SCHRÖDINGER OPERATORS WITH δ-INTERACTIONS SUPPORTED ON CONICAL SURFACES

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Abstract. We investigate the spectral properties of self-adjoint Schrödinger operators with attractive δ-interactions of constant strength $\alpha > 0$ supported on conical surfaces in $\mathbb{R}^3$. It is shown that the essential spectrum is given by $[-\alpha^2/4, +\infty)$ and that the discrete spectrum is infinite and accumulates to $-\alpha^2/4$. Furthermore, an asymptotic estimate of these eigenvalues is obtained.

1. Introduction

The purpose of this paper is to analyse the spectrum of the three-dimensional Schrödinger operator $-\Delta_{\alpha,\mathcal{C}_\theta}$ with an attractive δ-interaction of constant strength $\alpha > 0$ supported on the conical surface

$$\mathcal{C}_\theta := \Big\{(x,y,z) \in \mathbb{R}^3 \colon z := \cot(\theta)\sqrt{x^2+y^2}\Big\}, \qquad \theta \in (0, \pi/2).$$

The Schrödinger operator $-\Delta_{\alpha,\mathcal{C}_\theta}$ is defined via the first representation theorem [K, Theorem VI.2.1] as the unique self-adjoint operator in $L^2(\mathbb{R}^3)$ which is associated to the closed, densely defined, semibounded quadratic form

$$\mathfrak{a}_{\alpha,\mathcal{C}_\theta}[\psi] = \|\nabla\psi\|^2_{L^2(\mathbb{R}^3;\mathbb{C}^3)} - \alpha \int_{\mathcal{C}_\theta} |\psi|^2 \,\mathrm{d}\sigma \qquad \mathrm{dom}\,\mathfrak{a}_{\alpha,\mathcal{C}_\theta} = H^1(\mathbb{R}^3); \tag{1.1}$$

cf. [BEL13, BEKS94]. In a short form the main result of this note is the following theorem.

Theorem. *For any $\theta \in (0, \pi/2)$ and $\alpha > 0$ the essential spectrum of the operator $-\Delta_{\alpha,\mathcal{C}_\theta}$ is $[-\alpha^2/4, +\infty)$, the discrete spectrum is infinite and accumulates to $-\alpha^2/4$.*

In addition, we obtain an asymptotic estimate of the eigenvalues of $-\Delta_{\alpha,\mathcal{C}_\theta}$ lying below $-\alpha^2/4$, and the results also extend to local deformations of the conical surface $\mathcal{C}_\theta$, see Theorem 3.2 and Theorem 3.3 for details. The proof of our main result is based on standard techniques in spectral theory of self-adjoint operators: we construct singular sequences and use Neumann bracketing in the spirit of [EN03] to show the assertion on the essential spectrum; for the infiniteness of the discrete spectrum we employ variational principles. The same approach was applied in [S70] in the context of Schrödinger operators with slowly decaying negative regular potentials, see also [RS-IV, §XIII.3]. Similar arguments were also used in [DEK01, ET10] for the closely related question of infiniteness of the discrete spectrum for the Dirichlet Laplacian in a conical layer, see also [CEK04, J13, KV08, LL07, LR12] for further progress in this problem. We also point out [BEW09, DR13, EK02] for related spectral problems for Schrödinger operators with δ-potentials.

2. Essential spectrum of $-\Delta_{\alpha,\mathcal{C}_\theta}$

In this section we show that the essential spectrum of the operator $-\Delta_{\alpha,\mathcal{C}_\theta}$ is given by $[-\alpha^2/4,+\infty)$. The proof of the inclusion $\sigma_{\rm ess}(-\Delta_{\alpha,\mathcal{C}_\theta}) \supseteq [-\alpha^2/4,+\infty)$ makes use of singular sequences and for the other inclusion a specially chosen Neumann bracketing is used. A similar type of argument was also used in [BEL13, EN03] for δ and δ'-interactions on broken lines in the two-dimensional setting. For completeness we mention that the theorem (and its proof) below is also valid for $\theta = \pi/2$, in which case the conical surface is a half-plane, and the result is well-known.

Theorem 2.1. *Let $-\Delta_{\alpha,\mathcal{C}_\theta}$ be the self-adjoint operator in $L^2(\mathbb{R}^3)$ associated to the form* (1.1) *and let $\alpha > 0$ and $\theta \in (0,\pi/2)$. Then*

$$\sigma_{\rm ess}(-\Delta_{\alpha,\mathcal{C}_\theta}) = [-\alpha^2/4,+\infty).$$

Proof. Step 1. We verify the inclusion $\sigma_{\rm ess}(-\Delta_{\alpha,\mathcal{C}}) \supseteq [-\alpha^2/4,+\infty)$ by constructing singular sequences for the operator $-\Delta_{\alpha,\mathcal{C}_\theta}$ for every point of the interval $[-\alpha^2/4,+\infty)$. Let us start by fixing a function $\chi_1 \in C_0^\infty(1,2)$ such that

$$\|\chi_1\|_{L^2(1,2)} = 1, \tag{2.1}$$

and a function $\chi_2 \in C_0^\infty(-\varepsilon,\varepsilon)$ with some fixed $\varepsilon \in (0,\tan\theta)$, which satisfies

$$0 \le \chi_2 \le 1 \quad \text{and} \quad \chi_2(t) = 1 \text{ for } |t| < \varepsilon/2. \tag{2.2}$$

Define for all $p \in \mathbb{R}$ and $n \in \mathbb{N}$ the functions $\omega_{n,p}\colon \mathbb{R}^2_+ \to \mathbb{C}$ as

$$\omega_{n,p}(s,t) := \tfrac{1}{\sqrt{n}}\Big(\chi_1(\tfrac{s}{n})\exp(\mathrm{i}ps)\Big)\Big(\chi_2(\tfrac{t}{n})\exp(-\tfrac{\alpha}{2}|t|)\Big) \in C(\mathbb{R}^2_+)$$

in the coordinate system (s,t) in Figure 2.1. Here $\mathbb{R}^2_+$ denotes open right half-plane $\{(r,z) \in \mathbb{R}^2 : r > 0\}$.

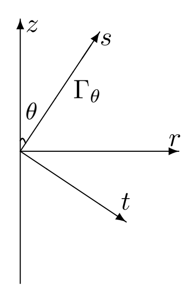

Figure 2.1. The right half-space $\mathbb{R}^2_+$ with the coordinate system (r,z). The ray Γ_θ emerges from the origin and constitutes the angle $\theta \in (0,\pi/2)$ with the z-axis. The coordinate system (s,t) is associated with Γ_θ.

Note that because of the choice $\varepsilon \in (0,\tan\theta)$ we have $\operatorname{supp}\omega_{n,p} \subset \mathbb{R}^2_+$ for all $n \in \mathbb{N}$ and, moreover, the distances between the z-axis and the supports of $\omega_{n,p}$ satisfy

$$\rho_n := \inf\{r\colon (r,z) \in \operatorname{supp}\omega_{n,p}\} \to +\infty, \qquad n \to \infty. \tag{2.3}$$

By dominated convergence, using (2.1) and (2.2), we get

$$\begin{aligned}\|\omega_{n,p}\|^2_{L^2(\mathbb{R}^2_+)} &= \left(\frac{1}{n}\int_n^{2n}\left|\chi_1(\tfrac{s}{n})e^{\mathrm{i}ps}\right|^2\mathrm{d}s\right)\left(\int_{-\varepsilon n}^{\varepsilon n}\left|\chi_2(\tfrac{t}{n})\right|^2 e^{-\alpha|t|}\mathrm{d}t\right)\\ &= \int_{-\varepsilon n}^{\varepsilon n}\left|\chi_2(\tfrac{t}{n})\right|^2 e^{-\alpha|t|}\mathrm{d}t \to \int_{-\infty}^{\infty} e^{-\alpha|t|}\mathrm{d}t = \frac{2}{\alpha}, \quad n\to\infty.\end{aligned} \tag{2.4}$$

We denote by $\omega_{n,p,\pm}$ the restrictions of $\omega_{n,p}$ onto the open subsets

$$S_+ = \{(r,z)\in\mathbb{R}^2_+ : z > r\cot\theta\} \quad \text{and} \quad S_- = \{(r,z)\in\mathbb{R}^2_+ : z < r\cot\theta\}$$

of $\mathbb{R}^2_+$. The partial derivatives of $\omega_{n,p,\pm}$ with respect to s and t are given by

$$\begin{aligned}\partial_s\omega_{n,p,\pm} &= \tfrac{1}{\sqrt{n}}\left(\tfrac{1}{n}\chi_1'(\tfrac{s}{n})e^{\mathrm{i}ps} + \mathrm{i}p\chi_1(\tfrac{s}{n})e^{\mathrm{i}ps}\right)\left(\chi_2(\tfrac{t}{n})e^{\pm\frac{\alpha}{2}t}\right),\\ \partial_t\omega_{n,p,\pm} &= \tfrac{1}{\sqrt{n}}\left(\chi_1(\tfrac{s}{n})e^{\mathrm{i}ps}\right)\left(\tfrac{1}{n}\chi_2'(\tfrac{t}{n})e^{\pm\frac{\alpha}{2}t} \pm \tfrac{\alpha}{2}\chi_2(\tfrac{t}{n})e^{\pm\frac{\alpha}{2}t}\right).\end{aligned}$$

Similarly as in (2.4), using dominated convergence, we get

$$\|\nabla\omega_{n,p}\|^2_{L^2(\mathbb{R}^2_+;\mathbb{C}^2)} = \int_{\mathbb{R}^2_+}\left(|\partial_s\omega_{n,p}|^2 + |\partial_t\omega_{n,p}|^2\right)\mathrm{d}s\mathrm{d}t \to \left(p^2 + \frac{\alpha^2}{4}\right)\frac{2}{\alpha}, \quad n\to\infty. \tag{2.5}$$

Let us define the sequence of functions $\psi_{n,p}\colon \mathbb{R}^3 \to \mathbb{C}$ as

$$\psi_{n,p}(r,\varphi,z) := \frac{\omega_{n,p}(r,z)}{\sqrt{2\pi r}}, \qquad n\in\mathbb{N}, \tag{2.6}$$

where the functions $\omega_{n,p}\colon \mathbb{R}^2_+ \to \mathbb{C}$ are interpreted as rotationally invariant functions on $\mathbb{R}^3$ in the cylindrical coordinate system (r,φ,z). The hypersurface $\mathcal{C}_\theta$ separates the Euclidean space $\mathbb{R}^3$ into two unbounded Lipschitz domains Ω_+ and Ω_-, where

$$\begin{aligned}\Omega_+ &= \left\{(x,y,z)\in\mathbb{R}^3 : z > \cot(\theta)\sqrt{x^2+y^2}\right\},\\ \Omega_- &= \left\{(x,y,z)\in\mathbb{R}^3 : z < \cot(\theta)\sqrt{x^2+y^2}\right\}.\end{aligned}$$

We use the notation $\psi_{n,p,\pm} := \psi_{n,p}|_{\Omega_\pm}$. Then $\psi_{n,p,\pm}\in C^\infty(\Omega_\pm)$ and from (2.4) we obtain

$$\|\psi_{n,p}\|^2_{L^2(\mathbb{R}^3)} = \|\omega_{n,p}\|^2_{L^2(\mathbb{R}^2_+)} \to \frac{2}{\alpha}, \qquad n\to\infty. \tag{2.7}$$

We claim that $\psi_{n,p}\in\mathrm{dom}\,(-\Delta_{\alpha,\mathcal{C}_\theta})$. For this it remains to check that the boundary conditions

$$\psi_{n,p,+}|_\Sigma = \psi_{n,p,-}|_\Sigma \quad \text{and} \quad \partial_{\nu_+}\psi_{n,p,+}|_\Sigma + \partial_{\nu_-}\psi_{n,p,-}|_\Sigma = \alpha\psi_{n,p}|_\Sigma \tag{2.8}$$

are satisfied; cf. [BEL13, Theorem 3.3 (i)]. In fact, by the definition of $\omega_{n,p}$ we have $\omega_{n,p,+}|_\Sigma = \omega_{n,p,-}|_\Sigma$, which implies that the first condition in (2.8) holds. Furthermore, one computes

$$\partial_{\nu_+}\omega_{n,p,+}|_\Sigma + \partial_{\nu_-}\omega_{n,p,-}|_\Sigma = \alpha\tfrac{1}{\sqrt{n}}\left(\chi_1(\tfrac{s}{n})\exp(\mathrm{i}ps)\right) = \alpha\omega_{n,p}|_\Sigma. \tag{2.9}$$

The gradient of $\psi_{n,p,\pm}$ can be expressed as

$$\nabla\psi_{n,p,\pm} = \tfrac{1}{\sqrt{2\pi r}}\nabla\omega_{n,p,\pm} + \omega_{n,p,\pm}\nabla\left(\tfrac{1}{\sqrt{2\pi r}}\right),$$

where ∇ acts on the functions $(r,\varphi,z)\mapsto\omega_{n,p,\pm}(r,z)$ and $(r,\varphi,z)\mapsto\frac{1}{\sqrt{2\pi r}}$. Hence, we obtain

$$\begin{aligned}\partial_{\nu_+}\psi_{n,p,+}|_\Sigma+\partial_{\nu_-}\psi_{n,p,-}|_\Sigma &= \big(\tfrac{1}{\sqrt{2\pi r}}\big|_\Sigma\big)\big(\partial_{\nu_+}\omega_{n,p,+}|_\Sigma+\partial_{\nu_-}\omega_{n,p,-}|_\Sigma\big)\\ &\quad+\big(\omega_{n,p}|_\Sigma\big)\big(\partial_{\nu_+}\big(\tfrac{1}{\sqrt{2\pi r}}\big)\big|_\Sigma+\partial_{\nu_-}\big(\tfrac{1}{\sqrt{2\pi r}}\big)\big|_\Sigma\big)\\ &= \big(\tfrac{1}{\sqrt{2\pi r}}\big|_\Sigma\big)\alpha\big(\omega_{n,p}|_\Sigma\big)=\alpha\psi_{n,p}|_\Sigma,\end{aligned}$$

where (2.9) was used in the second equality. Thus we have verified (2.8) and therefore $\psi_{n,p}\in\operatorname{dom}(-\Delta_{\alpha,\mathcal{C}_\theta})$. Moreover, according to [BEL13, Theorem 3.3 (i)] we also have

$$-\Delta_{\alpha,\mathcal{C}_\theta}\psi_{n,p}=(-\Delta\psi_{n,p,+})\oplus(-\Delta\psi_{n,p,-}). \tag{2.10}$$

Using the expression for the three-dimensional Laplacian in cylindrical coordinates we find

$$-\Delta\psi_{n,p,\pm}=-\tfrac{1}{r}\partial_r(r\partial_r\psi_{n,p,\pm})-\partial_z^2\psi_{n,p,\pm},$$

where the angular term is absent since the functions $\psi_{n,p,\pm}$ do not depend on φ. The above expression can be rewritten as

$$-\Delta\psi_{n,p,\pm}=-\partial_r^2\psi_{n,p,\pm}-\partial_z^2\psi_{n,p,\pm}-\tfrac{1}{r}(\partial_r\psi_{n,p,\pm}). \tag{2.11}$$

Next we compute the first and second order partial derivatives of $\psi_{n,p,\pm}$ with respect to r:

$$\begin{aligned}\partial_r\psi_{n,p,\pm}&=\frac{\partial_r\omega_{n,p,\pm}}{\sqrt{2\pi r}}-\frac{\omega_{n,p,\pm}}{2\sqrt{2\pi}r^{3/2}},\\ \partial_r^2\psi_{n,p,\pm}&=\frac{\partial_r^2\omega_{n,p,\pm}}{\sqrt{2\pi r}}-\frac{\partial_r\omega_{n,p,\pm}}{\sqrt{2\pi}r^{3/2}}+\frac{3}{4}\frac{\omega_{n,p,\pm}}{\sqrt{2\pi}r^{5/2}}.\end{aligned} \tag{2.12}$$

The last two summands in the expression for $\partial_r^2\psi_{n,p,\pm}$ can be estimated in L^2-norm as

$$\begin{aligned}\left\|\frac{\partial_r\omega_{n,p,\pm}}{\sqrt{2\pi}r^{3/2}}\right\|^2_{L^2(\mathbb{R}^3)}&\le\frac{1}{\rho_n^2}\|\nabla\omega_{n,p}\|^2_{L^2(\mathbb{R}^2_+;\mathbb{C}^2)}\to 0,\qquad n\to\infty,\\ \frac{9}{16}\left\|\frac{\omega_{n,p,\pm}}{\sqrt{2\pi}r^{5/2}}\right\|^2_{L^2(\mathbb{R}^3)}&\le\frac{9}{16\rho_n^4}\|\omega_{n,p}\|^2_{L^2(\mathbb{R}^2_+)}\to 0,\qquad n\to\infty,\end{aligned} \tag{2.13}$$

where we have used (2.3), (2.4) and (2.5). The second order partial derivatives of $\psi_{n,p,\pm}$ with respect to z are

$$\partial_z^2\psi_{n,p,\pm}=\frac{\partial_z^2\omega_{n,p,\pm}}{\sqrt{2\pi r}}. \tag{2.14}$$

Using (2.12), (2.13), (2.14) and the invariance of the Laplacian under rotation of the coordinate system we obtain that

$$-\partial_r^2\psi_{n,p,\pm}-\partial_z^2\psi_{n,p,\pm}=-\frac{1}{\sqrt{2\pi r}}\big(\partial_s^2\omega_{n,p,\pm}+\partial_t^2\omega_{n,p,\pm}\big)+o(1),\quad n\to\infty; \tag{2.15}$$

here and in the following we understand $o(1)$ in the strong sense with respect to the corresponding L^2-norm. With the help of (2.12) the norm of the last summand on the right hand side in (2.11) can be estimated as

$$\left\| \frac{\partial_r \psi_{n,p,\pm}}{r} \right\|^2_{L^2(\mathbb{R}^3)} \leq \left\| \frac{\partial_r \omega_{n,p,\pm}}{\sqrt{2\pi} r^{3/2}} \right\|^2_{L^2(\mathbb{R}^3)} + \left\| \frac{\omega_{n,p,\pm}}{2\sqrt{2\pi} r^{5/2}} \right\|^2_{L^2(\mathbb{R}^3)},$$

and from (2.13) we conclude

$$\left\| \frac{\partial_r \psi_{n,p,\pm}}{r} \right\|^2_{L^2(\mathbb{R}^3)} = o(1), \qquad n \to \infty.$$

From (2.11), the latter result and (2.15) we obtain

$$-\Delta \psi_{n,p,\pm} = -\frac{1}{\sqrt{2\pi r}} \big(\partial_s^2 \omega_{n,p,\pm} + \partial_t^2 \omega_{n,p,\pm} \big) + o(1), \quad n \to \infty. \tag{2.16}$$

Again using dominated convergence we compute

$$\begin{aligned} \partial_s^2 \omega_{n,p,\pm} &= \tfrac{1}{\sqrt{n}} \Big(\chi_2(\tfrac{t}{n}) e^{\pm \frac{\alpha}{2} t} \Big) \Big(\tfrac{1}{n^2} \chi_1''(\tfrac{s}{n}) e^{ips} + \tfrac{2ip}{n} \chi_1'(\tfrac{s}{n}) e^{ips} - p^2 \chi_1(\tfrac{s}{n}) e^{ips} \Big) \\ &= -p^2 \omega_{n,p,\pm} + o(1), \qquad n \to \infty, \end{aligned} \tag{2.17}$$

and

$$\begin{aligned} \partial_t^2 \omega_{n,p,\pm} &= \tfrac{1}{\sqrt{n}} \Big(\chi_1(\tfrac{s}{n}) e^{ips} \Big) \Big(\tfrac{1}{n^2} \chi_2''(\tfrac{t}{n}) e^{\pm \frac{\alpha}{2} t} \pm \tfrac{\alpha}{n} \chi_2'(\tfrac{t}{n}) e^{\pm \frac{\alpha}{2} t} + \tfrac{\alpha^2}{4} \chi_2(\tfrac{t}{n}) e^{\pm \frac{\alpha}{2} t} \Big) \\ &= \tfrac{\alpha^2}{4} \omega_{n,p,\pm} + o(1), \qquad n \to \infty. \end{aligned} \tag{2.18}$$

Finally, employing (2.10), (2.16), the definition of $\psi_{n,p}$ in (2.6) and (2.17), (2.18) we arrive at

$$-\Delta_{\alpha,\mathcal{C}_\theta} \psi_{n,p} = \Big(-\frac{\alpha^2}{4} + p^2 \Big) \psi_{n,p} + o(1), \qquad n \to \infty. \tag{2.19}$$

Since the supports of $\psi_{2^k,p}$ and $\psi_{2^{k'},p}$, $k \neq k'$, are disjoint the sequence $\{\psi_{2^k,p}\}_k$ converges weakly to zero. Moreover, by (2.7) we have $\liminf \|\psi_{2^k,p}\|_{L^2(\mathbb{R}^3)} > 0$ and hence (2.19) implies that $\{\psi_{2^k,p}\}_k$ is a singular sequence for the operator $-\Delta_{\alpha,\mathcal{C}_\theta}$ corresponding to the point $-\alpha^2/4 + p^2$. Therefore, $-\alpha^2/4 + p^2 \in \sigma_{\rm ess}(-\Delta_{\alpha,\mathcal{C}_\theta})$ for all $p \in \mathbb{R}$ (see, e.g. [BS87, Theorem 9.1.2] or [S, Proposition 8.11]) and it follows that $[-\alpha^2/4, +\infty) \subseteq \sigma_{\rm ess}(-\Delta_{\alpha,\mathcal{C}_\theta})$.

Step 2. In this step we show the inclusion $\sigma_{\rm ess}(-\Delta_{\alpha,\mathcal{C}_\theta}) \subseteq [-\alpha^2/4, +\infty)$ using form decomposition methods. For sufficiently large $n \in \mathbb{N}$ we define three subsets of the closed half-plane $\overline{\mathbb{R}^2_+} := \{(r,z) \in \mathbb{R}^2 \colon r \geq 0,\, z \in \mathbb{R}\}$

$$\begin{aligned} \pi_n^1 &:= \{(r(s,t), z(s,t)) \in \overline{\mathbb{R}^2_+} \colon s > n, |t| < \sqrt{n}\} \subset \overline{\mathbb{R}^2_+}, \\ \pi_n^2 &:= \big\{(r(s,t), z(s,t)) \in \overline{\mathbb{R}^2_+} \colon s < n, |t| < \sqrt{n}\big\} \subset \overline{\mathbb{R}^2_+}, \\ \pi_n^3 &:= \big\{(r(s,t), z(s,t)) \in \overline{\mathbb{R}^2_+} \colon |t| > \sqrt{n}\big\} \subset \overline{\mathbb{R}^2_+}, \end{aligned}$$

as shown in Figure 2.2.

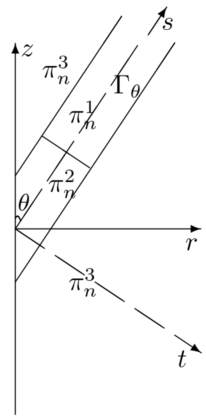

FIGURE 2.2. The subsets π_n^1, π_n^2 and π_n^3 of the closed half-plane $\overline{\mathbb{R}^2_+}$.

The ray Γ_θ, which emerges from the origin and constitutes the angle θ with z-axis, is decomposed into

$$\Gamma^1_{\theta,n} := \{((r(s,t), z(s,t)) \in \Gamma_\theta \colon s > n\},$$
$$\Gamma^2_{\theta,n} := \{((r(s,t), z(s,t)) \in \Gamma_\theta \colon s < n\}.$$

The splitting $\{\pi_n^k\}_{k=1}^3$ of $\overline{\mathbb{R}^2_+}$ induces the splitting of $\mathbb{R}^3$ into three domains

$$\Omega_n^k := \big\{(r,\varphi,z) \colon (r,z) \in \pi_n^k,\ \varphi \in [0,2\pi)\big\} \subset \mathbb{R}^3, \qquad k = 1,2,3,$$

and the splitting of the conical surface $\mathcal{C}_\theta$ into two parts

$$\mathcal{C}^1_{\theta,n} := \{(r,\varphi,z) \colon (r,z) \in \Gamma^1_{\theta,n},\ \varphi \in [0,2\pi)\} \subset \mathcal{C}_\theta,$$
$$\mathcal{C}^2_{\theta,n} := \{(r,\varphi,z) \colon (r,z) \in \Gamma^2_{\theta,n},\ \varphi \in [0,2\pi)\} \subset \mathcal{C}_\theta.$$

We agree to denote the restriction of $\psi \in L^2(\mathbb{R}^3)$ onto Ω_n^k with $k = 1,2,3$ by ψ_k.

Consider the quadratic form

$$\mathfrak{a}_{\alpha,\mathcal{C}_\theta,n}[\psi] := \sum_{k=1}^3 \|\nabla\psi_k\|^2_{L^2(\Omega_n^k;\mathbb{C}^3)} - \alpha\|\psi_1|_{\mathcal{C}^1_{\theta,n}}\|^2_{L^2(\mathcal{C}^1_{\theta,n})} - \alpha\|\psi_2|_{\mathcal{C}^2_{\theta,n}}\|^2_{L^2(\mathcal{C}^2_{\theta,n})},$$
$$\operatorname{dom}\mathfrak{a}_{\alpha,\mathcal{C},n} = \bigoplus_{k=1}^3 H^1(\Omega_n^k).$$

As in the proof of [BEL13, Proposition 3.1] one verifies that the form $\mathfrak{a}_{\alpha,\mathcal{C}_\theta,n}$ is closed, densely defined, symmetric and semibounded from below. Hence $\mathfrak{a}_{\alpha,\mathcal{C}_\theta,n}$ induces a self-adjoint operator $-\Delta_{\alpha,\mathcal{C}_\theta,n}$ in $L^2(\mathbb{R}^3)$ via the first representation theorem [K, Theorem VI.2.1]. The operator $-\Delta_{\alpha,\mathcal{C}_\theta,n}$ can be decomposed into an orthogonal sum $\oplus_{k=1}^3 H_{n,k}$ of self-adjoint operators $H_{n,k}$ in $L^2(\Omega_n^k)$ with respect to the orthogonal decomposition $L^2(\mathbb{R}^3) = \oplus_{k=1}^3 L^2(\Omega_n^k)$, where $H_{n,1}$ and $H_{n,2}$ correspond to the quadratic forms

$$\mathfrak{a}_{n,1}[\psi_1] = \|\nabla\psi_1\|^2_{L^2(\Omega_n^1;\mathbb{C}^3)} - \alpha\|\psi_1|_{\mathcal{C}^1_{\theta,n}}\|^2_{L^2(\mathcal{C}^1_{\theta,n})}, \qquad \operatorname{dom}\mathfrak{a}_{n,1} = H^1(\Omega_n^1),$$
$$\mathfrak{a}_{n,2}[\psi_2] = \|\nabla\psi_2\|^2_{L^2(\Omega_n^2;\mathbb{C}^3)} - \alpha\|\psi_2|_{\mathcal{C}^2_{\theta,n}}\|^2_{L^2(\mathcal{C}^2_{\theta,n})}, \qquad \operatorname{dom}\mathfrak{a}_{n,2} = H^1(\Omega_n^2),$$

respectively, and $H_{n,3}$ corresponds to the quadratic form $\mathfrak{a}_{n,3}[\psi_3] = \|\nabla\psi_3\|^2_{L^2(\Omega_n^3;\mathbb{C}^3)}$, $\operatorname{dom}\mathfrak{a}_{n,3} = H^1(\Omega_n^3)$.

Let us first estimate the spectrum of $H_{n,1}$. For this note that $C^\infty(\Omega_n^1)\cap H^1(\Omega_n^1)$ is a core of $\mathfrak{a}_{n,1}$ and thus it suffices to use functions from this set in the estimates below (see, e.g. [D95, Theorem 4.5.3]). For any $\psi_1 \in C^\infty(\Omega_n^1)\cap H^1(\Omega_n^1)$ normalized as $\|\psi_1\|_{L^2(\Omega_n^1)} = 1$ we obtain

$$\mathfrak{a}_{n,1}[\psi_1] \geq \int_0^{2\pi}\left(\int_n^{+\infty}\int_{-\sqrt{n}}^{\sqrt{n}} r(s,t)|\partial_t\psi_1(s,t,\varphi)|^2\mathrm{d}t\mathrm{d}s \right. \\ \left. - \alpha\int_n^{+\infty} r(s,0)|\psi_1(s,0,\varphi)|^2\mathrm{d}s\right)\mathrm{d}\varphi,$$

where we have used the form of the gradient in cylindrical coordinates and the invariance of the gradient with respect to rotations of the coordinate system, and the nonnegative terms corresponding to the partial derivatives of ψ_1 with respect to φ and s where estimated from below by zero. Note that for simple geometric reasons we have $r(s,t) \geq r(s,-\sqrt{n})$ for all $(s,t)\in\pi_n^1$. Using this observation we get

(2.20)
$$\mathfrak{a}_{n,1}[\psi_1] \geq \int_0^{2\pi}\int_n^{+\infty} r(s,-\sqrt{n})\left(\int_{-\sqrt{n}}^{\sqrt{n}}|\partial_t\psi_1(s,t,\varphi)|^2\mathrm{d}t \right. \\ \left. - \tfrac{\alpha r(s,0)}{r(s,-\sqrt{n})}|\psi_1(s,0,\varphi)|^2\right)\mathrm{d}s\mathrm{d}\varphi.$$

Consider the closed, densely defined, symmetric and semibounded form

$$\mathfrak{b}[h] = \int_{-\sqrt{n}}^{\sqrt{n}}|h'(t)|^2\mathrm{d}t - \beta|h(0)|^2, \qquad \operatorname{dom}\mathfrak{b} = H^1((-\sqrt{n},\sqrt{n})),$$

and denote by $\mu(\beta, 2\sqrt{n}) < 0$ the lower bound of the spectrum of the associated 1-D Schrödinger operator on the interval $(-\sqrt{n},\sqrt{n})$ with Neumann boundary conditions at the endpoints and attractive δ-interaction of strength $\beta > 0$ located at 0. Then

$$\mathfrak{b}[h] \geq \mu(\beta, 2\sqrt{n})\int_{-\sqrt{n}}^{\sqrt{n}}|h(t)|^2\mathrm{d}t$$

holds for all $h \in H^1((-\sqrt{n},\sqrt{n}))$ and hence (2.20) can be further estimated as

(2.21)
$$\mathfrak{a}_{n,1}[\psi_1] \geq \int_0^{2\pi}\int_n^{+\infty}\mu\Big(\tfrac{\alpha r(s,0)}{r(s,-\sqrt{n})}, 2\sqrt{n}\Big)\int_{-\sqrt{n}}^{\sqrt{n}} r(s,-\sqrt{n})|\psi_1(s,t,\varphi)|^2\mathrm{d}t\mathrm{d}s\mathrm{d}\varphi.$$

By the definition of π_n^1 one has

$$r(s,-\sqrt{n}) = r(s,t)\Big(1+\mathcal{O}(\tfrac{1}{\sqrt{n}})\Big), \qquad n\to\infty, \tag{2.22}$$

for $(s,t)\in\pi_n^1$, where the remainder is uniform in s. Hence, we obtain from (2.21) and (2.22)

$$\mathfrak{a}_{n,1}[\psi_1] \geq \mu\Big(\alpha\big(1+\mathcal{O}(\tfrac{1}{\sqrt{n}})\big), 2\sqrt{n}\Big)\Big(1+\mathcal{O}(\tfrac{1}{\sqrt{n}})\Big), \qquad n\to\infty, \tag{2.23}$$

where we used that

$$\int_0^{2\pi}\int_n^{+\infty}\int_{-\sqrt{n}}^{\sqrt{n}} r(s,t)|\psi_1(s,t,\varphi)|^2 \mathrm{d}t\mathrm{d}s\mathrm{d}\varphi = \|\psi_1\|^2_{L^2(\Omega_n^1)} = 1.$$

According to [EY02, Proposition 2.5] the following estimate

$$\mu(\beta, 2\sqrt{n}) \geq -\tfrac{\beta^2}{4} - C\beta^2\exp(-\tfrac{1}{2}\beta\sqrt{n})$$

holds with some constant $C > 0$ and n sufficiently large. Hence,

$$\mu\Big(\alpha\big(1 + O(\tfrac{1}{\sqrt{n}})\big), 2\sqrt{n}\Big) \geq -\tfrac{\alpha^2}{4} + \mathcal{O}(\tfrac{1}{\sqrt{n}}), \qquad n\to\infty.$$

Plugging the above estimate into (2.23) we arrive at

$$\mathfrak{a}_{n,1}[\psi_1] \geq -\tfrac{\alpha^2}{4} + \mathcal{O}(\tfrac{1}{\sqrt{n}}), \qquad n\to\infty.$$

Hence, for any $\varepsilon > 0$ there exists a sufficiently large n for which

$$\inf\sigma(H_{n,1}) \geq -\tfrac{\alpha^2}{4} - \varepsilon. \tag{2.24}$$

As $H^1(\Omega_n^2)$ is compactly embedded into $L^2(\Omega_n^2)$ the essential spectrum of $H_{n,2}$ is empty. The operator $H_{n,3}$ is non-negative and hence $\sigma(H_{n,3}) \subseteq [0,+\infty)$. Due to the orthogonal decomposition $-\Delta_{\alpha,\mathcal{C}_\theta,n} = \oplus_{k=1}^3 H_{n,k}$ the property (2.24) implies that for any $\varepsilon > 0$ there exists a sufficiently large n for which

$$\inf\sigma_{\mathrm{ess}}(-\Delta_{\alpha,\mathcal{C}_\theta,n}) \geq -\tfrac{\alpha^2}{4} - \varepsilon. \tag{2.25}$$

Finally, we apply a Neumann bracketing argument. Notice that the ordering $\mathfrak{a}_{\alpha,\mathcal{C}_\theta,n} \leq \mathfrak{a}_{\alpha,\mathcal{C}_\theta}$ holds in the sense of quadratic forms; cf. [K, §VI.5]. Hence by [BS87, Theorem 10.2.4]

$$\inf\sigma_{\mathrm{ess}}(-\Delta_{\alpha,\mathcal{C}_\theta,n}) \leq \inf\sigma_{\mathrm{ess}}(-\Delta_{\alpha,\mathcal{C}_\theta}). \tag{2.26}$$

In view of (2.26) the estimate (2.25) implies that for any $\varepsilon > 0$

$$\inf\sigma_{\mathrm{ess}}(-\Delta_{\alpha,\mathcal{C}_\theta}) \geq -\tfrac{\alpha^2}{4} - \varepsilon$$

and thus passing to the limit $\varepsilon \to 0+$ we arrive at

$$\inf\sigma_{\mathrm{ess}}(-\Delta_{\alpha,\mathcal{C}_\theta}) \geq -\tfrac{\alpha^2}{4},$$

which shows the inclusion $\sigma_{\mathrm{ess}}(-\Delta_{\alpha,\mathcal{C}_\theta}) \subseteq [-\alpha^2/4, +\infty)$ and finishes the proof of Theorem 2.1. □

3. Discrete spectrum of $-\Delta_{\alpha,\mathcal{C}_\theta}$

In this section we show that the discrete spectrum of the self-adjoint operator $-\Delta_{\alpha,\mathcal{C}_\theta}$ below the bottom $-\alpha^2/4$ of the essential spectrum is infinite for all angles $\theta \in (0,\pi/2)$ and we estimate the rate of the convergence of these eigenvalues to $-\alpha^2/4$ with the help of variational principles. The following lemma will be useful.

Lemma 3.1. *Let $\mathfrak{a}_{\alpha,\mathcal{C}_\theta}$ be the form in (1.1). For $\omega \in H^1(\mathbb{R}^2_+)$ with compact support $\operatorname{supp}\omega \subset \mathbb{R}^2_+$ define the function $\psi(r,\varphi,z) := \frac{\omega(r,z)}{\sqrt{2\pi r}}$. Then $\psi \in H^1(\mathbb{R}^3)$ and*

$$\mathfrak{a}_{\alpha,\mathcal{C}_\theta}[\psi] = \|\nabla\omega\|^2_{L^2(\mathbb{R}^2_+;\mathbb{C}^2)} - \int_{\mathbb{R}^2_+}\frac{1}{4r^2}|\omega(r,z)|^2\mathrm{d}r\mathrm{d}z - \alpha\|\omega|_{\Gamma_\theta}\|^2_{L^2(\Gamma_\theta)}, \tag{3.1}$$

where Γ_θ is the ray in Figure 2.1.

Proof. First of all observe that

$$\|\psi\|^2_{L^2(\mathbb{R}^3)} = \int_{\mathbb{R}}\int_{\mathbb{R}_+}\int_0^{2\pi} \tfrac{|\omega(r,z)|^2}{2\pi r}\, r\,\mathrm{d}\varphi\mathrm{d}r\mathrm{d}z = \|\omega\|^2_{L^2(\mathbb{R}^2_+)} < \infty. \tag{3.2}$$

Moreover, we compute

$$\partial_r\psi = \frac{\partial_r\omega}{\sqrt{2\pi r}} - \frac{\omega}{2r\sqrt{2\pi r}} \quad \text{and} \quad \partial_z\psi = \frac{\partial_z\omega}{\sqrt{2\pi r}}, \tag{3.3}$$

and setting $\rho := \inf\{r\colon (r,z)\in \operatorname{supp}\omega\} > 0$ we obtain

$$\begin{aligned}\|\nabla\psi\|^2_{L^2(\mathbb{R}^3)} &= \|\partial_r\psi\|^2_{L^2(\mathbb{R}^3)} + \|\partial_z\psi\|^2_{L^2(\mathbb{R}^3)}\\ &\le 2\big\|\tfrac{\partial_r\omega}{\sqrt{2\pi r}}\big\|^2_{L^2(\mathbb{R}^3)} + 2\big\|\tfrac{\omega}{2r\sqrt{2\pi r}}\big\|^2_{L^2(\mathbb{R}^3)} + \big\|\tfrac{\partial_z\omega}{\sqrt{2\pi r}}\big\|^2_{L^2(\mathbb{R}^3)}\\ &\le 2\|\partial_r\omega\|^2_{L^2(\mathbb{R}^2_+)} + \tfrac{1}{2\rho^2}\|\omega\|^2_{L^2(\mathbb{R}^2_+)} + \|\partial_z\omega\|^2_{L^2(\mathbb{R}^2_+)} < \infty.\end{aligned} \tag{3.4}$$

Hence (3.2) and (3.4) imply $\psi\in H^1(\mathbb{R}^3)$. Next we substitute ψ in the form $\mathfrak{a}_{\alpha,\mathcal{C}_\theta}$ in (1.1). It follows from the form of $\partial_z\psi$ in (3.3) and $\|\psi|_{\mathcal{C}_\theta}\|^2_{L^2(\mathcal{C}_\theta)} = \|\omega|_{\Gamma_\theta}\|^2_{L^2(\Gamma_\theta)}$ that

$$\begin{aligned}\mathfrak{a}_{\alpha,\mathcal{C}_\theta}[\psi] &= \int_{\mathbb{R}}\int_{\mathbb{R}_+}|\partial_r\psi|^2 2\pi r\mathrm{d}r\mathrm{d}z + \int_{\mathbb{R}}\int_{\mathbb{R}_+}|\partial_z\psi|^2 2\pi r\mathrm{d}r\mathrm{d}z - \alpha\|\psi|_{\mathcal{C}_\theta}\|^2_{L^2(\mathcal{C}_\theta)}\\ &= \int_{\mathbb{R}}\int_{\mathbb{R}_+}|\partial_r\psi|^2 2\pi r\mathrm{d}r\mathrm{d}z + \int_{\mathbb{R}}\int_{\mathbb{R}_+}|\partial_z\omega|^2\mathrm{d}r\mathrm{d}z - \alpha\|\omega|_{\Gamma_\theta}\|^2_{L^2(\Gamma_\theta)}.\end{aligned} \tag{3.5}$$

Denote the first integral by I_ψ. Making use of $\partial_r\psi$ in (3.3) we rewrite I_ψ as

$$I_\psi = \int_{\mathbb{R}}\int_{\mathbb{R}_+}|\partial_r\omega|^2\mathrm{d}r\mathrm{d}z + \int_{\mathbb{R}}\int_{\mathbb{R}_+}\frac{1}{4r^2}|\omega|^2\mathrm{d}r\mathrm{d}z - \int_{\mathbb{R}}\int_{\mathbb{R}_+}\frac{1}{r}\mathrm{Re}(\partial_r\omega\overline{\omega})\mathrm{d}r\mathrm{d}z \tag{3.6}$$

and the last term can be further rewritten as

$$\int_{\mathbb{R}}\int_{\mathbb{R}_+}\frac{1}{r}\mathrm{Re}(\partial_r\omega\overline{\omega})\mathrm{d}r\mathrm{d}z = \int_{\mathbb{R}}\int_{\mathbb{R}_+}\frac{1}{2r}\partial_r\big(|\omega|^2\big)\mathrm{d}r\mathrm{d}z = \int_{\mathbb{R}}\int_{\mathbb{R}_+}\frac{1}{2r^2}|\omega|^2\mathrm{d}r\mathrm{d}z, \tag{3.7}$$

where we integrated by parts and used the fact that $\operatorname{supp}\omega$ is contained in the open half-plane $\mathbb{R}^2_+$. Hence, (3.6) and (3.7) imply

$$I_\psi = \int_{\mathbb{R}}\int_{\mathbb{R}_+}|\partial_r\omega|^2\mathrm{d}r\mathrm{d}z - \int_{\mathbb{R}}\int_{\mathbb{R}_+}\frac{1}{4r^2}|\omega|^2\mathrm{d}r\mathrm{d}z.$$

Substituting this expression for the first integral in (3.5) we obtain (3.1). □

Now we are ready to formulate and prove our main result on the infiniteness of the discrete spectrum of $-\Delta_{\alpha,\mathcal{C}_\theta}$ below the bottom of the essential spectrum for all $\alpha>0$ and $\theta\in(0,\pi/2)$. Recall that $-\Delta_{\alpha,\mathcal{C}_\theta}$ is bounded from below, and hence it also follows that the discrete spectrum has a single accumulation point, namely $-\alpha^2/4$.

Theorem 3.2. *Let $-\Delta_{\alpha,\mathcal{C}_\theta}$ be the self-adjoint operator in $L^2(\mathbb{R}^3)$ associated to the form (1.1) and let $\alpha>0$ and $\theta\in(0,\pi/2)$. Then the discrete spectrum of $-\Delta_{\alpha,\mathcal{C}_\theta}$ below $-\alpha^2/4$ is infinite, accumulates at $-\alpha^2/4$, and the eigenvalues $\lambda_k < -\alpha^2/4$ (enumerated in non-decreasing order with multiplicities taken into account) satisfy the estimate*

$$\lambda_k \le -\frac{\alpha^2}{4} - \frac{\gamma(\theta)}{n_k^4}, \qquad k\in\mathbb{N}, \tag{3.8}$$

holds, where $\gamma(\theta) > 0$, $n_{k+1} := n_k^2 + n_k$ *for* $k \in \mathbb{N}$, *and* $n_1 = N$ *with* $N \in \mathbb{N}$ *sufficiently large.*

Proof. Let us pick a function $\chi_1 \in H_0^1(0,1)$ with $\|\chi_1\|_{L^2(0,1)} = 1$ such that

$$\|\chi_1'\|^2_{L^2(0,1)} < \frac{1}{4\sin^2\theta}\int_0^1 \frac{|\chi_1(t)|^2}{t^2}\mathrm{d}t \tag{3.9}$$

holds; cf. [BM97, Lemma in §1]. Let us fix $\varepsilon > 0$ and choose $\chi_2 \in C_0^\infty(-\varepsilon,\varepsilon)$ such that $0 \le \chi_2 \le 1$ and $\chi_2(t) = 1$ for $|t| \le \varepsilon/2$. In the coordinate system (s,t) in Figure 2.1 we define the sequence of functions

$$\omega_n(s,t) := \tfrac{1}{n}\chi_1\big(\tfrac{s-n}{n^2}\big)\chi_2\big(\tfrac{t}{\sqrt{n}}\big)\exp\big(-\tfrac{\alpha}{2}|t|\big) \in H_0^1(\mathbb{R}^2_+).$$

For sufficiently large $n \in \mathbb{N}$ the functions ω_n satisfy the conditions of Lemma 3.1. The function ω_n can also be viewed as a function in r and z; cf. Figure 2.1. Then we define

$$\psi_n(r,\varphi,z) := \frac{\omega_n(r,z)}{\sqrt{2\pi r}}, \qquad n \in \mathbb{N}. \tag{3.10}$$

Using Lemma 3.1 we compute the values

$$\begin{aligned} S_n :=& \mathfrak{a}_{\alpha,\mathcal{C}_\theta}[\psi_n] + \frac{\alpha^2}{4}\|\psi_n\|^2_{L^2(\mathbb{R}^3)} \\ =& \|\nabla\omega_n\|^2_{L^2(\mathbb{R}^2_+;\mathbb{C}^2)} - \int_{\mathbb{R}^2_+}\frac{1}{4r^2}|\omega_n|^2\mathrm{d}r\mathrm{d}z - \alpha\|\omega_n|_{\Gamma_\theta}\|^2_{L^2(\Gamma_\theta)} + \frac{\alpha^2}{4}\|\omega_n\|^2_{L^2(\mathbb{R}^2_+)}. \end{aligned} \tag{3.11}$$

It is not difficult to check the asymptotics

$$\int_{-\varepsilon\sqrt{n}}^{\varepsilon\sqrt{n}} \big|\chi_2\big(\tfrac{t}{\sqrt{n}}\big)\big|^2 e^{-\alpha|t|}\mathrm{d}t = \frac{2}{\alpha} + \mathcal{O}(e^{-c\sqrt{n}}), \qquad n \to \infty, \tag{3.12}$$

$$\int_{-\varepsilon\sqrt{n}}^{\varepsilon\sqrt{n}} \big|\chi_2'\big(\tfrac{t}{\sqrt{n}}\big)\big|^2 e^{-\alpha|t|}\mathrm{d}t = \mathcal{O}(e^{-c\sqrt{n}}), \qquad n \to \infty, \tag{3.13}$$

$$\int_{-\varepsilon\sqrt{n}}^{\varepsilon\sqrt{n}} \chi_2\big(\tfrac{t}{\sqrt{n}}\big)\chi_2'\big(\tfrac{t}{\sqrt{n}}\big) e^{-\alpha|t|}\mathrm{d}t = \mathcal{O}(e^{-c\sqrt{n}}), \qquad n \to \infty, \tag{3.14}$$

with some constant $c > 0$. Using (3.12) we get

$$\begin{aligned} \frac{\alpha^2}{4}\|\omega_n\|^2_{L^2(\mathbb{R}^2_+)} &= \frac{\alpha^2}{4}\left(\frac{1}{n^2}\int_n^{n+n^2} |\chi_1\big(\tfrac{s-n}{n^2}\big)|^2\mathrm{d}s\right)\left(\int_{-\varepsilon\sqrt{n}}^{\varepsilon\sqrt{n}}\big|\chi_2\big(\tfrac{t}{\sqrt{n}}\big)\big|^2 e^{-\alpha|t|}\mathrm{d}t\right) \\ &= \frac{\alpha}{2} + \mathcal{O}(e^{-c\sqrt{n}}), \qquad n \to \infty, \end{aligned} \tag{3.15}$$

and

$$\begin{aligned} \|\partial_s\omega_n\|^2_{L^2(\mathbb{R}^2_+)} &= \left(\int_{-\varepsilon\sqrt{n}}^{\varepsilon\sqrt{n}}\big|\chi_2\big(\tfrac{t}{\sqrt{n}}\big)\big|^2 e^{-\alpha|t|}\mathrm{d}t\right)\left(\frac{1}{n^4}\frac{1}{n^2}\int_n^{n+n^2}\big|\chi_1'\big(\tfrac{s-n}{n^2}\big)\big|^2\mathrm{d}s\right) \\ &= \frac{2}{\alpha}\frac{1}{n^4}\|\chi_1'\|^2_{L^2(0,1)} + \mathcal{O}(e^{-c\sqrt{n}}), \qquad n \to \infty, \end{aligned}$$

and from (3.13) and (3.14) we obtain

$$\begin{aligned}\|\partial_t\omega_n\|^2_{L^2(\mathbb{R}^2_+)} &= \left(\frac{1}{n^2}\int_n^{n+n^2}\left|\chi_1\left(\tfrac{s-n}{n^2}\right)\right|^2\mathrm{d}s\right)\left(\int_{-\varepsilon\sqrt{n}}^{\varepsilon\sqrt{n}}\left|\frac{\chi_2'\left(\frac{t}{\sqrt{n}}\right)}{\sqrt{n}}-\frac{\alpha\mathrm{sign}\,(t)\chi_2\left(\frac{t}{\sqrt{n}}\right)}{2}\right|^2 e^{-\alpha|t|}\mathrm{d}t\right)\\ &= \frac{\alpha}{2}+\mathcal{O}(e^{-c\sqrt{n}}), \qquad n\to\infty,\end{aligned}$$

that is,

$$\|\nabla\omega_n\|^2_{L^2(\mathbb{R}^2_+;\mathbb{C}^2)} = \frac{2}{\alpha}\frac{1}{n^4}\|\chi_1'\|^2_{L^2(0,1)}+\frac{\alpha}{2}+\mathcal{O}(e^{-c\sqrt{n}}), \qquad n\to\infty. \tag{3.16}$$

It is simple to see that

$$\alpha\|\omega_n|_{\Gamma_\theta}\|^2_{L^2(\Gamma_\theta)} = \frac{\alpha}{n^2}\int_n^{n+n^2}\left|\chi_1\left(\tfrac{s-n}{n^2}\right)\right|^2\mathrm{d}s = \alpha\|\chi_1\|^2_{L^2(0,1)} = \alpha, \tag{3.17}$$

and hence it remains to estimate the term $\int_{\mathbb{R}^2_+}\frac{1}{4r^2}|\omega_n|^2$ in (3.11). For that we make the following splitting

$$\int_{\mathbb{R}^2_+}\frac{1}{4r^2}|\omega_n(r,z)|^2 dr dz = \int_{-\varepsilon\sqrt{n}}^{\varepsilon\sqrt{n}}\int_n^{n+n^2}\frac{1}{4r(s,t)^2}|\omega_n(s,t)|^2\mathrm{d}s\mathrm{d}t = I_n+J_n, \tag{3.18}$$

where

$$I_n := \int_{-\varepsilon\sqrt{n}}^{\varepsilon\sqrt{n}}\int_n^{n+n^2}\frac{1}{4r(s,0)^2}|\omega_n(s,t)|^2\mathrm{d}s\mathrm{d}t \tag{3.19}$$

and

$$J_n := \int_{-\varepsilon\sqrt{n}}^{\varepsilon\sqrt{n}}\int_n^{n+n^2}\left(\frac{1}{4r(s,t)^2}-\frac{1}{4r(s,0)^2}\right)|\omega_n(s,t)|^2\mathrm{d}s\mathrm{d}t.$$

The term J_n can be further rewritten as

$$J_n = \int_{-\varepsilon\sqrt{n}}^{\varepsilon\sqrt{n}}\int_n^{n+n^2}\frac{(r(s,0)-r(s,t))(r(s,0)+r(s,t))}{4r(s,t)^2r(s,0)^2}|\omega_n(s,t)|^2\mathrm{d}s\mathrm{d}t. \tag{3.20}$$

For geometric reasons we have $|r(s,0)-r(s,t)|\le a\sqrt{n}$ with some $0<a\le\varepsilon$ and $r(s,t)>bn$ with some $b>0$ for all $(s,t)\in\operatorname{supp}\omega_n$. We first conclude from (3.20) that

$$|J_n|\le a\sqrt{n}\int_{-\varepsilon\sqrt{n}}^{\varepsilon\sqrt{n}}\int_n^{n+n^2}\left|\frac{2}{4r(s,t)r(s,0)^2}+\frac{r(s,0)-r(s,t)}{4r(s,t)^2r(s,0)^2}\right||\omega_n(s,t)|^2\mathrm{d}s\mathrm{d}t$$

and hence

$$|J_n|\le\left(\frac{2a}{b\sqrt{n}}+\frac{a^2}{b^2n}\right)I_n \tag{3.21}$$

follows together with (3.19). For I_n we have

$$\begin{aligned}I_n &= \left(\frac{1}{n^2}\int_n^{n+n^2}\frac{|\chi_1(\frac{s-n}{n^2})|^2}{4s^2\sin^2\theta}\mathrm{d}s\right)\left(\int_{-\varepsilon\sqrt{n}}^{\varepsilon\sqrt{n}}\left|\chi_2\left(\tfrac{t}{\sqrt{n}}\right)\right|^2e^{-\alpha|t|}\mathrm{d}t\right)\\ &= \left(\frac{1}{n^4}\int_0^1\frac{|\chi_1(u)|^2}{4\sin^2(\theta)(u+1/n)^2}\mathrm{d}u\right)\left(\int_{-\varepsilon\sqrt{n}}^{\varepsilon\sqrt{n}}\left|\chi_2\left(\tfrac{t}{\sqrt{n}}\right)\right|^2e^{-\alpha|t|}\mathrm{d}t\right),\end{aligned} \tag{3.22}$$

and the choice of χ_1 (see (3.9)) together with monotone convergence yields

$$\int_0^1 \frac{|\chi_1(u)|^2}{(u+1/n)^2}\mathrm{d}u = \int_0^1 \frac{|\chi_1(u)|^2}{u^2}\mathrm{d}u + o(1), \qquad n\to\infty.$$

Hence we conclude from (3.12) and (3.22) that

$$I_n = \frac{2}{\alpha}\frac{1}{n^4}\frac{1}{4\sin^2(\theta)}\int_0^1 \frac{|\chi_1(u)|^2}{u^2}\mathrm{d}u + o\left(\frac{1}{n^4}\right), \qquad n\to\infty,$$

and from (3.21) we find

$$J_n = o\left(\frac{1}{n^4}\right), \qquad n\to\infty.$$

It follows that (3.18) becomes

(3.23)
$$\int_{\mathbb{R}^2_+} \frac{1}{4r^2}|\omega_n(r,z)|^2\mathrm{d}r\mathrm{d}z = \frac{2}{\alpha}\frac{1}{n^4}\frac{1}{4\sin^2(\theta)}\int_0^1 \frac{|\chi_1(u)|^2}{u^2}\mathrm{d}u + o\left(\frac{1}{n^4}\right), \qquad n\to\infty.$$

Finally, (3.15), (3.16), (3.17) and (3.23) yield

$$S_n = \frac{2}{\alpha}\frac{1}{n^4}\left(\|\chi_1'\|^2_{L^2(0,1)} - \int_0^1 \frac{|\chi_1(u)|^2}{4\sin^2(\theta)u^2}\mathrm{d}u\right) + o\left(\frac{1}{n^4}\right), \qquad n\to\infty, \tag{3.24}$$

for S_n in (3.11). In view of the above asymptotics and according to (3.9) there exists $N\in\mathbb{N}$ such that for all $n\geq N$ we have

$$S_n \leq -\frac{2\gamma(\theta)}{\alpha n^4} \tag{3.25}$$

for some constant $\gamma(\theta)>0$. Let us consider a sequence $\{n_k\}_k$, where $n_1 := N$ and $n_{k+1} := n_k^2 + n_k$ for $k\in\mathbb{N}$. Then the functions ψ_{n_k} in (3.10) have disjoint supports for all $k\in\mathbb{N}$ and hence are orthogonal in $L^2(\mathbb{R}^3)$. The space

$$F_k := \mathrm{span}\big\{\psi_{n_1},\psi_{n_2},\dots,\psi_{n_k}\big\} \subset H^1(\mathbb{R}^3),$$

has dimension k and for an arbitrary $\psi = \sum_{l=1}^k a_l\psi_{n_l}\in F_k$, $a_l\in\mathbb{C}$, we get

$$\|\psi\|^2_{L^2(\mathbb{R}^3)} = \sum_{l=1}^k |a_l|^2\|\psi_{n_l}\|^2_{L^2(\mathbb{R}^3)} = \sum_{l=1}^k |a_l|^2\|\omega_{n_l}\|^2_{L^2(\mathbb{R}^2_+)} \leq \frac{2}{\alpha}\sum_{l=1}^k |a_l|^2, \tag{3.26}$$

where we have also used the estimate $\|\omega_{n_l}\|^2_{L^2(\mathbb{R}^2_+)} \leq \frac{2}{\alpha}$. Employing (3.25) we obtain

$$\mathfrak{a}_{\alpha,\mathcal{C}_\theta}[\psi] + \frac{\alpha^2}{4}\|\psi\|^2_{L^2(\mathbb{R}^3)} = \sum_{l=1}^k |a_l|^2 S_{n_l} \leq -\frac{2\gamma(\theta)}{\alpha n_k^4}\sum_{l=1}^k |a_l|^2,$$

where we have again used the disjointness of the supports of $\{\psi_{n_l}\}_{l=1}^k$. Combining the above estimate with (3.26) we get

$$\frac{\mathfrak{a}_{\alpha,\mathcal{C}_\theta}[\psi]}{\|\psi\|^2_{L^2(\mathbb{R}^3)}} = -\frac{\alpha^2}{4} + \frac{\mathfrak{a}_{\alpha,\mathcal{C}_\theta}[\psi] + (\alpha^2/4)\|\psi\|^2_{L^2(\mathbb{R}^3)}}{\|\psi\|^2_{L^2(\mathbb{R}^3)}} \leq -\frac{\alpha^2}{4} - \frac{\gamma(\theta)}{n_k^4} < -\frac{\alpha^2}{4}. \tag{3.27}$$

Hence, according to [BS87, Theorem 10.2.3] the operator $-\Delta_{\alpha,\mathcal{C}_\theta}$ has at least k eigenvalues below the bottom of the essential spectrum $-\alpha^2/4$. The above construction works for any $k\in\mathbb{N}$, so that the operator $-\Delta_{\alpha,\mathcal{C}_\theta}$ has infinitely many eigenvalues below $-\alpha^2/4$. The eigenvalue estimate (3.8) follows from [BS87, Theorem 10.2.3] and (3.27). □

Let $\theta \in (0, \pi/2)$ and $\mathcal{C}_\theta$ be the conical surface as above. A hypersurface $\Sigma \subset \mathbb{R}^3$, which for some compact set $K \subset \mathbb{R}^3$ satisfies the condition $\Sigma \setminus K = \mathcal{C}_\theta \setminus K$ and which splits the space $\mathbb{R}^3$ into two unbounded Lipschitz domains, is called *a local deformation* of $\mathcal{C}_\theta$; cf. [BEL13, Section 4.2]. Below we consider the self-adjoint Schrödinger operator $-\Delta_{\alpha,\Sigma}$ with an attractive δ-interaction of constant strength $\alpha > 0$ supported on the Lipschitz hypersurface Σ. This Schrödinger operator is defined via the quadratic form

$$\mathfrak{a}_{\alpha,\Sigma}[\psi] = \|\nabla\psi\|^2_{L^2(\mathbb{R}^3;\mathbb{C}^3)} - \alpha \int_\Sigma |\psi|^2 \,\mathrm{d}\sigma \qquad \mathrm{dom}\,\mathfrak{a}_{\alpha,\Sigma} = H^1(\mathbb{R}^3). \tag{3.28}$$

The assertion on the essential spectrum in the next theorem is a consequence of [BEL13, Theorem 4.7]; the infiniteness of the discrete spectrum can be shown as in the proof of Theorem 3.2 using the same functions ψ_n in (3.10) and $n \in \mathbb{N}$ sufficiently large.

Theorem 3.3. *Let $\theta \in (0, \pi/2)$ and $\alpha > 0$. Let Σ be a local deformation of the cone $\mathcal{C}_\theta$ and let $-\Delta_{\alpha,\Sigma}$ be the self-adjoint operator in $L^2(\mathbb{R}^3)$ associated to* (3.28). *Then*

$$\sigma_{\mathrm{ess}}(-\Delta_{\alpha,\Sigma}) = [-\alpha^2/4, +\infty),$$

the discrete spectrum below $-\alpha^2/4$ is infinite, accumulates at $-\alpha^2/4$, and the eigenvalues $\lambda_k < -\alpha^2/4$ (enumerated in non-decreasing order with multiplicities taken into account) satisfy the estimate

$$\lambda_k \le -\frac{\alpha^2}{4} - \frac{\gamma(\theta)}{n_k^4}, \qquad k \in \mathbb{N},$$

holds, where $\gamma(\theta) > 0$, $n_{k+1} := n_k^2 + n_k$ for $k \in \mathbb{N}$, and $n_1 = N$ with $N \in \mathbb{N}$ sufficiently large.

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