Existence of stationary pulses for nonlocal reaction-diffusion equations

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Abstract. A nonlocal reaction-diffusion equation and a system of equations from population dynamics are considered on the whole axis. Existence of solutions in the form of stationary pulses is proved by a perturbation method. It is based on spectral properties of the linearized operators and on the implicit function theorem.

Key words: reaction-diffusion equation, existence of pulse solutions, perturbation methods

AMS subject classification: 35K57, 35A16, 92D15

1 Introduction

Nonlocal reaction-diffusion equation

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + au^2(1 - J(u)) - \sigma u, \tag{1.1}$$

where

$$J(u) = \int_{-\infty}^{\infty} \phi(x-y)u(y,t)dy$$

describes various biological phenomena such as emergence and evolution of biological species and the process of speciation in a more general context [19], [20]. An important property of nonlocal reaction-diffusion equations equations is that they have solutions in the form of periodic travelling waves [5], [9], [10], [12]. Such solutions do not exist for the usual (scalar) reaction-diffusion equations.

In this work we will prove the existence of a new type of solutions of this equation in the form of stationary pulses. We will consider equation (1.1) in the stationary case,

$$w'' + aw^2(1 - J(w)) - \sigma w = 0. \tag{1.2}$$

Here $x \in \mathbb{R}$, a and σ are positive constants,

$$J(w) = \int_{-\infty}^{\infty} \phi(x-y)w(y)dy, \qquad \phi(x) = \begin{cases} 1 & , \quad |x| \le N \\ 0 & , \quad |x| > N \end{cases},$$

N is a positive number. We will prove that for N sufficiently large equation (1.2) has a positive solution $w(x) \in C^2(\mathbb{R})$ with the limits

$$w(\pm\infty) = 0. \tag{1.3}$$

Instead of a step-wise constant function $\phi(x)$ we can consider any other bounded even non-negative integrable function such that it depends on a parameter and locally converges to 1 as the parameter tends to some given value.

Let us note that if the kernel $\phi(x)$ is replaced by $\psi(x) = \phi(x)/(2N)$, then it converges to the δ-function as $N \to 0$. In the limiting case equation (1.2) becomes a usual reaction-diffusion equation for which existence of pulses can be easily proved analytically. Existence of pulses for all sufficiently small N can be proved by a perturbation technique similarly to travelling waves [2], [3]. Thus, existence can be proved for all sufficiently small N. However pulse solutions are unstable in this case because they are unstable for the limiting reaction-diffusion equation. We will consider here their existence for sufficiently large N. In this case they can be stable.

Nonlocal Fisher-KPP equation, which is similar to equation (1.1) with nonlinearity $u(1-J(u))$, also has solutions in the form of simple and periodic travelling waves [1], [4], [6], [8] - [12], [14], [16], [21]. However solutions in the form of standing pulses are unlikely to exist for this equation. Spike solutions are studied for some reaction-diffusion systems [13], [15], [17], [22], [23].

The method of proof of the existence of pulses is also based on the perturbation technique. If we formally replace the integral $J(w)$ in (1.2) by the integral $I(w) = \int_{\infty}^{\infty} w(y)dy$, then the existence of solutions for this limiting equation can be easily proved. Hence we can expect that there exists a solution for sufficiently large values of N. We will use the implicit function theorem which implies invertibility of the linearized operator. We will prove it using the Fredholm property, index and solvability conditions of the operators under consideration [18], [19]. These properties of the operators will be used in the last section to study existence of stationary pulses of a system of two equations arising in population dynamics.

2 Existence of pulses for the scalar equation with non-local consumption

2.1 Existence in the case of the global consumption

In order to prove the existence of solutions of problem (1.2), (1.3) we will consider the equation

$$w'' + aw^2(1 - I(w)) - \sigma w = 0, \tag{2.1}$$

where

$$I(w) = \int_{-\infty}^{\infty} w(y)dy.$$

By the change of variables $w \to w/a$, $h = 1/a$ we can reduce it to the equation

$$w'' + w^2(1 - hI(w)) - \sigma w = 0. \tag{2.2}$$

We will analyze the existence of classical solutions $w(x)$ which satisfy the following properties:

$$w(x) > 0, \ x \in \mathbb{R}, \ \ w(x) \to 0, \ \ x \to \pm\infty, \ \ w(x) = w(-x). \tag{2.3}$$

Set

$$c = 1 - h \int_{-\infty}^{\infty} w(y)dy \tag{2.4}$$

and consider the equation

$$w'' + cw^2 - \sigma w = 0. \tag{2.5}$$

For each fixed positive c, there exists a unique solution of this equation satisfying (2.3). Its existence can be easily proved by the analysis of the phase plane of the system of two first-order equations,

$$w' = p, \quad p' = -cw^2 + \sigma w$$

or by the explicit integration of the equation

$$\frac{dp}{dw} = \frac{1}{p}\left(-cw^2 + \sigma w\right)$$

(see below). Let us note that since $\sigma > 0$, then this solution exponentially decays at infinity. Denote this solution by $w_c(x)$. Substituting it into (2.4), we obtain the equation

$$c = 1 - h \int_{-\infty}^{\infty} w_c(y)dy. \tag{2.6}$$

Denote by w_1 the solution of (2.5) with $c = 1$. Then $w_c = w_1/c$ and we can write (2.6) as

$$c^2 - c + h \int_{-\infty}^{\infty} w_1(y)dy = 0. \tag{2.7}$$

This equation has two solutions if

$$h \int_{-\infty}^{\infty} w_1(y)dy < \frac{1}{4} \, . \tag{2.8}$$

We note that for every σ fixed, solution $w_1(x)$ of (2.5) with $c = 1$ exists and it is independent of h. Let us take a positive value of h which satisfies condition (2.8). Then equation (2.7) has two solutions, c_1 and c_2, such that $0 < c_1 < 1/2 < c_2 < 1$. If $h \to 0$, then $c_1 \to 0, c_2 \to 1$. Therefore,

$$w_{c_1}(x) \to \infty, \quad w_{c_2}(x) \to w_1(x), \quad h \to 0.$$

The first convergence occurs uniformly on every bounded interval, the second uniformly in $\mathbb{R}$.

Denote $h_0 = 1/(4 \int_{-\infty}^{\infty} w_1(y)dy)$. Then condition (2.8) is satisfied for $h < h_0$, and there are two solutions of equation (2.2).

Theorem 2.1. *For any value of h such that $0 < h < h_0$, there are two positive solutions of equation (2.2) exponentially decaying at infinity.*

In the case of $a > 24\sqrt{\sigma}$ we have the two pulse solutions of equation (2.1) given by the formula

$$w_{1,2}(x) = \frac{3\sigma}{2ac_{1,2}\, cosh^2\left(\frac{\sqrt{\sigma}}{2}x\right)} \, , \quad c_{1,2} = \frac{1}{2} \pm \sqrt{\frac{1}{4} - \frac{6}{a}\sqrt{\sigma}} \, .$$

If $a = 24\sqrt{\sigma}$, then the two solutions coincide. Finally, for $0 < a < 24\sqrt{\sigma}$, there are no real valued pulse solutions.

2.2 Operators and spaces

Consider Hölder spaces $E = C^{2+\alpha}(\mathbb{R})$ and $F = C^{\alpha}(\mathbb{R})$, $0 < \alpha < 1$ and weighted spaces E_μ and F_μ defined as follows:

$$E_\mu = \{u : u\mu \in E, \|u\|_{E_\mu} = \|u\mu\|_E\}, \qquad F_\mu = \{u : u\mu \in F, \|u\|_{F_\mu} = \|u\mu\|_F\}.$$

As a weight function we take $\mu(x) = 1 + x^2$. Set

$$A_\epsilon(w) = w'' + aw^2(1 - J_\epsilon(w)) - \sigma w,$$

$$A_0(w) = w'' + aw^2(1 - J_0(w)) - \sigma w,$$

where

$$J_\epsilon(w) = \int_{x-1/\epsilon}^{x+1/\epsilon} w(y)dy, \quad J_0(w) = \int_{-\infty}^{\infty} w(y)dy.$$

We will consider the operator A_ϵ for $\epsilon > 0$ and $\epsilon = 0$ as defined above. We can extend it for negative ϵ by symmetry, $A_\epsilon = A_{-\epsilon}, \epsilon < 0$. It is a bounded and continuous operator acting from E_μ into F_μ. We will show that it is continuous with respect to ϵ.

Lemma 2.2. *For any $\delta > 0$ there exists ϵ_0 such that*

$$\|A_\epsilon(w) - A_0(w)\|_{F_\mu} < \delta, \quad \forall \epsilon, w, \;\; 0 < \epsilon \leq \epsilon_0, \|w\|_{E_\mu} \leq M, \tag{2.9}$$

where ϵ_0 can depend on M.

Proof. We have

$$A_0(w) - A_\epsilon(w) = aw^2(J_\epsilon(w) - J_0(w)).$$

Set

$$g(x) = \mu(x)w^2(x)(J_\epsilon(w) - J_0(w)).$$

We should estimate the Hölder norm of the function g. Let us begin with the uniform norm. Since

$$|\mu(x)w(x)| \leq M, \quad |w(x)| \leq \frac{M}{\mu(x)} \;,$$

then we have the estimate

$$|J_\epsilon(w)|, |J_0(w)| \leq M_1$$

with some positive constant M_1. Hence for any $\delta > 0$ we can choose $x_0 > 0$ such that

$$|g(x)| \leq \delta \quad \text{for} \quad |x| \geq x_0.$$

We will now obtain a similar estimate for $|x| < x_0$. We have

$$g(x) = -\mu(x)w^2(x) \left(\int_{-\infty}^{x-1/\epsilon} w(y)dy + \int_{x+1/\epsilon}^{\infty} w(y)dy \right) . \tag{2.10}$$

We can choose ϵ_0 such that for any $\epsilon \leq \epsilon_0$ the estimates

$$\left| \mu(x)w^2(x) \int_{x+1/\epsilon}^{\infty} w(y)dy \right| < \frac{\delta}{2} \;, \quad \left| \mu(x)w^2(x) \int_{-\infty}^{x-1/\epsilon} w(y)dy \right| < \frac{\delta}{2} \;, \quad \forall |x| \leq x_0$$

hold. Hence we have the estimate

$$\sup_{x\in\mathbb{R}} |g(x)| \le \delta. \tag{2.11}$$

Next, we should estimate the expression

$$H = \sup_{x_1,x_2\in\mathbb{R}} \frac{|g(x_1) - g(x_2)|}{|x_1 - x_2|^\alpha} \; .$$

Obviously, it is sufficient to consider the case where $|x_1 - x_2| < 1$. We can proceed as before and consider either $|x| > x^*$ for some x^* sufficiently large or $|x| \le x^*$. In the first case, H is small since $\mu(x)w(x)$ and the integrals in (2.10) are bounded in the Hölder norm while $w(x) \to 0$ in the Hölder norm as $|x| \to \infty$. In the second case, we use the fact that the integrals converge to 0 in the Hölder norm as $\epsilon \to 0$. The lemma is proved.

□

In what follows we will also consider the subspaces of the spaces E_μ and F_μ which consist of even functions:

$$E^0_\mu = \{u \in E_\mu, u(x) = u(-x), \ \forall x \in \mathbb{R}\}, \quad F^0_\mu = \{u \in F_\mu, u(x) = u(-x), \ \forall x \in \mathbb{R}\}.$$

If $w \in E^0_\mu$, then $J_\epsilon(w)$ is also an even function and $A_\epsilon(w) \in F^0_\mu$. Therefore we can consider this operator acting from E^0_μ into F^0_μ.

2.3 Linearized operator

Consider the linearized operator to the operator $A_0(w)$,

$$Lu = u'' + 2auw_0(1 - I(w_0)) - aw_0^2 I(u) - \sigma u$$

for some $w_0 \in E_\mu$, and the formally adjoint operator

$$L^* v = v'' + 2avw_0(1 - I(w_0)) - aI^*(v) - \sigma v,$$

where

$$I^*(v) = \int_{-\infty}^{\infty} w_0^2(x) v(x) dx.$$

We will consider the operator L acting from E_μ into F_μ and the operator L^* from $H^2_\infty(\mathbb{R})$ into $L^2_\infty(\mathbb{R})$ [18]. The norms in these spaces are given by the equality:

$$\|u\|_{L^2_\infty(\mathbb{R})} = \sup_i \|\phi_i u\|_{L^2(\mathbb{R})}, \quad \|u\|_{H^2_\infty(\mathbb{R})} = \sup_i \|\phi_i u\|_{H^2(\mathbb{R})},$$

where ϕ_i is a partition of unity. The operator L and L^* are linear bounded operators in the corresponding spaces. They satisfy the relation

$$\int_{-\infty}^{\infty} v(x)(Lu)dx = \int_{-\infty}^{\infty} u(x)(L^*v)dx.$$

Let $w_0(x)$ be an even positive solution of equation (2.1). Set $u_0 = -w_0'$. Differentiating this equation, we obtain

$$L_0 u_0 \equiv u_0'' + 2au_0 w_0(1 - I(w_0)) - \sigma u_0 = 0.$$

Since $I(u_0) = I^*(u_0) = 0$, then $Lu_0 = L^*u_0 = 0$. Hence u_0 is the eigenfunction corresponding to the zero eigenvalue of both operators.

The eigenvalue $\lambda = 0$ of the operators $L_0 : E \to F$ is simple. Indeed, if there are two linearly independent bounded eigenfunctions, then all solutions of the equation $L_0 u = 0$ are bounded as their linear combination. However it has exponentially growing solutions.

We can now summarize the spectral properties of the operator L_0. Its essential spectrum lies in the left-half plane. Its principal eigenvalue is simple and positive, and the corresponding eigenfunction is positive, according to the standard Sturm-Liouville theory. It has a simple zero eigenvalue with the eigenfunction $u_0(x) = -w_0'(x)$ which is positive for positive x and negative for negative x. It can be verified that it does not have other positive eigenvalues except for the principal eigenvalue since $u_0(x)$ has a unique zero at the the origin. These properties remain true for more general nonlinearities.

Lemma 2.3. *If $I(w_0) \neq 1/2$ ($a > 24\sqrt{\sigma}$), then the equation $L^*v = 0$ has a unique, up to a constant factor, solution u_0.*

Proof. Suppose that v_0 is an eigenfunction corresponding to the zero eigenvalue of the operator L^*. Then

$$v_0'' + 2av_0 w_0(1 - I(w_0)) - aI^*(v_0) - \sigma v_0 = 0.$$

Multiplying this equality by w_0 and integrating, we obtain

$$-\int_{-\infty}^{\infty} v_0' w_0' dx + 2aI^*(v_0)(1 - I(w_0)) - aI(w_0)I^*(v_0) - \sigma \int_{-\infty}^{\infty} v_0 w_0 dx = 0. \tag{2.12}$$

Since w_0 is a solution of equation (2.1), we multiply the equation

$$w_0'' + aw_0^2(1 - I(w_0)) - \sigma w_0 = 0 \tag{2.13}$$

by v_0 and integrate:

$$-\int_{-\infty}^{\infty} v_0' w_0' dx + aI^*(v_0)(1 - I(w_0)) - \sigma \int_{-\infty}^{\infty} v_0 w_0 dx = 0. \tag{2.14}$$

Subtracting this equation from equation (2.12), we get

$$I^*(v_0)(1 - I(w_0)) - I(w_0)I^*(v_0) = 0.$$

If $I(w_0) \neq 1/2$, then $I^*(v_0) = 0$. Hence v_0 is an eigenfunction of the operator L_0 corresponding to the zero eigenvalue. Since this eigenfunction is unique up to a constant factor, we get $v_0 = u_0$.

□

Remark 2.4. We proved in Section 2.1 that $I(w_0) = 1/2$ corresponds to the bifurcation point where solutions of equation (2.1) appear due to a subcritical bifurcation. For these values of parameters, the zero eigenvalue of the operator L^* is double, because of the bifurcation and of the invariance with respect to translation. The previous lemma affirms that outside of the bifurcation point this eigenvalue is simple.

Lemma 2.5. *If $I(w_0) \neq 1/2$ ($a > 24\sqrt{\sigma}$), then equation $Lu = 0$ has a unique solution $u = u_0$ in E_μ.*

Proof. The operators $L : E_\mu \to F_\mu$ and $L^* : H^2_\infty(\mathbb{R}) \to L^2_\infty(\mathbb{R})$ satisfy the Fredholm property and have the zero index. It follows from Lemma 2.3 that the equation $L^*v = 0$ has a unique solution. Therefore, since the index equals zero, the nonhomogeneous equation $L^*v = f$ has a unique solvability condition.

Suppose that equation $Lu = 0$ has two linearly independent solutions $u_0, u_1 \in E_\mu$. Then equation $L^*v = f$ has at least two solvability conditions. Indeed, we can choose a function $f \in L^2_\infty(\mathbb{R})$ such that

$$\int_{-\infty}^{\infty} f(x)u_0(x)dx = 0, \qquad \int_{-\infty}^{\infty} f(x)u_1(x)dx \neq 0.$$

If equation $L^*v = f$ has a solution, then we multiply this equation by u_1 and integrate over $\mathbb{R}$. We get

$$\int_{-\infty}^{\infty} (L^*v)u_1 dx = \int_{-\infty}^{\infty} v(Lu_1)dx = 0, \qquad \int_{-\infty}^{\infty} (L^*v)u_1 dx = \int_{-\infty}^{\infty} f u_1 dx \neq 0.$$

This contradiction proves the lemma.

□

It follows from the lemma that a real eigenvalue of the linearized operator cannot cross the origin and change stability of the solution.

2.4 Existence in the case of nonlocal consumption

We will prove existence of solutions of equation (1.2) by the implicit function theorem. We consider the operator $A_\epsilon(w) : E^0_\mu \to F^0_\mu$. It is bounded and continuous. We suppose that the equation $A_0(w) = 0$ has a solution w_0. Conditions of the existence of solutions are given in Section 2.1.

Let us consider the Fréchet derivative of the operator $A_\epsilon(w)$:

$$A'_\epsilon(w)u = u'' + 2aw(x)(1 - J_\epsilon(w))u - \sigma u - aw^2(x)J_\epsilon(u).$$

Lemma 2.6. *The operator $A'_\epsilon(w)$ is continuous with respect to w and ϵ in the operator norm.*

The proof of the lemma is standard and we omit it.

Lemma 2.7. *If $I(w_0) \neq 1/2$ ($a > 24\sqrt{\sigma}$), then the operator $A'_0(w_0) : E^0_\mu \to F^0_\mu$ is invertible.*

Proof. Consider the equation

$$A'_0(w_0)u = f \tag{2.15}$$

for an arbitrary $f \in F^0_\mu$. Since f is an even function and u_0 is odd, $u_0(x) = -w'_0(x)$, then equality

$$\int_{-\infty}^{\infty} f(x)u_0(x)dx = 0$$

holds. It is the unique solvability condition for equation (2.15). Indeed, since it is a Fredholm operator with the zero index and its kernel has dimension 1 (Lemma 2.5), then the codimension of its image is also one-dimensional. Therefore equation (2.15) has a solution $u_1 \in E_\mu$.

Since $A'_0(w_0)u_0 = 0$, then any function $v_k(x) = u_1(x) + ku_0(x)$ is a solution of this equation for any real k. Let us verify that only one of them belongs to E^0_μ. Since $f(x)$ and $w_0(x)$ are even functions, then along with solution $u_1(x)$, the function $u_1(-x)$ is also a solution of this equation. Set $z(x) = u_1(x) - u_1(-x)$. Since $z(x)$ is a solution of the homogeneous equation, then

$$u_1(x) - u_1(-x) = k_1 u_0(x), \tag{2.16}$$

where k_1 is a constant. Then it is possible to choose a number k_2 such that the function $v_{k_2}(x) = u_1(x) + k_2 u_0(x)$ is even. Indeed, from the equality $v_{k_2}(x) = v_{k_2}(-x)$ we get

$$u_1(x) + k_2 u_0(x) = u_1(-x) + k_2 u_0(-x).$$

Since $u_0(x)$ is an odd function, from the last equality and (2.16) we obtain $k_2 = -k_1/2$. Hence we proved that there exists an even solution of equation (2.15). Let us verify that it is unique. If there are two such solutions $z_1(x)$ and $z_2(x)$, then their difference satisfies the homogeneous equation. Hence $z_1(x) - z_2(x) = k_3 u_0(x)$. Since the difference of two even function is an even function, and $u_0(x)$ is an odd function, then this equality can hold only for $k_3 = 0$. Hence the two even solutions coincide.

Thus equation (2.15) has a unique solution in E^0_μ for any $f \in F^0_\mu$. By the Banach theorem, the operator $A'_0(w_0)$ has a bounded inverse.

□

The main result of this section is given by the following theorem.

Theorem 2.8. *Let $a > 1/h_0$, where h_0 is defined in Theorem 2.1. Then equation (1.2) has an even positive solution decaying at infinities for all N sufficiently large.*

Proof. Consider the operator $A_\epsilon(w) : E_\mu^0 \to F_\mu^0$. It is bounded, continuous and equation $A_0(w) = 0$ has a solution w_0. The Fréchet derivative $A'_\epsilon(w)$ is a bounded linear operator, continuous with respect to w and ϵ in the operator norm. Finally, the operator $A'_0(w_0)$ is invertible. By the implicit function theorem equation $A_\epsilon(w) = 0$ has a unique solution from E_μ^0 in the vicinity of the function w_0 for all ϵ sufficiently small.

□

Let us note that under the conditions of the theorem, equation (2.1) has two pulse solutions. Theorem 2.8 affirms the existence of pulse solutions of equation (1.2) in the vicinity of these solutions.

3 System of nonlocal equations

In this section we consider the system of equations

$$d_1 u'' + auv(1 - I(u) - I(v)) - \sigma u = 0, \tag{3.1}$$

$$d_2 v'' + auv(1 - I(u) - I(v)) - \sigma v = 0 \tag{3.2}$$

which describes the distribution of a population in the space of phenotypes. Here u is the density of males, v is the density of females. The second term in the left-hand sides of these equations is the reproduction rate which is proportional to the product uv and to available resources $(1 - I(u) - I(v))$. The last terms are their mortality. It is assumed that both parents have the same phenotype. We will look for a positive solution of this system with the limits at infinities

$$u(\pm\infty) = v(\pm\infty) = 0. \tag{3.3}$$

Diffusion terms in these equations correspond to genetic variability which shows how the phenotypes of offsprings differ from the phenotype of parents. If $d_1 = d_2$, then taking the difference of two equations, we get $u = v$. In this case we can reduce the system of equations to the scalar equation (2.1). However these two coefficients can differ from each other since genetic variability of males is usually greater than that of females. We will prove here the existence of solutions of problem (3.1)-(3.3) in the case where the difference between diffusion coefficients is sufficiently small.

We write system (3.1), (3.2) in the form

$$u'' + a_0 uv(1 - I(u) - I(v)) - \sigma_0 u = 0, \tag{3.4}$$

$$v'' + a_\epsilon uv(1 - I(u) - I(v)) - \sigma_\epsilon v = 0, \tag{3.5}$$

where $a_0 = a/d_1, \sigma_0 = \sigma/d_1$, $a_\epsilon = a_0 + \epsilon, \sigma_\epsilon = \sigma_0 + \epsilon$. If $\epsilon = 0$, then $u = v = w/2$, where w is a solution of the equation

$$w'' + \frac{a_0}{2} w^2(1 - I(w)) - \sigma_0 w = 0, \quad w(\pm\infty) = 0. \tag{3.6}$$

Consider next the system linearized about w for $\epsilon = 0$:

$$\tilde{u}'' + \frac{a_0}{2} \tilde{u}w(1 - I(w)) + \frac{a_0}{2} \tilde{v}w(1 - I(w)) - \frac{a_0}{4} w^2(I(\tilde{u}) + I(\tilde{v})) - \sigma_0\tilde{u} = 0, \tag{3.7}$$

$$\tilde{v}'' + \frac{a_0}{2} \tilde{u}w(1 - I(w)) + \frac{a_0}{2} \tilde{v}w(1 - I(w)) - \frac{a_0}{4} w^2(I(\tilde{u}) + I(\tilde{v})) - \sigma_0\tilde{v} = 0. \tag{3.8}$$

Set $z = \tilde{u} - \tilde{v}$. Taking the difference of these two equations, we get the equation for z:

$$z'' - \sigma_0 z = 0, \quad z(\pm\infty) = 0.$$

Therefore $z \equiv 0$ and $\tilde{u} \equiv \tilde{v}$. Hence system (3.7), (3.8) can be reduced to the single equation:

$$\tilde{u}'' + a_0\tilde{u}w(1 - I(w)) - \frac{a_0}{2} w^2 I(\tilde{u}) - \sigma_0\tilde{u} = 0. \tag{3.9}$$

It coincides with the equation obtained as the linearization of equation (3.6). Due to Lemma 2.5 it has a unique solution $\tilde{u}_0 \in E_\mu$ if $a_0 > 2/h_0$. Hence system (3.7), (3.8) also has a unique even solution $\tilde{u} = \tilde{v} = \tilde{u}_0$. Similar to Lemma 2.7 we can now conclude that the corresponding operator is invertible on the subspace of even functions. We can now formulate the existence theorem.

Theorem 3.1. *Let $a_0 > 2/h_0$, where h_0 is defined in Theorem 2.1. Then system (3.4), (3.5) has an even positive solution decaying at infinities for all ϵ sufficiently small.*

The proof of this theorem is similar to the proof of Theorem 2.8.

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