}(x_n,y_n) dx_n dy_n, \end{aligned}$$

where we changed variables by setting

$$w' = \frac{x'}{|x_n-y_n|}, \qquad z' = \frac{y'}{|x_n-y_n|},$$

and the quantity

$$\tilde{J}(x_n,y_n) := J_{\frac{n+2s}{2},n-1}\left(\frac{\sqrt{R^2-|y_n|^2}}{|x_n-y_n|}, \frac{\sqrt{R^2-|x_n|^2}}{|x_n-y_n|}\right),$$

is as defined in (6.2.45). Applying then Corollary 6.18, we get[19]

$$S_{n,s}(R) \leqslant c_\delta \left(S^{(1)}_{n,s,\delta}(R) + S^{(2)}_{n,s}(R)\right), \tag{6.5.9}$$

where

$$\begin{aligned} S^{(1)}_{n,s,\delta}(R) &:= \int_{-R}^{R}\int_{-R}^{R} \frac{|u_0(\varpi x_n)-u_0(\varpi y_n)|^2}{|x_n-y_n|^{(1-\delta)(n+2s)}} (R^2-|y_n|^2)^{n-1-\delta\frac{n+2s}{2}} dx_n dy_n \\ S^{(2)}_{n,s}(R) &:= \int_{-R}^{R}\int_{\{|y_n|<|x_n|\}} \frac{|u_0(\varpi x_n)-u_0(\varpi y_n)|^2}{|x_n-y_n|^{1+2s}} \\ &\qquad\qquad \times \left[(R^2-|y_n|^2)^{\frac{n-1}{2}} - (R^2-|x_n|^2)^{\frac{n-1}{2}}\right] dx_n dy_n, \end{aligned}$$

the value δ satisfies

$$\delta \in \left(\frac{n-1}{n+2s}, \frac{n}{n+2s}\right), \tag{6.5.10}$$

and c_δ is a positive constant which may depend on n, s and δ.

[19] Observe that we use the estimate for J provided by Corollary 6.18 with $\alpha = (n+2s)/2$ and $n-1$ in place of n.

To estimate the first integral, we take

$$\delta = \frac{2n-1}{2(n+2s)},$$

which is clearly admissible for (6.5.10). Accordingly, taking advantage of Hölder's inequality and the fact that $|u_0| \leqslant 1$,

$$\begin{aligned}
& S^{(1)}_{n,s,\frac{2n-1}{2(n+2s)}}(R) \\
&= \int_{-R}^{R}\int_{-R}^{R} \frac{|u_0(\varpi x_n) - u_0(\varpi y_n)|^2}{|x_n - y_n|^{\frac{1+4s}{2}}} (R^2 - |y_n|^2)^{\frac{2n-3}{4}}\, dx_n dy_n \\
&\leqslant cR^{\frac{2n-3}{2}} \int_{-R}^{R}\int_{-R}^{R} \left[\frac{|u_0(\varpi x_n) - u_0(\varpi y_n)|^2}{|x_n - y_n|^{1+2s}}\right]^{\frac{1+4s}{2(1+2s)}} |u_0(\varpi x_n) - u_0(\varpi y_n)|^{\frac{1}{1+2s}}\, dx_n dy_n \\
&\leqslant cR^{\frac{2n-3}{2}} [u_0]^{\frac{1+4s}{1+2s}}_{H^s([-\varpi R, \varpi R])} \left[\int_{-R}^{R}\int_{-R}^{R} |u_0(\varpi x_n) - u_0(\varpi y_n)|^2 dx_n dy_n\right]^{\frac{1}{2(1+2s)}} \\
&\leqslant cR^{\frac{2n-3}{2} + \frac{1}{1+2s}} [u_0]^{\frac{1+4s}{1+2s}}_{H^s([-\varpi R, \varpi R])}.
\end{aligned}$$

Recalling (6.4.43), we compute

$$[u_0]^2_{H^s([-\varpi R, \varpi R])} \leqslant \int_{-\varpi R}^{\varpi R} \left[\int_{\mathbb{R}} \frac{|u_0(r) - u_0(t)|^2}{|r-t|^{1+2s}}\, dr\right] dt \leqslant c\int_0^{\varpi R} \frac{dt}{1+t^{2s}} \leqslant c\Psi_s(R). \quad (6.5.11)$$

Accordingly,

$$S^{(1)}_{n,s,\frac{2n-1}{2(n+2s)}}(R) \leqslant c \begin{cases} R^{n-2s} & \text{if } s \in (0, 1/2) \\ R^{n-1} (\log R)^{\frac{3}{4}} & \text{if } s = 1/2 \\ R^{n-1-\frac{1}{2}\frac{2s-1}{1+2s}} & \text{if } s \in (1/2, 1). \end{cases} \quad (6.5.12)$$

The term $S^{(2)}_{n,s}$ is more delicate. We start supposing $n \geqslant 3$. Notice that, if $0 \leqslant a \leqslant b$ and $\beta \geqslant 1$, then

$$b^\beta - a^\beta = \beta \int_a^b t^{\beta-1}\, dt \leqslant \beta b^{\beta-1}(b-a).$$

Applying this formula with $\beta = (n-1)/2$, we get

$$\begin{aligned}
(R^2 - |y_n|^2)^{\frac{n-1}{2}} - (R^2 - |x_n|^2)^{\frac{n-1}{2}} &\leqslant \frac{n-1}{2}(R^2 - |y_n|^2)^{\frac{n-3}{2}} \left(|x_n|^2 - |y_n|^2\right) \\
&\leqslant (n-1)R^{n-2}|x_n - y_n|,
\end{aligned}$$

if $|y_n| \leqslant |x_n| \leqslant R$. Using the above estimate in combination with Hölder's inequality and (6.5.11),

$$S^{(2)}_{n,s}(R) \leqslant c \begin{cases} R^{n-2s} & \text{if } s \in (0, 1/2) \\ R^{n-1}\sqrt{\log R} & \text{if } s = 1/2 \\ R^{n-1-\frac{2s-1}{1+2s}} & \text{if } s \in (1/2, 1). \end{cases} \quad (6.5.13)$$

We address the case $n = 2$ in a slightly different way. First, fix any $\mu \in (1, 2)$ and notice that, for any $0 \leqslant a \leqslant b$,

$$\begin{aligned}
\sqrt{b} - \sqrt{a} = \frac{1}{2}\int_a^b \frac{dt}{\sqrt{t}} &\leqslant \left(\int_a^b t^{-\frac{\mu}{2}} dt\right)^{\frac{1}{\mu}} \left(\int_a^b dt\right)^{\frac{\mu-1}{\mu}} \\
&= \frac{1}{2}\left[\frac{2}{2-\mu}\left(b^{\frac{2-\mu}{2}} - a^{\frac{2-\mu}{2}}\right)\right]^{\frac{1}{\mu}} (b-a)^{\frac{\mu-1}{\mu}}.
\end{aligned}$$

Hence, by choosing e.g. $\mu = 3/2$ we deduce that

$$\sqrt{R^2 - |y_2|^2} - \sqrt{R^2 - |x_2|^2} \leqslant cR^{\frac{2}{3}} |x_2 - y_2|^{\frac{1}{3}},$$

and thus, arguing as for (6.5.13),

$$S_{2,s}^{(2)}(R) \leqslant c \begin{cases} R^{2-2s} & \text{if } s \in (0, 1/2) \\ R \,(\log R)^{\frac{5}{6}} & \text{if } s = 1/2 \\ R^{1-\frac{1}{3}\frac{2s-1}{1+2s}} & \text{if } s \in (1/2, 1). \end{cases} \tag{6.5.14}$$

By combining (6.5.12) and either (6.5.13) or (6.5.14), by (6.5.9) we conclude that

$$\lim_{R\to+\infty} \frac{S_{n,s}(R)}{R^{n-1}\Psi_s(R)} = 0 \quad \text{if } s \in [1/2, 1), \tag{6.5.15}$$

and

$$S_{n,s}(R) \leqslant cR^{n-2s} \quad \text{if } s \in (0, 1/2). \tag{6.5.16}$$

Formulae (6.5.8), (6.5.15) and (6.5.16) imply claims (6.5.5) and (6.5.6).

We now show that (6.1.29) is true. Recall that we prove its validity under the stronger assumption (6.1.6) on K. To check (6.1.29), we use the identity displayed on the first line of (6.5.7) to write

$$\begin{aligned} I_{n,s}(R) &\geqslant T_{n,s}(R) \\ &= \frac{\alpha_{n-1}}{\varpi^{2s}} \int_{-R}^{R} \int_{\{|x_n|>R\}} \frac{|u_0(\varpi x_n) - u_0(\varpi y_n)|^2}{|x_n - y_n|^{1+2s}} \left[R^2 - |y_n|^2\right]^{\frac{n-1}{2}} dx_n dy_n. \end{aligned}$$

By restricting the above integral to the values $|x_n| \leqslant R/2$ and recalling (6.4.47), we get

$$I_{n,s}(R) \geqslant cR^{n-1} \int_{-\frac{R}{2}}^{\frac{R}{2}} \int_{\{|x_n|>R\}} \frac{|u_0(\varpi x_n) - u_0(\varpi y_n)|^2}{|x_n - y_n|^{1+2s}} dx_n dy_n \geqslant cR^{n-2s}.$$

By (6.1.6), the left-hand inequality of (6.1.28) then follows.

Finally, we head to the proof of (6.1.30) and (6.1.34). Let now $s \in [1/2, 1)$. Arguing as in (6.5.4) and changing variables appropriately, we get

$$\begin{aligned} &\int_{\mathbb{R}^n} \int_{B_R} |u^*(x) - u^*(y)|^2 K(x-y)\, dx dy \\ &\quad = \frac{\alpha_{n-1} R^{n-1}}{\varpi} \int_{-\varpi R}^{\varpi R} \left(\int_{\mathbb{R}} |u_0(t) - u_0(r)|^2 k\,(t-r)\; dr \right) \left(1 - \frac{t^2}{\varpi^2 R^2}\right)^{\frac{n-1}{2}} dt. \end{aligned}$$

Moreover, we easily compute

$$\int_{B_R} W(u^*(x))\, dx = \frac{\alpha_{n-1} R^{n-1}}{\varpi} \int_{-\varpi R}^{\varpi R} W(u_0(t)) \left(1 - \frac{t^2}{\varpi^2 R^2}\right)^{\frac{n-1}{2}} dt.$$

Hence, we write

$$\begin{aligned} &\frac{\mathscr{E}_K(u^*, B_R)}{R^{n-1}} \\ &\quad = \frac{\alpha_{n-1}}{\varpi} \left[\frac{1}{4} \int_{-\varpi R}^{\varpi R} \int_{\mathbb{R}} |u_0(t) - u_0(r)|^2 k(t-r)\, dr dt + \int_{-\varpi R}^{\varpi R} W(u_0(t))\, dt \right] \\ &\quad\quad + \theta_1(R) - \theta_2(R), \end{aligned} \tag{6.5.17}$$

where

$$\theta_1(R) = \frac{1}{4R^{n-1}} \int_{\mathbb{R}^n \setminus B_R} \int_{B_R} |u^*(x) - u^*(y)|^2 K(x-y)\, dxdy$$
$$\theta_2(R) = \frac{\alpha_{n-1}}{\varpi} \int_{-\varpi R}^{\varpi R} \alpha(t, \varpi R)\beta(t)\, dt,$$

with

$$\alpha(t, R') = 1 - \left(1 - \frac{t^2}{R^2}\right)^{\frac{n-1}{2}},$$

and β as in (6.4.42), with k in place of K.

Notice that

$$\lim_{R \to +\infty} \frac{\theta_1(R)}{\Psi_s(R)} = 0, \tag{6.5.18}$$

by (6.1.31) or (6.1.35). Furthermore, we claim that it also holds

$$\lim_{R \to +\infty} \frac{\theta_2(R)}{\Psi_s(R)} = 0. \tag{6.5.19}$$

In order to check that (6.5.19) is valid, we distinguish between the two possibilities $s = 1/2$ and $s > 1/2$.

The latter case is easier. Indeed, when $s > 1/2$, we know by (6.4.42) that $\beta \in L^1(\mathbb{R})$. Since $\alpha \leqslant 1$, we may simply employ the Dominated Convergence Theorem to deduce (6.5.19).

Conversely, when $s = 1/2$ we need a more refined argument, inspired by [PSV13, Lemma 4]. Write $R' := \varpi R$. First we claim that, for any fixed $\kappa \in (0, 1)$,

$$\lim_{R' \to +\infty} \frac{1}{\log R'} \int_{\{\kappa R' < |t| \leqslant R'\}} \beta(t)\, dt = 0. \tag{6.5.20}$$

Indeed, by (6.4.42), for any $R' \geqslant 1$ we have

$$\int_{\{\kappa R' < |t| \leqslant R'\}} \beta(t)\, dt \leqslant c_1 \int_{\kappa R'}^{R'} \frac{1}{1+t}\, dt = c_1 \log \frac{1+R'}{1+\kappa R'} \leqslant c_1 \log \frac{2}{\kappa},$$

for some $c_1 > 0$. From this, (6.5.20) clearly follows. In view of (6.5.20), the way α is defined and, again, (6.4.42),

$$\begin{aligned}
&\lim_{R \to +\infty} \frac{\theta_2(R)}{\log R} \\
&= \frac{\alpha_{n-1}}{\varpi} \lim_{R' \to +\infty} \frac{1}{\log R'} \left[\int_{-\kappa R'}^{\kappa R'} \alpha(t, R')\beta(t)\, dt + \int_{\{\kappa R' < |t| \leqslant R'\}} \alpha(t, R')\beta(t)\, dt \right] \\
&\leqslant c_2 \left[2 \lim_{R' \to +\infty} \frac{1}{\log R'} \left(\int_0^{\kappa R'} \frac{dt}{1+t} \right) \sup_{|\tau| \leqslant \kappa R} \alpha(\tau, R') + \lim_{R' \to +\infty} \frac{1}{\log R'} \int_{\{\kappa R' < |t| \leqslant R'\}} \beta(t)\, dt \right] \\
&= 2c_2 \left[1 - \left(1 - \kappa^2\right)^{\frac{n-1}{2}} \right],
\end{aligned}$$

where $c_2 > 0$ is independent of κ. Since we may take κ as small as we like, we deduce that (6.5.19) is true also in this case.

By using (6.5.18) and (6.5.19) in (6.5.17), it is easy to see that (6.1.30) and (6.1.34) are valid. Also, (6.1.32) follows from (6.1.22) in Theorem 6.1, by noticing that if K satisfies (6.1.7), then the one dimensional kernel k defined by (6.1.25) is of the type (6.1.8), with $\lambda_\star$ given by (6.1.33). Indeed, using (6.1.7) we compute

$$k(t) = \frac{1}{\varpi} \int_{\mathbb{R}^{n-1}} K\left(z', \frac{t}{\varpi}\right) \, dz' = \frac{\varpi^{n-1+2s}}{|t|^{n+2s}} \int_{\mathbb{R}^{n-1}} K\left(\frac{\varpi z'}{t}, 1\right) \, dz',$$

for a.a. $t \neq 0$. Changing now coordinates by setting $y' = \varpi z'/t$, we get

$$k(t) = \frac{\varpi^{2s}}{|t|^{1+2s}} \int_{\mathbb{R}^{n-1}} K\left(y', 1\right) \, dy' = \lambda_\star |t|^{-1-2s},$$

and we are done. This concludes the proof of Theorem 6.5.

Chapter 7

Fractional mean curvature and $C^{1,\alpha}$ perturbations

7.1 Introduction and statement of the result

In the seminal work [CRS10], Caffarelli, Roquejoffre and Savin introduced the concept of fractional perimeter of a measurable set $E \subset \mathbb{R}^n$ inside a fixed open bounded set $\Omega \subset \mathbb{R}^n$. More precisely, they defined[20]

$$\mathrm{Per}_s(E,\Omega) := \mathcal{L}_s(E\cap\Omega, \mathcal{C}E\cap\Omega) + \mathcal{L}_s(E\cap\Omega, \mathcal{C}E\cap\mathcal{C}\Omega) + \mathcal{L}_s(E\cap\mathcal{C}\Omega, \mathcal{C}E\cap\Omega),$$

where $s \in (0,1)$ is a fixed parameter and $\mathcal{L}_s$ is the integral functional defined for any two non-overlapping measurable sets $A, B \subset \mathbb{R}^n$ as

$$\mathcal{L}_s(A,B) := \int_A \int_B \frac{dxdy}{|x-y|^{n+s}}.$$

In contrast with the classical notion of De Giorgi perimeter, this is *nonlocal*, as it also takes into account interactions with the complements of E and Ω in $\mathbb{R}^n$.

A nonlocal s-minimal surface in Ω is, hence, the boundary of a set E of finite s-perimeter for which

$$\mathrm{Per}_s(E,\Omega) \leqslant \mathrm{Per}_s(F,\Omega) \quad \text{for any measurable } F \subset \mathbb{R}^n \text{ with } E\cap\mathcal{C}\Omega = F\cap\mathcal{C}\Omega.$$

In [CRS10], the existence of such minimizers is proved, together with other results concerning their regularity, the Hausdorff dimension of the singular set and the relation with nonlocal equations. In particular, they proved that the rescaled characteristic function

$$\widetilde{\chi}_E(x) := \chi_E(x) - \chi_{\mathcal{C}E}(x) = \begin{cases} 1 & \text{if } x \in E, \\ -1 & \text{if } x \in \mathcal{C}E, \end{cases}$$

of a minimizer E satisfies the Euler-Lagrange equation

$$(-\Delta)^{s/2}\, \widetilde{\chi}_E = 0, \qquad \text{on } \partial E \cap \Omega,$$

in a suitable viscosity sense.

Similarly to the local framework, a natural notion of fractional mean curvature has been introduced, so that s-minimal surfaces are precisely those having vanishing s-mean curvature. The result is the assignment, for $x \in \partial E$,

$$H_s[E](x) := \mathrm{P.V.} \int_{\mathbb{R}^n} \frac{\widetilde{\chi}_E(y)}{|x-y|^{n+s}}\, dy. \tag{7.1.1}$$

[20]Note that in this chapter we often write $\mathcal{C}E$ to indicate the complement $\mathbb{R}^n \setminus E$ of a given set $E \subseteq \mathbb{R}^n$.

Notice that this definition is well-posed if ∂E is of class C^2 at x (see, e.g., [AV14, Lemma 7] and also Corollary 7.6 in the present chapter).

Over the last few years, an increasing interest has risen around nonlocal minimal surfaces and the related fractional mean curvature operator. Nice surveys on the topic can be found in [V13] and [AV14]. In particular, the latter proposes a definition of nonlocal principal curvatures and establishes a relation with the s-mean curvature reminiscent to what happens in the classical setting. See also the thesis [L15], where the author proposes a thorough presentation of the results obtained in [CRS10], as well as some original contributions to the theory.

In the next few paragraphs we will give a brief overview of the main developments in the field of nonlocal minimal surfaces.

In [BFV14, SV13, FV13] and [CV13] improvements concerning the regularity of s-minimal surfaces are obtained.

The problem of determining the asymptotic behaviours of the s-perimeter is successfully addressed in [DFPV13], as $s \to 0^+$, and in [ADPM11, CV11], as $s \to 1^-$. We also mention [L14], where the author presents analogous results obtained for a class of anisotropic nonlocal perimeters.

Finally, a Bernstein-type conjecture has been proposed for entire s-minimal graphs of $\mathbb{R}^{n+1}$. In [SV13] it has been proved to be true in the case $n = 1$ and for $n = 2$ the problem has been solved in [FV13]. In particular, in the latter contribution a De Giorgi-type lemma is stated: the validity of the conjecture in $n + 1$ dimensions is ensured by the non-existence of singular n-dimensional s-minimal cones. In higher dimensions the conjecture is still open, while in the classical case the result is true up to $n = 7$.

At a technical point of [FV13], the two authors needed to establish a relation between the s-mean curvature of a subgraph and that of its image under a C^2 graph diffeomorphism.

More in general, it is natural to conjecture that given a set E of class C^2 in a neighbourhood of a point $\bar{x} \in \partial E$ and a global C^2 diffeomorphism Ψ of $\mathbb{R}^n$, the difference between the nonlocal mean curvature of E and that of its transformed $\Psi(E)$, at $\bar{x}$ and $\Psi(\bar{x})$ respectively, can be controlled by means of the C^2 norm of Ψ.

In the present work we give a proof of this fact in full details. Indeed, we prove something slightly stronger, since we lower the regularity assumptions on both the sets and the diffeomorphism to $C^{1,\alpha}$, with $\alpha \in (s, 1]$.

The precise statement of the result is the content of the following

Theorem 7.1. *Let $\eta_0, R > 0$ and $s \in (0, 1)$. Let E be an open subset of $\mathbb{R}^n$, take a point $\bar{x} \in \partial E$ and assume ∂E to be of class $C^{1,\alpha}$ in $B_R(\bar{x})$, for some $\alpha \in (s, 1]$. Let Ψ be a global diffeomorphism of $\mathbb{R}^n$ of class $C^1(\mathbb{R}^n, \mathbb{R}^n) \cap C^{1,\alpha}(B_R(\bar{x}), \mathbb{R}^n)$ and set*

$$F := \Psi(E), \qquad \bar{y} := \Psi(\bar{x}).$$

Decomposing Ψ and its inverse Ψ^{-1} as

$$\Psi(x) = x + \Phi(x), \qquad \textit{for any } x \in \mathbb{R}^n, \tag{7.1.2}$$

and

$$\Psi^{-1}(y) = y + \Xi(y), \qquad \textit{for any } y \in \mathbb{R}^n, \tag{7.1.3}$$

for suitable functions Φ and Ξ, suppose that

$$\|J\Phi\|_{L^\infty(\mathbb{R}^n)}, [J\Phi]_{C^{0,\alpha}(B_R(\bar{x}))}, \|J\Xi\|_{L^\infty(\mathbb{R}^n)}, [J\Xi]_{C^{0,\alpha}(\Psi(B_R(\bar{x})))} \leqslant \eta, \tag{7.1.4}$$

for some $0 < \eta < \eta_0$. Then,

$$\big|H_s[E](\bar{x}) - H_s[F](\bar{y})\big| \leqslant C\eta, \tag{7.1.5}$$

for some constant $C > 0$ depending on n, s, η_0, R, α and the $C^{1,\alpha}$ norm of E at $\bar{x}$.

Notice that the s-mean curvature is well-defined not only for C^2 sets, but also for those being just $C^{1,\alpha}$ regular, provided $\alpha > s$. This fact is probably well-known to the experts but we nevertheless include a proof of it in Subsection 7.3.3.

Needless to say, the decompositions defined by formulae (7.1.2)-(7.1.3) are not restrictive at all. In fact, we employ this notation to the sole purpose of making more evident the role of Φ and Ξ as perturbations of the identity. The relation $\Xi = -\Phi \circ \Psi^{-1}$ clearly holds.

Notice that, if η_0 is suitably small, in dependence of n, then we can require condition (7.1.4) to hold a priori for $J\Phi$ only. Indeed, if this is the case, it can be shown that also the corresponding bound on $J\Xi$ is satisfied.

Finally, we stress that the hypotheses of the theorem are obviously satisfied by C^2 diffeomorphisms. In this case, one may be interested in the precise dependence of the constant C in (7.1.5) on s. To this scope, we took care of this dependence all along the proof and we finally made it explicit in formula (7.2.15).

As a result, one may observe that C diverges, while taking its limit as $s \to 1^-$. This is not surprising at all, since - at least regarding the asymptotic analysis with respect to the parameter s - the *right* normalization for the s-mean curvature is obtained by correcting the quantity described in (7.1.1) with the factor $1 - s$. Indeed, after this modification we see that the new constant C does not diverge anymore and, thus, the result is stable as s approaches 1 from below. Furthermore, by [AV14, Theorem 12] or [CV13, Lemma 9], we know that

$$(1 - s)H_s[E](\bar{x}) \longrightarrow c_n H[E](\bar{x}), \qquad \text{as } s \to 1^-,$$

where $H[E](\bar{x})$ denotes the classical mean curvature of ∂E at $\bar{x}$ and c_n is some dimensional constant. Therefore, using estimate (7.2.15) we may recover the standard version of Theorem 7.1 for the classical mean curvature (see also Section 7.4).

The heart of the proof of Theorem 7.1 is contained in Section 7.2, while we postpone some useful auxiliary computations to the subsequent Section 7.3. In the conclusive Section 7.4 we recall the corresponding well-known result for the classical mean curvature.

Notation

Next is a list of the less standard notations and conventions employed in the course of the chapter.

- Points of the $\mathbb{R}^n$ will be denoted with small letters, as x and y, while primed ones will indicate $(n-1)$-dimensional points, as in Chapter 6. In general, we will make no difference between elements and sets of $\mathbb{R}^{n-1}$ and those of the hyperplane $\mathbb{R}^{n-1} \times \{0\}$ of $\mathbb{R}^n$. Hence, we will often refer to a point of $\mathbb{R}^n$ as $x = (x', x_n) \in \mathbb{R}^{n-1} \times \mathbb{R}$. Similarly, $B'_R(x')$ and B'_R stand for $(n-1)$-dimensional balls.

  We will also use primed notations for differential operators applied to functions defined on subsets of $\mathbb{R}^{n-1}$. So, gradients of such functions will be denoted by ∇' and Laplacians by Δ'. No confusion should arise from the fact that the symbol Δ will also be used at times for the symmetric difference between two sets, i.e. $E \Delta F = (E \setminus F) \cup (F \setminus E)$, for $E, F \subseteq \mathbb{R}^n$.

- Given a point $x \in \mathbb{R}^n$, a hyperplane $\pi \ni x$ orthogonal to $\nu \in S^{n-1}$ and two numbers $r, H > 0$, we will write $K_{\pi,r,H}(x)$ to denote the open cylinder of radius r and height $2H$, centered at x, directed along ν. In symbols,

$$K_{\pi,r,H}(x) = \{y \in \mathbb{R}^n : |y - x - [(y - x) \cdot \nu]\, \nu| < r,\ |(y - x) \cdot \nu| < H\}.$$

We will use $K_{r,H}(x)$ to identify the cylinder directed along the n-th axis

$$K_{r,H}(x) = B'_r(x') \times (-H, H),$$

and set $K_{r,H} = K_{r,H}(0)$.

- The components of a vector valued function will be indicated with superscripts. Thus, if $F : \mathbb{R}^n \to \mathbb{R}^m$, we will write

$$F(x) = \big(F^\ell(x)\big)^{\ell=1,\dots,m} = \big(F^1(x), \dots, F^m(x)\big).$$

To avoid confusion, we will never use short notations for the derivatives of vector functions. Hence, the Jacobian matrix and Hessian tensor of F will be referred to as

$$JF(x) = \big(\partial_i F^\ell(x)\big)^{\ell=1,\dots,m}_{i=1,\dots,n}, \qquad J^2 F = \big(\partial^2_{ij} F^\ell(x)\big)^{\ell=1,\dots,m}_{i,j=1,\dots,n}.$$

- Latin letters, like i, j, k, will be used for indices running from 1 to n, while Greek letters, such as μ, ν, κ, identify those that range between 1 and $n-1$.

- We will understand the matrices as endowed with the Frobenius norm

$$\|A\|_F := \sqrt{A^T A} = \sqrt{\sum_{i,j=1}^n |A_{ij}|^2}, \qquad \text{for } A = [A_{ij}] \in \mathrm{Mat}_n(\mathbb{R}),$$

where A^T is the transpose of A. Any other norm works pretty much the same, but then some attention to the constants involved in the various computations should be paid.

- Sometimes we will use the big O notation. Indeed, saying that a function f is $O(\eta)$ will mean that there exists a constant $C > 0$ independent of η such that

$$|f(x)| \leqslant C\eta,$$

for any x in the domain of f.

7.2 Proof of Theorem 7.1

First, denote by $\nu_F \in S^{n-1}$ the normal vector to the tangent hyperplane π_F to ∂F at $\bar{y}$ pointing inside F. Also, denote by L_F the half-space determined by π_F containing ν_F. We adopt the same notation with respect to E at the point $\bar{x}$.

Let $r > 0$ be some fixed number, whose value will be specified later. We begin with the computation inside the ball of radius r with center $\bar{x}$. We observe that, by symmetry,

$$\mathrm{P.V.} \int_{B_r(\bar{y})} \frac{\widetilde{\chi}_{L_F}(y)}{|y - \bar{y}|^{n+s}}\, dy = 0. \tag{7.2.1}$$

Using (7.2.1) and applying the change of variables induced by Ψ, we compute

$$\begin{aligned}
\mathrm{P.V.} \int_{B_r(\bar{y})} \frac{\widetilde{\chi}_F(y)}{|y - \bar{y}|^{n+s}}\, dy &= \int_{B_r(\bar{y})} \frac{\widetilde{\chi}_F(y) - \widetilde{\chi}_{L_F}(y)}{|y - \bar{y}|^{n+s}}\, dy \\
&= \int_{\Psi^{-1}(B_r(\bar{y}))} \frac{\widetilde{\chi}_E(x) - \widetilde{\chi}_{\Psi^{-1}(L_F)}(x)}{|\Psi(x) - \Psi(\bar{x})|^{n+s}} |\det J\Psi(x)|\, dx.
\end{aligned}$$

Now, Lemma 7.3 tells us that

$$\begin{aligned} |\det J\Psi(x)| &= 1 + O(\eta), \\ |\Psi(x) - \Psi(\bar{x})|^{-n-s} &= |x - \bar{x}|^{-n-s}(1 + O(\eta)). \end{aligned} \tag{7.2.2}$$

We remark that the functions defining the big O's only depend on n and η_0, besides x. Indeed, one can choose e.g. $\bar{\lambda} = n + 1$, in the notation of Lemma 7.3, to obtain estimates independent of s. Thence, we obtain

$$\text{P.V.} \int_{B_r(\bar{y})} \frac{\widetilde{\chi}_F(y)}{|y - \bar{y}|^{n+s}}\, dy = \int_{\Psi^{-1}(B_r(\bar{y}))} \frac{\widetilde{\chi}_E(x) - \widetilde{\chi}_{\Psi^{-1}(L_F)}(x)}{|x - \bar{x}|^{n+s}} (1 + O(\eta))\, dx. \tag{7.2.3}$$

Now we prove that, up to choosing r small enough, it holds

$$\int_{\Psi^{-1}(B_r(\bar{y}))} \frac{\left|\widetilde{\chi}_E(x) - \widetilde{\chi}_{\Psi^{-1}(L_F)}(x)\right|}{|x - \bar{x}|^{n+s}}\, dx \leqslant C/(\alpha - s), \tag{7.2.4}$$

for some constant $C > 0$ depending only on n, η_0 and E.

To this scope, notice that we can select a radius $\tilde{r} > 0$ and a height $\tilde{H} > 0$, depending on n, α, R, η_0 and E, such that both

$$\partial E \cap K_{\pi_E, \tilde{r}, \tilde{H}}(\bar{x}) \qquad \text{and} \qquad \partial \Psi^{-1}(L_F) \cap K_{\pi_E, \tilde{r}, \tilde{H}}(\bar{x}),$$

can be written as graphs of $C^{1,\alpha}$ functions with respect to π_E. The assertion relative to ∂E is a direct consequence of its regularity properties in a neighbourhood of $\bar{x}$. On the other hand, we may employ Proposition 7.4 to obtain that the same is true also for $\partial \Psi^{-1}(L_F)$. Furthermore, if $x \in \Psi^{-1}(B_r(\bar{y}))$, then

$$|x - \bar{x}| = |\Psi^{-1}(\Psi(x)) - \Psi^{-1}(\bar{y})| \leqslant |\Psi(x) - \bar{y}| + |\Xi(\Psi(x)) - \Xi(\bar{y})| \leqslant (1 + \eta_0) r,$$

and so

$$\Psi^{-1}(B_r(\bar{y})) \subset B_{(1+\eta_0)r}(\bar{x}) \subset K_{\pi_E, (1+\eta_0)r, (1+\eta_0)r}(\bar{x}). \tag{7.2.5}$$

Thus, we take

$$r < \min\left\{ \frac{\tilde{r}}{1 + \eta_0}, \frac{\tilde{H}}{1 + \eta_0}, 1 \right\}. \tag{7.2.6}$$

Now, observe that both ∂E and $\partial \Psi^{-1}(L_F)$ are tangent to π_E at $\bar{x}$. We take advantage of this fact, together with Lemma 7.5 and (7.2.5), (7.2.6), to obtain that

$$\int_{\Psi^{-1}(B_r(\bar{y}))} \frac{|\widetilde{\chi}_E(x) - \widetilde{\chi}_{L_E}(x)|}{|x - \bar{x}|^{n+s}}\, dx \leqslant C_1/(\alpha - s),$$

and

$$\int_{\Psi^{-1}(B_r(\bar{y}))} \frac{\left|\widetilde{\chi}_{\Psi^{-1}(L_F)}(x) - \widetilde{\chi}_{L_E}(x)\right|}{|x - \bar{x}|^{n+s}}\, dx \leqslant C_2 \eta/(\alpha - s), \tag{7.2.7}$$

where $C_1 = C_1(n, E)$ and $C_2 = C_2(n, \eta_0)$ are positive constants. The combination of these two inequalities immediately leads to (7.2.4). Notice that we employed (7.3.10) to recover the bound for the $C^{1,\alpha}$ norm of $\Psi^{-1}(L_F)$ necessary to apply Lemma 7.5. Moreover, we simply controlled r with 1, since (7.2.6) is in force.

By this, (7.2.3) may be read as

$$\begin{aligned} \text{P.V.} \int_{B_r(\bar{y})} \frac{\widetilde{\chi}_F(y)}{|y - \bar{y}|^{n+s}}\, dy = & \int_{\Psi^{-1}(B_r(\bar{y}))} \frac{\widetilde{\chi}_E(x) - \widetilde{\chi}_{\Psi^{-1}(L_F)}(x)}{|x - \bar{x}|^{n+s}}\, dx \\ & + (\alpha - s)^{-1} O(\eta). \end{aligned} \tag{7.2.8}$$

Now we only need to estimate the quantity

$$\text{P.V.} \int_{\Psi^{-1}(B_r(\bar{y}))} \frac{\widetilde{\chi}_{\Psi^{-1}(L_F)}(x)}{|x-\bar{x}|^{n+s}}\,dx.$$

To do so, we first add and subtract $\widetilde{\chi}_{L_E}$ to the numerator. With the aid of (7.2.7), we compute

$$\begin{aligned}\left|\text{P.V.} \int_{\Psi^{-1}(B_r(\bar{y}))} \frac{\widetilde{\chi}_{\Psi^{-1}(L_F)}(x)}{|x-\bar{x}|^{n+s}}\,dx\right| \leqslant{}& \left|\text{P.V.} \int_{\Psi^{-1}(B_r(\bar{y}))} \frac{\widetilde{\chi}_{L_E}(x)}{|x-\bar{x}|^{n+s}}\,dx\right| \\ &+ C(\alpha-s)^{-1}\eta,\end{aligned} \tag{7.2.9}$$

with $C>0$ depending only on n and η_0. Furthermore, by symmetry we have

$$\begin{aligned}\left|\text{P.V.} \int_{\Psi^{-1}(B_r(\bar{y}))} \frac{\widetilde{\chi}_{L_E}(x)}{|x-\bar{x}|^{n+s}}\,dx\right| &\leqslant \int_{\Psi^{-1}(B_r(\bar{y}))\Delta B_r(\bar{x})} \frac{dx}{|x-\bar{x}|^{n+s}} \\ &\quad + \text{P.V.} \int_{B_r(\bar{x})} \frac{\widetilde{\chi}_{L_E}(x)}{|x-\bar{x}|^{n+s}}\,dx \\ &= \int_{\Psi^{-1}(B_r(\bar{y}))\Delta B_r(\bar{x})} \frac{dx}{|x-\bar{x}|^{n+s}}.\end{aligned}$$

Now, if $x \in \Psi^{-1}(B_r(\bar{y}))\Delta B_r(\bar{x})$, then we either have $x \notin B_r(\bar{x})$ or $x \notin \Psi^{-1}(B_r(\bar{y}))$. While in the first case it clearly holds $|x-\bar{x}| \geqslant r$, the latter yields

$$r \leqslant |\Psi(x)-\Psi(\bar{x})| \leqslant |x-\bar{x}| + |\Phi(x)-\Phi(\bar{x})| \leqslant (1+\eta)|x-\bar{x}|.$$

That is

$$\text{if} \quad x \notin \Psi^{-1}(B_r(\bar{y})) \ \text{ or } \ x \notin B_r(\bar{x}), \quad \text{then} \quad |x-\bar{x}| \geqslant \frac{r}{1+\eta}. \tag{7.2.10}$$

A similar argument leads to the upper bound

$$|x-\bar{x}| \leqslant (1+\eta)r.$$

Thanks to these two inequalities, we compute

$$\begin{aligned}\left|\text{P.V.} \int_{\Psi^{-1}(B_r(\bar{y}))} \frac{\widetilde{\chi}_{L_E}(x)}{|x-\bar{x}|^{n+s}}\,dx\right| &\leqslant \int_{B_{(1+\eta)r}(\bar{x})\setminus B_{r/(1+\eta)}(\bar{x})} \frac{dx}{|x-\bar{x}|^{n+s}} \\ &= \alpha_{n-1} \int_{r/(1+\eta)}^{(1+\eta)r} \rho^{-1-s}\,d\rho \\ &= \frac{\alpha_{n-1}}{sr^s(1+\eta)^s}\left[(1+\eta)^{2s}-1\right] \\ &\leqslant Cs^{-1}\eta,\end{aligned} \tag{7.2.11}$$

for some positive constant C depending on n, η_0, R, α and E. Notice that in the last line we used (7.2.6) and Lemma 7.2 with $\lambda = 2s$, $\bar{\lambda} = 2$. Combining (7.2.9) and (7.2.11) we get

$$\left|\text{P.V.} \int_{\Psi^{-1}(B_r(\bar{y}))} \frac{\widetilde{\chi}_{\Psi^{-1}(L_F)}(x)}{|x-\bar{x}|^{n+s}}\,dx\right| \leqslant C(s(\alpha-s))^{-1}\eta.$$

Consequently, (7.2.8) finally becomes

$$\begin{aligned}\text{P.V.} \int_{B_r(\bar{y})} \frac{\widetilde{\chi}_F(y)}{|y-\bar{y}|^{n+s}}\,dy ={}& \text{P.V.} \int_{\Psi^{-1}(B_r(\bar{y}))} \frac{\widetilde{\chi}_E(x)}{|x-\bar{x}|^{n+s}}\,dx \\ &+ (s(\alpha-s))^{-1}O(\eta).\end{aligned} \tag{7.2.12}$$

The computation outside $B_r(\bar{y})$ is much simpler. Here we do not have to deal with the singularity of the kernel and, indeed, the estimates are almost immediate. Nevertheless, we provide all the details.

Making the same substitution performed at the start of the proof and using (7.2.2) we recover

$$\begin{aligned}\int_{\mathcal{C}B_r(\bar{y})} \frac{\widetilde{\chi}_F(y)}{|y-\bar{y}|^{n+s}}\,dy &= \int_{\Psi^{-1}(\mathcal{C}B_r(\bar{y}))} \frac{\widetilde{\chi}_E(x)}{|\Psi(x)-\Psi(\bar{x})|^{n+s}}|\det JF(x)|\,dx \\ &= \int_{\mathcal{C}\Psi^{-1}(B_r(\bar{y}))} \frac{\widetilde{\chi}_E(x)}{|x-\bar{x}|^{n+s}}(1+O(\eta))\,dx.\end{aligned} \tag{7.2.13}$$

Using now (7.2.10), we estimate

$$\begin{aligned}\int_{\mathcal{C}\Psi^{-1}(B_r(\bar{y}))} \frac{|\widetilde{\chi}_E(x)|}{|x-\bar{x}|^{n+s}}\,dx &\leqslant \int_{\mathcal{C}B_{r/(1+\eta)}(\bar{x})} \frac{dx}{|x-\bar{x}|^{n+s}} \\ &= \alpha_{n-1}\int_{r/(1+\eta)}^{+\infty} \rho^{-1-s}d\rho \\ &= \frac{\alpha_{n-1}(1+\eta)^s}{sr^s} \\ &\leqslant \frac{\alpha_{n-1}(1+\eta_0)}{sr}.\end{aligned}$$

Thus, we can bring the big O in (7.2.13) out of the integral to write

$$\int_{\mathcal{C}B_r(\bar{y})} \frac{\widetilde{\chi}_F(y)}{|y-\bar{y}|^{n+s}}\,dy = \int_{\mathcal{C}\Psi^{-1}(B_r(\bar{y}))} \frac{\widetilde{\chi}_E(x)}{|x-\bar{x}|^{n+s}}\,dx + s^{-1}O(\eta). \tag{7.2.14}$$

Combining equations (7.2.12) and (7.2.14), we finally conclude that there exists a positive constant C, depending on n, η_0, R, α and E, such that

$$|H_s[F](\Psi(\bar{x})) - H_s[E](\bar{x})| \leqslant C(s(\alpha-s))^{-1}\eta, \tag{7.2.15}$$

and hence (7.1.5) is proved.

7.3 Auxiliary results

We collect here some minor results which have been used to prove Theorem 7.1. The section is divided into three parts. The first subsection contains an estimate for a one-dimensional function, the second is devoted to some general facts about diffeomorphisms of $\mathbb{R}^n$ and the third to singular integrals.

7.3.1 One-dimensional analysis

In this short paragraph we include a technical computation involving a scalar function.

Lemma 7.2. *Fix $\eta_0 > 0$ and $\bar{\lambda} > 0$. Then, there exists a constant $C > 0$ depending only on $\bar{\lambda}$ and η_0 for which*

$$|(1+\eta)^\lambda - 1| \leqslant C\eta, \tag{7.3.1}$$

for any $|\lambda| \leqslant \bar{\lambda}$ and $\eta \in [0, \eta_0)$.

Proof. First notice that we can restrict to the case $\lambda > 0$. Indeed, when $\lambda = 0$ the result is obvious, while if $\lambda < 0$ we may recover it from the positive case, observing that

$$|(1+\eta)^{\lambda} - 1| = 1 - (1+\eta)^{-|\lambda|} = \frac{(1+\eta)^{|\lambda|} - 1}{(1+\eta)^{|\lambda|}} \leqslant |(1+\eta)^{|\lambda|} - 1|.$$

Thus, assume $\lambda > 0$ and define

$$\varphi(t) := (1+t)^{\lambda}, \qquad \text{for any } t \in [0, \eta_0).$$

We have

$$\varphi'(t) = \lambda(1+t)^{\lambda-1}, \qquad \varphi''(t) = \lambda(\lambda-1)(1+t)^{\lambda-2}.$$

Then, we consider separately the two cases $\lambda \in (0,1)$ and $\lambda \geqslant 1$.

In the first situation, we have $\varphi'' < 0$ so that

$$\varphi'(t) \leqslant \varphi'(0) = \lambda,$$

and thus

$$|(1+\eta)^{\lambda} - 1| = \varphi(\eta) - \varphi(0) = \int_0^{\eta} \varphi'(t)\, dt \leqslant \lambda\eta \leqslant \eta.$$

If $\lambda \geqslant 1$, then $\varphi'' \geqslant 0$ and hence

$$\varphi'(t) \leqslant \varphi'(\eta_0) = \lambda(1+\eta_0)^{\lambda-1}.$$

By this we get

$$|(1+\eta)^{\lambda} - 1| = \varphi(\eta) - \varphi(0) = \int_0^{\eta} \varphi'(t)\, dt \leqslant \lambda(1+\eta_0)^{\lambda-1}\eta \leqslant \bar{\lambda}(1+\eta_0)^{\bar{\lambda}-1}\eta,$$

and in either cases (7.3.1) is proved. □

7.3.2 Facts concerning diffeomorphisms

We collect here a pair of general results about diffeomorphisms of $\mathbb{R}^n$. In the first lemma we control some quantities related to a diffeomorphism with its C^1 norm.

Lemma 7.3. *Let $\eta_0 > 0$, U be a domain of $\mathbb{R}^n$ and $\Psi : U \to \mathbb{R}^n$ be a C^1 diffeomorphism. Decomposing Ψ and Ψ^{-1} as in (7.1.2)-(7.1.3), suppose that*

$$\|J\Phi\|_{L^\infty(U)}, \|J\Xi\|_{L^\infty(\Psi(U))} \leqslant \eta, \tag{7.3.2}$$

for some $0 < \eta < \eta_0$. Then,

$$\big| |\det J\Psi(x)| - 1 \big| \leqslant C\eta, \qquad \textit{for any } x \in U, \tag{7.3.3}$$

for some constant $C > 0$ depending only on n and η_0. Moreover, given $0 < \lambda < \bar{\lambda}$, then

$$\left| \left[\frac{|\Psi(x) - \Psi(y)|}{|x-y|} \right]^{-\lambda} - 1 \right| \leqslant C\eta, \qquad \textit{for any } x, y \in U \textit{ such that } x \neq y, \tag{7.3.4}$$

for some constant $C > 0$ depending only on η_0 and $\bar{\lambda}$.

Proof. Recalling Leibniz formula for the determinant of a matrix, we compute for any $x \in U$

$$\begin{aligned} \det J\Psi(x) &= \sum_{\sigma \in S_n} \operatorname{sgn}(\sigma) \prod_{i=1}^n \partial_{\sigma(i)} \Psi^i(x) \\ &= \sum_{\sigma \in S_n} \operatorname{sgn}(\sigma) \prod_{i=1}^n \left(\delta_{i\sigma(i)} + \partial_{\sigma(i)} \Phi^i(x)\right), \end{aligned} \tag{7.3.5}$$

where S_n is the symmetric group on $\{1, \ldots, n\}$ and $\operatorname{sgn}(\sigma)$ denotes the sign of the permutation σ. Notice now that if $\sigma \neq I$ - the identical permutation - then there exists an index j for which $\sigma(j) \neq j$ and so, with the aid of (7.3.2),

$$\begin{aligned} \left| \operatorname{sgn}(\sigma) \prod_{i=1}^n \left(\delta_{i\sigma(i)} + \partial_{\sigma(i)} \Phi^i(x)\right) \right| &= \left| \partial_{\sigma(j)} \Phi^j(x) \prod_{\substack{i=1 \\ i \neq j}}^n \left(\delta_{i\sigma(i)} + \partial_{\sigma(i)} \Phi^i(x)\right) \right| \\ &\leqslant \left|\partial_{\sigma(j)} \Phi^j(x)\right| \prod_{\substack{i=1 \\ i \neq j}}^n \left(1 + \left|\partial_{\sigma(i)} \Phi^i(x)\right|\right) \\ &\leqslant \eta(1+\eta)^{n-1} \\ &\leqslant (1+\eta_0)^{n-1}\eta, \end{aligned} \tag{7.3.6}$$

On the other hand, the term relative to the identical permutation I can be written as

$$\begin{aligned} \operatorname{sgn}(I) \prod_{i=1}^n \left(\delta_{iI(i)} + \partial_{I(i)} \Phi^i(x)\right) &= \prod_{i=1}^n \left(1 + \partial_i \Phi^i(x)\right) \\ &= 1 + \sum_{j=1}^n \sum_{1 \leqslant i_1 < \ldots < i_j \leqslant n} \prod_{k=1}^j \partial_{i_k} \Phi^{i_k}(x). \end{aligned}$$

Since, using (7.3.2) and Lemma 7.2, it holds

$$\left| \sum_{j=1}^n \sum_{1 \leqslant i_1 < \ldots < i_j \leqslant n} \prod_{k=1}^j \partial_{i_k} \Phi^{i_k}(x) \right| \leqslant \sum_{j=1}^n \binom{n}{j} \eta^j = (1+\eta)^n - 1 \leqslant C\eta,$$

we are then able to deduce that

$$\left| \left| \operatorname{sgn}(I) \prod_{i=1}^n \left(\delta_{iI(i)} + \partial_{I(i)} \Phi^i(x)\right) \right| - 1 \right| \leqslant C\eta, \tag{7.3.7}$$

for some constant $C > 0$ depending only on n and η_0. Putting together inequalities (7.3.6) and (7.3.7), recalling (7.3.5) we finally conclude that

$$\big| |\det J\Psi(x)| - 1 \big| \leqslant C\eta, \qquad \text{for any } x \in U,$$

for some constant $C > 0$ depending only on n and η_0, which is (7.3.3).

Now we turn to (7.3.4). Notice that, for any $x, y \in U$,

$$\frac{|\Psi(x) - \Psi(y)|}{|x-y|} \leqslant \frac{|x-y| + |\Phi(x) - \Phi(y)|}{|x-y|} \leqslant 1 + \eta,$$

and

$$\begin{aligned}\frac{|\Psi(x)-\Psi(y)|}{|x-y|}&=\frac{|\Psi(x)-\Psi(y)|}{|\Psi^{-1}(\Psi(x))-\Psi^{-1}(\Psi(y))|}\\&\geqslant\frac{|\Psi(x)-\Psi(y)|}{|\Psi(x)-\Psi(y)|+|\Xi(\Psi(x))-\Xi(\Psi(y))|}\\&\geqslant\frac{1}{1+\eta},\end{aligned}$$

by (7.3.2). Furthermore, by Lemma 7.2 there exists a constant $C>0$ depending only on η_0 and $\bar{\lambda}$ for which

$$\left|(1+\eta)^{\pm\lambda}-1\right|\leqslant C\eta.$$

Hence, we deduce that

$$\left|\left[\frac{|\Psi(x)-\Psi(y)|}{|x-y|}\right]^{-\lambda}-1\right|\leqslant C\eta,$$

and the proof is complete. □

Next is the following proposition, where we address the problem of estimating the size of the domain over which the perturbation of a hyperplane is a graph. Moreover, we give an estimate of its norm as a graph in terms of the norm of the diffeomorphism.

Proposition 7.4. *Fix $\eta_0,R>0$, $\alpha\in(0,1]$, $\bar{x}\in\mathbb{R}^n$ and $e\in S^{n-1}$. Denote by π the hyperplane orthogonal to e which passes through $\bar{x}$. Let $\Psi:B_R(\bar{x})\subset\mathbb{R}^n\to\mathbb{R}^n$ be a $C^{1,\alpha}$ diffeomorphism and, decomposing Ψ and Ψ^{-1} as in (7.1.2)-(7.1.3), assume that, for some $0<\eta<\eta_0$,*

$$\|J\Phi\|_{C^{0,\alpha}(B_R(\bar{x}))},\|J\Xi\|_{C^{0,\alpha}(\Psi(B_R(\bar{x})))}\leqslant\eta. \tag{7.3.8}$$

Then, there exists a radius $r_\star>0$ and a height $H_\star>0$, depending only on n, α, η_0 and R, such that the hypersurface

$$\Psi\left(\pi\cap B_R(\bar{x})\right)\cap K_{\Psi_*\pi,r_\star,H_\star}(\Psi(\bar{x})), \tag{7.3.9}$$

is a $C^{1,\alpha}$ graph with respect to the tangent hyperplane $\Psi_*\pi$ to $\Psi\left(\pi\cap B_R(\bar{x})\right)$ at $\Psi(\bar{x})$.

Moreover, denoting by h the $C^{1,\alpha}$ function defining (7.3.9) as a graph and by B' the $(n-1)$-dimensional ball of center $\Psi(\bar{x})$ and radius $r_\star$ contained in $\Psi_*\pi$ on which h is defined, we have

$$\|\nabla' h\|_{C^{0,\alpha}(B')}\leqslant C\eta, \tag{7.3.10}$$

for some constant $C>0$ depending only on n and η_0.

Proof. We remark that it is enough to prove the proposition for $e=e_n$ and $\bar{x}=0$. Moreover, by composing Ψ with a translation, we may also assume $\Psi(0)=0$.

We restrict for the moment to prove the result under the additional hypothesis

$$\Psi_*\pi=\pi \qquad\text{and}\qquad \langle J\Psi(0)e_n,e_n\rangle>0. \tag{7.3.11}$$

At a second stage we will show that the general case boils down to this one.

First, observe that (7.3.11) is equivalent to asking

$$\partial_\mu\Psi^n(0)=\partial_\mu(\Psi^{-1})^n(0)=0,\qquad\text{for any }\mu=1,\dots,n-1,$$

and

$$\partial_n\Psi^n(0),\partial_n(\Psi^{-1})^n(0)>0.$$

By this and (7.3.8) we then obtain

$$\partial_n(\Psi^{-1})^n(0) = \frac{1}{\partial_n \Psi^n(0)} = \frac{1}{1+\partial_n \Phi^n(0)} \geqslant \frac{1}{1+\eta_0}. \tag{7.3.12}$$

Now, we claim that there exists a radius $R_\star \in (0, R]$, depending only on α, η_0 and R, such that

$$\partial_n(\Psi^{-1})^n(y) \geqslant \frac{1}{2(1+\eta_0)}, \qquad \text{for any } y \in B_{R_\star}. \tag{7.3.13}$$

Indeed, by (7.3.12) and (7.3.8) we get

$$\partial_n(\Psi^{-1})^n(y) \geqslant \partial_n(\Psi^{-1})^n(0) - \eta|y|^\alpha \geqslant \frac{1}{1+\eta_0} - \eta_0|y|^\alpha,$$

which gives (7.3.13), by taking $R_\star = \min\{[2\eta_0(1+\eta_0)]^{-1/\alpha}, R\}$.

Consequently, we may apply the Implicit Function Theorem to deduce the existence of two numbers $r, H \in (0, R_\star]$ and a C^1 function $h : B'_r \to [-H, H]$ for which

$$(y', h(y')) \in B_{R_\star}, \qquad \text{for any } y' \in B'_r, \tag{7.3.14}$$

and

$$(\Psi^{-1})^n(y', h(y')) = 0, \qquad \text{for any } y' \in B'_r. \tag{7.3.15}$$

We recover the $C^{0,\alpha}$ bound on the gradient of h. By differentiating (7.3.15) we get

$$\partial_\mu(\Psi^{-1})^n(y', h(y')) + \partial_n(\Psi^{-1})^n(y', h(y'))h_\mu(y') = 0,$$

for any $\mu = 1, \dots, n-1$, and so

$$h_\mu(y') = -\frac{\partial_\mu(\Psi^{-1})^n(y', h(y'))}{\partial_n(\Psi^{-1})^n(y', h(y'))}, \qquad \text{for any } y' \in B'_r. \tag{7.3.16}$$

Then, combining (7.3.8) and (7.3.13), we have

$$\|\nabla' h\|_{L^\infty(B'_r)} \leqslant \left\| \frac{\nabla'\Xi^n}{\partial_n(\Psi^{-1})^n} \right\|_{L^\infty(B_{R_\star})} \leqslant 2(1+\eta_0)\eta. \tag{7.3.17}$$

From this bound and (7.3.14), we see that the choice

$$r = \frac{R_\star}{\sqrt{1+4(1+\eta_0)^2\eta_0^2}}, \qquad H = \frac{2(1+\eta_0)\eta_0 R_\star}{\sqrt{1+4(1+\eta_0)^2\eta_0^2}},$$

is admissible. Moreover, by the Implicit Function Theorem there exists a number $\kappa \in (0, 1]$ such that $\Psi(\pi \cap B_R) \cap K_{\kappa r, \kappa H}$ is *entirely* parametrized by the graph of h restricted to $B'_{\kappa r}$. We claim that κ may be chosen to depend only on η_0 and α. Indeed, assume that there exists $y \in K_{\kappa r, H}$ such that $(\Psi^{-1})^n(y) = 0$, but y does not belong to the graph of h. Hence, by (7.3.8) and the fact that $\Psi^{-1}(y)$ should lay outside of the set $\Psi^{-1}(\{(z', h(z')) : z' \in B'_r\})$, we have

$$|\Psi^{-1}(y)| \geqslant \inf_{z' \in \partial B'_r} |\Psi^{-1}(z', h(z'))| \geqslant \inf_{z' \in \partial B'_r} \frac{\sqrt{|z'|^2 + h(z')^2}}{1+\eta_0} \geqslant \frac{r}{1+\eta_0},$$

so that

$$|y_n|^2 = |y|^2 - |y'|^2 \geqslant \frac{|\Psi^{-1}(y)|^2}{(1+\eta_0)^2} - |y'|^2 \geqslant \left[\frac{1}{(1+\eta_0)^4} - \kappa^2\right] r^2.$$

In order to have $|y_n| > \kappa H$ it is enough to take $\kappa < (1+\eta_0)^{-2}\left(1+4(1+\eta_0)^2\eta_0^2\right)^{-1/2}$. Note that, if we set $r_\star := \kappa r$ and $H_\star := \kappa H$, then h defines $\Psi(\pi \cap B_R)$ as a graph in the cylinder $K_{r_\star,H_\star}$.

Finally, we turn to the $C^{0,\alpha}$ seminorm of h. In order to simplify the exposition, we will adopt the shorter notation

$$\psi_i(y') := \partial_i(\Psi^{-1})^n(y', h(y')).$$

We stress that (7.3.16) now reads as

$$h_\mu(y') = -\frac{\psi_\mu(y')}{\psi_n(y')}.$$

Moreover, we have that

$$\frac{1}{2(1+\eta_0)} \leqslant |\psi_n(y')| \leqslant 1+\eta \quad \text{and} \quad |\psi_\mu(y')| \leqslant \eta, \qquad \text{for any } y' \in B'_{r_\star}.$$

Given $y', z' \in B'_{r_\star}$, we also notice that, using (7.3.17), we may estimate

$$\begin{aligned}
|\psi_i(y') - \psi_i(z')| &= |\partial_i(\Psi^{-1})^n(y', h(y')) - \partial_i(\Psi^{-1})^n(z', h(z'))| \\
&\leqslant \left[\partial_i(\Psi^{-1})^n\right]_{C^{0,\alpha}(B_{r_\star})} \left(|y'-z'|^2 + |h(y') - h(z')|^2\right)^{\alpha/2} \\
&\leqslant (\delta_{in} + \eta)\left(1+4(1+\eta_0)^2\eta^2\right)^{\alpha/2} |y'-z'|^\alpha.
\end{aligned}$$

Using these inequalities we compute

$$\begin{aligned}
|h_\mu(y') - h_\mu(z')| &= \frac{|\psi_n(z')\psi_\mu(y') - \psi_n(y')\psi_\mu(z')|}{|\psi_n(y')||\psi_n(z')|} \\
&\leqslant \frac{|\psi_\mu(y')||\psi_n(z') - \psi_n(y')| + |\psi_n(y')||\psi_\mu(y') - \psi_\mu(z')|}{|\psi_n(y')||\psi_n(z')|} \\
&\leqslant 4\left[\eta(1+\eta) + (1+\eta)\eta\right]\left(1+4(1+\eta_0)^2\eta^2\right)^{\alpha/2}(1+\eta_0)^2|y'-z'|^\alpha \\
&\leqslant 8(1+\eta_0)^3\left(1+4(1+\eta_0)^2\eta_0^2\right)^{1/2}\eta\,|y'-z'|^\alpha,
\end{aligned}$$

that is $h \in C^{1,\alpha}(B'_{r_\star})$ and

$$\left[\nabla' h\right]_{C^{0,\alpha}(B'_{r_\star})} \leqslant C\eta, \tag{7.3.18}$$

for some constant $C > 0$ depending only on η_0. The combination of (7.3.17) and (7.3.18) leads to (7.3.10).

To conclude, we show that hypothesis (7.3.11) may be dropped.

Let $v \in S^{n-1}$ be a vector orthogonal to $\Psi_*\pi$ and consider a rotation $Q \in \mathrm{SO}(n)$ such that $Qv = e_n$. Up to an orthogonal change of variables in the hyperplane π, we may indeed assume v to be spanned by e_{n-1} and e_n. Hence, we write

$$v = \frac{1}{\sqrt{1+t^2}}\left(e_{n-1} + te_n\right),$$

for some $t \in \mathbb{R}$. Moreover we may take Q of the form

$$Q = \begin{pmatrix} I_{n-2} & 0 \\ 0 & R \end{pmatrix}, \tag{7.3.19}$$

where I_{n-2} is the identity matrix of $\mathrm{Mat}_{n-2}(\mathbb{R})$ and $R \in SO(2)$ is defined by

$$R = \frac{1}{\sqrt{1+t^2}}\begin{pmatrix} t & -1 \\ 1 & t \end{pmatrix}. \tag{7.3.20}$$

Then, we introduce the function

$$\Psi_Q(x) := Q\Psi(x), \qquad \text{for any } x \in B_R.$$

Notice that Ψ_Q is a $C^{1,\alpha}$ diffeomorphism. In addition, for any $w \in \pi$ we have

$$\langle J\Psi_Q(0)w, e_n\rangle = \langle QJ\Psi(0)w, e_n\rangle = \langle J\Psi(0)w, Q^T e_n\rangle = \langle J\Psi(0)w, v\rangle = 0,$$

since $J\Psi(0)w \in \Psi_*\pi$ and v is orthogonal to $\Psi_*\pi$ by definition. Hence, $(\Psi_Q)_*\pi = \pi$. Furthermore, we can choose v in a way that

$$\langle J_Q\Psi(0)e_n, e_n\rangle = \langle J\Psi(0)e_n, v\rangle > 0.$$

Thus, assumption (7.3.11) holds true for Ψ_Q.

Now, we prove that the Jacobians $J\Phi_Q$ and $J\Xi_Q$, defined as in (7.1.2)-(7.1.3), satisfy a bound similar to (7.3.8). We claim that it is enough to show that there exists a dimensional constant $C > 0$ such that

$$\|Q - I\| = \|Q^T - I\| \leqslant C\eta. \tag{7.3.21}$$

Indeed, we compute

$$J\Phi_Q = J\Psi_Q - I = QJ\Psi - I = QJ\Phi + Q - I,$$

and similarly

$$J\Xi_Q = J\Xi Q^T + Q^T - I.$$

Notice that formula $\Psi_Q^{-1} = \Psi^{-1} \circ Q^T$ has been used to recover the last identity. Thus, since $\|Q\| = \|Q^T\| = \sqrt{n}$, if (7.3.21) holds, then we immediately deduce that

$$\|J\Phi_Q\|_{C^{0,\alpha}(B_R)}, \|J\Xi_Q\|_{C^{0,\alpha}(B_R)} \leqslant (\sqrt{n} + C)\eta.$$

Now we prove (7.3.21). Observe that we may restrict to consider $\eta \leqslant 1/2$. Indeed, if this is not the case we simply estimate

$$\|Q - I\| \leqslant \|Q\| + \|I\| = 2\sqrt{n} \leqslant 4\sqrt{n}\eta.$$

Thus, we assume $\eta \leqslant 1/2$ in what follows. By (7.3.19) and (7.3.20), we have

$$\|Q - I\|^2 = \|R - I\|^2 = 2\left(\frac{t}{\sqrt{1+t^2}} - 1\right)^2 + \frac{2}{1+t^2} = 4\left(1 - \frac{t}{\sqrt{1+t^2}}\right). \tag{7.3.22}$$

Note that, by (7.3.8) and the definition of v, we get

$$\begin{aligned}
0 < 1 - \eta &\leqslant |v|^2 + \langle J\Phi(0)v, v\rangle = \langle J\Psi(0)v, v\rangle \\
&= \frac{1}{\sqrt{1+t^2}}\left[\langle J\Psi(0)e_{n-1}, v\rangle + t\langle J\Psi(0)e_n, v\rangle\right] = \frac{t}{\sqrt{1+t^2}}\langle J\Psi(0)e_n, v\rangle.
\end{aligned}$$

Moreover, it holds $\langle J\Psi(0)e_n, v\rangle > 0$, so that $t > 0$. On the other hand,

$$\begin{aligned}
0 = \langle J\Psi(0)e_{n-1}, v\rangle &= \frac{1}{\sqrt{1+t^2}}\left[\langle J\Psi(0)e_{n-1}, e_{n-1}\rangle + t\langle J\Psi(0)e_{n-1}, e_n\rangle\right] \\
&= \frac{1}{\sqrt{1+t^2}}\left[1 + \langle J\Phi(0)e_{n-1}, e_{n-1}\rangle + t\langle J\Phi(0)e_{n-1}, e_n\rangle\right].
\end{aligned}$$

Hence, we obtain that

$$\frac{1}{2} \leqslant 1 - \eta \leqslant 1 + \langle J\Phi(0)e_{n-1}, e_{n-1}\rangle = -t\langle J\Phi(0)e_{n-1}, e_n\rangle \leqslant \eta t,$$

that is, $t \geqslant 1/(2\eta)$. Then, after a simple computation, from (7.3.22) we finally deduce the bound

$$\|Q - I\|^2 \leqslant 8\eta^2,$$

which immediately implies (7.3.21).

By the previous results, it is now clear that Ψ_Q satisfies the hypotheses of the proposition and (7.3.11). Consequently, the first part of the argument applies, yielding the thesis for Ψ_Q. But then the proof is complete, since $\Psi(\pi \cap B_R)$ is the rotation of $\Psi_Q(\pi \cap B_R)$ by means of Q^T. □

7.3.3 Integral computations

In this subsection we report a couple of straightforward results concerning singular integrals. The first one provides an estimate for the detachment of a $C^{1,\alpha}$ graph from its tangent hyperplane inside a ball.

Lemma 7.5. *Let $\eta, r > 0$, $s \in (0,1)$, $\alpha \in (s,1]$ and $\bar{x} \in \mathbb{R}^n$. Let $h : B'_r(\bar{x}') \to \mathbb{R}$ be a given $C^{1,\alpha}$ function, with $h(\bar{x}') = \bar{x}_n$ and*

$$\left[\nabla' h\right]_{C^{0,\alpha}(B'_r(\bar{x}'))} \leqslant \eta.$$

Then, denoting by

$$G := \{(x', x_n) \in B'_r(\bar{x}') \times \mathbb{R} : x_n < h(x')\},$$

the subgraph of h, and by

$$L := \{(x', x_n) \in \mathbb{R}^{n-1} \times \mathbb{R} : x_n < h(\bar{x}') + \nabla' h(\bar{x}') \cdot (x' - \bar{x}')\},$$

the lower half-space determined by the tangent hyperplane of h at $\bar{x}$, we have that

$$\int_{B_r(\bar{x})} \frac{|\widetilde{\chi}_G(x) - \widetilde{\chi}_L(x)|}{|x - \bar{x}|^{n+s}}\, dx \leqslant C(\alpha - s)^{-1} r^{\alpha - s} \eta, \tag{7.3.23}$$

for some constant $C > 0$ depending only on n.

Proof. We assume without loss of generality that $\bar{x} = 0$, i.e. $h(0) = 0$, and $\nabla' h(0) = 0$. Observe that the function $\mathcal{P}$ defined by

$$\mathcal{P}(x') := \eta |x'|^{1+\alpha}, \qquad \text{for any } x' \in \mathbb{R}^{n-1}$$

is such that

$$-\mathcal{P}(x') \leqslant h(x') \leqslant \mathcal{P}(x'), \qquad \text{for any } x' \in B'_r.$$

Therefore, setting

$$P := \left\{(x', x_n) \in \mathbb{R}^{n-1} \times \mathbb{R} : |x_n| < \mathcal{P}(x')\right\},$$

we have

$$|\widetilde{\chi}_G - \widetilde{\chi}_L| \leqslant 2\chi_{G \Delta L} \leqslant 2\chi_P, \qquad \text{in } B_r.$$

Thus, me may conclude that

$$\begin{aligned} \int_{B_r} \frac{|\widetilde{\chi}_G(x) - \widetilde{\chi}_L(x)|}{|x|^{n+s}}\, dx &\leqslant 2 \int_{B_r} \frac{\chi_P(x)}{|x|^{n+s}}\, dx \\ &\leqslant 4 \int_{B'_r} \left(\int_0^{\eta |x'|^{1+\alpha}} \frac{dx_n}{|x|^{n+s}} \right) dx' \\ &\leqslant 4\eta \int_{B'_r} \frac{|x'|^{1+\alpha}}{|x'|^{n+s}}\, dx' \\ &= \frac{4\alpha_{n-2}}{\alpha - s} r^{\alpha - s} \eta, \end{aligned}$$

which yields (7.3.23). □

As a consequence, we deduce that the s-mean curvature is a well-defined quantity for $C^{1,\alpha}$ sets, if $\alpha > s$.

Corollary 7.6. *Let $s \in (0,1)$ and $\alpha \in (s,1]$. Let $E \subset \mathbb{R}^n$ be an open set and take $\bar{x} \in \partial E$. If ∂E is of class $C^{1,\alpha}$ at $\bar{x}$, then $H_s[E](\bar{x})$ is well-defined in the principal value sense.*

Proof. By definition, we know that E may be written as the subgraph of a $C^{1,\alpha}$ function, locally in $B_r(\bar{x})$, for some small radius $r > 0$. Thus, denoting by L the lower half-space determined by the tangent hyperplane to ∂E at $\bar{x}$, we may apply Lemma 7.5 to deduce that

$$\begin{aligned}\left|\mathrm{P.V.}\int_{B_r(\bar{x})} \frac{\widetilde{\chi}_E(x)}{|x-\bar{x}|^{n+s}}\,dx\right| &\leqslant \int_{B_r(\bar{x})} \frac{|\widetilde{\chi}_E(x)-\widetilde{\chi}_L(x)|}{|x-\bar{x}|^{n+s}}\,dx \\ &\quad + \left|\mathrm{P.V.}\int_{B_r(\bar{x})} \frac{\widetilde{\chi}_L(x)}{|x-\bar{x}|^{n+s}}\,dx\right| \\ &= \int_{B_r(\bar{x})} \frac{|\widetilde{\chi}_E(x)-\widetilde{\chi}_L(x)|}{|x-\bar{x}|^{n+s}}\,dx \\ &< +\infty.\end{aligned}$$

Notice that the integral on the second line vanishes by symmetry, in the principal value sense. Furthermore, outside of $B_r(\bar{x})$ we simply estimate

$$\left|\int_{\mathcal{C}B_r(\bar{x})} \frac{\widetilde{\chi}_E(x)}{|x-\bar{x}|^{n+s}}\,dx\right| \leqslant \alpha_{n-1}\int_r^{+\infty} \rho^{-1-s}\,d\rho = \frac{\alpha_{n-1}}{sr^s} < +\infty.$$

These two estimates yield the thesis. $\square$

7.4 The result in the classical framework

In this appendix we present a straightforward computation showing the validity of the counterpart of Theorem 7.1 for the classical mean curvature. By so doing, we extend our result, formally including the case $s = 1$. Notice that this conclusion may be rigorously obtained as a limiting case of Theorem 7.1, as discussed in the introduction. Nevertheless, we provide here a direct proof.

Let E be an open set of $\mathbb{R}^n$ and $\bar{x} \in \partial E$. Assume E to have C^2 boundary at $\bar{x}$. Let $R > 0$ and $\Psi : B_R(\bar{x}) \to \mathbb{R}^n$ be a C^2 diffeomorphism. Define $F \subset \mathbb{R}^n$ and $\bar{y} \in \partial F$ by setting

$$F := \Psi(E \cap B_R(\bar{x})), \qquad \bar{y} := \Psi(\bar{x}).$$

Decomposing Ψ as in (7.1.2) and assuming

$$\|J\Psi(\bar{x})\|, \|J^2\Psi(\bar{x})\| \leqslant \eta,$$

for some small $\eta > 0$, we will show that the mean curvatures of ∂E and ∂F, at $\bar{x}$ and $\bar{y}$ respectively, are linked by the relation

$$\big|H[E](\bar{x}) - H[F](\bar{y})\big| \leqslant C\eta.$$

Notice that, without any loss of generality, we may assume both ∂E and ∂F to be smooth graphs with respect to the hyperplane $\{x_n = 0\}$, locally around $\bar{x}$ and $\bar{y}$ respectively. That is

$$\begin{aligned}E \cap B_\varepsilon(\bar{x}) &= \big\{x = (x', x_n) \in \mathbb{R}^{n-1} \times \mathbb{R} : u(x') < x_n\big\} \cap B_\varepsilon(\bar{x}),\\ F \cap B_\varepsilon(\bar{y}) &= \big\{y = (y', y_n) \in \mathbb{R}^{n-1} \times \mathbb{R} : v(y') < y_n\big\} \cap B_\varepsilon(\bar{y}),\end{aligned}$$

for some C^2 functions $u : B'_\varepsilon(\bar{x}') \to \mathbb{R}$ and $v : B'_\varepsilon(\bar{y}') \to \mathbb{R}$, satisfying

$$\Psi^n(x', u(x')) = v(\Psi'(x', u(x'))), \qquad \text{for any } x' \in B'_\varepsilon(\bar{x}').$$

When we differentiate this equation we get

$$\partial_\mu \Psi^n + \partial_n \Psi^n u_\mu = v_\kappa \left(\partial_\mu \Psi^\kappa + \partial_n \Psi^\kappa u_\mu\right).$$

Taking one more derivative, we find

$$\begin{aligned}
&\partial^2_{\mu\nu}\Psi^n + \partial^2_{\mu n}\Psi^n u_\nu + \left(\partial^2_{n\nu}\Psi^n + \partial^2_{nn}\Psi^n u_\nu\right) u_\mu + \partial_n \Psi^n u_{\mu\nu} \\
&\quad = v_{\kappa\xi}\Big(\partial_\mu \Psi^\kappa + \partial_n \Psi^\kappa u_\mu\Big)\Big(\partial_\nu \Psi^\xi + \partial_n \Psi^\xi u_\nu\Big) \\
&\qquad + v_\kappa \left(\partial^2_{\mu\nu}\Psi^\kappa + \partial^2_{\mu n}\Psi^\kappa u_\nu + \left(\partial^2_{n\nu}\Psi^\kappa + \partial^2_{nn}\Psi^\kappa u_\nu\right) u_\mu + \partial_n \Psi^\kappa u_{\mu\nu}\right).
\end{aligned}$$

Supposing then for simplicity that $\bar{x} = \bar{y} = 0$, $u(0) = v(0) = 0$ and $\nabla' v(0) = 0$, we deduce from the above relations

$$u_\mu(0) = -\frac{\partial_\mu \Psi^n(0)}{1 + \partial_n \Phi^n(0)} = O(\eta),$$

and

$$\begin{aligned}
u_{\mu\nu}(0) = &\Big[v_{\kappa\xi}(0)\Big(\partial_\mu \Psi^\kappa(0) + \partial_n \Psi^\kappa(0) u_\mu(0)\Big)\Big(\partial_\nu \Psi^\xi(0) + \partial_n \Psi^\xi(0) u_\nu(0)\Big) \\
&- \partial^2_{\mu\nu}\Psi^n(0) - \partial^2_{\mu n}\Psi^n(0) u_\nu(0) \\
&- \left(\partial^2_{n\nu}\Psi^n(0) + \partial^2_{nn}\Psi^n(0) u_\nu(0)\right) u_\mu(0)\Big] [1 + \partial_n \Phi^n(0)]^{-1} \\
= &\; v_{\mu\nu}(0) + O(\eta).
\end{aligned}$$

Hence, we may finally conclude that

$$\begin{aligned}
H[E](0) &= \operatorname{div}\left(\frac{\nabla u}{\sqrt{1 + |\nabla u|^2}}\right)(0) \\
&= \left(1 + |\nabla u|^2\right)^{-1/2} \Delta u(0) - \left(1 + |\nabla u|^2\right)^{-3/2} u_{\mu\nu}(0) u_\mu(0) u_\nu(0) \\
&= \Delta v(0) + O(\eta) \\
&= H[F](0) + O(\eta),
\end{aligned}$$

which is what we wanted to prove.

Appendix B

A remark on separability in L^p_{loc} spaces

We discuss here some separability properties of the subsets of the space $L^p_{\mathrm{loc}}(\mathbb{R}^n)$ of locally p-summable functions, for $1 \leqslant p < +\infty$. While the literature on the standard Lebesgue spaces $L^p(\mathbb{R}^n)$ is large and exhaustive, $L^p_{\mathrm{loc}}(\mathbb{R}^n)$ classes are somehow rarely considered as functional spaces. As we have not been able to find precise references for the few facts about $L^p_{\mathrm{loc}}(\mathbb{R}^n)$ that we took advantage of in Proposition 5.13 of Chapter 5, we provide directly here a proof of such results.

First, with the aid of the following proposition, we endow $L^p_{\mathrm{loc}}(\mathbb{R}^n)$ with a separable metric made up on the exhaustion of balls $\bigcup\limits_{k\in\mathbb{N}} B_k$ of $\mathbb{R}^n$.

Proposition B.1. *Let $1 \leqslant p < +\infty$ and define*

$$d(u,v) := \sum_{\ell=1}^{+\infty} \frac{1}{2^\ell} \frac{\|u-v\|_{L^p(B_\ell)}}{1+\|u-v\|_{L^p(B_\ell)}},$$

for any $u, v \in L^p_{\mathrm{loc}}(\mathbb{R}^n)$. Then, $\big(L^p_{\mathrm{loc}}(\mathbb{R}^n), d\big)$ is a separable metric space.

Proof. It is straightforward to check that d is a metric. Thus, we only focus on the proof of the separability.

Since $L^p(\mathbb{R}^n)$ is separable, we may select a sequence $\{u_j\}_{j\in\mathbb{N}}$ which is dense in this space. We claim that $\{u_j\}$ is dense in $\big(L^p_{\mathrm{loc}}(\mathbb{R}^n), d\big)$, too. For a general function $v \in L^p_{\mathrm{loc}}(\mathbb{R}^n)$ and any $k \in \mathbb{N}$, write

$$\bar{v}^k := \begin{cases} v & \text{in } B_k \\ 0 & \text{in } \mathbb{R}^n \setminus B_k. \end{cases}$$

Thus, $\bar{v}^k \in L^p(\mathbb{R}^n)$. Fix now $u \in L^p_{\mathrm{loc}}(\mathbb{R}^n)$. For any $k \in \mathbb{N}$, let u_{j_k} be such that

$$\|u - u_{j_k}\|_{L^p(B_k)} \leqslant \|\bar{u}^k - u_{j_k}\|_{L^p(\mathbb{R}^n)} \leqslant 2^{-k}.$$

Of course, such u_{j_k} exists in view of the density of $\{u_j\}$ in $L^p(\mathbb{R}^n)$. Moreover, we can choose $\{j_k\}$ to be increasing in k, so that $\{u_{j_k}\}$ is a subsequence of $\{u_j\}$. For any k, we then have

$$
\begin{aligned}
d(u_{j_k}, u) &= \sum_{\ell=1}^{k} \frac{1}{2^\ell} \frac{\|u_{j_k} - u\|_{L^p(B_\ell)}}{1 + \|u_{j_k} - u\|_{L^p(B_\ell)}} + \sum_{\ell=k+1}^{+\infty} \frac{1}{2^\ell} \frac{\|u_{j_k} - u\|_{L^p(B_\ell)}}{1 + \|u_{j_k} - u\|_{L^p(B_\ell)}} \\
&\leqslant \|u_{j_k} - u\|_{L^p(B_k)} \sum_{\ell=1}^{k} \frac{1}{2^\ell} + \sum_{\ell=k+1}^{+\infty} \frac{1}{2^\ell} \\
&\leqslant \frac{1}{2^{k-1}},
\end{aligned}
$$

and hence $d(u_{j_k}, u) \to 0$ as $k \to +\infty$. It follows that $\{u_j\}$ is dense in $\big(L^p_{\text{loc}}(\mathbb{R}^n), d\big)$. $\square$

Now that we have established this property, we can proceed to the kind of separability we are most interested in.

Proposition B.2. *Let $1 \leqslant p < +\infty$. Then, any subset X of $L^p_{\text{loc}}(\mathbb{R}^n)$ is separable with respect to pointwise a.e. convergence. That is, there exists a sequence $\{u_j\}_{j\in\mathbb{N}} \subseteq X$ such that, for any $u \in X$, a subsequence $\{u_{j_k}\}$ of $\{u_j\}$ converges to u a.e. in $\mathbb{R}^n$.*

Proof. First of all, we point out that if $v_j \to v$ in $\big(L^p_{\text{loc}}(\mathbb{R}^n), d\big)$, then v_j also converges to v in $L^p(B_k)$, for any $k \in \mathbb{N}$. Indeed,

$$
\frac{1}{2^k} \frac{\|v_j - v\|_{L^p(B_k)}}{1 + \|v_j - v\|_{L^p(B_k)}} \leqslant d(v_j, v) \longrightarrow 0,
$$

as $j \to +\infty$ and thence the claim follows by noticing that, given a sequence of non-negative real numbers $\{a_j\}_{j\in\mathbb{N}}$ and $a \in [0, +\infty)$,

$$
a_j \longrightarrow a \quad \text{if and only if} \quad \frac{a_j}{1 + a_j} \longrightarrow \frac{a}{1 + a},
$$

as $j \to +\infty$.

After this preliminary observation, we can now head to the actual proof of the proposition. Note that, since it is a subset of $L^p_{\text{loc}}(\mathbb{R}^n)$, X is itself a separable metric space with respect to d. This follows by applying Proposition B.1 and, for instance, Proposition 3.25 of [B11]. Let then $\{u_j\}_{j\in\mathbb{N}} \subseteq X$ be a dense sequence. Fixed an element $u \in X$, by the initial remark we know that there exists a subsequence $\{v_j\}$ of $\{u_j\}$ such that $v_j \to u$ in $L^p(B_k)$, for any $k \in \mathbb{N}$.

We now perform a diagonal argument in order to extract a further subsequence $\{v_j^*\}$ from $\{v_j\}$ which converges to u a.e. in $\mathbb{R}^n$.

Since $\{v_j\}$ converges to u in $L^p(B_1)$, we may select a subsequence $\{v_j^1\}$ from $\{v_j\}$ which converges to u a.e. in B_1. Then, $\{v_j^1\}$ still converges to u in $L^p(B_2)$, as it is a subsequence of $\{v_j\}$, and hence there exists another subsequence $\{v_j^2\}$ of $\{v_j^1\}$ converging to u a.e. in B_2. We keep extracting nested subsequences and obtain, for any k, a subsequence $\{v_j^k\} \subseteq \{v_j^{k-1}\}$ converging to u a.e. in B_k. Set $v_j^* := v_j^j$ for any $j \in \mathbb{N}$. This new sequence $\{v_j^*\}$ is eventually a subsequence of each of the previous sequences. Thus, it converges to u a.e. in B_k, for any $k \in \mathbb{N}$, that is a.e. in $\mathbb{R}^n$ $\square$

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