Family of Smale-Williams Solenoid Attractors as Orbits of Differential Equations: Exact Solution and Conjugacy

Yi-Chiuan Chen[1] and Wei-Ting Lin[2]

[1] *Institute of Mathematics, Academia Sinica, Taipei 10617, Taiwan*[a]

[2] *Department of Physics, National Taiwan University, Taipei 10617, Taiwan*[b]

(Dated: 25 April 2014)

We show that the family of the Smale-Williams solenoid attractors parameterized by its contraction rate can be characterized as a solution of a differential equation. The exact formula describing the attractor can be obtained by solving the differential equation subject to an explicitly given initial condition. Using the formula, we present a simple and explicit proof that the dynamics on the solenoid is topologically conjugate to the shift on the inverse limit space of the expanding map $t \mapsto mt$ mod 1 for some integer $m \geq 2$ and to a suspension over the adding machine.

[a] Electronic mail: Author to whom correspondence should be addressed. YCChen@math.sinica.edu.tw

[b] Electronic mail: r99222027@ntu.edu.tw

Fractals are usually described through infinite intersection of sets by means of iterated function systems. Although such descriptions are mathematically elegant, it may still be helpful and desirable to have explicit formulae describing the fractals. For instance, the famous middle-third Cantor set $C = \bigcap_{i\geq 0} C_i$, often constructed by the infinite intersection of a sequence of nested sets C_i, where $C_0 = [0, 1]$ and each C_{i+1} is obtained by removing the middle-third open interval of every interval in C_i, can be expressed explicitly by $C = \{x \,|\, x = \sum_{k=1}^{\infty} a_k/3^k,\ a_k = 0 \text{ or } 2\ \forall k \in \mathbb{N}\}$. This expression has some advantages. For example, the dynamics of the tent map $x \to 3/2 - 3|x - 1/2|$ on C can be understood in an algebraically easy way. In general, it is difficult to find an explicit formula for a fractal. Here, we study the Smale-Williams solenoid attractor, which is a fractal attractor, and occupies a prominent place in the development of dynamical systems and fractals. We find that an explicit formula, which describes the attractor, can be obtained as an exact solution of a differential equation. With the formula, the dynamics of the solenoid diffeomorphism on the attractor can be understood straightforwardly and completely by direct computation. One of the main points of this paper is to contribute such computation accessible to general readers.

I. INTRODUCTION

One of manifesting features of chaotic systems is the possession of fractal attractors. For an *attractor* Λ of a map f, we mean that there exists a neighborhood N such that $f(N) \subset N$ and $\Lambda = \bigcap_{i\geq 0} f^i(N)$. Hyperbolic chaotic attractors are, in particular, important in the sense that they are structurally stable and exhibit complicated dynamical behavior. Here, an attractor Λ is said to be *chaotic* if the restriction of f to Λ is topologically transitive and has sensitive dependence on initial conditions. One popular mathematical model of hyperbolic chaotic fractal attractors is the Smale-Williams solenoid attractor[10,13,14].

The Smale-Williams solenoid has appeared in many typical textbooks on chaotic dynamical systems, for instance, Devaney[1], Katok and Hasselblatt[3], or Robinson[8]. Besides, a physical realistic system of the solenoid has been proposed. In 2005, Kuznetsov[4] constructed a non-autonomous flow of two coupled van der Pol oscillators whose Poincaré map demonstrates the attractor of Smale-Williams type. The flow constructed can be implemented as an electronic device[6]. (See also Kuznetsov and Sataev[5], and Wilczak[12] for numerical examination of uniformly hyperbolicity for the attractor.)

The Smale-Williams attractor has both expanding (1-dimensional) and contracting (2-dimensional) directions. By virtue of its hyperbolicity, as the contraction rate varies, the attractor forms a continuous family of solenoids. We show that this family is the solution of a differential equation with an explicitly prescribed initial condition. The solution itself turns out to be an exact formula representing the solenoid attractor with a given contraction rate.

Recall the definition of Smale-Williams solenoid diffeomorphism. Let $D^2 = \{z \in \mathbb{C} | \, |z| \le 1\}$ be the unit disk on the complex plane and $S^1 = \partial D^2 = \{z \in \mathbb{C} | \, |z| = 1\}$ its boundary. The solid torus $S^1 \times D^2$ is the domain of interest. Let $g : S^1 \to S^1$ be an expanding circle map defined by $g(s) = s^m$, where $m \ge 2$ is an integer. The solenoid map $q : S^1 \times D^2 \to S^1 \times D^2$ is define by

$$q(s, z) = (g(s), \varepsilon z + \frac{1}{2}s),$$

where the positive real number ε has to satisfy

$$\varepsilon < \frac{1}{2} \sin \frac{\pi}{m} \le \frac{1}{2} \tag{1}$$

so that q is an invertible map. Note $q^{k+1}(S^1 \times D^2) \subset q^k(S^1 \times D^2)$ for any integer $k \ge 0$. The Smale-Williams solenoid for the map q is

$$\Lambda_q = \bigcap_{k=0}^{\infty} q^k(S^1 \times D^2).$$

Denote Σ to be the space of sequences:

$$\Sigma = \{\boldsymbol{a} = (a_j)_{j \le -1} | \; a_j \in \{0, 1, 2, \dots, m-1\} \; \forall j \le -1\}.$$

One of the main results of this paper is to show that Λ_q satisfies a differential equation.

Theorem 1. *A point* $(\exp(i2\pi t), z)$ *belongs to* Λ_q *if and only if* $z = \zeta(\varepsilon)$ *and* ζ *is a solution of the following differential equation*

$$\frac{d\zeta}{d\varepsilon} = \sum_{k \le -2} \varepsilon^{-k-2} \frac{1}{2} \exp\left(i2\pi \left(tm^k + \sum_{j=k}^{-1} a_j m^{k-j-1} \right) \right) \tag{2}$$

subject to the initial condition

$$\zeta(0) = \frac{1}{2}\exp(i2\pi\frac{t + a_{-1}}{m}) \tag{3}$$

for some $(a_k)_{k\leq -1} \in \Sigma$.

As a matter of fact, there exists an exact formula for Λ_q.

Theorem 2. *The Smale-Williams solenoid can be expressed explicitly as*

$$\Lambda_q = \bigcup_{t\in[0,1)} \bigcup_{\boldsymbol{a}\in\Sigma} \left(\exp(i2\pi t), \sum_{k\leq -1} \varepsilon^{-k-1}\frac{1}{2}\exp\left(i2\pi\left(tm^k + \sum_{j=k}^{-1} a_j m^{k-j-1} \right)\right)\right). \tag{4}$$

The inverse limit $\varprojlim(S^1, g)$ (see, for example, Devaney[1] or Robinson[8]) of the map g on S^1 is defined by

$$\varprojlim(S^1, g) := \{\boldsymbol{s} = (s_k)_{k\leq 0}|\ g(s_{k-1}) = s_k\ \forall k \leq 0\}.$$

There is a natural shift map σ on $\varprojlim(S^1, g)$ defined by

$$\sigma((\dots, s_{-2}, s_{-1}, s_0)) = (\dots, s_{-1}, s_0, g(s_0)).$$

The *adding machine map* $A : \Sigma \to \Sigma$, or called the *odometer map*, is defined as

$$A(\boldsymbol{a}) = (\dots, a_{-3}, a_{-2}, a_{-1}) + (\dots, 0, 0, 1) \bmod m. \tag{5}$$

For points in the product space $[0, 1] \times \Sigma$, denote by $\sim$ the equivalence relation that $(1, \boldsymbol{a})$ is identified with $(0, A(\boldsymbol{a}))$. For any given $\boldsymbol{a} = (\dots, a_{-3}, a_{-2}, a_{-1}) \in \Sigma$ and $e \in \{0, 1, \dots, m-1\}$, we use the following notation:

$$\begin{aligned}\boldsymbol{a}e &:= (\dots, a_{-2}, a_{-1})e \\ &:= (\dots, a_{-2}, a_{-1}, e).\end{aligned}$$

Define a map τ of the quotient space $([0, 1] \times \Sigma)/\sim$ by

$$\tau(t, \boldsymbol{a}) := (mt - e, \boldsymbol{a}e) \qquad \text{if } \frac{e}{m} \leq t \leq \frac{e+1}{m} \tag{6}$$

for some $e \in \{0, 1, \dots, m - 1\}$. Then, we have

Theorem 3. *Let* $p : ([0,1]\times\Sigma)/\sim\ \to \varprojlim(S^1,g)$, $(t,(\ldots,a_{-2},a_{-1}))\mapsto(\ldots,s_{-1},s_0)$, *and* $h:\varprojlim(S^1,g)\to\Lambda_q$, $(\ldots,s_{-1},s_0)\mapsto(s_0,z_0)$, *be defined by*

$$s_k=\begin{cases}\exp\left(i2\pi\left(tm^k+\sum_{j=k}^{-1}a_jm^{k-j-1}\right)\right) & \text{if } k\le -1\\ \exp(i2\pi t) & \text{if } k=0\end{cases}\tag{7}$$

and

$$z_0=\sum_{k\le -1}\varepsilon^{-k-1}\frac{1}{2}s_k,\tag{8}$$

respectively. Then, the following diagram

$$\begin{array}{ccc}([0,1]\times\Sigma)/\sim & \xrightarrow{\tau} & ([0,1]\times\Sigma)/\sim\\ {\scriptstyle p}\big\downarrow & & \big\downarrow{\scriptstyle p}\\ \varprojlim(S^1,g) & \xrightarrow{\sigma} & \varprojlim(S^1,g)\\ {\scriptstyle h}\big\downarrow & & \big\downarrow{\scriptstyle h}\\ \Lambda_q & \xrightarrow{q} & \Lambda_q\end{array}\tag{*}$$

is commutative.

Actually, formulae (7) and (8) can be derived by straightforward observation from formula (4), and vice versa. It has been known that the restriction of q to Λ_q is topologically conjugate to the two-sided shift σ on the inverse limit space $\varprojlim(S^1,g)$ (see, for example, Devaney[1], Robinson[8], Shub[9]). It is also known that it is topologically conjugate to the map τ on the suspension $([0,1]\times\Sigma)/\sim$ over the adding machine (for example, Takens[11]). Moreover, formulae similar to (4) for solenoids in more abstract mathematical settings, though little-known, are also known (see Hewitt and Ross[2], or Kwapisz[7]). One point of Theorem 3 is to show that there are explicit formulae for the topological conjugacies, meaning that p and h are the topological conjugacies with appropriate topologies. The main point is to utilize these formulae to provide an algebraically explicit, clear, and less abstract proof of the commutativity of the diagram (*). Our proof is accessible for general readers.

Remark 4. The fact that Λ_q is homeomorphic to the product $[0,1]\times\Sigma$ with $(1,\boldsymbol{a})$ identified with $(0,A(\boldsymbol{a}))$ can be understood as follows: For each $t\in\mathbb{R}/\mathbb{Z}$, the intersection $\Lambda_{q,t}=\Lambda_q\cap(\{\exp(i2\pi t)\}\times D^2)$ of Λ_q and the section $\{\exp(i2\pi t)\}\times D^2$ is a Cantor set. (Thus, $\Lambda_{q,t}$ is homeomorphic to Σ.) So, for small positive c, the intersection of Λ_q and

$\bigcup_{t-c\le t'\le t+c}\exp(i2\pi t')\times D^2$ can be described as the product $\bigcup_{t-c\le t'\le t+c}\exp(i2\pi t')\times \Lambda_{q,t}$. Following t, in a natural way, one can define a return map $\mathcal{R}_t:\Lambda_{q,t}\to\Lambda_{q,t}$. Then Λ_q is homeomorphic to the product $[0,1]\times\Lambda_{q,t}$ with $(1,z)$ identified with $(0,\mathcal{R}_t(z))$.

The rest of the paper is mainly devoted to proving and elaborating Theorem 3. In addition to verifying the commutativity of the diagram (*), to show the bijections p and h are topological conjugacies we need to prove they are homeomorphisms when $\varprojlim(S^1,g)$ and $([0,1]\times\Sigma)/\sim$ are endowed with appropriate topologies. Propositions 7 and 8 in Section II address this issue. In Section III, we prove Theorems 1 and 3. For completeness sake, we derive formula (4) for the Smale-Williams solenoid in the Appendix.

II. USEFUL LEMMAS AND HOMEOMORPHISMS p AND h

A. Useful lemmas

The precise meaning of the adding machine map A is as follows:

$$A(\boldsymbol{a})=\begin{cases}(\ldots,a_{-2},a_{-1}+1) & \text{if } a_{-1}\in\{0,1,\ldots,m-2\} \quad (9)\\ (\ldots,a_{l-2},a_{l-1}+1,0,\ldots,0) & \text{if } a_{l-1}\in\{0,1,\ldots,m-2\},\\ & \quad a_l=\ldots=a_{-2}=a_{-1}=m-1,\ l\le -1 \quad (10)\\ (\ldots,0,0) & \text{if } \boldsymbol{a}=(\ldots,m-1,m-1). \quad (11)\end{cases}$$

Note that A is a one-to-one and onto, with its inverse

$$A^{-1}(\boldsymbol{a})=\begin{cases}(\ldots,a_{-2},a_{-1}-1) & \text{if } a_{-1}\in\{1,2,\ldots,m-1\} \quad (12)\\ (\ldots,a_{l-2},a_{l-1}-1,m-1,\ldots,m-1) & \text{if } a_{l-1}\in\{1,2,\ldots,m-1\},\\ & \quad a_l=\ldots=a_{-2}=a_{-1}=0,\\ & \quad l\le -1 \quad (13)\\ (\ldots,m-1,m-1) & \text{if } \boldsymbol{a}=(\ldots,0,0). \quad (14)\end{cases}$$

The following lemma tells us how the adding machine is incorporated in formula (4) and the diagram (*).

Lemma 5. *Assume $m\ge 2$ is an integer.*

$$\begin{aligned}&\exp\left(i2\pi\left((t+n)m^k+a_{-1}m^k+a_{-2}m^{k+1}+\ldots+a_km^{-1}\right)\right)\\ &=\exp\left(i2\pi\left(tm^k+A^n(\boldsymbol{a})_{-1}m^k+A^n(\boldsymbol{a})_{-2}m^{k+1}+\ldots+A^n(\boldsymbol{a})_km^{-1}\right)\right)\end{aligned} \tag{15}$$

for any $t\in\mathbb{R}$, $n\in\mathbb{Z}$, $\boldsymbol{a}\in\Sigma$, and integer $k\le -1$.

Proof. We prove the $n = 1$ case first. From (9) and (11), it is clear that the lemma holds if $a_{-1} \in \{0, 1, \ldots, m-2\}$ or if $\boldsymbol{a} = (\ldots, m-1, m-1)$. If $a_{-1} = a_{-2} = \ldots = a_l = m-1$ and $a_{l-1} \in \{0, 1, \ldots, m-2\}$ for some $l \leq -1$, then from (10) the right hand side of (15) becomes

$$\exp\left(i2\pi t m^k\right) \quad \text{if } k \geq l,$$

or

$$\exp\left(i2\pi\left(tm^k + (a_{l-1}+1)m^{-1}\right)\right) \quad \text{if } k = l-1,$$

or

$$\exp\left(i2\pi\left(tm^k + (a_{l-1}+1)m^{k-l} + a_{l-2}m^{k-l+1} + a_{l-3}m^{k-l+2} + \ldots + a_k m^{-1}\right)\right) \quad \text{if } k \leq l-2.$$

It is easy to see that the lemma holds for cases $k \geq l$ or $k = l-1$. The case $k \leq l-2$ is also easy to verify:

$$\begin{aligned}
&(t+1)m^k + (m-1)m^k + (m-1)m^{k+1} + \ldots + (m-1)m^{k-l-1} + a_{l-1}m^{k-l} \\
&+a_{l-2}m^{k-l+1} + a_{l-3}m^{k-l+2} + \ldots + a_k m^{-1} \\
=\ &tm^k + 0 \cdot m^k + 0 \cdot m^{k+1} + \ldots + 0 \cdot m^{k-l-1} + (a_{l-1}+1)m^{k-l} \\
&+a_{l-2}m^{k-l+1} + a_{l-3}m^{k-l+2} + \ldots + a_k m^{-1}.
\end{aligned}$$

Having proved the $n = 1$ case, the $n = -1$ case can be proved by letting $t = t' - 1$ and $\boldsymbol{a} = A^{-1}(\boldsymbol{b})$. Thence, we can verify all $n \in \mathbb{Z}$ cases inductively. The verification is straightforward, and we omit it. ☐

Lemma 6.
(i) $A(\boldsymbol{a}e) = \boldsymbol{a}(e+1)$ *if* $e \in \{0, 1, \ldots, m-2\}$.
(ii) $A(\boldsymbol{a}(m-1)) = A(\boldsymbol{a})0$.
(iii) $A^{-1}(\boldsymbol{a}0) = A^{-1}(\boldsymbol{a})(m-1)$.
(iv) $A^{-1}(\boldsymbol{a}e) = \boldsymbol{a}(e-1)$ *if* $e \in \{1, 2, \ldots, m-1\}$.

Proof. Asserts (i) and (iv) are clearly true. The proof of (ii) and (iii) is straightforward by letting $\boldsymbol{a} = (\ldots, a_{-2}, a_{-1})$ and considering the definition of A and A^{-1} for three different cases:

$$(9) \Rightarrow \qquad \begin{aligned} A(\ldots, a_{-2}, a_{-1}, m-1) &= (\ldots, a_{-2}, a_{-1}+1, 0) \\ &= (\ldots, a_{-2}, a_{-1}+1)0; \end{aligned}$$

$$(10) \Rightarrow \quad A(\ldots, a_{l-2}, a_{l-1}, m-1, \ldots, \underset{\substack{\| \\ a_{-1}}}{m-1}, m-1) = (\ldots, a_{l-2}, a_{l-1}+1, 0, \ldots, 0, 0)$$

$$= (\ldots, a_{l-2}, a_{l-1}+1, 0, \ldots, 0)0;$$

$$(11) \Rightarrow \quad A(\ldots, m-1, \underset{\substack{\| \\ a_{-1}}}{m-1}, m-1) = (\ldots, 0, 0, 0)$$

$$= (\ldots, 0, 0)0.$$

And,

$$(12) \Rightarrow \quad A^{-1}(\ldots, a_{-2}, a_{-1}, 0) = (\ldots, a_{-2}, a_{-1}-1, m-1)$$

$$= (\ldots, a_{-2}, a_{-1}-1)(m-1);$$

$$(13) \Rightarrow \quad A(\ldots, a_{l-2}, a_{l-1}, 0, \ldots, \underset{\substack{\| \\ a_{-1}}}{0}, 0) = (\ldots, a_{l-2}, a_{l-1}-1, m-1, \ldots, m-1, m-1)$$

$$= (\ldots, a_{l-2}, a_{l-1}-1, m-1, \ldots, m-1)(m-1);$$

$$(14) \Rightarrow \quad A(\ldots, 0, \underset{\substack{\| \\ a_{-1}}}{0}, 0) = (\ldots, m-1, m-1, m-1)$$

$$= (\ldots, m-1, m-1)(m-1).$$

□

B. Homeomorphisms p and h

In this subsection, we equip spaces $\varprojlim(S^1, g)$ and $([0,1]\times\Sigma)/\sim$ with metrics, then show that p and h are homeomorphisms.

Note that the shift map σ is one-to-one and onto. Its inverse is given by

$$\sigma^{-1}((\ldots, s_{-2}, s_{-1}, s_0)) = (\ldots, s_{-3}, s_{-2}, s_{-1}).$$

The shift σ becomes a homeomorphism if the inverse limit space $\varprojlim(S^1, g)$ is equipped with the following metric

$$d(\boldsymbol{s}, \tilde{\boldsymbol{s}}) := \sum_{j\le 0} |s_l - \tilde{s}_l| 2^j, \tag{16}$$

where $|\cdot - \cdot|$ is the usual Euclidean distance in the complex plane.

Assume the bijectivity of p in Theorem 3, with the metric d on $\varprojlim(S^1, g)$, we can equip a metric δ on $([0,1] \times \Sigma)/\sim$ by

$$\delta((t, \boldsymbol{a}), (t', \boldsymbol{a}')) = d(p(t, \boldsymbol{a}), p(t', \boldsymbol{a}')).$$

Then p is an isometry and hence an homeomorphism.

Proposition 7. *The map $p : ([0,1] \times \Sigma)/\sim \to \varprojlim(S^1, g)$ defined in Theorem 3 is a homeomorphism.*

Proof. In view of the last paragraph, we only need to show p is a bijection.

Suppose there are $0 \le t' \le t \le 1$ and $\boldsymbol{a}', \boldsymbol{a} \in \Sigma$ such that $p((t, \boldsymbol{a})) = p((t', \boldsymbol{a}')) = (s_k)_{k \le 0}$. Since $s_0 = \exp(i2\pi t)$, it follows that t must equal t' or that $t' = 0$ and $t = 1$. If $t = t'$, then

$$s_{-1} = \exp(i2\pi(\frac{t}{m} + \frac{a_{-1}}{m})) = \exp(i2\pi(\frac{t'}{m} + \frac{a'_{-1}}{m})) = \exp(i2\pi(\frac{t}{m} + \frac{a'_{-1}}{m})).$$

Because $a_{-1},\ a'_{-1} \in \{0, 1, \dots, m-1\}$, we have $a_{-1} = a'_{-1}$. Assume $a_j = a'_j$ for $k < j \le -1$ and $k \le -2$. Then,

$$\begin{aligned} s_k &= \exp\left(i2\pi\left(tm^k + \sum_{j=k+1}^{-1} a_j m^{k-j-1}\right)\right)\exp(i2\pi\frac{a_k}{m}) \\ &= \exp\left(i2\pi\left(tm^k + \sum_{j=k+1}^{-1} a_j m^{k-j-1}\right)\right)\exp(i2\pi\frac{a'_k}{m}). \end{aligned}$$

This leads to $a_k = a'_k$. By induction, $(t, \boldsymbol{a}) = (t', \boldsymbol{a}')$. If $t = 1$ and $t' = 0$, let $s_k = p((1, \boldsymbol{a}))_k = p((0, \boldsymbol{a}'))_k$, $k \le 0$. Now, by Lemma 5,

$$\begin{aligned} p((1, \boldsymbol{a}))_k &= \exp\left(i2\pi\left((0+1)m^k + \sum_{j=k}^{-1} a_j m^{k-j-1}\right)\right) \\ &= \exp\left(i2\pi\left(0 \cdot m^k + \sum_{j=k}^{-1} A(\boldsymbol{a})_j m^{k-j-1}\right)\right) \\ &= p((0, A(\boldsymbol{a})))_k. \end{aligned}$$

Using the just proved $t = t'$ case, we have $\boldsymbol{a}' = A(\boldsymbol{a})$. Since $(1, \boldsymbol{a})$ is identified with $(0, A(\boldsymbol{a}))$, this proves that p is one-to-one.

To show p is onto, given $\boldsymbol{s} \in \varprojlim(S^1, g)$, we have to find $(t, \boldsymbol{a}) \in ([0,1] \times \Sigma)/\sim$ such that $p((t, \boldsymbol{a})) = \boldsymbol{s}$. By expressing $s_0 = \exp(i2\pi t)$ with $t \in [0, 1)$, we have the desired t. The relation $g(s_{-1}) = s_{-1}^m = s_0$ gives $s_{-1} = \exp(i2\pi(\frac{t}{m} + \frac{a_{-1}}{m}))$ for a unique $a_{-1} \in \{0, 1, \dots, m-1\}$.

Assume $s_k = \exp(i2\pi(tm^k + \sum_{j=k}^{-1} a_j m^{k-j-1}))$ with $a_j \in \{0, 1, \ldots, m-1\}$ for $k \leq j \leq -1$. Then $s_{k-1}^m = s_k$ leads to

$$s_{k-1} = \exp\left(i2\pi\left(tm^{k-1} + \sum_{j=k}^{-1} a_j m^{(k-1)-j-1} + a_{k-1}m^{-1}\right)\right)$$
$$= \exp\left(i2\pi\left(tm^{k-1} + \sum_{j=k-1}^{-1} a_j m^{(k-1)-j-1}\right)\right)$$

with a unique $a_{k-1} \in \{0, 1\}$. Inductively, we obtain an $\boldsymbol{a} = (\ldots, a_{-2,}a_{-1}) \in \Sigma$ with which $p((t, \boldsymbol{a})) = \boldsymbol{s}$. □

Having proved that p is a homeomorphism, the diagram (*) tells that τ is a composition of homeomorphisms, $\tau = p^{-1} \circ \sigma \circ p$, thus a homeomorphism. Indeed, τ is a bijection, with

$$\tau^{-1}(t, \boldsymbol{a}) = (\frac{t+e}{m}, \boldsymbol{w}) \qquad \text{if } \boldsymbol{a} = \boldsymbol{w}e.$$

Note that τ is well-defined since $(1, \boldsymbol{a}) \sim (0, A(\boldsymbol{a}))$, and the well-definedness can be verified by using Lemma 6.

Proposition 8. *The map $h : \varprojlim(S^1, g) \to \Lambda_q$ defined in Theorem 3 is a homeomorphism.*

Proof. We show h is a bijection first. The fact that h is onto is clear by (4) and (7). In our derivation of (4), to come in the Appendix, we shall see that there is a one-to-one correspondence between $([0,1] \times \Sigma)/\sim$ and Λ_q. Because we have proved that p is one-to-one, h is one-to-one. Nonetheless, we provide an algebraically explicit proof below.

Assume there exist distinct $\boldsymbol{s}, \boldsymbol{s}' \in \varprojlim(S^1, g)$ such that $h(\boldsymbol{s}) = h(\boldsymbol{s}')$. We get $s_0 = s_0'$, and by (8),

$$\sum_{k \leq -1} \epsilon^{-k-1} s_k = \sum_{k \leq -1} \epsilon^{-k-1} s_k'.$$

Since $\boldsymbol{s} \neq \boldsymbol{s}'$, there exists $k_0 \leq -1$ such that $s_{k_0} \neq s_{k_0}'$ and $s_k = s_k'$ for $k_0 < k \leq 0$. Thus the above equation becomes

$$\sum_{k \leq k_0} \epsilon^{-k-1} s_k = \sum_{k \leq k_0} \epsilon^{-k-1} s_k'.$$

Because the series is absolutely convergent, the equation can be written as

$$\epsilon^{-k_0-1}(s_{k_0} - s_{k_0}') = \sum_{k < k_0} \epsilon^{-k-1}(s_k' - s_k).$$

The absolute value gives rise to

$$\begin{aligned}\left|\epsilon^{-k_0-1}(s_{k_0}-s'_{k_0})\right| &= \left|\sum_{k<k_0}\epsilon^{-k-1}(s'_k-s_k)\right| \\ &\le \sum_{k<k_0}\epsilon^{-k-1}\left|s'_k-s_k\right| \\ &\le \sum_{k<k_0}\epsilon^{-k-1}\cdot 2 \\ &= 2\frac{\epsilon^{-(k_0-1)-1}}{1-\epsilon}. \end{aligned} \tag{17}$$

Since $g(s_{k_0}) = s_{k_0+1} = g(s'_{k_0})$ and $s_{k_0} \neq s'_{k_0}$, we have $2\sin\frac{\pi}{m} \le \left|s_{k_0}-s'_{k_0}\right| \le 2$. Hence (17) leads to

$$\frac{1}{2}\sin\frac{\pi}{m} \le \frac{1}{1+\sin\frac{\pi}{m}}\sin\frac{\pi}{m} \le \epsilon.$$

But this contradict (1), which requires $\epsilon < \frac{1}{2}\sin\frac{\pi}{m}$. We thus conclude $\boldsymbol{s} = \boldsymbol{s}'$.

To show that h is an homeomorphism, suppose $\boldsymbol{s}, \boldsymbol{s}' \in \varprojlim(S^1, g)$, $h(\boldsymbol{s}) = (s_0, z_0)$, and $h(\boldsymbol{s}') = (s'_0, z'_0)$. From (8), we see that (s'_0, z'_0) can be made arbitrarily close to (s_0, z_0) provided that $s'_k = s_k$ for $k \ge -n$ and n large enough. From (16), this means (s'_0, z'_0) can be made arbitrarily close to (s_0, z_0) in Λ_q provided that $\boldsymbol{s}'$ and $\boldsymbol{s}$ are close enough in $\varprojlim(S^1, g)$. Therefore, h is continuous. Consequently, h is a continuous bijection from the compact space $\varprojlim(S^1, g)$ into $S^1 \times D^2$, hence, a homeomorphism. □

III. PROOFS OF THEOREMS 1 AND 3

A. Proof of Theorem 1

From Theorem 2, a point $(\exp(i2\pi t), z)$ belongs to Λ_q if and only if

$$z = \sum_{k\le -1}\varepsilon^{-k-1}\frac{1}{2}\exp\left(i2\pi\left(tm^k + \sum_{j=k}^{-1}a_j m^{k-j-1}\right)\right) \tag{18}$$

for some $(a_k)_{k\le -1} \in \Sigma$. Let $z = \zeta(\varepsilon)$. Immediately, $\frac{d\zeta}{d\varepsilon}$ has to satisfy (2) and $\zeta(0)$ satisfy (3). Conversely, integration of (2) by ε gives rise to

$$\zeta(\varepsilon) = C + \sum_{k\le -2}\varepsilon^{-k-1}\frac{1}{2}\exp\left(i2\pi\left(tm^k + \sum_{j=k}^{-1}a_j m^{k-j-1}\right)\right),$$

where C is the integration constant. The initial condition (3) requires that $\zeta(0) = C = \frac{1}{2}\exp(i2\pi\frac{t+a_{-1}}{m})$. Thus $\zeta(\varepsilon)$ satisfies the right hand side of (18). □

B. Proof of Theorem 3

Propositions 7 and 8 show that p and h are homeomorphisms. To prove the theorem, it remains to verify that $p \circ \tau = \sigma \circ p$ and $h \circ \sigma = q \circ h$. Using (6), we get $p \circ \tau(t, \boldsymbol{a}) = (\ldots, s_k, \ldots, s_0)$, where

$$s_k = \begin{cases} \exp\left(i2\pi\left(mtm^k + \sum_{j=k}^{-2} a_{j+1} m^{k-j-1}\right)\right) & \text{if } k \le -2 \\ \exp(i2\pi t) & \text{if } k = -1 \\ \exp(i2\pi mt) & \text{if } k = 0 \end{cases}$$

On the other hand, $\sigma \circ p(t, \boldsymbol{a}) = (\ldots, s_k, \ldots, s_0)$, where

$$s_k = \begin{cases} \exp\left(i2\pi\left(tm^{k+1} + \sum_{j=k+1}^{-1} a_j m^{(k+1)-j-1}\right)\right) & \text{if } k \le -2 \\ \exp(i2\pi t) & \text{if } k = -1 \\ (\exp(i2\pi t))^m & \text{if } k = 0. \end{cases}$$

Hence, it is not difficult to see that $p \circ \tau$ is equal to $\sigma \circ p$. Now,

$$h \circ \sigma(\boldsymbol{s}) = (g(s_0), \sum_{k \le -1} \epsilon^{-k-1} \frac{1}{2} s_{k+1}).$$

In comparison with

$$\begin{aligned} q \circ h(\boldsymbol{s}) &= q((s_0, \sum_{k \le -1} \epsilon^{-k-1} \frac{1}{2} s_k)) \\ &= (g(s_0), (\sum_{k \le -1} \epsilon^{-k} \frac{1}{2} s_k) + \frac{1}{2} s_0) \\ &= (g(s_0), \sum_{k \le -1} \epsilon^{-k-1} \frac{1}{2} s_{k+1}), \end{aligned}$$

we get $h \circ \sigma = q \circ h$. □

ACKNOWLEDGEMENTS

Chen was partially supported by NSC 101-2115-M-001-010, NSC 102-2115-M-001-009, and the NCTS of Taiwan.

APPENDIX: PROOF OF THEOREM 2

For convenience, let $D(t) = \{\exp(i2\pi t)\} \times D^2$ be the section at 'angle' $t \in [0, 1)$ of the solid torus. What the map q does is sending the disk $D(t)$ into the disk $D(mt \bmod 1)$ with a scaling ε in radius and a translation $\frac{1}{2}\exp(i2\pi t)$ from the center of $D(mt \bmod 1)$.

$\Lambda_q \cap D(t)$ is the intersection of a sequence of nested disks. To label these disks, define $D_{a_{-1}} = q(D(\frac{t}{m} + \frac{a_{-1}}{m})) \subset D(t)$ for $a_{-1} \in \{0, 1, \ldots, m-1\}$. Note that $0 \le \frac{t}{m} + \frac{a_{-1}}{m} < 1$ if $0 \le t < 1$. For each a_{-1}, the disk $D_{a_{-1}}$ contains small disks $D_{a_{-2}a_{-1}}$, $a_{-2} \in \{0, 1, \ldots, m-1\}$ satisfying $q^{-2}(D_{a_{-2}a_{-1}}) = D(\frac{t}{m^2} + \frac{a_{-1}}{m^2} + \frac{a_{-2}}{m})$. Inductively, we get disks $D_{a_{-n}\cdots a_{-2}a_{-1}}$ of radius ε^n with $a_j \in \{0, 1, \ldots, m-1\}$ for every $-n \le j \le -1$. Note that $D_{a_{-n-1}\cdots a_{-2}a_{-1}} \subset D_{a_{-n}\cdots a_{-2}a_{-1}}$. Therefore, we have $\Lambda_q \cap D(t) = \bigcap_{n \le -1} \bigcup_{\boldsymbol{a} \in \Sigma} D_{a_{-n}\cdots a_{-1}}$.

Let $x \in \Lambda_q \cap D(t)$. From the above paragraph, there exists $\boldsymbol{a} \in \Sigma$ such that $x \in D_{a_{-n}} \cap D_{a_{-2}a_{-1}} \cap \cdots \cap D_{a_{-n}\cdots a_{-1}} \cap \cdots$. On the other hand, for given $\boldsymbol{a}$, the intersection $D_{a_{-1}} \cap D_{a_{-2}a_{-1}} \cap \cdots \cap D_{a_{-n}\cdots a_{-1}} \cap \cdots$ contains only one point. So there is a one-to-one correspondence between x and $\boldsymbol{a}$.

Suppose $x \in D_{a_{-1}} \cap D_{a_{-2}a_{-1}} \cap \cdots$. Let the center of $D_{a_{-n}\cdots a_{-1}}$ be $c_{a_{-n}\cdots a_{-1}}$. Since $x \in D_{a_{-n}\cdots a_{-1}}$ and the radius of $D_{a_{-n}\cdots a_{-1}}$ is ε^n,

$$|x - c_{a_{-n}\cdots a_{-1}}| < \varepsilon^n. \tag{19}$$

We have seen that $D_{a_{-1}}$ is the image of $D(\frac{t}{m} + \frac{a_{-1}}{m})$ under q, and that $D_{a_{-2}a_{-1}}$ is the image of $D(\frac{t}{m^2} + \frac{a_{-1}}{m^2} + \frac{a_{-2}}{m})$ under q^2. Similarly, by induction, we have that $D_{a_{-n}\cdots a_{-1}}$ is the image of $D(\frac{t}{m^n} + \sum_{i=1}^{n} \frac{a_{-i}}{m^{n+1-i}})$ under q^n. Let

$$t_k := \begin{cases} tm^k + \displaystyle\sum_{j=k}^{-1} a_j m^{k-j-1} & (k \le -1) \\ t & (k = 0). \end{cases}$$

Note $t_{k+1} = mt_k \bmod 1$ for $k \le -1$. The image of $D(t_k)$ is a disk in $D(t_{k+1})$ with center at $(t_{k+1}, \frac{1}{2}\exp(i2\pi t_k))$, and after $-k-1$ times iteration, $D(t_{k+1})$ becomes $D_{a_{k+1}\cdots a_{-1}}$ in $D(t)$. Therefore, the vector $\frac{1}{2}\exp(i2\pi t_k)$ from the center of $D(t_{k+1})$ pointing to the center of $q(D(t_k))$ is shrunk by the factor ε^{-k-1} to a vector from the center of $D_{a_k\cdots a_{-1}}$ to the center of $D_{a_{k+1}\cdots a_{-1}}$. So, $c_{a_k\cdots a_{-1}} - c_{a_{k+1}\cdots a_{-1}} = \varepsilon^{-k-1}\frac{1}{2}\exp(i2\pi t_k)$. Therefore,

$$
\begin{aligned}
c_{a_{-k}\cdots a_n} &= (c_{a_{-1}} - 0) + \sum_{j=k}^{-2}(c_{a_j\cdots a_{-1}} - c_{a_{j-1}\cdots a_{-1}}) \\
&= \frac{1}{2}\exp(i2\pi t_{-1}) + \sum_{j=k}^{-2}\varepsilon^{-k-1}\frac{1}{2}\exp(i2\pi t_k) \\
&= \sum_{j=k}^{-1}\varepsilon^{-k-1}\frac{1}{2}\exp(i2\pi t_k).
\end{aligned}
$$

Since $\varepsilon < \frac{1}{2}$, the series $\sum_{k\le -1}\varepsilon^{-k-1}\frac{1}{2}\exp(i2\pi t_k) = \lim_{k\to-\infty} c_{a_k\cdots a_{-1}}$ is absolutely convergent and then by (19)

$$
x = (\exp(i2\pi t), \sum_{k\le -1}\varepsilon^{-k-1}\frac{1}{2}\exp(i2\pi t_k))
$$

in Λ_q. Hence, we obtain formula (4). □

Remark 9. Suppose $(s,z) \in \Lambda_q$. Let $(s_k, z_k) = q^k(s,z)$ for every integer k. Because q is invertible, there are sequences $(t_k)_{k\in\mathbb{Z}} \in [0,1)^{\mathbb{Z}}$ and $(a_k)_{k\in\mathbb{Z}} \in \{0,1,\ldots,m-1\}^{\mathbb{Z}}$ such that $s = \exp(i2\pi t_0)$, $t_k + a_{k-1} = mt_{k-1}$, and $z_k = \varepsilon z_{k-1} + \frac{1}{2}\exp(i2\pi t_{k-1})$ for all k. Then,

$$
\begin{aligned}
z &= \varepsilon z_{-1} + \frac{1}{2}\exp\left(i2\pi\frac{t_0 + a_{-1}}{m}\right) \\
&= \varepsilon\left(\varepsilon z_{-2} + \frac{1}{2}\exp\left(i2\pi\frac{t_{-1} + a_{-2}}{m}\right)\right) + \frac{1}{2}\exp\left(i2\pi\frac{t_0 + a_{-1}}{m}\right) \\
&= \varepsilon^2 z_{-2} + \varepsilon\frac{1}{2}\exp\left(i2\pi\frac{t_0 + a_{-1}}{m^2} + \frac{a_{-2}}{m}\right) + \frac{1}{2}\exp\left(i2\pi\frac{t_0 + a_{-1}}{m}\right) \\
&= \vdots \\
&= \sum_{k\le -1}\varepsilon^{-k-1}\frac{1}{2}\exp\left(i2\pi\left(t_0 m^k + \sum_{j=k}^{-1} a_j m^{k-j-1}\right)\right),
\end{aligned}
$$

which coincides with the expression of z in Theorem 2.

REFERENCES

[1]R. L. Devaney, *An introduction to chaotic dynamical systems* (Westview Press, 1989).

[2]E. Hewitt, K. A. Ross, *Abstract harmonic analysis I* (Springer, 1963).

[3]A. Katok, B. Hasselblatt, *Introduction to the modern theory of dynamical systems* (Cambridge university press, 1996).

[4]S. P. Kuznetsov, "Example of a physical system with a hyperbolic attractor of the Smale-Williams type," Phys. Rev. Lett. **95**(14), 144101 (2005).

[5]S. P. Kuznetsov, I. R. Sataev, "Hyperbolic attractor in a system of coupled non-autonomous van der Pol oscillators: Numerical test for expanding and contracting cones," Phys. Lett. A **365**(1), 97-104 (2007).

[6]S. P. Kuznetsov, E. P. Seleznev, "A strange attractor of the Smale-Williams type in the chaotic dynamics of a physical system," J. Exp. Theor. Phys. **102**(2), 355-364 (2006).

[7]J. Kwapisz, "Homotopy and dynamics for homeomorphisms of solenoids and Knaster continua," Fund. Math. **168**(3), 251-278 (2001).

[8]C. Robinson, *Dynamical systems: stability, symbolic dynamics, and chaos* 2nd ed. (CRC Press, 1999).

[9]M. Shub, *Global stability of dynamical systems* (Springer-Verlag, 1987).

[10]S. Smale, "Differentiable dynamical systems," Bull. Amer. Math. Soc. **7**3, 747-817 (1967).

[11]F. Takens, "Multiplications in solenoids as hyperbolic attractors," Topology Appl. **1**52(3), 219-225 (2005).

[12]D. Wilczak, "Uniformly hyperbolic attractor of the Smale-Williams type for a Poincaré map in the Kuznetsov system," SIAM J. Applied Dynamical Systems **9**(4), 1263-1283 (2010).

[13]R. F. Williams, "One-dimensional non-wandering sets," Topology **6**, 473-487 (1967).

[14]R. F. Williams, "Expanding attractors," Inst. Hautes Études Sci. Publ. Math. **43**, 169-203 (1974).