Topological Horseshoe in Travelling Waves of Discretized KdV-Burgers-KS type Equations

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Abstract

Applying the concept of anti-integrable limit to space-time discretized KdV-Burgers-KS type equations, we show that there exist topological horseshoes in the phase space formed by the initial states of travelling wave solutions of the resulted coupled map lattices. In particular, the coupled map lattices display spatio-temporal chaos on the horseshoes.

Key words. horseshoe, spatio-temporal chaos, travelling wave, anti-integrable limit

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1 Introduction and main result

In this paper, we investigate the travelling wave solutions of discretized KdV-Burgers-KS type Equations. More precisely, we are concerned with the time-evolution and space-translation dynamics for the travelling wave solutions of the space-time discretization of the following partial differential equation (PDE)

$$\partial_t u + \partial_x \left(V(u) + \varepsilon_1 \partial_x u + \varepsilon_2 \partial_x^2 u + \varepsilon_3 \partial_x^3 u + \varepsilon_4 \partial_x^4 u \right) = 0, \tag{1.1}$$

where V is a degree-$(l+1)$ polynomial function of $\mathbb{R}$ with $l \geq 1$. In the equation, ε_1 and ε_3 denote the constant dissipation coefficients, while ε_2 and ε_4 stand for constant dispersion coefficients. The discretized PDE can be treated as a coupled map lattice (CML). We shall show that the CML admits spatio-temporal chaotic travelling waves if $\varepsilon = (\varepsilon_1, \varepsilon_2, \varepsilon_3, \varepsilon_4)$ is sufficiently small.

When $V(u) = u^{l+1}/(l+1)$ and $\varepsilon_1 = \varepsilon_3 = 0$, the equation reduces to a fifth-order KdV equation, which has been employed to model shallow water waves with surface tension[13, 16, 28], magneto-acoustic waves in plasmas [15], and oscillatory tails of non-vanishing amplitude waves in optics [7]. In shallow water waves, ε_2 is related to the Bond number in the presence of surface tension and $\varepsilon_2 = 0$ corresponds to the critical Bond number $\frac{1}{3}$ [28]. When $\varepsilon_1 = \varepsilon_3 = \varepsilon_4 = 0$, (1.1) leads to the generalized KdV equation. It is called the modified KdV equation when $l = 2$. In the case $l = 1$, (1.1) is the celebrated Korteweg-de Vries (KdV) equation [17] of the form

$$\partial_t u + u \partial_x u + \varepsilon_2 \partial_x^3 u = 0. \tag{1.2}$$

Note that both KdV and modified KdV are integrable and there is Miura transformation between them [1, 11], and that the generalized KdV equation is not integrable for $l \geq 3$ as shown by the inverse scattering transformation [22].

When $V(u) = u^2/2$, $\varepsilon_1 = \varepsilon_3 = 1$ and $\varepsilon_2 = \varepsilon_4 = 0$, it represents the simplest form of the Kuramoto-Sivashinsky (KS) equation

$$\partial_t u + u \partial_x u + \partial_x^2 u + \partial_x^4 u = 0, \tag{1.3}$$

which models pattern formations in different physical contexts, e.g. [14, 23]. When $V(u) = u^2/2$ and $\varepsilon_2 = \varepsilon_3 = \varepsilon_4 = 0$, (1.1) stands for the Burgers equation of the form

$$\partial_t u + u\partial_x u + \varepsilon_1 \partial_x u = 0. \tag{1.4}$$

This paper is motivated by the work of [29]. Using the approach of anti-integrable limit (see e.g. [4, 9, 20] for an account) and the contraction mapping theorem, Zheng *et al.* proved existence of solitary-wave-like solutions of a space-time discretization system of (1.2), and mentioned the possibility of various type of travelling wave solutions, as is apparently implied by the theory of anti-integrable limit. The main purpose of this paper is three-fold:

1) To prove the existence of such various type solutions, in particular chaotic solutions, not only for discretized system of (1.2) but also for discretized system of (1.1) .

2) Although it has been known that the method of anti-integrability can be employed to prove the existence of chaotic solutions for discretized partial differential equations, or coupled map lattices (CMLs) in general, (see e.g. [19, 21, 26, 27]), the chaos displayed is spatial, not temporal nor spatio-temporal. Here, we go beyond this, showing that the discretized system manifests not only the spatial but also the temporal and spatio-temporal chaos.

3) In reaction-diffusion, sine-Gordon, as well as KdV equations, strong nonlinearity and weak dissipation or dispersion play the same role up to a rescaling of magnitude, space and time. (See for example, (3.4), (3.10), and (3.14).) There are similar rescalings for their discretized counterparts. Whether the limits of infinite nonlinearity or zero dissipation or dispersion correspond to the anti-integrable limit will be fully discussed. We shall see that the case of discretized KdV equations is different from that of discretized Fisher-KPP and sine-Gordon equations: the discretized KdV equations possess a conservation law (namely (1.11)), but the other two do not.

Our approach is again the anti-integrable limit, but in contrast with the contraction mapping theorem used in [29], we use the implicit function theorem.

It is well-known that the KdV equation (1.2) possesses a family of solitary wave solutions parametrized by wave speed. Korteweg and de Vries also found that there is a two-parameter family of spatially periodic travelling wave solutions, called *cnoidal waves*, described by the Jacobian elliptic functions of modulus m, with $0 < m < 1$, and parametrized by m and wave speed. They showed that the cnoidal waves become the solitary waves as $m \to 1$, and collapses to sinusoidal waves of zero amplitude as $m \to 0$ [17]. In our discretized KdV system, not only solitary-wave-like but also spatially periodic travelling wave solutions are found. In fact, the solitary-wave-like solutions we found are of multi-peak, and the number of peaks can be arbitrary. Furthermore, our result implies that for any solitary-wave-like solution found, there is a family of spatially periodic travelling waves converges to it with the product topology.

In this paper, we focus on travelling waves propagating to the left. The case of waves propagating to the right can be treated in the same manner.

Let $\Delta x > 0$ and $\Delta t > 0$ be small real numbers. We discretize the space and time into lattice points x_j and t_n, respectively, with $j \in \mathbb{Z}$, $n \in \mathbb{N}$, and use

$$\begin{aligned} x_{j+1} &:= x_j + \Delta x, \\ t_{n+1} &:= t_n + \Delta t. \end{aligned}$$

In this paper, we employ the discretization below for the derivative with respective to time:

$$\partial_t u(x_j, t_n) \longrightarrow [u(x_j, t_n + \Delta t) - u(x_j, t_n)]/\Delta t,$$

and the following discretization for the derivatives with respect to space:

$$\begin{aligned} \partial_x u(x_j, t_n) &\longrightarrow [u(x_j + \Delta x, t_n) - u(x_j, t_n)]/\Delta x, \\ \partial_x^2 u(x_j, t_n) &\longrightarrow [u(x_j + \Delta x, t_n) - 2u(x_j, t_n) + u(x_j - \Delta x, t_n)]/(\Delta x)^2, \\ \partial_x^3 u(x_j, t_n) &\longrightarrow [u(x_j + 2\Delta x, t_n) - 3u(x_j + \Delta t, t_n) + 3u(x_j, t_n) \\ &\qquad -u(x_j - \Delta x, t_n)]/(\Delta x)^3, \\ \partial_x^4 u(x_j, t_n) &\longrightarrow [u(x_j + 2\Delta x, t_n) - 4u(x_j + \Delta t, t_n) + 6u(x_j, t_n) \\ &\qquad -4u(x_j - \Delta x, t_n) + u(x_j - 2\Delta x, t_n)]/(\Delta x)^4, \\ \partial_x^5 u(x_j, t_n) &\longrightarrow [u(x_j + 3\Delta x, t_n) - 5u(x_j + 2\Delta x, t_n) + 10u(x_j + \Delta t, t_n) \\ &\qquad -10u(x_j, t_n) + 5u(x_j - \Delta x, t_n) - u(x_j - 2\Delta x, t_n)]/(\Delta x)^5. \end{aligned}$$

For every integer k, use the notation

$$\begin{aligned} u_{j+k,n} &:= u(x_j + k\Delta x, t_n), \\ u_{j,n+k} &:= u(x_j, t_n + k\Delta t). \end{aligned}$$

Thereby, we have the following equation in our discretization of (1.1):

$$\begin{aligned} &\frac{u_{j,n+1} - u_{j,n}}{\Delta t} + \frac{V(u_{j+1,n}) - V(u_{j,n})}{\Delta x} \\ &\quad + \varepsilon_1 \frac{u_{j+1,n} - 2u_{j,n} + u_{j-1,n}}{(\Delta x)^2} + \varepsilon_2 \frac{u_{j+2,n} - 3u_{j+1,n} + 3u_{j,n} - u_{j-1,n}}{(\Delta x)^3} \\ &\quad + \varepsilon_3 \frac{u_{j+2,n} - 4u_{j+1,n} + 6u_{j,n} - 4u_{j-1,n} + u_{j-2,n}}{(\Delta x)^4} \\ &\quad + \varepsilon_4 \frac{u_{j+3,n} - 5u_{j+2,n} + 10u_{j+1,n} - 10u_{j,n} + 5u_{j-1,n} - u_{j-2,n}}{(\Delta x)^5} = 0. \end{aligned} \tag{1.5}$$

The discretized KdV-Burgers-KS type equation (1.5) is the CML system we study in this paper.

Let $c > 0$. In PDEs, a travelling wave solution propagating to the left having the form $u(x,t) = v(x + ct)$ gives rise to

$$\partial_x u = \partial_t u / c. \tag{1.6}$$

From (1.6) we require that

$$\frac{u_{j+1,n} - u_{j,n}}{\Delta x} = \frac{u_{j,n+1} - u_{j,n}}{c\Delta t}. \tag{1.7}$$

Hence, associated with $\Delta x = c\Delta t$, by a travelling wave solution propagating to the left with speed c for a discretized PDE, we mean a solution $(u_{j,n})_{(j,n)\in\mathbb{Z}\times\mathbb{N}}$ satisfying

$$u_{j+1,n} = u_{j,n+1} \qquad \text{(wave propagating to the left)} \tag{1.8}$$

or equivalently

$$u_{j,n} = u_{j-1,n+1} \qquad \text{(wave propagating to the left)} \tag{1.9}$$

for all $j \in \mathbb{Z}$, $n \in \mathbb{N}$.

Remark 1. In CMLs, a travelling wave solution propagating to the left with speed $c \in \mathbb{Q}$ means a solution of the form

$$u_{j,n} = \Gamma(qj + pn) \quad \text{with } p/q = c,$$

p, $q \in \mathbb{N}$, coprime, and $\Gamma : \mathbb{R} \to \mathbb{R}$ a continuous function. (See [2, 3, 24] and the definition therein for $c \in \mathbb{R}$.) This gives rise to

$$u_{j,n} = u_{j-p,n+q} \tag{1.10}$$

One possible situation leading to (1.10) is to set the space-step Δx equal to the time-step Δt in the discretization of PDEs so that a wave profile propagates p steps in space when time elapses q steps. If we define a travelling wave as in this remark, our approach by the anti-integrable limit still works for $c = 1$ case. Because in this case, (1.10) is just (1.9). For general c see subsection 3.2.

Substituting (1.7) into (1.5), we obtain

$$\begin{aligned}
& cu_{j+1,n} + V(u_{j+1,n}) + \varepsilon_1 \frac{u_{j+1,n} - u_{j,n}}{\Delta x} + \varepsilon_2 \frac{u_{j+2,n} - 2u_{j+1,n} + u_{j,n}}{(\Delta x)^2} \\
& + \varepsilon_3 \frac{u_{j+2,n} - 3u_{j+1,n} + 3u_{j,n} - u_{j-1,n}}{(\Delta x)^3} \\
& + \varepsilon_4 \frac{u_{j+3,n} - 4u_{j+2,n} + 6u_{j+1,n} - 4u_{j,n} + u_{j-1,n}}{(\Delta x)^4} \\
= \; & cu_{j,n} + V(u_{j,n}) + \varepsilon_1 \frac{u_{j,n} - u_{j-1,n}}{\Delta x} + \varepsilon_2 \frac{u_{j+1,n} - 2u_{j,n} + u_{j-1,n}}{(\Delta x)^2} \\
& + \varepsilon_3 \frac{u_{j+1,n} - 3u_{j,n} + 3u_{j-1,n} - u_{j-2,n}}{(\Delta x)^3} \\
& + \varepsilon_4 \frac{u_{j+2,n} - 4u_{j+1,n} + 6u_{j,n} - 4u_{j-1,n} + u_{j-2,n}}{(\Delta x)^4}.
\end{aligned}$$

Therefore, a solution must fulfill the following conservation law

$$\begin{aligned}
& cu_{j,n} + V(u_{j,n}) + \varepsilon_1 \frac{u_{j,n} - u_{j-1,n}}{\Delta x} + \varepsilon_2 \frac{u_{j+1,n} - 2u_{j,n} + u_{j-1,n}}{(\Delta x)^2} \\
& + \varepsilon_3 \frac{u_{j+1,n} - 3u_{j,n} + 3u_{j-1,n} - u_{j-2,n}}{(\Delta x)^3} \\
& + \varepsilon_4 \frac{u_{j+2,n} - 4u_{j+1,n} + 6u_{j,n} - 4u_{j-1,n} + u_{j-2,n}}{(\Delta x)^4} = b
\end{aligned} \tag{1.11}$$

for some constant b for all $j \in \mathbb{Z}$ and $n \in \mathbb{N}$. Notice that, with fixed n, the equation (1.11) of the conservation law is a recurrence relation.

Remark 2. Rather than use $V'(u_{j,n})(u_{j+1,n}-u_{j,n})/\Delta x$ to discretize $\partial_x V(u(x_{j,n}))$, we use $(V(u_{j+1,n}) - V(u_{j,n}))/\Delta x$. The virtue is that it gives (1.11). That is, it corresponds to preserving the following conservation law of the KdV-Burgers-KS type PDE:

$$\pm cv + V(v) + \varepsilon_1 v' + \varepsilon_2 v'' + \varepsilon_3 v''' + \varepsilon_4 v'''' = \text{constant}$$

for travelling waves of the form $u(x,t) = v(x \pm ct)$.

Our main result is the following.

Theorem 3. *There is an open set $\mathcal{CB} \subset \mathbb{R}^+ \times \mathbb{R}$ such that for any $(c,b) \in \mathcal{CB}$ there exists $\delta > 0$ such that the space-time discretized KdV-Burgers-KS type equation* (1.5) *possesses a set $\mathcal{U}$ of bounded travelling wave solutions of the form $u_{j,n+1} = u_{j+1,n}$ for all $j \in \mathbb{Z}$ and $n \in \mathbb{N}$ satisfying $\Delta x = c\Delta t$ and the conservation law* (1.11) *provided* $0 \le \max\{|\varepsilon_1|, |\varepsilon_2|, |\varepsilon_3|, |\varepsilon_4|\} < \delta$. *Moreover, let $\mathcal{I}$ be the set of all initial states from $\mathcal{U}$, then $\mathcal{I}$ is a Cantor set in the product topology of $\mathbb{R}^{\mathbb{Z}}$. The set $\mathcal{I}$ is invariant under the time-evolution T of* (1.5)*, and the system* (1.5) *is spatio-temporally chaotic on $\mathcal{I}$. In fact, the restriction $T|\mathcal{I}$ of T on $\mathcal{I}$ is a topological horseshoe.*

The open set $\mathcal{CB}$ in Theorem 3 can be very huge. For example, if (1.5) is the discretized system of the KdV equation (1.2), then $\mathcal{CB}$ is the set $\{(c,b)|\ c^2 + 2b > 0\}$. The proof of Theorem 3 will be postponed to Section 5, and the precise meaning of the theorem will be clear after Theorem 18, also located in Section 5. In next section, we review the notion of spatio-temporal chaos for CMLs. In Section 3, using the concept of anti-integrable limit, we prove the existence of the topological horseshoe $T|\mathcal{I}$ in the phase space of CML (1.5). We discuss and compare how the concept can be applied to discretized equations of the Fisher-KPP, sine-Gordon and KdV equations for steady-state and travelling wave solutions. Section 4 is devoted to the space-translation dynamics on $\mathcal{I}$. We show that for generic ε_1, ε_2, ε_3 and ε_4, the dynamics is topologically conjugate to the restriction of a four-dimensional Hénon-like map G_4 to a Cantor set $\mathcal{A} \subset \mathbb{R}^4$.

We emphasize that $T|\mathcal{I}$ is a topological horseshoe embedded as a subsystem in the infinite dimensional dynamical system (1.5), whereas $G_4|\mathcal{A}$ is a one in a finite dimensional dynamical system. We also remark that our results and approach can be extended easily to a space-time discretization of the following partial differential equations

$$\partial_t u + \partial_x V(u) + \sum_{i=2}^{k} \varepsilon_{i-1} \partial_x^i u = 0,$$

for any positive integer k.

2 Spatio-temporal chaos for CMLs

Let $T : (\mathbb{R}^d)^{\mathbb{Z}} \to (\mathbb{R}^d)^{\mathbb{Z}}$ be the *time-evolution operator* of a CML $(T, (\mathbb{R}^d)^{\mathbb{Z}})$ with phase space $(\mathbb{R}^d)^{\mathbb{Z}}$ defined by

$$T(\mathbf{v}) = \mathbf{w} = (\ldots, w_{-1}, w_0, w_1, \ldots)$$

with

$$w_j = \phi(v_{j-r}, v_{j-r+1}, \ldots, v_{j+r}) \qquad \forall j \in \mathbb{Z}$$

for some smooth function ϕ and positive integer r. Given $n \in \mathbb{N}$, suppose $\mathbf{u}_n \in (\mathbb{R}^d)^{\mathbb{Z}}$. We write

$$T(\mathbf{u}_n) = \mathbf{u}_{n+1} \quad \text{or} \quad u_{j,n+1} = \phi(u_{j-r,n}, u_{j-r+1,n}, \ldots, u_{j+r,n}).$$

Let $\sigma : (\mathbb{R}^d)^{\mathbb{Z}} \to (\mathbb{R}^d)^{\mathbb{Z}}$ be the usual shift operator, i.e.

$$\sigma(\mathbf{v}) = \mathbf{w} \quad \text{with} \quad w_j = v_{j+1} \; \forall j \in \mathbb{Z}.$$

When the space which σ acts on is a phase space, σ is also called the *space-translation operator*.

A definition of **spatio-temporal chaos** (or **space-time chaos**) for CMLs has been introduced by Bunimovich, Pesin and Sinai [6, 25], and applied to lattices of weakly interacting chaotic systems. It is based on invariant measures. The measure should be invariant with respect to both the time-evolution and space-translation operations, and should be mixing and be a physical one. In their definition, a CML is said to display

- **temporal chaos** if there exists a measure which is mixing and invariant under the $\mathbb{Z}^1$-action generated by the time-evolution operator T;
- **spatial chaos** if there exists a measure which is mixing and invariant under the $\mathbb{Z}^1$-action generated by the space-translation operator σ;
- **spatio-temporal chaos** if there is a measure which is mixing and invariant under the $\mathbb{Z}^2$-action generated by T and S.

In the current framework, a lattice of weakly interacting systems or a weakly coupled CML $(T_\epsilon, (\mathbb{R}^d)^{\mathbb{Z}})$ has the form

$$u_{j,n+1} = f(u_{j,n}) + \varepsilon g(u_{j-r,n}, u_{j-r+1,n}, \dots, u_{j+r,n}), \tag{2.1}$$

in which f, called the *local map*, and g are smooth maps, and ε is a small coupling parameter. Providing that ε is small and the map f is hyperbolic, it has been known, e.g. [24], that the system (2.1) displays the temporal chaos in the set of spatially-homogeneous solutions, i.e., solutions of the form $u_{j+1,n} = u_{j,n}$ $\forall j \in \mathbb{Z}$, $n \in \mathbb{N}$. In [10], Chow and Shen studied the following discretized Nagumo equation,

$$u_{j,n+1} = u_{j,n} + \beta u_{j,n}(u_{j,n} - a)(1 - u_{j,n}) + \varepsilon(u_{j-1,n} - 2u_{j,n} + u_{j+1,n}), \tag{2.2}$$

where $u_{j,n} \in \mathbb{R}$ for all $j \in \mathbb{Z}$, $n \in \mathbb{N}$, and showed that there are parameters β, $\varepsilon > 0$ and $0 < a < 1$ such that the system displays the spatial chaos on the set of steady-state solutions, i.e., solutions of the form $u_{j,n+1} = u_{j,n}$.

As a matter of fact, for the CML (2.2) there is a set of steady-state solutions on which the space-translation operator is topologically conjugate to the Bernoulli shift with two symbols.

A topological aspect of [6, 25] for spatio-temporal chaos has been given by Afraimovich and Fernandez [2] (see also [5]). In the current content, it is described by the following two-dimensional Markov scheme.

Assume f has a hyperbolic local maximal set Λ of chaotic orbits. So, the uncoupled system T_0 possesses an invariant set $\mathcal{I}_0 = \bigotimes_{\mathbb{Z}} \Lambda$. Because of the hyperbolicity, there corresponds a topological Markov chain (σ, Σ_A) with A the transition matrix. Let $\boldsymbol{\sigma}_T := \bigotimes_{\mathbb{Z}} \sigma$ and $\Omega := \bigotimes_{\mathbb{Z}} \Sigma_A$. Then, $(\boldsymbol{\sigma}_T, \Omega)$ is topologically conjugate to the time-evolution T_0 on $\mathcal{I}_0$. Note that $\boldsymbol{\sigma}_T((e_{j,n})_{(j,n)\in\mathbb{Z}\times\mathbb{N}}) = (\bar{e}_{j,n})_{(j,n)\in\mathbb{Z}\times\mathbb{N}}$ with $\bar{e}_{j,n} = e_{j,n+1}$. Define $\boldsymbol{\sigma}_S$ by $\boldsymbol{\sigma}_S((e_{j,n})_{(j,n)\in\mathbb{Z}\times\mathbb{N}}) = (\bar{e}_{j,n})_{(j,n)\in\mathbb{Z}\times\mathbb{N}}$ with $\bar{e}_{j,n} = e_{j+1,n}$. Then, one also has that $(\boldsymbol{\sigma}_S, \Omega)$ is topologically conjugate to the space-translation σ on $\mathcal{I}_0$.

There is a structural stability result for weakly interacting CMLs. Under some hyperbolicity conditions for the local map, it was proved that for small ε there is a T_ε-invariant set $\mathcal{I}_\varepsilon$ on which the above structure survives.

It has been shown that CMLs restricted to the set of travelling wave solutions may display spatio-temporal chaos [3] (see also [24]). In contrast to [2, 6, 25], where the set displaying spatio-temporal chaos is infinite dimensional, the spatio-temporal chaotic set constructed from travelling waves in [3, 24] is of finite or zero dimension. From the viewpoint of the symbolic dynamics, the set of travelling wave solutions corresponds to a subset in the symbolic space: For example, the product $\Omega = \bigotimes_{\mathbb{Z}}\{0,1\}^{\mathbb{Z}} = \{0,1\}^{\mathbb{Z}^2}$ is invariant and spatio-temporally chaotic under the $\mathbb{Z}^2$-action generated by $\boldsymbol{\sigma}_T$ and $\boldsymbol{\sigma}_S$. Consider the subspace $\widetilde{\Omega} = \{(e_{j,n})_{(j,n)\in\mathbb{Z}\times\mathbb{N}} |\ e_{j,n+1} = e_{j+1,n}\ \forall (j,n) \in \mathbb{Z}\times\mathbb{N}\}$. It is clear that $\widetilde{\Omega}$ is also invariant under the $\mathbb{Z}^2$-action of $\boldsymbol{\sigma}_T$ and $\boldsymbol{\sigma}_S$. Furthermore, the restriction of $\boldsymbol{\sigma}_T$ as well as $\boldsymbol{\sigma}_S$ to $\widetilde{\Omega}$ are both topologically conjugate to the Bernoulli shift $(\sigma, \{0,1\}^{\mathbb{Z}})$. In virtue of this, in this paper we employ the following definitions:

Definition 4. A CML is said to admit

- **temporal chaos** if there is a set $\mathcal{I} \subset (\mathbb{R}^d)^{\mathbb{Z}}$ invariant under the $\mathbb{Z}^2$-action of T and σ and if there exists a topological Markov chain Σ_T of positive topological entropy such that the following diagram commutes

$$\begin{array}{ccc} \Sigma_T & \xrightarrow{\sigma} & \Sigma_T \\ h_T \Big\downarrow & & \Big\downarrow h_T \\ \mathcal{I} & \xrightarrow{T^l} & \mathcal{I} \end{array} \tag{2.3}$$

  for some positive integer l and a homeomorphism h_T with the product topology;

- **spatial chaos** if there is a set $\mathcal{I} \subset (\mathbb{R}^d)^{\mathbb{Z}}$ invariant under the $\mathbb{Z}^2$-action of T and σ and if there exists a topological Markov chain Σ_S of positive topological entropy such that the following diagram commutes

$$\begin{array}{ccc} \Sigma_S & \xrightarrow{\sigma} & \Sigma_S \\ h_S \big\downarrow & & \big\downarrow h_S \\ \mathcal{I} & \xrightarrow{\sigma^k} & \mathcal{I} \end{array} \tag{2.4}$$

for some positive integer k and a homeomorphism h_S with the product topology;

- **spatio-temporal chaos** if there is a set $\mathcal{I} \subset (\mathbb{R}^d)^{\mathbb{Z}}$ invariant under the $\mathbb{Z}^2$-action of T and σ and if there exist topological Markov chains Σ_T and Σ_S of positive topological entropy such that both diagrams (2.3) and (2.4) are commutative for some positive integers l, k and homeomorphisms h_T, h_S with the product topology.

Let $(\mathbf{u}_n)_{n\in\mathbb{N}}$ be a solution of a CML. Define P to be the projection to its initial state

$$P : (\mathbf{u}_n)_{n\in\mathbb{N}} \mapsto \mathbf{u}_0.$$

The example below points out that there are subsets consisting of bounded solutions of a CML on which the CML displays either spatial or temporal chaos, but not both.

Example 5. An uncoupled identical logistic map lattice $u_{j,n+1} = 5u_{j,n}(1 - u_{j,n})$, $j \in \mathbb{Z}$, $n \in \mathbb{N}$, has a set $\mathcal{U}_S$ of bounded steady-state solutions:

$$\mathcal{U}_S = \left\{ (u_{j,n})_{(j,n)\in\mathbb{Z}\times\mathbb{N}} \middle|\, u_{j,n+1} = u_{j,n} \in \{0, 4/5\} \right\}.$$

The system is spatially chaotic on the set $\mathcal{I}_S = P(\mathcal{U}_S)$. Although the set $\mathcal{I}_S$ is invariant under T, but every element in $\mathcal{I}_S$ is a fixed point of T. The system has also a set $\mathcal{U}_T$ of bounded spatially homogeneous solutions:

$$\mathcal{U}_T = \left\{ (u_{j,n})_{(j,n)\in\mathbb{Z}\times\mathbb{N}} \middle|\, u_{j+1,n} = u_{j,n} \in \Lambda_5 \right\},$$

where Λ_5 is the invariant set consisting of the initial points of bounded orbits of the logistic map $w \mapsto 5w(1-w)$. The system is temporally chaotic on the set $\mathcal{I}_T = P(\mathcal{U}_T)$. Although the set $\mathcal{I}_T$ is invariant under σ, every element in $\mathcal{I}_T$ is a fixed point of σ.

Example 6. The uncoupled logistic map lattice in the above example is spatio-temporally chaotic on the set $\mathcal{I} = P(\mathcal{U})$ of initial states of bounded solutions

$$\mathcal{U} = \left\{(u_{j,n})_{(j,n)\in\mathbb{Z}\times\mathbb{N}} |\ u_{j,n} \in \Lambda_5\right\}$$

in the sense of [2, 6, 25]. Let

$$\widetilde{\mathcal{U}} = \left\{(u_{j,n})_{(j,n)\in\mathbb{Z}\times\mathbb{N}} |\ u_{j,n+1} = u_{j+1,n} \in \Lambda_5\right\},$$

be a set of bounded travelling wave solutions, then the system is spatio-temporally chaotic on $P(\widetilde{\mathcal{U}})$ in the sense of [3, 24] and Definition 4.

3 Topological horseshoes via the anti-integrable limit

3.1 Existence of travelling wave solutions

A solution of CML (1.5) can be obtained once a solution of the conservation law (1.11) is found: If the sequence $(u_{j,n_0})_{j\in\mathbb{Z}}$ solves (1.11) for some $n = n_0 \in \mathbb{N}$, then

$$u_{j,n_0+1} = u_{j+1,n_0} \quad \text{and} \quad u_{j,n_0-1} = u_{j-1,n_0} \quad \forall j \in \mathbb{Z}$$

by (1.8). Automatically, the sequence $(u_{j,n_0+1})_{j\in\mathbb{Z}}$ or $(u_{j,n_0-1})_{j\in\mathbb{Z}}$ fulfills (1.11) for $n = n_0+1$ or n_0-1, respectively. Hence, the question of finding travelling wave solutions $(u_{j,n})_{(j,n)\in\mathbb{Z}\times\mathbb{N}}$ of (1.5) amounts to finding solutions $(u_{j,n_0})_{j\in\mathbb{Z}}$ of (1.11) for an arbitrary fixed n_0. We set $n_0 = 0$.

Let $(\mathbb{R}^d)^{\mathbb{Z}}$ be the space of sequences $\mathbf{v} = (\ldots, v_{-1}, v_0, v_1, \ldots) \in \mathbb{R}^d$, and $\sigma : (\mathbb{R}^d)^{\mathbb{Z}} \to (\mathbb{R}^d)^{\mathbb{Z}}$ be the shift, i.e. $\sigma(\mathbf{v}) = (\sigma(\mathbf{v})_i)_{i\in\mathbb{Z}}$ with $\sigma(\mathbf{v})_i = v_{i+1}$ for all $i \in \mathbb{Z}$. Let $\|\cdot\|_\infty$ be the supremum norm, $\|\mathbf{v}\|_\infty = \sup_{i\in\mathbb{Z}} |v_i|$, and $\|\cdot\|$ be the norm defined by $\|\mathbf{v}\| = \sum_{i\in\mathbb{Z}} |v_i|/2^{|i|}$. Let

$$l_\infty(\mathbb{R}^d) = \{\mathbf{v} \in (\mathbb{R}^d)^{\mathbb{Z}} |\ \|\mathbf{v}\|_\infty < \infty\}$$

and

$$\mathcal{M}(\mathbb{R}^d) := \{\mathbf{v} \in (\mathbb{R}^d)^{\mathbb{Z}} |\ \|\mathbf{v}\| < \infty\}.$$

Note that both $l_\infty(\mathbb{R}^d)$ and $\mathcal{M}(\mathbb{R}^d)$ are Banach spaces, and that the former has the uniform topology, while the latter has the product topology. Let

$\mu > 0$. Denote by $B(0, \mu)$ and $U(0, \mu)$ the closed balls of radius μ centered at the origin in $l_\infty(\mathbb{R}^d)$ and $\mathcal{M}(\mathbb{R}^d)$, respectively. As subsets of $(\mathbb{R}^d)^{\mathbb{Z}}$, it is clear that $B(0, \mu) \subset U(0, 3\mu)$.

For given c and b, define a map $F(\cdot, \cdot; c, b) : l_\infty(\mathbb{R}) \times \mathbb{R}^4 \to l_\infty(\mathbb{R})$ by

$$F(\mathbf{v}, \varepsilon; c, b) = (F_j(\mathbf{v}, \varepsilon; c, b))_{j \in \mathbb{Z}},$$

with

$$\varepsilon = (\varepsilon_1, \varepsilon_2, \varepsilon_3, \varepsilon_4)$$

and

$$\begin{aligned} F_j(\mathbf{v}, \varepsilon; c, b) &= cv_j + V(v_j) + \varepsilon_1 \frac{v_j - v_{j-1}}{\Delta x} + \varepsilon_2 \frac{v_{j+1} - 2v_j + v_{j-1}}{(\Delta x)^2} \\ &+ \varepsilon_3 \frac{v_{j+1} - 3v_j + 3v_{j-1} - v_{j-2}}{(\Delta x)^3} \\ &+ \varepsilon_4 \frac{v_{j+2} - 4v_{j+1} + 6v_j - 4v_{j-1} + v_{j-2}}{(\Delta x)^4} - b \end{aligned} \tag{3.1}$$

for all $j \in \mathbb{Z}$. Let $\mathbf{u}_n \in l_\infty(\mathbb{R})$ with $\mathbf{u}_n = (\ldots, u_{-1,n}, u_{0,n}, u_{1,n}, \ldots)$. From (1.11), we have the following result. Its proof is easy, thus we omit it.

Proposition 7. *Assume $\varepsilon \neq 0$. The discretized KdV-Burgers-KS type equation* (1.5) *admits a bounded travelling wave solution $(\mathbf{u}_n)_{n \in \mathbb{N}}$ propagating to the left with speed c if and only if there is a constant b such that $F(\mathbf{u}_0, \varepsilon; c, b) = 0$.*

Becuase $V(v)$ is a polynomial, there is a nonempty open set $\mathcal{CB} \subset \mathbb{R}^+ \times \mathbb{R}$ such that for all $(c, b) \in \mathcal{CB}$ the algebraic equation $cv + V(v) = b$ for v has at least two distinct roots and all of the roots are non-degenerate. For given $(c, b) \in \mathcal{CB}$, let $S_{c,b}$ be the sets consisting of non-degenerate roots of $cv + V(v) = b$. Define a subset $\Sigma_{c,b} \subset l_\infty(\mathbb{R})$ by

$$\Sigma_{c,b} := \{\mathbf{v} = (v_j)_{j \in \mathbb{Z}} |\ v_j \in S_{c,b}\ \forall j \in \mathbb{Z}\}.$$

Example 8. For the KdV, KS, and Burgers equations, (1.2)-(1.4), we have

$$\begin{aligned} \mathcal{CB} &= \{(c, b) |\ c^2 + 2b > 0\}, \\ \Sigma_{c,b} &= \{(v_j)_{j \in \mathbb{Z}} |\ v_j = c - \sqrt{c^2 + 2b} \text{ or } c + \sqrt{c^2 + 2b} \text{ with } c^2 + 2b > 0\}. \end{aligned}$$

Proposition 9. $\mathbf{u}_0^\dagger \in l_\infty(\mathbb{R})$ *satisfies* $F(\mathbf{u}_0^\dagger, 0; c, b) = 0$ *if* $\mathbf{u}_0^\dagger \in \Sigma_{c,b}$.

Proof. $F_j(\mathbf{v}, 0; c, b) = cv_j + V(v_j) - b$ for any integer j. Thus, $F(\mathbf{u}_0^\dagger, 0; c, b) = 0$ if $u_{j,0}^\dagger \in S_{c,b}$ for all $j \in \mathbb{Z}$. □

Lemma 10. *Providing* $(c, b) \in \mathcal{CB}$, *the linear operators* $D_{\mathbf{v}}F(\mathbf{u}_0^\dagger, 0; c, b)$ *are invertible whenever* $\mathbf{u}_0^\dagger \in \Sigma_{c,b}$. *Moreover,*

$$\begin{aligned} \inf\left\{|c + V'(u_{j,0}^\dagger)|^{-1} :\ u_{j,0}^\dagger \in S_{c,b}\right\} &\leq \|D_{\mathbf{v}}F(\mathbf{u}_0^\dagger, 0; c, b)^{-1}\| \\ &\leq \sup\left\{|c + V'(u_{j,0}^\dagger)|^{-1} :\ u_{j,0}^\dagger \in S_{c,b}\right\}. \end{aligned}$$

Proof. $D_{\mathbf{v}}F(\mathbf{v}, \varepsilon; c, b)$ is a linear operator of $l_\infty(\mathbb{R})$,

$$\boldsymbol{\xi} = (\xi_k)_{k\in\mathbb{Z}} \mapsto \left(\sum_{k\in\mathbb{Z}} D_{v_k}F_j(\mathbf{v}, \varepsilon; c, b)\xi_k\right)_{j\in\mathbb{Z}}.$$

Simple calculation shows that

$$D_{v_k}F_j(\mathbf{v}, \varepsilon; c, b) = \begin{cases} \frac{\varepsilon_4}{(\Delta x)^4} & \text{if } k = j+2 \\ \frac{\varepsilon_2}{(\Delta x)^2} + \frac{\varepsilon_3}{(\Delta x)^3} - 4\frac{\varepsilon_4}{(\Delta x)^4} & \text{if } k = j+1 \\ c + V'(v_j) + \frac{\varepsilon_1}{\Delta x} - 2\frac{\varepsilon_2}{(\Delta x)^2} - 3\frac{\varepsilon_3}{(\Delta x)^3} + 6\frac{\varepsilon_4}{(\Delta x)^4} & \text{if } k = j \\ -\frac{\varepsilon_1}{\Delta x} + \frac{\varepsilon_2}{(\Delta x)^2} + 3\frac{\varepsilon_3}{(\Delta x)^3} - 4\frac{\varepsilon_4}{\Delta x^4} & \text{if } k = j-1 \\ -\frac{\varepsilon_3}{(\Delta x)^3} + \frac{\varepsilon_4}{(\Delta x)^4} & \text{if } k = j-2 \\ 0 & \text{otherwise.} \end{cases}$$

It is then easy to see that the linear operators $D_{\mathbf{v}}F(\mathbf{u}_0^\dagger, 0; c, b)$ are invertible if $\mathbf{u}_0^\dagger \in \Sigma_{c,b}$, because each operator can be represented as an infinite diagonal matrix with non-zero entries $c + V'(u_{j,0}^\dagger)$ for all $j \in \mathbb{Z}$. □

Example 11. For the KdV equation (1.2),

$$D_{\mathbf{v}}F(\mathbf{v}, \varepsilon; c, b) = \\ \begin{pmatrix} & \vdots & \vdots & \vdots & \\ \cdots & c + v_{-1} - 2\frac{\varepsilon_2}{(\Delta x)^2} & \frac{\varepsilon_2}{(\Delta x)^2} & 0 & \cdots \\ \cdots & \frac{\varepsilon_2}{(\Delta x)^2} & c + v_0 - 2\frac{\varepsilon_2}{(\Delta x)^2} & \frac{\varepsilon_2}{(\Delta x)^2} & \cdots \\ \cdots & 0 & \frac{\varepsilon_2}{(\Delta x)^2} & c + v_1 - 2\frac{\varepsilon_2}{(\Delta x)^2} & \cdots \\ & \vdots & \vdots & \vdots & \end{pmatrix},$$

as it is realized in matrix form. And, $D_{\mathbf{v}}F(\mathbf{u}_0^\dagger, 0; c, b)$ is invertible if $\mathbf{u}_0^\dagger \in \Sigma_{c,b}$ with $c^2 + 2b > 0$ because the infinite matrix above is diagonal with entries $-\sqrt{c^2+2b}$ or $\sqrt{c^2+2b}$.

By virtue of Lemma 10, using the theory of anti-integrability (see [4, 8, 20] and Theorem 19 in the Appendix), we conclude that

Proposition 12. *Assume $(c, b) \in \mathcal{CB}$. There exists $\delta > 0$ and a unique function $\Phi_\varepsilon : \Sigma_{c,b} \to \mathbb{R}^{\mathbb{Z}}$ such that $\Phi_\varepsilon(\mathbf{u}_0^\dagger)$ is a bounded sequence satisfying $\Phi_0(\Sigma_{c,b}) = \Sigma_{c,b}$ and $F(\Phi_\varepsilon(\mathbf{u}_0^\dagger), \varepsilon; c, b) = 0$ for any $\mathbf{u}_0^\dagger \in \Sigma_{c,b}$ provided $0 \leq |\varepsilon| < \delta$. Let*

$$\mathcal{I} := \Phi_\varepsilon(\Sigma_{c,b}).$$

With the metric $\|\cdot\|$, the map Φ_ε depends C^1 on ε and is a homeomorphism from $\Sigma_{c,b}$ to $\mathcal{I}$. Moreover, the following diagram commutes

$$\begin{array}{ccc} \Sigma_{c,b} & \xrightarrow{\sigma} & \Sigma_{c,b} \\ {\scriptstyle \Phi_\varepsilon}\big\downarrow & & \big\downarrow{\scriptstyle \Phi_\varepsilon} \\ \mathcal{I} & \xrightarrow{\sigma} & \mathcal{I}. \end{array}$$

Proof. Because $S_{c,b}$ is a finite set, Lemma 10 implies that $\|D_{\mathbf{v}}F(\mathbf{u}_0^\dagger, 0; c, b)^{-1}\|$ is bounded above on $\Sigma_{c,b} \times \{0\}$. From the definition (3.1) of F, it is easy to verify all other assumptions in (i) and (ii) of Theorem 19. For assumptions in (iii) of Theorem 19, note that all points in $\Sigma_{c,b}$ are isolated with the metric $\|\cdot\|_\infty$, but $\Sigma_{c,b}$ is a Cantor set thus compact with the metric $\|\cdot\|$. Hence, to prove the proposition, it remains to check the continuity of $F(\cdot, \varepsilon; c, b)$, or equivalently to check the continuity of $F_j(\cdot, \varepsilon; c, b)$ for all $j \in \mathbb{Z}$. But $F_j(\cdot, \varepsilon; c, b)$ certainly depends continuously on $\mathbf{v}$ for every $j \in \mathbb{Z}$ because it depends on only finitely many components of $\mathbf{v}$. Namely, it is a function of the form $F_j(\cdot, \varepsilon; c, b) = \chi(u_{j-r}, \ldots, u_j, \ldots, u_{j+r})$ for some continuous function $\chi : \mathbb{R}^{2r+1} \to \mathbb{R}$ and fixed $r \in \mathbb{N}$. □

3.2 Applicability: strong nonlinearity and weak diffusion or dispersion

The method of anti-integrable limit has been successfully applied to prove the existence of spatially chaotic steady-state or travelling wave solutions in a variety of CMLs. In this subsection, we discuss and compare how this method is applied to discretized Fisher-KPP, sine-Gordon, and KdV equations. We shall see that, in some situations, strong nonlinearity of local map is equivalent to weak coupling, whereas in some situations, they are not.

Consider first the following CML

$$\frac{\alpha}{\beta}\frac{u_{j,n+1}-u_{j,n}}{\Delta t}+u_{j,n}(1-u_{j,n})+\frac{\varepsilon_1}{\beta}\frac{u_{j+1,n}-2u_{j,n}+u_{j-1,n}}{(\Delta x)^2}=0, \qquad (3.2)$$

which is a space-time discretization of the Fisher-Kolmogorov-Petrovskii-Piscounov (Fisher-KPP) equation [12, 18] of the form

$$\alpha\partial_t u+\beta u(1-u)+\varepsilon_1\partial_x^2 u=0. \qquad (\alpha,\beta,\varepsilon_1\neq 0) \qquad (3.3)$$

Note that (3.2) has the form of (2.1) and that (3.3) can be transformed to

$$\partial_t u+u(1-u)+\partial_x^2 u=0$$

by rescaling

$$x\to(\varepsilon_1/\beta)^{1/2}x \quad \text{and} \quad t\to(\alpha/\beta)t. \qquad (3.4)$$

With fixed Δt and Δx, when restricted to the steady-state solutions (i.e. $u_{j,n+1}=u_{j,n}\ \forall j\in\mathbb{Z},\ n\in\mathbb{N}$), (3.2) reduces to the algebraic equation $u_{j,n}(1-u_{j,n})=0$ as $\varepsilon_1/\beta\to 0$. Because all the roots of the algebraic equation are simple, the theory of anti-integrable limit implies that the CML (3.2) possesses spatially chaotic steady-state solutions when ε_1/β is sufficiently small. We emphasize that, with finite α in this case, the situation of large β and moderate ε_1 is equivalent to the situation of moderate β and small ε_1.

With (1.7) for travelling wave solutions, (3.2) gives rise to

$$\frac{c\alpha}{\beta}\frac{u_{j+1,n}-u_{j,n}}{\Delta x}+u_{j,n}(1-u_{j,n})+\frac{\varepsilon_1}{\beta}\frac{u_{j+1,n}-2u_{j,n}+u_{j-1,n}}{(\Delta x)^2}=0. \qquad (3.5)$$

If we employ the definition of travelling wave with speed $c=p/q$ as in Remark 1, from (3.2) we obtain

$$\frac{\alpha}{\beta}\frac{w_{i+p}-w_i}{\Delta t}+w_i(1-w_i)+\frac{\varepsilon_1}{\beta}\frac{w_{i+q}-2w_i+w_{i-q}}{(\Delta x)^2}=0, \qquad (3.6)$$

where

$$qj+pn=i \quad \text{and} \quad u_{j,n}=\Gamma(qj+pn)=w_i. \qquad (3.7)$$

It is plain to see that the limit $(\alpha/\beta, \varepsilon_1/\beta) \to (0,0)$ is a non-degenerate anti-integrable limits for both recurrence relations (3.5) and (3.6). Hence, providing $(\alpha/\beta, \varepsilon_1/\beta)$ is sufficiently small, via (3.5), CML (3.2) admits spatially chaotic travelling waves of speed c having the form $u_{j,n+1} = u_{j+1,n}$, while via (3.6), it admits spatially chaotic travelling waves of speed p/q having the form $u_{j,n} = u_{j-p,n+q}$.

In our discretization scheme, the sine-Gordon equation of the form

$$\alpha \partial_t^2 u + \beta \sin u + \varepsilon_1 \partial_x^2 u = 0 \qquad (\alpha, \beta, \varepsilon_1 \neq 0) \tag{3.8}$$

becomes

$$\frac{\alpha}{\beta} \frac{u_{j,n+1} - 2u_{j,n} + u_{j,n-1}}{(\Delta t)^2} + \sin u_{j,n} + \frac{\varepsilon_1}{\beta} \frac{u_{j+1,n} - 2u_{j,n} + u_{j-1,n}}{(\Delta x)^2} = 0. \tag{3.9}$$

Note that (3.8) can be transformed to

$$\partial_t^2 u + \sin u - \partial_x^2 u = 0$$

by rescaling

$$x \to (-\varepsilon_1/\beta)^{1/2} x \quad \text{and} \quad t \to (\alpha/\beta)^{1/2} t, \tag{3.10}$$

provided $\varepsilon_1 < 0$. With (1.7) for travelling waves, (3.9) leads to

$$\sin u_{j,n} + \frac{c^2\alpha + \varepsilon_1}{\beta} \frac{u_{j+1,n} - 2u_{j,n} + u_{j-1,n}}{(\Delta x)^2} = 0. \tag{3.11}$$

Note that the case $c = 0$ is the case of steady-state solutions. With (1.10) and (3.7) for travelling waves, (3.9) gives

$$\frac{\alpha}{\beta} \frac{w_{i+p} - 2w_i + w_{i-p}}{(\Delta t)^2} + \sin w_i + \frac{\varepsilon_1}{\beta} \frac{w_{i+q} - 2w_i + w_{i-q}}{(\Delta x)^2} = 0. \tag{3.12}$$

The recurrence relation (3.11), which may also be obtained from the steady-state solutions of the Frenkel-Kontorova model, corresponds to the standard map, and is the one studied in [4, 20] (see also [30]). The limit $(c^2\alpha + \varepsilon_1)/\beta \to 0$ is an anti-integrable limit and (3.11) admits chaotic solutions when $(c^2\alpha + \varepsilon_1)/\beta$ is small enough. In a similar manner, the recurrence relation (3.12) possesses chaotic solutions if α/β and ε_1/β are sufficiently small.

Let us return back to KdV equations. Note first that all KdV equations of the form

$$\alpha\partial_t u + \beta u\partial_x u + \varepsilon_2\partial_x^3 u = 0 \qquad (\alpha, \beta, \varepsilon_2 \neq 0) \tag{3.13}$$

can be transformed to

$$\partial_t u + u\partial_x u + \partial_x^3 u = 0$$

by rescaling

$$u \to (\varepsilon_2/\beta^3)^{1/5}u, \quad x \to (\varepsilon_2^2/\beta)^{1/5}x, \quad \text{and} \quad t \to \alpha(\varepsilon_2/\beta^3)^{1/5}t. \tag{3.14}$$

For the KdV equation (3.13), equation (1.5) ought to be

$$\frac{\alpha}{\beta}\frac{u_{j,n+1} - u_{j,n}}{\Delta t} + \frac{u_{j+1,n}^2 - u_{j,n}^2}{2\Delta x} + \frac{\varepsilon_2}{\beta}\frac{u_{j+2,n} - 3u_{j+1,n} + 3u_{j,n} - u_{j-1,n}}{(\Delta x)^3} = 0, \tag{3.15}$$

and the conservation law (1.11) becomes

$$\frac{c\alpha}{\beta}u_{j,n} + \frac{1}{2}u_{j,n}^2 + \frac{\varepsilon_2}{\beta}\frac{u_{j+1,n} - 2u_{j,n} + u_{j-1,n}}{(\Delta x)^2} = b. \tag{3.16}$$

If we use the notion of travelling wave as described in Remark 1 and use (3.7), then (1.5) gives

$$\frac{\alpha}{\beta}\frac{w_{i+p} - w_i}{\Delta t} + \frac{w_{i+q}^2 - w_i^2}{2\Delta x} + \frac{\varepsilon_2}{\beta}\frac{w_{i+2q} - 3w_{i+q} + 3w_i - w_{i-q}}{(\Delta x)^3} = 0, \tag{3.17}$$

which has the following conservation law

$$\frac{\alpha}{\beta}\frac{w_i}{\Delta t} + \frac{1}{2}\frac{w_i^2}{\Delta x} + \frac{\varepsilon_2}{\beta}\frac{w_{i+1} - 2w_i + w_{i-1}}{(\Delta x)^3} = b \qquad \forall i \in \mathbb{Z} \tag{3.18}$$

if $p = q = 1$. Like the discretized reaction-diffusion equation case (3.5), the limit $(\alpha/\beta, \varepsilon_2/\beta) \to (0, 0)$ is a non-degenerate anti-integrable limit if $b \neq 0$. However, what unlike (3.5) is that the limit $\varepsilon_2/\beta \to 0$ can also be a non-degenerate anti-integrable limit with suitable choices of c, α, β and b. For recurrence relation (3.5), both α and ε_1 must be small so as to use the method of anti-integrable limit to prove the existence of solutions for moderate β, whereas for CML (3.15) or recurrence relation (3.17), with the help of conservation law (3.16) or (3.18) respectively, travelling waves can exist for arbitrarily large α (as implied by Example 8).

4 Spatial profile and space-translation dynamics

We have seen that bounded travelling wave solutions of (1.5) at time $t = t_n$ must fulfill the conservation law (1.11). In other words, they must fulfill $F(\mathbf{u}_n, \varepsilon; c, b) = 0$. In this section, we shall concentrate our study on the spatial profile $\mathbf{u}_n = (u_{j,n})_{j\in\mathbb{Z}}$ for a fixed n. Without loss of generality, we let $n = 0$. For the sake of simplicity of notation, we let $u_{j,0} = v_j$ for all $j \in \mathbb{Z}$.

Let

$$\bar{\varepsilon}_1 = \frac{\varepsilon_1}{\Delta x}, \quad \bar{\varepsilon}_2 = \frac{\varepsilon_2}{(\Delta x)^2}, \quad \bar{\varepsilon}_3 = \frac{\varepsilon_3}{(\Delta x)^3}, \quad \bar{\varepsilon}_4 = \frac{\varepsilon_4}{(\Delta x)^4}.$$

Subsequently, it is clear from the conservation law (1.11) that the bounded travelling wave solutions are equivalent to the bounded orbits of the following four-dimensional Hénon-like map $G_4 : \mathbb{R}^4 \to \mathbb{R}^4$, $(v_{j-1}, v_j, v_{j+1}, v_{j+2}) \mapsto (v_{j-2}, v_{j-1}, v_j, v_{j+1})$, with

$$\begin{aligned}(\bar{\varepsilon}_3 - \bar{\varepsilon}_4)v_{j-2} &= -b + cv_j + V(v_j) - (\bar{\varepsilon}_1 - \bar{\varepsilon}_2 - 3\bar{\varepsilon}_3 + 4\bar{\varepsilon}_4)v_{j-1} \\ &\quad +(\bar{\varepsilon}_1 - 2\bar{\varepsilon}_2 - 3\bar{\varepsilon}_3 + 6\bar{\varepsilon}_4)v_j + (\bar{\varepsilon}_2 - \bar{\varepsilon}_3 - 4\bar{\varepsilon}_4)v_{j+1} + \bar{\varepsilon}_4 v_{j+2}\end{aligned}$$

if $\bar{\varepsilon}_3 - \bar{\varepsilon}_4 \neq 0$. If $\bar{\varepsilon}_3 \neq 0$ and $\bar{\varepsilon}_4 = 0$, (1.11) is equivalent to the following three-dimensional Hénon-like map $G_3 : \mathbb{R}^3 \to \mathbb{R}^3$, $(v_{j-1}, v_j, v_{j+1}) \mapsto (v_{j-2}, v_{j-1}, v_j)$, with

$$\begin{aligned}\bar{\varepsilon}_3 v_{j-2} &= -b + cv_j + V(v_j) - (\bar{\varepsilon}_1 - \bar{\varepsilon}_2 - 3\bar{\varepsilon}_3)v_{j-1} \\ &\quad +(\bar{\varepsilon}_1 - 2\bar{\varepsilon}_2 - 3\bar{\varepsilon}_3)v_j + (\bar{\varepsilon}_2 - \bar{\varepsilon}_3)v_{j+1}.\end{aligned}$$

If $\bar{\varepsilon}_1 - \bar{\varepsilon}_2 \neq 0$ and $\bar{\varepsilon}_3 = \bar{\varepsilon}_4 = 0$, it is equivalent to the following Hénon-like map $G_2 : \mathbb{R}^2 \to \mathbb{R}^2$,$(v_j, v_{j+1}) \mapsto (v_{j-1}, v_j)$, with

$$(\bar{\varepsilon}_1 - \bar{\varepsilon}_2)v_{j-1} = -b + cv_j + V(v_j) + (\bar{\varepsilon}_1 - 2\bar{\varepsilon}_2)v_j + \bar{\varepsilon}_2 v_{j+1}.$$

If $\bar{\varepsilon}_1 \neq 0$ and $\bar{\varepsilon}_2 = \bar{\varepsilon}_3 = \bar{\varepsilon}_4 = 0$, then it is equivalent to the following one dimensional map $G_1 : \mathbb{R}^1 \to \mathbb{R}^1$, $v_j \mapsto v_{j-1}$, with

$$\bar{\varepsilon}_1 v_{j-1} = -b + (\bar{\varepsilon}_1 + c)v_j - V(v_j).$$

Example 13. For the KdV equation, with $\bar{\varepsilon}_1 = \bar{\varepsilon}_3 = \bar{\varepsilon}_4 = 0$ and $V(v) = v^2/2$, by setting $b = 1/2$ and $c = 2\bar{\varepsilon}_2$, the map G_2 is an area-preserving Hénon map of the form

$$(v_j, v_{j+1}) \mapsto (\frac{1}{2\bar{\varepsilon}_2}(1 - v_j^2) - v_{j+1}, v_j).$$

Example 14. For the Burgers equation, with $\bar{\varepsilon}_2 = \bar{\varepsilon}_3 = \bar{\varepsilon}_4 = 0$ and $V(v) = v^2/2$, setting $b = -1/2$ and $c = -\bar{\varepsilon}_1$, we obtain a quadratic map of the form

$$v_j \mapsto \frac{1}{2\bar{\varepsilon}_1}(1 - v_j^2)$$

for the map G_1.

We call the map G_1, G_2, G_3 or G_4 the *travelling wave map* associated with the conservation law (1.11) (see also [3, 24]). The above analysis shows that the travelling wave map is, though generically, not always defined.

Remark 15. If $\bar{\varepsilon}_3 - \bar{\varepsilon}_4 \neq 0$ and $\bar{\varepsilon}_4 \neq 0$, the map G_4 is a diffeomorphism of $\mathbb{R}^4$. If $\bar{\varepsilon}_2 - \bar{\varepsilon}_3 \neq 0$ and $\bar{\varepsilon}_3 \neq 0$, the map G_3 is a diffeomorphism of $\mathbb{R}^3$. If $\bar{\varepsilon}_1 - \bar{\varepsilon}_2 \neq 0$ and $\bar{\varepsilon}_2 \neq 0$, the map G_2 is a diffeomorphism of $\mathbb{R}^2$. The map G_1 cannot be a diffeomorphism because we have assumed that the polynomial $V(v)$ is at least quadratic in v.

When $\bar{\varepsilon}_1 = \bar{\varepsilon}_2 = \bar{\varepsilon}_3 = \bar{\varepsilon}_4 = 0$, the recurrence relations $F_j(\mathbf{v}, \varepsilon; c, b) = 0$ reduce to algebraic equations $cv_j + V(v_j) - b = 0$ for all integer j. When $(c, b) \in \mathcal{CB}$, one can solve the algebraic equations to get $v_j \in S_{c,b}$ for all $j \in \mathbb{Z}$. Hence, the limiting situation $\bar{\varepsilon}_1 = \bar{\varepsilon}_2 = \bar{\varepsilon}_3 = \bar{\varepsilon}_4 = 0$ (or $\epsilon = 0$) is the anti-integrable limit. Given $\mathbf{v} = (v_j)_{j\in\mathbb{Z}}$ or $(v_j)_{j\in\mathbb{N}}$, let π_d be the projection

$$\pi_d : \mathbf{v} \mapsto \begin{cases} v_0 & \text{if } d = 1 \\ (v_0, v_1) & \text{if } d = 2 \\ (v_0, v_1, v_2) & \text{if } d = 3 \\ (v_0, v_1, v_2, v_3) & \text{if } d = 4 \end{cases}$$

and define

$$\mathcal{A} = \pi_d \circ \Phi_\varepsilon(\Sigma_{c,b}).$$

Proposition 16. *Suppose the travelling wave map G_d of $\mathbb{R}^d$ ($1 \le d \le 4$) is well-defined. For any $(c, b) \in \mathcal{CB}$, there is a positive δ such that G_d admits a horseshoe $\mathcal{A}$ of N symbols as a subsystem, with N equal to the cardinality of $S_{c,b}$, provided $0 < |\varepsilon| < \delta$. More precisely, with the metric $\|\cdot\|$, the composition $\pi_d \circ \Phi_\varepsilon$ is a topological conjugacy between $\Sigma_{c,b}$ and $\mathcal{A}$, and the following diagram commutes*

$$\begin{array}{ccc} \Sigma_{c,b} & \xrightarrow{\sigma} & \Sigma_{c,b} \\ {\scriptstyle \pi_d\circ\Phi_\varepsilon}\downarrow & & \downarrow{\scriptstyle \pi_d\circ\Phi_\varepsilon} \\ \mathcal{A} & \xrightarrow{G_d} & \mathcal{A}. \end{array}$$

Proof. Proposition 16 is a consequence of Proposition 12. Let $\mathbf{u}_0 \in \mathcal{I} = \Phi_\epsilon(\Sigma_{c,b})$. Because an orbit of G_d can be generated from $\mathbf{u}_0$ and is uniquely determined by its initial point $\pi_d(\mathbf{u}_0)$, the projection π_d is a continuous bijection thus a homeomorphism from $\mathcal{I}$ to $\mathcal{A}$. It is clear that $G_d \circ \pi_d(\mathbf{u}_0) = \pi_d \circ \sigma(\mathbf{u}_0)$. □

5 Time-evolution dynamics and spatio-temporal chaos

The discretized KdV-Burgers-KS type equation (1.5) is a CML: The time-evolution operator $T : \mathbb{R}^{\mathbb{Z}} \to \mathbb{R}^{\mathbb{Z}}$ is defined by $(u_{j,n})_{j\in\mathbb{Z}} \mapsto (u_{j,n+1})_{j\in\mathbb{Z}}$. The following proposition is easy to check, and we omit its proof.

Proposition 17. *The time-evolution operator T and the space-translation operator σ commute.*

We arrive at

Theorem 18. *For any $(c, b) \in \mathcal{CB}$, let δ, Φ_ε and $\mathcal{I}$ be as those of Proposition 12. Then, $\mathcal{I}$ is invariant under the $\mathbb{Z}^2$-action of T and σ, and the following diagram commutes*

$$\begin{array}{ccc} \Sigma_{c,b} & \xrightarrow{\sigma} & \Sigma_{c,b} \\ {\scriptstyle \Phi_\varepsilon}\downarrow & & \downarrow{\scriptstyle \Phi_\varepsilon} \\ \mathcal{I} & \xrightarrow{T} & \mathcal{I} \end{array}$$

provided $0 \le |\varepsilon| < \delta$.

Proof. The invariance follows from Propositions 12 and the above, so it is enough to show that $\sigma(\mathbf{v}) = T(\mathbf{v})$ for every $\mathbf{v} \in \mathcal{I}$. If $\mathbf{v} \in \mathcal{I}$, then $v_{j+1,n} = v_{j,n+1}$ for all $j \in \mathbb{Z}$, $n \in \mathbb{N}$. Therefore, $\sigma((v_{j,n})_{(j,n)\in\mathbb{Z}\times\mathbb{N}}) = (v_{j+1,n})_{(j,n)\in\mathbb{Z}\times\mathbb{N}} = (v_{j,n+1})_{(j,n)\in\mathbb{Z}\times\mathbb{N}} = T((v_{j,n})_{(j,n)\in\mathbb{Z}\times\mathbb{N}})$. □

Proof of Theorem 3.

Take $\mathcal{CB}$, δ, and $\mathcal{I}$ as those of Theorem 18. Then $\mathcal{U} = (T^n(\mathbf{u}_0))_{n\in\mathbb{N}}$ with $\mathbf{u}_0 \in \mathcal{I}$ and T the time-evolution operator of CML (1.5). Proposition 12 and Theorem 18 imply respectively that the restriction of the CML to $\mathcal{I}$ is spatially and temporally chaotic, thereby is spatio-temporally chaotic. □

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Appendix

Let $V \subset l_\infty(\mathbb{R}^d)$ and $\mathcal{E} \subset \mathbb{R}^n$ be open sets and $F : V \times \mathcal{E} \to l_\infty(\mathbb{R}^d)$ a C^1 function. Let $\Sigma \subset V$ and $\varepsilon^\dagger \in \mathcal{E}$ be such that $F(\mathbf{v}^\dagger, \varepsilon^\dagger) = 0$ for all $\mathbf{v}^\dagger \in \Sigma$ and that every point in Σ is an isolated point of Σ . Assume that the continuous linear operator $D_{\mathbf{v}}F(\mathbf{v}^\dagger, \varepsilon^\dagger) : l_\infty(\mathbb{R}^d) \to l_\infty(\mathbb{R}^d)$ is invertible and its inverse is likewise continuous for all $\mathbf{v}^\dagger \in \Sigma$. By the implicit function theorem, for every $\mathbf{v}^\dagger$ there exists $\hat{\delta}(\mathbf{v}^\dagger)$ and a unique C^1 function $\theta(\cdot; \mathbf{v}^\dagger) : \mathcal{E} \to V$ such that $F(\theta(\varepsilon; \mathbf{v}^\dagger), \varepsilon) = 0$ and $\theta(\varepsilon^\dagger; \mathbf{v}^\dagger) = \mathbf{v}^\dagger$ provided $0 \le |\varepsilon - \varepsilon^\dagger| < \hat{\delta}(\mathbf{v}^\dagger)$. In [8], a version of the following theorem was presented (see also [19] for a similar result and proof). For the sake of completeness, we include a self-contained proof for the current version.

Theorem 19.
(i) Assume $\|D_{\mathbf{v}}F(\mathbf{v}^\dagger, \varepsilon^\dagger)^{-1}\|_\infty$ *is bounded above on* $\Sigma \times \{\varepsilon^\dagger\}$*, and assume*

for any $\gamma > 0$ there exist $\lambda_0 > 0$ and $\delta_0 > 0$ such that for all $\mathbf{v}^\dagger \in \Sigma$ we have $\|F(\mathbf{v},\varepsilon)\|_\infty < \gamma$ and $\|D_\mathbf{v}F(\mathbf{v},\varepsilon) - D_\mathbf{v}F(\mathbf{v}^\dagger,\varepsilon^\dagger)\|_\infty < \gamma$ whenever $\mathbf{v} \in B(\mathbf{v}^\dagger,\lambda_0)$ and $|\varepsilon - \varepsilon^\dagger| < \delta_0$. Then $\delta := \inf_{\mathbf{v}^\dagger\in\Sigma} \hat{\delta}(\mathbf{v}^\dagger) \geq \delta_1 > 0$ for some $\delta_1 < \delta_0$ and the map

$$\begin{aligned} \Phi_\varepsilon : \Sigma &\to \mathcal{I} := \bigcup_{\mathbf{v}^\dagger\in\Sigma} \theta(\varepsilon^\dagger,\mathbf{v}^\dagger) \\ \mathbf{v}^\dagger &\mapsto \theta(\varepsilon^\dagger;\mathbf{v}^\dagger), \end{aligned}$$

is a bijection provided $0 \leq |\varepsilon - \varepsilon^\dagger| < \delta$.
(ii) In addition to the assumptions in (i), if Σ is σ-invariant, i.e. $\sigma(\Sigma) = \Sigma$, and if $F(\cdot,\varepsilon)$ commutes with σ for all $\varepsilon \in \mathcal{E}$, i.e. $\sigma \circ F(\mathbf{v},\varepsilon) = F(\sigma(\mathbf{v}),\varepsilon)$, then the following diagram commutes

$$\begin{array}{ccc} \Sigma & \xrightarrow{\sigma} & \Sigma \\ {\scriptstyle \Phi_\varepsilon}\downarrow & & \downarrow{\scriptstyle \Phi_\varepsilon} \\ \mathcal{I} & \xrightarrow{\sigma} & \mathcal{I} \end{array}$$

provided $0 \leq |\varepsilon - \varepsilon^\dagger| < \delta$.
(iii) With the product topology, regard Σ and $\mathcal{I}$ as subsets of $\mathcal{M}(\mathbb{R}^d)$. In addition to the assumptions in (i), if $\Sigma \subset B(0,\mu)$ for some $\mu > 0$, Σ is compact, and $F(\cdot,\varepsilon)$ is continuous on $B(0,\mu+\lambda_0/2)$ for every $\varepsilon \in \mathcal{E}$, then there exists $0 < \delta_2 \leq \delta_1$ such that Φ_ε is a continuous function thus a homeomorphism on Σ provided $0 \leq |\varepsilon - \varepsilon^\dagger| < \delta_2$.

Proof. (i) For each $\mathbf{v}^\dagger \in \Sigma$ and $\varepsilon \in \mathcal{E}$, define a map $G(\cdot;\mathbf{v}^\dagger,\varepsilon) : V \to l_\infty(\mathbb{R}^d)$, $\mathbf{v} \mapsto \mathbf{v} - D_\mathbf{v}F(\mathbf{v}^\dagger,\varepsilon^\dagger)^{-1}F(\mathbf{v},\epsilon)$. By assumptions in (i), there are $0 < \lambda_1 < \lambda_0$ and $0 < \delta_1 < \delta_0$ such that for $\mathbf{u}$ in closed ball $B(\mathbf{v}^\dagger,\lambda_1)$ and $|\varepsilon - \varepsilon^\dagger| \leq \delta_1$, we have

$$\|D_\mathbf{v}F(\mathbf{v}^\dagger,\varepsilon^\dagger)^{-1}\|_\infty \, \|D_\mathbf{v}F(\mathbf{v}^\dagger,\varepsilon^\dagger) - D_\mathbf{v}F(\mathbf{u},\varepsilon)\|_\infty \leq 1/2$$

and

$$\|D_\mathbf{v}F(\mathbf{v}^\dagger,\varepsilon^\dagger)^{-1}\|_\infty \, \|F(\mathbf{v}^\dagger,\varepsilon)\|_\infty < \lambda_1/2. \tag{5.1}$$

Thus, for $\mathbf{v}$, $\mathbf{w} \in B(\mathbf{v}^\dagger, \lambda_1)$ and $|\varepsilon - \varepsilon^\dagger| \le \delta_1$, we get

$$\begin{aligned} & \|(G(\mathbf{v};\mathbf{v}^\dagger,\varepsilon) - G(\mathbf{w};\mathbf{v}^\dagger,\varepsilon)\|_\infty \\ = \ & \|D_{\mathbf{v}}F(\mathbf{v}^\dagger,\varepsilon^\dagger)^{-1}\left(D_{\mathbf{v}}F(\mathbf{v}^\dagger,\varepsilon^\dagger) - D_{\mathbf{v}}F(\mathbf{u},\varepsilon)\right)(\mathbf{v}-\mathbf{w})\|_\infty \\ & (\text{for some } \mathbf{u} \in B(\mathbf{v}^\dagger,\lambda_1)) \\ \le \ & \|\mathbf{v}-\mathbf{w}\|_\infty/2 \end{aligned}$$

and

$$\begin{aligned} & \|G(\mathbf{v};\mathbf{v}^\dagger,\varepsilon) - \mathbf{v}^\dagger\|_\infty \\ \le \ & \|G(\mathbf{v};\mathbf{v}^\dagger,\varepsilon) - G(\mathbf{v}^\dagger;\mathbf{v}^\dagger,\varepsilon)\|_\infty + \|G(\mathbf{v}^\dagger;\mathbf{v}^\dagger,\varepsilon) - \mathbf{v}^\dagger\|_\infty \\ \le \ & \|\mathbf{v}-\mathbf{v}^\dagger\|_\infty/2 + \|D_{\mathbf{v}}F(\mathbf{v}^\dagger,\varepsilon^\dagger)^{-1}\|_\infty \, \|F(\mathbf{v}^\dagger,\varepsilon)\|_\infty \\ < \ & \lambda_1. \end{aligned}$$

This implies that $G(\cdot;\mathbf{v}^\dagger,\varepsilon)$ is a contraction map with contraction constant at least $1/2$ on $B(\mathbf{v}^\dagger,\lambda_1)$ for any $\mathbf{v}^\dagger \in \Sigma$ and $|\varepsilon-\varepsilon^\dagger| \le \delta_1$. Hence $\inf_{\mathbf{v}^\dagger\in\Sigma}\hat{\delta}(\mathbf{v}^\dagger) \ge \delta_1 > 0$.

The radius λ_1 is independent of $\mathbf{v}^\dagger$ and ε, and $\Phi_\varepsilon(\mathbf{v}^\dagger)$ is the unique fixed point in $B(\mathbf{v}^\dagger,\lambda_1)$ for $G(\cdot;\mathbf{v}^\dagger,\varepsilon)$. Because $\mathbf{v}^\dagger$'s are isolated points in Σ, the balls $B(\mathbf{v}^\dagger,\lambda_1)$'s are disjoint in $l_\infty(\mathbb{R}^d)$. It follows that Φ_ε is bijective on Σ.

(ii) $F(\cdot,\varepsilon)$ has a unique zero at $\Phi_\varepsilon(\mathbf{v}^\dagger)$ in $B(\mathbf{v}^\dagger,\lambda_1)$, so do $\sigma \circ F(\cdot,\varepsilon)$ and $F(\sigma(\cdot),\varepsilon)$ because $\sigma \circ F(\cdot,\varepsilon) = F(\sigma(\cdot),\varepsilon)$. This implies that $F(\cdot,\varepsilon)$ has a unique zero at $\sigma\circ\Phi_\varepsilon(\mathbf{v}^\dagger)$ in $B(\sigma(\mathbf{v}^\dagger),\lambda_1)$. Because $F(\cdot,\varepsilon)$ has been shown to have a unique zero at $\Phi_\varepsilon(\sigma(\mathbf{v}^\dagger))$ in $B(\sigma(\mathbf{v}^\dagger),\lambda_1)$, it must $\sigma\circ\Phi_\varepsilon(\mathbf{v}^\dagger) = \Phi_\varepsilon(\sigma(\mathbf{v}^\dagger))$ by the uniqueness.

(iii) In view of (5.1), we see that there exists $0 < \delta_2 \le \delta_1$ such that $\|\Phi_\varepsilon(\mathbf{v}^\dagger) - \mathbf{v}^\dagger\|_\infty < \lambda_1/2$ for all $\mathbf{v}^\dagger \in \Sigma$ and $0 \le |\varepsilon-\varepsilon^\dagger| < \delta_2$. Suppose $\mathbf{v}^{\dagger(k)} \to \mathbf{v}^\dagger$ in Σ as $k \to \infty$, and suppose $\Phi_\varepsilon(\mathbf{v}^{\dagger(k)})$ converges, via a subsequence if necessary, to a point $\mathbf{v}^*_\varepsilon$ in $B(0,\mu+\lambda_1/2)$. (Because $\Phi_\varepsilon(\mathbf{v}^{\dagger(k)}) \in B(0,\mu+\lambda_1/2)$ and $B(0,\mu+\lambda_1/2)$ is compact in $\mathcal{M}(\mathbb{R}^d)$.) Assume the subsequence. For any $N \in \mathbb{N}$, there is $k \in \mathbb{N}$ such that

$$|v_i^{\dagger(k)} - v_i^\dagger| < \lambda_1/2$$

and then

$$\begin{aligned} |\Phi_\varepsilon(\mathbf{v}^{\dagger(k)})_i - v_i^\dagger| \ &\le \ |\Phi_\varepsilon(\mathbf{v}^{\dagger(k)})_i - v_i^{\dagger(k)}| + |v_i^{\dagger(k)} - v_i^\dagger| \\ &< \ \lambda_1 \end{aligned}$$

for all $k > K$, $|i| < N$, and $0 \leq |\varepsilon - \varepsilon^\dagger| < \delta_2$. Passing to $N \to \infty$, we have

$$\|\mathbf{v}_\varepsilon^* - \mathbf{v}^\dagger\|_\infty \leq \lambda_1.$$

Besides,

$$F(\mathbf{v}_\varepsilon^*, \varepsilon) = F(\lim_{k\to\infty} \Phi_\varepsilon(\mathbf{v}^{\dagger(k)}), \varepsilon) = \lim_{k\to\infty} F(\Phi_\varepsilon(\mathbf{v}^{\dagger(k)}), \varepsilon) = 0$$

because $F(\cdot, \varepsilon)$ is continuous on $B(0, \mu + \lambda_0/2)$. This means that $\mathbf{v}_\epsilon^*$ is a zero of $F(\cdot, \varepsilon)$ in $B(\mathbf{v}^\dagger, \lambda_1)$. From the proof of (i), we conclude that $\mathbf{v}_\varepsilon^*$ must be $\Phi_\varepsilon(\mathbf{v}^\dagger)$. □

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