On certain types of point symmetries of systems of second-order ordinary differential equations

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Abstract

Existence criteria for some generic types of point symmetries of systems of n-second order ordinary differential equations are studied, specially in connection with the generation of semisimple subalgebras of symmetries belonging to the simple linear and orthogonal types, as well as their maximal dimension and rank. The structure of certain time-dependent symmetries, in particular scaling symmetries, are also studied, and the structure of the subalgebras they span determined. Generic examples illustrating the procedure are given.

Key words: Lie symmetry method, second order systems of ODEs, Lie algebra,
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1 Introduction

Far beyond its original motivation and conception to reduce and classify ordinary differential equations (ODEs), as well as to analyze certain properties of their solutions, the Lie symmetry method has been generalized and enlarged to become a powerful and sometimes indispensable tool in Geometry and Mechanics [1,2]. Among the various questions analyzed, the linearization of ordinary differential equations is relevant problem, as it reduces heavily the computational complexity. While the linearization problem for scalar second order ODEs has been solved satisfactorily [3–5], for the case of systems there still remains some work to do. In this context, it is known that, under certain symmetry conditions, systems

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of nonlinear ordinary differential equations can be locally mapped onto linear systems [6], enabling the derivation of additional criteria to analyze the properties of their solutions and their numerical implementation.

Systems of second order ordinary differential equations constitute a particularly important case in applications, as they naturally arise in Classical Mechanics, the theory of integrable systems and General Relativity [1,7–9]. Although the symmetry analysis provides some simplifications for the study of generic mechanical systems, the analysis is mostly devoted to special types of systems, like the harmonic oscillators, the Kepler problem, the Hénon-Heiles system and other integrable systems admitting point symmetries in connection with Noether symmetries [10–13]. In this context, a direct procedure for associating conserved quantities with dynamical symmetries of Lagrangian systems was developed in [10], and further extended to general systems of second-order ordinary differential equations.

Although for the case of systems of n second-order equations various general results exist, like reducibility criteria or upper bound formulae for the dimension of the symmetry algebras [3], little is known about the structure of such Lie algebras in the arbitrary case (see e.g. [4,14,15] and references therein). The problem of n-systems having maximal symmetry, corresponding to a simple Lie algebra of point symmetries isomorphic to $\mathfrak{sl}(n+2,\mathbb{R})$, is well understood, and such systems are known to be linearizable to the n-free particle system $\ddot{\mathbf{x}} = 0$ [16]. This result, however, does not exhaust the possibilities of linearizing systems, as show the criteria developed in recent years to linearize systems to constant coefficient systems and determine their equivalence classes [14,17]. For linear systems of second order equations it has been shown that there are essentially four types of symmetries (see e.g. [3]), the analysis of which allows the deduction of qualitative properties of the system, like their autonomous/nonautonomous character or the increase/decrease of anisotropy [18]. Special cases as that of linear systems with constant coefficients have also been studied extensively [19–23], with detailed analysis of the symmetry generators for low orders of n, as well as various criteria for the general case [22]. These results have been further enlarged recently to the case of generic linear systems with $n = 2$ equations which cannot be significantly simplified by means of the Jordan canonical form [24].

The main objective of this work is to analyze some general features concerning concrete types of symmetries of (non-linear) systems of n second-order ODEs $\ddot{\mathbf{x}} = \omega^{\alpha}(t,\mathbf{x})$. We first analyze the symmetry condition in order to derive some sufficient conditions to ensure that a system possesses maximal symmetry, and further to obtain the generic shape for the components of points symmetries for systems that are non-maximally symmetric. [1] We then focus on some special types of symmetries, developing some criteria to ensure either their existence or absence, as well as to compute the maximal possible number of independent symmetries of these kinds. Further, it is analyzed whether these symmetry types generate a subalgebra of the Lie algebra $\mathfrak{L}$ of point symmetries of a system. This specifically allows to determine the structure of semisimple subalgebras spanned by symmetries of these types, and to obtain an upper bound for its rank. The case of subalgebras of special linear or orthogonal type

[1] I.e., n-systems where the Lie algebra of point symmetries is not isomorphic to $\mathfrak{sl}(n+2,\mathbb{R})$.

is considered in detail. In addition, two kinds of time-dependent symmetries are considered, giving rise to either Abelian subalgebras of symmetries or subalgebras isomorphic to $\mathfrak{sl}(2,\mathbb{R})$.

Unless otherwise stated, the Einstein summation convention is used. All Lie algebras and vector fields used in this work are assumed to be real.

1.1 Point symmetries of systems of ordinary differential equations

Among the various methods to compute symmetries of differential equations, those of most common use are the direct prolongation method and the formulation in terms of differential operators [6]. The latter approach will be more convenient for our analysis, for which purpose we review it briefly. It is well known (see e.g. [25]) that a system of n second-order ordinary differential equations

$$\ddot{\mathbf{x}} = \omega^{\alpha}\left(t,\mathbf{x}\right),\ 1\leq \alpha \leq n, \tag{1}$$

where $\ddot{\mathbf{x}},\mathbf{x} \in\mathbb{R}^{n}$, can be formulated in equivalent form in terms of the partial differential equation

$$\mathbf{A}f = \left(\frac{\partial}{\partial t} + \dot{x}_i \frac{\partial}{\partial x_i} + \omega^{i}\left(t,\mathbf{x}\right)\frac{\partial}{\partial \dot{x}_i}\right) f = 0. \tag{2}$$

In this context, a vector field $X = \xi\left(t,x_i\right)\frac{\partial}{\partial t} + \eta_i\left(t,x_i\right)\frac{\partial}{\partial x_i} \in \mathfrak{X}\left(\mathbb{R}^{n+1}\right)$ is a Lie point symmetry generator of the system (1) whenever the extended vector field $\dot{X} = X + \dot{\eta}_i\left(t,x_i,\dot{x}_i\right)\frac{\partial}{\partial \dot{x}_i}$ satisfies the commutator

$$\left[\dot{X},\mathbf{A}\right] = -\frac{d\xi}{dt}\mathbf{A}, \tag{3}$$

where $\dot{\eta}_i = -\frac{d\xi}{dt}\dot{x}_i + \frac{d\eta_i}{dt}$. The advantage of this approach is that the condition on the prolongation of the symmetry generator X is automatically given by the commutator. The n resulting equations of the components of the bracket are identities in t, x_i and $\dot{x}_i$. As the components ξ and η_i of X do not depend on the variables $\dot{x}_i$, these n equations can be further separated into more equations, the solutions of which provide the analytical expression of the symmetry generator X.

A notably important case is given for the free particle system $\ddot{\mathbf{x}} = 0$ in $n \geq 2$, which constitutes the prototype of linearizable system [3]. In this case, integration of the symmetry condition (3) is straightforward, leading to the n^2+4n+3 symmetry generators

$$\frac{\partial}{\partial t}, \frac{\partial}{\partial x_i}, t\frac{\partial}{\partial t}, t\frac{\partial}{\partial x_i}, x_i\frac{\partial}{\partial t}, t\left(t\frac{\partial}{\partial t} + \sum_{k=1}^{n} x_k \frac{\partial}{\partial x_k}\right), x_i\left(t\frac{\partial}{\partial t} + \sum_{k=1}^{n} x_k \frac{\partial}{\partial x_k}\right), x_i\frac{\partial}{\partial x_j} \tag{4}$$

for $1 \leq i,j \leq n$. It is well known that the Lie algebra generated by these vector fields is isomorphic to the simple Lie algebra $\mathfrak{sl}(n+2,\mathbb{R})$. For linear systems, it was proved in [3] that a necessary and sufficient condition for a system to be (locally) linearizable (i.e., to have a symmetry algebra $\mathfrak{L}$ isomorphic to $\mathfrak{sl}(n+2,\mathbb{R})$) is that $\dim\mathfrak{L} = n^2+4n+3$. This result was sharpened in [16] to hold for arbitrary systems of second order ODEs, where it was shown that the dimension condition $\dim\mathfrak{L} = n^2+4n+3$ implies that the system is equivalent to the free particle system [16]. In the following, according to the terminology introduced in [3], we call a system $\ddot{\mathbf{x}} = \omega^{\alpha}(t,\mathbf{x})$ maximally symmetric if $\dim \mathfrak{L} = n^2+4n+3$.

Let $\ddot{\mathbf{x}} = \omega_{\alpha}(t,\mathbf{x})$ be a system of n second order ODEs. If X is a symmetry, expanding the symmetry condition (3) and separating the resulting equations, it can be easily shown that the components of X can be written in the following generic form (see e.g. [3]):

$$\xi(t,\mathbf{x}) = \sum_{i=1}^{n} \phi_i(t)x_i + \psi_0(t), \tag{5}$$

$$\eta_j(t,\mathbf{x}) = \sum_{i=1}^{n} \left(\dot{\phi}_i(t)x_ix_j + F_{ij}(t)x_i\right) + \psi_j(t), \quad 1 \leq j \leq n \tag{6}$$

In order to simplify the expression of the point symmetries, we first analyze those symmetries which ensure that the system is maximally symmetric, so that they can be discarded in our general analysis.

Lemma 1 *If for some $f(t) \neq 0$ the system $\ddot{\mathbf{x}} = \omega_{\alpha}(t,\mathbf{x})$ $(1 \leq \alpha \leq n)$ admits the point symmetry*

$$X = f(t)\, x_{i_0} \frac{\partial}{\partial t} + \dot{f}(t)\, x_{i_0} x_j \frac{\partial}{\partial x_j} \tag{7}$$

for some index $i_0 \in \{1,\cdots,n\}$, then the system is maximally symmetric.

Proof. The symmetry condition (3) is given by

$$\left[\dot{X}, \mathbf{A}\right] = -\left(x_{i_0}\dot{f}(t) + \dot{x}_{i_0} f(t)\right)\mathbf{A}, \tag{8}$$

where the prolonged vector field $\dot{X}$ is given by

$$\dot{X} = X + \left(\ddot{f}(t)\, x_{i_0}x_j + \dot{f}(t)\left(\dot{x}_j x_{i_0} + \dot{x}_{i_0} x_j\right) - \dot{x}_{i_0}\left(\dot{f}(t)\, x_j + \dot{x}_j f(t)\right)\right)\frac{\partial}{\partial \dot{x}_j}.$$

Developing the commutator and separating the resulting expression for the variables $\dot{x}_j$s leads to the system $(1 \leq j \leq n)$

$$\begin{array}{l} \dot{f}(t)\, x_{i_0}\left(\omega^j + x_l \frac{\partial \omega^j}{\partial x_l}\right) + x_{i_0}\left(f(t)\,\frac{\partial \omega^j}{\partial t} - x_j\, f^{(3)}(t)\right) - \dot{x}_j\left(x_{i_0}\ddot{f}(t) - \dot{f}(t)\,\omega^{i_0}\right) + \\ +2\dot{x}_{i_0}\left(f(t)\,\omega^j - x_j\ddot{f}(t)\right) = 0. \end{array} \tag{9}$$

In particular, as the functions $\omega^{\alpha}(t,\mathbf{x})$ do not depend on the $\dot{x}_i$s, for any $1 \leq j \leq n$ the differential equations

$$f(t)\,\omega^j - x_j\,\ddot{f}(t) = 0 \tag{10}$$

must be satisfied. If $\ddot{f}(t) = 0$, then obviously $\omega^j(t,\mathbf{x}) = 0$ holds for all j, and the system corresponds to that of the free particle, hence possesses maximal symmetry. If $\ddot{f}(t) \neq 0$, then it follows at once that the component function ω^j is given by

$$\omega^j(t,\mathbf{x}) = \frac{\ddot{f}(t)}{f(t)}x_j, \quad 1 \leq j \leq n, \tag{11}$$

which implies that the system is of the form $\ddot{\mathbf{x}}_\alpha = \psi(t)\,\mathbf{x}_\alpha$. Such linear systems are well known to have a maximal symmetry algebra, and moreover can be locally transformed to the form $\ddot{\mathbf{x}} = 0$ (see e.g. [3,16]). ■

For systems having maximal symmetry, we deduce in particular that for a given index $i_0 \in \{1, \cdots, n\}$, at most two independent symmetries of the type $X = f(t)\,x_{i_0}\frac{\partial}{\partial t} + \dot{f}(t)\,x_{i_0}x_j\frac{\partial}{\partial x_j}$ exist. For the free particle case this is obvious, as $f(t) = a_0 + a_1 t$, while for the general case $\ddot{f}(t) \neq 0$, a second symmetry of this type would lead, by equation (11), to the second order equation

$$\frac{\ddot{f}(t)}{f(t)} = \frac{\ddot{g}(t)}{g(t)} \tag{12}$$

with general solution

$$g(t) = \left(C_1 \int \frac{dt}{f^2(t)} + C_2\right) f(t), \tag{13}$$

and merely $g(t) = f(t)\int\frac{dt}{f^2(t)}$ would provide an independent symmetry. [2]

This result can be formulated in a slightly generalized form, in order to allow us to simplify further the expression for the components of points symmetries in the case of non-maximally symmetric systems $\ddot{\mathbf{x}} = \omega^\alpha(t, \mathbf{x})$, and specifically for non-linear systems.

Proposition 1 *Suppose there exists an index* $\alpha \in \{1, \cdots, n\}$ *such that* $\omega^\alpha(t, \mathbf{x}) \neq g(t)\,x_\alpha + h(t)$*. Then the components of a point symmetry* $X = \xi(t, \mathbf{x})\frac{\partial}{\partial t} + \eta_j(t, \mathbf{x})\frac{\partial}{\partial x_j}$ *of the system have the generic form*

$$\begin{aligned} \xi(t, \mathbf{x}) &= \psi_0(t), \\ \eta_j(t, \mathbf{x}) &= \frac{1}{2}\dot{\psi}_0(t)\,x_j + \sum_{k=1}^{n} m_{jk}x_k + \psi_j(t),\ 1 \leq j \leq n,\ m_{jk} \in \mathbb{R}. \end{aligned} \tag{14}$$

Proof. Without loss of generality, and reordering the indices if necessary, we can suppose that $\alpha = n$. By assumption, $\omega^n(t, \mathbf{x}) \neq g(t)\,x_n + h(t)$. Let $X = \xi(t, \mathbf{x})\frac{\partial}{\partial t} + \eta_j(t, \mathbf{x})\frac{\partial}{\partial x_j}$ be a point symmetry of the system, the components of which are of the form (6). Expanding the symmetry condition $\left[\dot{X}, \mathbf{A}\right] + \frac{d\xi}{dt}\mathbf{A} = 0$ and focusing only on the component in $\frac{\partial}{\partial \dot{x}_n}$ leads to the expression

$$\left(\dot{x}_j G_n^j(t, \mathbf{x}) + x_j H_n^j(t, \mathbf{x})\right)\frac{\partial}{\partial \dot{x}_n}, \tag{15}$$

where [3] for $j \leq n-1$

$$G_n^j(t, \mathbf{x}) = \omega^n(t, \mathbf{x})\,\phi_j(t) - \ddot{\phi}_j(t)\,x_n - \dot{F}_{nj}(t) \tag{16}$$

[2] The maximal possible number of symmetries of this type would therefore be $2n$, two for any index $j \in \{1, \cdots, n\}$.

[3] The precise form of $H_n^j(t, \mathbf{x})$ is irrelevant for our argument, for which reason we omit it.

and

$$G_n^n(t,\mathbf{x}) = 3\omega^n(t,\mathbf{x})\,\phi_n(t) + \sum_{k\neq n}\left(x_k\ddot{\phi}_k(t) - \omega^k(t,\mathbf{x})\,\phi_k(t)\right) + 3x_n\ddot{\phi}_n(t) + 2\ddot{F}_{nn}(t) - \dddot{\psi}_0(t)\,. \tag{17}$$

Since the functions $\omega^j(t,\mathbf{x})$ do not depend on the $\dot{x}_k$s for $1 \leq j,k \leq n$, both expressions (16) and (17) must vanish identically. Now, as $\omega^n(t,\mathbf{x})$ is non-linear in x_n by assumption, it follows at once that the equations (16) are satisfied only for $\phi_1(t) = 0, \cdots, \phi_{n-1}(t) = 0$, which further implies that $\dot{F}_{nj}(t) = 0$ for $1 \leq j \leq n-1$. With this simplification, equation (17) reduces to

$$3\omega^n(t,\mathbf{x})\,\phi_n(t) + 3x_n\ddot{\phi}_n(t) + 2\dot{F}_{nn}(t) - \ddot{\psi}_0(t) = 0,$$

and again the equation is only satisfied if $\phi_n(t) = 0$ and $2\dot{F}_{nn}(t) - \ddot{\psi}_0(t) = 0$. Inserting this result into the coefficients of $\frac{\partial}{\partial \dot{x}_k}$ for indices $k \leq n-1$ allows us to rewrite the symmetry condition as

$$\left[\dot{X},\mathbf{A}\right] + \frac{d\xi}{dt}\mathbf{A} = \left(\sum_{k\neq j}\dot{x}_j\dot{F}_{kj}(t) + \left(2\dot{F}_{kk}(t) - \ddot{\psi}_0(t)\right)\dot{x}_k + x_j H_k^j(t,\mathbf{x})\right)\frac{\partial}{\partial \dot{x}_k}, \tag{18}$$

from which we easily deduce that

$$\dot{F}_{kj}(t) = 0,\ 2\dot{F}_{kk}(t) - \ddot{\psi}_0(t) = 0. \tag{19}$$

Solving these equations proves that the components of the symmetry X have the required form (14). ■

As a consequence of this reduction, the symmetry condition (3) simplifies to

$$\sum_{l=1}^{n}\frac{\partial\omega^k(t,\mathbf{x})}{\partial x_l}\left(\sum_{j=1}^{n} m_{lj}x_j + \frac{x_l}{2}\dot{\psi}_0(t) + \psi_l(t)\right) + \frac{\partial\omega^k(t,\mathbf{x})}{\partial t}\psi_0(t) - \ddot{\psi}_k(t) +$$
$$+\frac{3}{2}\omega^k(t,\mathbf{x})\,\dot{\psi}_0(t) - \sum_{l=1}^{n}\omega^l(t,\mathbf{x})\,m_{kl} - \frac{1}{2}x_k\psi_0^{(3)}(t) = 0 \tag{20}$$

for any $1 \leq k \leq n$. However, these equations are still too general to be solved for arbitrary systems, and further assumptions on the type of symmetries or the equations of motion must be introduced to obtain valid criteria. In any case, lemma 1 and the simplified symmetry condition leads us to the rough upper bound for the dimension of the symmetry algebra $\mathfrak{L}$ of non-maximally symmetric systems

$$\dim\mathfrak{L} < n^2 + 2n + 3. \tag{21}$$

2 Point symmetries with $\xi(t,\mathbf{x}) = 0$

In this paragraph we analyze some types of symmetries such that the condition $\xi(t,\mathbf{x}) = 0$ holds. Besides existence criteria, we show how to use such symmetries to construct systems with a prescribed semisimple subalgebra of symmetries. The cases of special linear and orthogonal algebras are studied in detail.

Proposition 2 *Let $n \geq 2$ and $1 \leq j_0 \leq n$. Suppose that the system $\ddot{\mathbf{x}} = \omega_\alpha(t, \mathbf{x})$ $(1 \leq \alpha \leq n)$ satisfies at least one of the following conditions:*

(1) there exists some $l \neq j_0$ such that $\frac{\partial \omega^l}{\partial x_{j_0}} \neq 0$,
(2) $\frac{\partial^2 \omega^{j_0}}{\partial x_{j_0}^2} \neq 0$.

Then the system cannot possess the point symmetries $Y_{j_0} = \psi_{j_0}(t) \frac{\partial}{\partial x_{j_0}}$ and $X_{j_0 k} = x_k \frac{\partial}{\partial x_{j_0}}$ for $1 \leq k \leq n$.

Proof. Suppose there exists an index $i_0 \in \{1, \cdots, n\}$ such that $X_{j_0 i_0} = x_{i_0} \frac{\partial}{\partial x_{j_0}}$ is a point symmetry of the system. As $\xi(t, \mathbf{x}) = 0$, the symmetry condition (3) reduces to $\left[\dot{X}_{j_0 i_0}, \mathbf{A}\right] = 0$. The precise bracket is given by

$$\left[\dot{X}_{j_0 i_0}, \mathbf{A}\right] = x_{i_0} \frac{\partial \omega^l}{\partial x_{j_0}} \frac{\partial}{\partial \dot{x}_l} - \delta_{i_0}^l \omega^l \frac{\partial}{\partial \dot{x}_{j_0}} = 0. \tag{22}$$

For indices $l \neq j_0$ this leads to the constraint $\frac{\partial \omega^l}{\partial x_{j_0}} = 0$, while for $j_0 = l$, (22) simplifies to

$$x_{i_0} \frac{\partial \omega^{j_0}}{\partial x_{j_0}} - \omega^{i_0} = 0. \tag{23}$$

Two possibilities are given: If $i_0 \neq j_0$, solving for ω^{i_0} we obtain

$$\omega^{i_0} = x_{i_0} \frac{\partial \omega^{j_0}}{\partial x_{j_0}}.$$

Now, taking the partial derivative with respect to x_{j_0}leads to

$$\frac{\partial \omega^{i_0}}{\partial_{x_{j_0}}} = 0 = x_{i_0} \frac{\partial^2 \omega^{j_0}}{\partial x_{j_0}},$$

contradicting the assumption $\frac{\partial^2 \omega^{j_0}}{\partial x_{j_0}^2} \neq 0$. For the case $i_0 = j_0$, we get the partial differential equation $x_{i_0} \frac{\partial \omega^{j_0}}{\partial x_{j_0}} - \omega^{j_0} = 0$ with general solution

$$\omega^{j_0}(t, \mathbf{x}) = x_{j_0} \theta^{j_0}(t, x_1, \cdots, \widehat{x}_{j_0}, \cdots, n),$$

implying that $\frac{\partial^2 \omega^{j_0}}{\partial x_{j_0}^2} = 0$, again against the hypothesis. This shows that whenever one of the conditions is satisfied, no vector fields $X_{j_0 i_0}$ can be symmetries of the system.
For the vector fields $Y_{j_0} = \psi_{j_0}(t) \frac{\partial}{\partial x_{j_0}}$, the argument is completely analogous, the symmetry condition being here

$$\left[\dot{Y}_{j_0}, A\right] = \psi_{j_0}(t) \frac{\partial \omega^l}{\partial x_{j_0}} \frac{\partial}{\partial \dot{x}_l} - \ddot{\psi}_{j_0}(t) \frac{\partial}{\partial \dot{x}_{j_0}} = 0. \tag{24}$$

For any index $l \neq j_0$, it is immediate that $\frac{\partial \omega^l}{\partial x_{j_0}} = 0$. If $\ddot{\psi}_{j_0}(t) \neq 0$ and $l = j_0$, the constraint (24) implies that

$$\omega^{j_0}(t, \mathbf{x}) = \frac{\ddot{\psi}_{j_0}(t)}{\psi_{j_0}(t)} x_{j_0} + \theta^{j_0}(t, x_1, \cdots, \widehat{x}_{j_0}, \cdots, x_n). \tag{25}$$

Since in this case $\omega^{j_0}(t, \mathbf{x})$ is linear in x_{j_0}, it follows that $\frac{\partial^2 \omega^{j_0}}{\partial x_{j_0}^2} = 0$, contradicting the assumption. The contradiction also follows whenever $\ddot{\psi}(t) = 0$ holds, as then immediately $\frac{\partial \omega^{j_0}}{\partial x_{j_0}} = 0$. ■

As follows from (3), the number of symmetries $X_{j_0 k} = x_k \frac{\partial}{\partial x_{j_0}}$ is obviously lower or equal than n^2, equality being given only in the case of maximal symmetry. On the other hand, for a fixed index $1 \leq j_0 \leq n$, there are at most two independent symmetries of the shape $Y_{j_0} = \psi_{j_0}(t) \frac{\partial}{\partial x_{j_0}}$. For $\ddot{\psi}_{j_0} = 0$, the assertion is trivial, whereas for $\ddot{\psi}_{j_0} \neq 0$, the condition (24) leads to an equation of type (12). Since any point symmetry satisfying the constraint $\xi(t, \mathbf{x}) = 0$ is a linear combination of symmetries of types X_{ij} and Y_l, the total number m_0 of symmetries subjected to the condition $\xi(t, \mathbf{x}) = 0$ must fulfil the bound $0 \leq m_0 \leq n^2 + 2n$.

Corollary 1 *Let $i_0, j_0 \in \{1, \cdots, n\}$ and $i_0 \neq j_0$. If $X_{i_0 j_0} = x_{j_0} \frac{\partial}{\partial x_{i_0}}$ and $X_{j_0 i_0} = x_{i_0} \frac{\partial}{\partial x_{j_0}}$ are symmetries of the system $\ddot{\mathbf{x}} = \omega^\alpha(t, \mathbf{x})$, then $x_{i_0} \frac{\partial}{\partial x_{i_0}}$ and $x_{j_0} \frac{\partial}{\partial x_{j_0}}$ are also symmetries.*

Proof. Arguing similarly as in the preceding proof, we have that the vector fields $X_{i_0 j_0}$ and $X_{j_0 i_0}$ impose the following constraints on the component functions ω^{i_0} and ω^{j_0} respectively:

$$\begin{aligned} \frac{\partial \omega^{i_0}}{\partial x_{j_0}} = 0,\ \omega^{i_0} = x_{i_0} \frac{\partial \omega^{j_0}}{\partial x_{j_0}}, \\ \frac{\partial \omega^{j_0}}{\partial x_{i_0}} = 0,\ \omega^{j_0} = x_{j_0} \frac{\partial \omega^{i_0}}{\partial x_{i_0}}. \end{aligned}$$

It follows in particular that $\omega^{i_0} = x_{i_0} \frac{\partial \omega^{i_0}}{\partial x_{i_0}}$ and $\omega^{j_0} = x_{j_0} \frac{\partial \omega^{j_0}}{\partial x_{j_0}}$, hence that

$$\begin{aligned} \omega^{i_0} &= x_{i_0} \Omega^{i_0}(t, x_1, \cdots, \widehat{x}_{i_0}, \cdots, \widehat{x}_{j_0}, \cdots, x_n), \\ \omega^{j_0} &= x_{j_0} \Omega^{j_0}(t, x_1, \cdots, \widehat{x}_{i_0}, \cdots, \widehat{x}_{j_0}, \cdots, x_n). \end{aligned} \tag{26}$$

From the properties of the brackets of Lie algebra [26], it follows at once that the commutator $[X_{i_0 j_0}, X_{j_0 i_0}] = X_{i_0 i_0} - X_{j_0 j_0}$ is a symmetry of the system. Now, for the vector field $X_{j_0 j_0}$, the symmetry condition reads

$$x_{j_0} \frac{\partial \omega^l}{\partial x_{j_0}} \frac{\partial}{\partial \dot{x}_l} - \delta^l_{j_0} \omega^l \frac{\partial}{\partial \dot{x}_{j_0}} = 0,$$

which is trivially satisfied by (26) and the fact that $\frac{\partial \omega^l}{\partial x_{j_0}} = \frac{\partial \omega^l}{\partial x_{i_0}} = 0$ holds for any $l \neq i_0, j_0$. Finally, by linearity we easily deduce that $X_{i_0 i_0}$ is also a symmetry of the system. ■

For indices $i_0 \neq j_0$, it is straightforward to verify that the vector fields $X_{i_0 j_0}$ and $X_{j_0 i_0}$ generate a Lie algebra isomorphic to $\mathfrak{sl}(2, \mathbb{R})$. The corollary states that whenever the system admits a $\mathfrak{sl}(2,\mathbb{R})$ algebra of symmetries generated by $X_{i_0j_0}$ and $X_{j_0i_0}$, it always admits the reductive extension $\mathfrak{sl}(2,\mathbb{R})\oplus\mathbb{R}$, [4] the copy of $\mathbb{R}$ being generated by $X_{i_0i_0}+X_{j_0j_0}$.

Lemma 2 *Let $\alpha_1\neq\alpha_2$ and $\alpha_3\neq\alpha_4$ be indices such that the sets $\{\alpha_1,\alpha_2\}$ and $\{\alpha_3,\alpha_4\}$ do not coincide. If $\{X_{\alpha_1\alpha_2}, X_{\alpha_2\alpha_1}, X_{\alpha_3\alpha_4}, X_{\alpha_4\alpha_3}\}$ are symmetries of the system $\ddot{\mathbf{x}}=\omega^{\alpha}(t,\mathbf{x})$, then one of the two following cases holds:*

(1) If $\{\alpha_1,\alpha_2\}\cap\{\alpha_3,\alpha_4\}=\emptyset$, the symmetries generate a Lie algebra isomorphic to $\mathfrak{sl}(2,\mathbb{R})\oplus\mathfrak{sl}(2,\mathbb{R})\oplus\mathbb{R}^2$.
(2) If $\{\alpha_1,\alpha_2\}\cap\{\alpha_3,\alpha_4\}\neq\emptyset$, the symmetries generate a Lie algebra isomorphic to $\mathfrak{sl}(3,\mathbb{R})\oplus\mathbb{R}$.

Proof. The first case follows immediately from the preceding result. Suppose that $\{\alpha_1,\alpha_2\}\cap\{\alpha_3,\alpha_4\}=\emptyset$. Then the vector fields clearly satisfy the commutators

$$[X_{\alpha_1\alpha_2},X_{\alpha_3\alpha_4}]=[X_{\alpha_2\alpha_1},X_{\alpha_3\alpha_4}]=[X_{\alpha_1\alpha_2},X_{\alpha_4\alpha_3}]=[X_{\alpha_2\alpha_1},X_{\alpha_4\alpha_3}]=0.$$

This implies that the copies of $\mathfrak{sl}(2,\mathbb{R})\oplus\mathbb{R}$ generated by $\{X_{\alpha_1\alpha_2},X_{\alpha_2\alpha_1}\}$ and $\{X_{\alpha_3\alpha_4},X_{\alpha_4\alpha_3}\}$ are independent, thus they generate the Lie algebra $\mathfrak{sl}(2,\mathbb{R})\oplus\mathfrak{sl}(2,\mathbb{R})\oplus\mathbb{R}^2$. [5]
Now let $\{\alpha_1,\alpha_2\}\cap\{\alpha_3,\alpha_4\}\neq\emptyset$. Without loss of generality, we can suppose that $\alpha_2=\alpha_3$. Computing the brackets of the vector fields we obtain that

$$\begin{aligned}&[X_{\alpha_1\alpha_2},X_{\alpha_2\alpha_4}]=-x_{\alpha_4}\frac{\partial}{\partial x_{\alpha_1}}=-X_{\alpha_1\alpha_4},\ [X_{\alpha_1\alpha_2},X_{\alpha_4\alpha_2}]=0,\\&[X_{\alpha_2\alpha_1},X_{\alpha_2\alpha_4}]=0,\ [X_{\alpha_2\alpha_1},X_{\alpha_4\alpha_2}]=x_{\alpha_1}\frac{\partial}{\partial x_{\alpha_4}}=X_{\alpha_4\alpha_1}.\end{aligned}\tag{27}$$

These commutators show that the system necessarily admits the six independent symmetries $B=\{X_{\alpha_1\alpha_2}, X_{\alpha_2\alpha_1}, X_{\alpha_2\alpha_4}, X_{\alpha_4\alpha_2}, X_{\alpha_1\alpha_4}, X_{\alpha_4\alpha_1}\}$. The preceding result further implies that the vector fields $X_{\alpha_1\alpha_1}, X_{\alpha_2\alpha_2}$ and $X_{\alpha_4\alpha_4}$ must also be symmetries, hence the Lie algebra generated by the vector fields of B is of dimension nine. Since $X_{\alpha_1\alpha_1}+X_{\alpha_2\alpha_2}+X_{\alpha_4\alpha_4}$ commutes with all generators, the algebra is clearly isomorphic to $\mathfrak{sl}(3,\mathbb{R})\oplus\mathbb{R}$. ■

As a consequence of this property we can inductively estimate the largest semisimple algebra of symmetries generated by symmetries of the type X_{ij} that a system can have. It is easily seen that a subalgebra isomorphic to $\mathfrak{sl}(l,\mathbb{R})$ requires l different indices. In fact, reordering the indices $\{1,\cdots,n\}$ if necessary, we can suppose without loss of generality that the generators of the subalgebra are the symmetries $x_i\frac{\partial}{\partial x_j}$ for $1\leq i,j\leq l$. Now we consider the subset A of $\{1,\cdots,n\}$

$$A=\{1,\cdots l_1\}\cup\{l_1+1,\cdots,l_2\}\cup\{l_{k-1}+1,\cdots,l_k\}\subseteq\{1,\cdots,n\}\tag{28}$$

[4] Recall that a reductive Lie algebra is the direct sum of a semisimple and an Abelian Lie algebra [26].
[5] It should be noted that this does not exclude the possibility that $X_{\alpha_1\alpha_3}, X_{\alpha_1\alpha_4}, X_{\alpha_2\alpha_3}$ or $X_{\alpha_2\alpha_4}$ is also a symmetry of the system.

where $l_1, \cdots, l_k$ are (not necessarily distinct) integers and such that the three following conditions are satisfied:

(1) for any $1 \leq a \leq k$ and any indices $i, j \in \{l_{a-1}+1, \cdots, l_a\}$, the vector field $X_{ij} = x_i \frac{\partial}{\partial x_j}$ is a symmetry of the system,
(2) for $a \neq b$ and indices $i \in \{l_{a-1}+1, \cdots, l_a\}$, $j \in \{l_{b-1}+1, \cdots, l_b\}$, X_{ij} is not a symmetry of the system.
(3) for $i, j \in \{1, \cdots, n\} - A$, either X_{ij} or X_{ji} is not a symmetry of the system.

A long but completely routine computation, like done before, shows that these conditions imply that the system admits the semisimple subalgebra of symmetries

$$\mathfrak{sl}(l_1, \mathbb{R}) \oplus \mathfrak{sl}(l_2, \mathbb{R}) \oplus \cdots \oplus \mathfrak{sl}(l_k, \mathbb{R}), \tag{29}$$

and by equation (28), the inequality $l_1 + \cdots + l_k \leq n$ is satisfied. Taking into account the possibility that some l_a coincide, the general case can be summarized in the following

Proposition 3 *Let $n \geq 2$. The maximal semisimple Lie algebra $\mathfrak{s}$ of the system $\ddot{\mathbf{x}} = \omega^{\alpha}(t, \mathbf{x}),\ 1 \leq \alpha \leq n$ generated by symmetries of the type $x_i \frac{\partial}{\partial x_j}$ has the form*

$$\mathfrak{s} = \mathfrak{sl}(k, \mathbb{R})^{a_k} \oplus \mathfrak{sl}(k-1, \mathbb{R})^{a_{k-1}} \oplus \cdots \oplus \mathfrak{sl}(3, \mathbb{R})^{a_3} \oplus \mathfrak{sl}(2, \mathbb{R})^{a_2}, \tag{30}$$

where $(a_{k_0}, \cdots, a_2) \in \mathbb{N}^{k-1} \cup \{0\}$ and

$$2a_2 + 3a_3 + 4a_4 + \cdots + k\, a_k \leq n. \tag{31}$$

In particular, the rank of $\mathfrak{s}$ satisfies

$$\mathrm{rank}(\mathfrak{s}) \leq n - 2 \tag{32}$$

if the system is not maximally symmetric.

Proof. The first part of the proof follows from the preceding remarks. We prove the second assertion concerning the rank of $\mathfrak{s}$. First of all, if $k = n$, then $a_n = 1$, $a_j = 0$ for $j \leq n-1$ and the subalgebra $\mathfrak{s}$ is isomorphic to $\mathfrak{sl}(n, \mathbb{R})$. In these conditions, the system is necessarily of maximal symmetry [13]. Hence suppose that $k \leq n-1$. For any $2 \leq k \leq n$ we have $\mathrm{rank}(\mathfrak{sl}(k, \mathbb{R})) = k-1$, and by equation (30), the rank of $\mathfrak{s}$ is given by $\mathrm{rank}(\mathfrak{s}) = a_2 + 2a_3 + \cdots + (k-1)\, a_k$. Inserting this into equation (31) leads to

$$\mathrm{rank}(\mathfrak{s}) + \sum_{i=2}^{k} a_i = 2a_2 + 3a_3 + 4a_4 + \cdots + k\, a_k \leq n. \tag{33}$$

Now, if the subalgebra $\mathfrak{s}$ does not reduce to zero, then $\sum_{i=2}^{k} a_i \geq 1$. Two cases must be distinguished: If $\sum_{i=2}^{k} a_i = 1$, let $a_{j_0} = 1$ be the only non-vanishing index. In this case, $\mathrm{rank}(\mathfrak{s}) = j_0 - 1$, and by the assumption $j_0 \leq n-1$ it follows at once that $j_0 - 1 \leq n-2$.

It remains the case where $\sum_{i=2}^{k} a_i \geq 2$, but using equation (33), we immediately obtain that

$$\operatorname{rank}(\mathfrak{s}) \leq n - \sum_{i=2}^{k} a_i \leq n - 2.$$

■

Example 1. Let $n \geq 2$ and consider the system

$$\begin{aligned} \ddot{x}_k &= x_k \mathrm{e}^{x_n}, \;\; 1 \leq k \leq n-1 \\ \ddot{x}_n &= \alpha\, x_n^k\, t^r, \;\; \alpha, k, r \in \mathbb{R}^*. \end{aligned} \tag{34}$$

By proposition 2, such a system can only have symmetries of the type $X_{ij} = x_i \frac{\partial}{\partial x_j}$. Now, by formula (23), all vector fields X_{ij} for $1 \leq i, j \leq n-1$ are symmetries of the system, while those for which either $i = n$ or $j = n$ holds do not satisfy the symmetry condition. As a consequence, for arbitrary values of α, k, r, the algebra of symmetries is isomorphic to $\mathfrak{sl}(n-1, \mathbb{R}) \oplus \mathbb{R}$.

It should be observed that, up to now, we have only considered the case of semisimple subalgebras generated by vectors fields of the type X_{ij}, as it is the easiest to describe. Obviously, not any semisimple subalgebra $\mathfrak{s}$ of point symmetries subjected to the condition $\xi(t, \mathbf{x}) = 0$ must be generated by vector fields of this form. In particular, the generators of $\mathfrak{s}$ can be given by linear combinations of these. This is shown for example by rotationally invariant systems like the n-dimensional oscillators or the Kepler problem [12,18], where the generators of the $\mathfrak{so}(n)$ subalgebra are the vector fields $X_{ij} - X_{ji}$ for $1 \leq i < j \leq n$. However, we can easily modify proposition 3 in order to adapt it to the case of semisimple subalgebras isomorphic to $\mathfrak{so}(n)$,

Proposition 4 *Let $n \geq 2$. The maximal semisimple Lie algebra $\mathfrak{s}$ of the system $\ddot{\mathbf{x}} = \omega^{\alpha}(t, \mathbf{x})$, $1 \leq \alpha \leq n$ generated by symmetries of the type $X_{[ij]} = x_i \frac{\partial}{\partial x_j} - x_j \frac{\partial}{\partial x_i}$ has the form*

$$\mathfrak{s} = \mathfrak{so}(3)^{b_1} \oplus \mathfrak{so}(4)^{b_2} \oplus \cdots \oplus \mathfrak{so}(m)^{b_{m-2}}, \tag{35}$$

where $(b_1, \cdots, b_{m-2}) \in \mathbb{N}^{m-2} \cup \{0\}$ and

$$3b_1 + 4b_2 + 5b_3 + \cdots + m\, b_{m-2} \leq n. \tag{36}$$

In particular, the rank of $\mathfrak{s}$ satisfies

$$\operatorname{rank}(\mathfrak{s}) \leq \left[\frac{n}{2}\right].$$

Proof. Suppose that the system admits a subalgebra $\mathfrak{s}$ with decomposition (35). It is clear that for any $b_k \neq 0$, we need $(k+2)\, b_k$ distinct indices in $\{1, \cdots, n\}$ such that the vector fields $X_{[ij]}$ generate $\mathfrak{so}(k)^{b_k}$. Hence, extracting from $\{1, \cdots, n\}$ all the indices for which $X_{[ij]}$

is a symmetry of the system, and reasoning as before, we obtain the subset B of $\{1,\cdots,n\}$

$$B=\{1,\cdots l_1\}\cup\{l_1+1,\cdots,l_2\}\cup\{l_{k-1}+1,\cdots,l_k\}\subseteq\{1,\cdots,n\} \tag{37}$$

where $l_1,\cdots,l_k$ are (not necessarily distinct) integers and the following conditions hold:

(1) for any $1\leq a\leq k$ and any indices $i,j\in\{l_{a-1}+1,\cdots,l_a\}$, the vector field $X_{[ij]}=x_i\frac{\partial}{\partial x_j}-x_j\frac{\partial}{\partial x_i}$ is a symmetry of the system,
(2) for $a\neq b$ and indices $i\in\{l_{a-1}+1,\cdots,l_a\}$, $j\in\{l_{b-1}+1,\cdots,l_b\}$, $X_{[ij]}$ is not a symmetry of the system.
(3) for $i,j\in\{1,\cdots,n\}-A$, $X_{[ij]}$ is not a symmetry of the system.

Clearly the symmetries $X_{[ij]}$ associated to the subset B give rise to a semisimple subalgebra (35). The rank of such a subalgebra $\mathfrak{s}$ is easily computed to be

$$\operatorname{rank}(\mathfrak{s})=b_1+2\,(b_2+b_3)+\cdots+\left[\frac{m}{2}\right]b_{m-2}\leq\left[\frac{n}{2}\right]. \tag{38}$$

■

Example 2. For $n\geq 2$ consider the system defined by the equations

$$\ddot{x}_l=x_l\theta\,(t)\left(g_{11}x_1^2+g_{22}x_2^2+\cdots+g_{nn}x_n^2\right)^k,\ 1\leq l\leq n, \tag{39}$$

where $g_{ii}=\pm 1$ for $1\leq i\leq n$, $k\in\mathbb{Z}$ and $\theta\,(t)\neq 0$. This system possesses the semisimple subalgebra $\mathfrak{s}$ generated by the symmetries

$$X_{[i,j]}=g_{ii}x_i\frac{\partial}{\partial x_j}-g_{jj}x_j\frac{\partial}{\partial x_i},\ 1\leq i,j\leq n. \tag{40}$$

It is routine to verify that $\mathfrak{s}\simeq\mathfrak{so}\,(p,q)$, where p is the number of indices such that $g_{ii}=1$, and q that of indices such that $g_{ii}=-1$. Moreover, depending on the choice of the function $\theta\,(t)$, the system can posses more symmetries. In any case, it can be easily verified that

$$\frac{n\,(n-1)}{2}\leq\dim\mathfrak{L}\leq\frac{n\,(n-1)+4}{2}. \tag{41}$$

If $\theta\,(t)$ is a constant, the system has the obvious symmetry $\frac{\partial}{\partial t}$, as well as a scaling symmetry $Z=\alpha t\frac{\partial}{\partial t}+x_i\frac{\partial}{\partial x_i}$, $\alpha\in\mathbb{R}$. In particular, if $k=-\frac{3}{2}$, $g_{ii}=1$ for all i and $\theta\,(t)=const.$, we recover the well known Kepler problem [12].

It would be conceivable to develop also a similar criterion for maximal semisimple subalgebras of symplectic type $\mathfrak{sp}(2n)$, but a detailed description is more complicated, as there is no single type of vector fields that generates these Lie algebras, in contrast to the special linear and orthogonal algebras [27,28]. For this reason we omit an explicit description here.

3 Point symmetries with $\xi(t,\mathbf{x}) \neq 0$

In this section we analyze another special type of symmetries, now having a nonzero coefficient in $\frac{\partial}{\partial t}$. From the structural point of view, symmetries of this kind are more restrictive, and their number will be considerably lower to those satisfying $\xi(t,\mathbf{x}) = 0$, giving rise to subalgebras of at most dimension three. In addition, some properties of systems admitting scaling symmetries are obtained.

Proposition 5 *Let $n \geq 2$ and $\ddot{\mathbf{x}} = \omega^{\alpha}(t,\mathbf{x})$ $(1 \leq \alpha \leq n)$. Suppose that there exists an index $j_0 \in \{1,\cdots,n\}$ such that $\frac{\partial \omega^{j_0}}{\partial t} \neq 0$. Then there exists at most one Lie point symmetry of the form*

$$X = f(t)\frac{\partial}{\partial t} + \frac{1}{2}\dot{f}(t)\, x_j \frac{\partial}{\partial x_j}, \tag{42}$$

unless $\omega^j = \theta(t)\, x_j$ for $1 \leq j \leq n$.

Proof. Let us consider first the case where $\ddot{\mathbf{x}} \neq \theta(t)\mathbf{x}_{\alpha}$. Consider a vector field X of type (42). Then the prolonged vector field

$$\dot{X} = f(t)\frac{\partial}{\partial t} + \frac{1}{2}\dot{f}(t)\, x_j \frac{\partial}{\partial x_j} + \frac{1}{2}\left(x_j \ddot{f}(t) - \dot{x}_j \dot{f}(t)\right)\frac{\partial}{\partial \dot{x}_j}$$

leads to the symmetry condition

$$2f(t)\frac{\partial \omega^j}{\partial t} - x_j\, f^{(3)}(t) + \left(x_k \frac{\partial \omega^j}{\partial x_k} + 3\,\omega^j\right)\dot{f}(t) = 0,\ 1 \leq j \leq n. \tag{43}$$

By assumption, for the index j_0 we have $\frac{\partial \omega^{j_0}}{\partial t} \neq 0$, hence $\dot{f}(t)$ cannot vanish identically for $f(t) \neq 0$. Let $R \subset \{1,\cdots,n\}$ be the set of indices l such that $\frac{\partial \omega^l}{\partial t} \neq 0$. The symmetry condition (43) is satisfied only if for any $l \in R$ the corresponding component function ω^l has the following form:

$$\omega^l(t,\mathbf{x}) = \frac{x_l}{2\, f^2(t)}\int^t f(s)\, f^{(3)}(s)\; ds + \frac{1}{f(t)\sqrt{f(t)}}\Omega^l\left(\frac{\mathbf{x}}{\sqrt{f(t)}}\right). \tag{44}$$

Failure for an index $l \in R$ to fulfil equation (44) implies that (43) is only satisfied for $f(t) = 0$. On the other hand, if $f(t) \neq 0$, equation (43) shows that only the multiples αX of X will be symmetries of the system, proving that it has at most one point symmetry of type (42).

Now let S be the set of indices $k \in \{1,\cdots,n\}$ such that $\frac{\partial \omega^k}{\partial t} = 0$, i.e., $S = \{1,\cdots,n\} - R$. In this case, equation (43) reduces to

$$-x_k\, f^{(3)}(t) + \left(x_r \frac{\partial \omega^k}{\partial x_r} + 3\,\omega^k\right)\dot{f}(t) = 0. \tag{45}$$

The equation can be easily separated, and we can rewrite it as

$$\left(x_r \frac{\partial \omega^k}{\partial x_r} + 3\,\omega^k\right) x_k^{-1} = \frac{f^{(3)}\left(t\right)}{\dot{f}\left(t\right)}) = \lambda, \tag{46}$$

where $\lambda \in \mathbb{R}$ is some constant. It follows in particular that for any $k \in S$ the component function $\omega^k(\mathbf{x})$ has the form

$$\omega^k\left(\mathbf{x}\right) = \frac{\lambda}{4} x_k + x_1^{-3}\,\Omega^k\left(\frac{x_2}{x_1}, \cdots, \frac{x_n}{x_1}\right). \tag{47}$$

Again, if for some $k_0 \in S$ the function $\omega^k\left(\mathbf{x}\right)$ has not the form (47), then (45) is only satisfied for $f\left(t\right) = 0$. We conclude that the system only admits a symmetry of type (42) if the component functions satisfy equations (44) and (47) for all indices in R and S respectively. Finally, for the (maximal symmetry) case $\omega^j = \theta\left(t\right) x_j$ $(1 \leq j \leq n)$, the symmetry condition (43) merely states that the third order ordinary differential equation

$$2f\left(t\right)\dot{\theta}\left(t\right) - \; f^{(3)}\left(t\right) + \;\left(n+1\right)\,\theta\left(t\right)\;\dot{f}\left(t\right) = 0 \tag{48}$$

is satisfied, and the solution to this ODE clearly provides three independent symmetries. ■

The interesting consequence of this result is that explicit time dependency of the system $\ddot{\mathbf{x}} = \omega^\alpha\left(t, \mathbf{x}\right)$ imposes severe restrictions on the existence of symmetries of type (42). Up to the well-known maximal symmetry case, such systems either have no symmetry of this type, or exactly one.

Corollary 2 *If the system $\ddot{\mathbf{x}} = \omega^\alpha\left(t, \mathbf{x}\right)$ is not maximally symmetric and admits more than one point symmetry of type (42), then $\frac{\partial \omega^\alpha}{\partial t} = 0$ for all $1 \leq \alpha \leq n$.*

The situation is somewhat different for time-independent (autonomous) systems

$$\ddot{\mathbf{x}} = \omega^\alpha\left(\mathbf{x}\right),\; 1 \leq \alpha \leq n, \tag{49}$$

which always admit the symmetry $X = \frac{\partial}{\partial t}$. We analyze now under which conditions additional symmetries of this form exist, and determine the maximal number of independent symmetries. Suppose that additional symmetries of type (42) do exist. Then equation (46) must be satisfied for any index $1 \leq \alpha \leq n$, and following the preceding proof we have that equations of motion have the form

$$\omega^\alpha\left(\mathbf{x}\right) = \frac{\lambda}{4} x_\alpha + x_1^{-3}\,\Omega^\alpha\left(\frac{x_2}{x_1}, \cdots, \frac{x_n}{x_1}\right) \tag{50}$$

for some real constant λ. In particular, since $\frac{\partial \omega^\alpha}{\partial t} = 0$ holds for any α, the (non-constant) function $f\left(t\right)$ is further subjected to the constraint (see eq. (46))

$$f^{(3)}\left(t\right) = \lambda\,\dot{f}\left(t\right). \tag{51}$$

Two possibilities can arise, depending on the derivatives of $f\left(t\right)$ and the value of λ:

(1) If $\lambda \neq 0$, then $f^{(3)}(t) \neq 0$ and integration of (51) gives

$$f(t) = \begin{cases} a_0 + a_1 e^{\sqrt{\lambda}t} + a_2 e^{-\sqrt{\lambda}t}, & \lambda > 0 \\ a_0 + a_1 \sin\left(\sqrt{-\lambda}t\right) + a_2 \cos\left(\sqrt{-\lambda}t\right), & \lambda < 0 \end{cases}$$

Let X_1, X_2 and X_3 be the point symmetries corresponding to the parameter values $(a_0, a_1, a_2) = (1,0,0)$, $(0,1,0)$ and $(0,0,1)$ respectively. A routine computation shows that the commutators are given by

$$\begin{aligned} &[X_1, X_2] = \sqrt{\lambda}\, X_2, \quad [X_1, X_3] = -\sqrt{\lambda} X_3, \quad [X_2, X_3] = -2\sqrt{\lambda} X_1, \text{ for } \lambda > 0, \\ &[X_1, X_2] = \sqrt{-\lambda}\, X_3, \; [X_1, X_3] = -\sqrt{-\lambda} X_2, \; [X_2, X_3] = -\sqrt{-\lambda} X_1, \text{ for } \lambda < 0. \end{aligned}$$

Let $\mathfrak{s}$ be the Lie algebra generated by X_1, X_2 and X_3. Computing the Killing form [6] of $\mathfrak{s}$ it is easily shown that, in both cases, $\mathfrak{s}$ is a simple Lie algebra isomorphic to $\mathfrak{sl}(2,\mathbb{R})$.

(2) If $\lambda = 0$, then $f^{(3)}(t) = 0$ and $f(t) = b_0 + b_1 t + b_2 t^2$. The three point symmetries are thus

$$X_1 = \frac{\partial}{\partial t}, \; X_2 = t\frac{\partial}{\partial t} + \frac{1}{2} x_k \frac{\partial}{\partial x_k}, \; X_3 = t^2 \frac{\partial}{\partial t} + t\, x_k \frac{\partial}{\partial x_k},$$

with commutators

$$[X_1, X_2] = X_1, \; [X_1, X_3] = 2X_2, \; [X_2, X_3] = X_3.$$

Again, it is routine to verify that the Lie algebra they generate is isomorphic to $\mathfrak{sl}(2,\mathbb{R})$.

With this analysis of the commutators, we have proved the following

Proposition 6 *Let $n \geq 2$ and $\ddot{\mathbf{x}} = \omega^\alpha(\mathbf{x})$, $(1 \leq \alpha \leq n)$ be a time-independent system. The Lie algebra $\mathfrak{s}$ generated by symmetries of type (42) is either Abelian or isomorphic to $\mathfrak{sl}(2,\mathbb{R})$. In particular, the number of independent symmetries does not exceed three.*

Example 3. Let $n \geq 2$ and consider the system

$$\begin{aligned} \ddot{x}_k &= (\lambda/\mu)\, x_k + x_{k+1} x_1^{-\mu}, \; 1 \leq k \leq n-1 \\ \ddot{x}_n &= (\lambda/\mu)\, x_n + \rho x_1^{1-\mu}, \end{aligned} \tag{52}$$

where $\lambda, \mu, \rho \in \mathbb{R}$ and $\mu \neq 0$. Using proposition 2, it follows at once that the system does not admit symmetries of type $X_{ij} = x_i \frac{\partial}{\partial x_j}$ and $Y_j = \varphi(t) \frac{\partial}{\partial x_j}$ for $1 \leq i, j, \leq n$, as well as linear combinations of them. Since the $\ddot{x}$s do not depend on t, the vector field $\frac{\partial}{\partial t}$ is always a symmetry. Now, for values $\mu \neq 4$, proposition 5 implies that this is the only symmetry, whereas for $\mu = 4$ the system possesses a symmetry algebra $\mathfrak{L}$ isomorphic to $\mathfrak{sl}(2,\mathbb{R})$.

---

[6] Recall that the Killing form is determined by the adjoint representation of the Lie algebra. See e.g. [26].

It is clear that symmetries with $\xi(t,\mathbf{x}) \neq 0$ must not be restricted to those of type (42), as linear combinations with symmetries of the type X_{ij} or Y_j analyzed previously are possible and not uncommon [19,22]. However, with full generality, the simplified symmetry condition (20) does not provide much information.

Finally, in addition to type (42), we consider the symmetries

$$X = \alpha t \frac{\partial}{\partial t} + \sum_{j=1}^{n} \beta_j x_j \frac{\partial}{\partial x_j}, \; \alpha, \beta_j \in \mathbb{R} \tag{53}$$

corresponding to scaling symmetries. It can be easily verified that, if $\beta_j = \frac{\alpha}{2}$ for all j, X coincides with a symmetry of type (42), while for any other choice of the parameters, we obtain a symmetry type different from the previous ones. Actually they can be seen as a linear combination of symmetries of type X_{ij} and (42). Scaling symmetries have shown their interest in connection with some of the most relevant integrable systems, as well with some properties of their constants of motion [11,12]. Actually the existence of a point symmetry (53) imposes restrictions on the equations of motion. For a system $\ddot{\mathbf{x}} = \omega^{\alpha}(t,\mathbf{x})$**,** the symmetry condition (20) implies that the functions $\omega^k(t,\mathbf{x})$ satisfy the linear PDE

$$\alpha t \frac{\partial \omega^k(t,\mathbf{x})}{\partial t} + \sum_{j=1}^{n} \beta_j x_j \frac{\partial \omega^k(t,\mathbf{x})}{\partial x_j} + (2\alpha - \beta_k)\, \omega^k(t,\mathbf{x}) = 0, \; 1 \leq k \leq n. \tag{54}$$

Essentially, three cases can be distinguished, depending on the values of the parameters $(\alpha, \beta_1, \cdots, \beta_n)$.

(1) If $\alpha \neq 0$, $\beta_j = 0$ for $1 \leq j \leq n$, we can put $\alpha = 1$. The solution to (54) is given by

$$\omega^k(t,\mathbf{x}) = t^{-2}\Omega^k(x_1, \cdots, x_n), \; 1 \leq k \leq n.$$

In particular, for autonomous systems the vector field $t\frac{\partial}{\partial t}$ cannot be a symmetry, unless $\omega^k(t,\mathbf{x}) = 0$ holds for all k. This again leads to the maximally symmetry case.

(2) Let $\alpha \neq 0$ and $\beta_{j_0} \neq 0$ for some j_0. Reordering the indices, we can suppose that $\beta_1 \neq 0$ and $\alpha = 1$. The equations of motion have the form

$$\omega^k(t,\mathbf{x}) = t^{(\beta_k - 2)}\Omega^k\left(\frac{x_1}{t^{\beta_1}}, \cdots, \frac{x_n}{t^{\beta_n}}\right) \quad \text{if } \frac{\partial \omega^k(t,\mathbf{x})}{\partial t} \neq 0,$$
$$\omega^k(t,\mathbf{x}) = x_1^{(\beta_k - 2)/\beta_1}\Omega^k\left(\frac{x_2^{\beta_1}}{x_1^{\beta_2}}, \cdots, \frac{x_n^{\beta_1}}{x_1^{\beta_n}}\right) \text{ if } \frac{\partial \omega^k(t,\mathbf{x})}{\partial t} = 0.$$

(3) Let $\alpha = 0$. Again, without loss of generality, we can assume that $\beta_1 \neq 0$. In this case, equation (54) has the solution

$$\omega^k(t,\mathbf{x}) = \Omega^k\left(t, \frac{x_2^{\beta_1}}{x_1^{\beta_2}}, \cdots, \frac{x_n^{\beta_1}}{x_1^{\beta_n}}\right) x_1^{\beta_k/\beta_1}.$$

From this case-by-case analysis we immediately deduce the following result:

Proposition 7 *For $n \geq 2$, the number of symmetries of type (53) of a system $\ddot{\mathbf{x}} = \omega^{\alpha}(t, \mathbf{x})$, $1 \leq \alpha \leq n$, is at most $n+1$. In particular, they generate an Abelian subalgebra of the symmetry algebra $\mathfrak{L}$.*

As an extreme case, if the system admits the symmetries $X_{ii} = x_i \frac{\partial}{\partial x_i}$ for $1 \leq i \leq n$, the preceding observations show that the system is linear and has the form

$$\omega^k(t, \mathbf{x}) = f_k(t)\, x_k,\ 1 \leq k \leq n.$$

However, as illustrates the following example, a system can be highly nonlinear and still have a maximal subalgebra of symmetries of type (53).

Example 4. Let $\sigma_1, \cdots, \sigma_n \in \mathbb{R} - \{0, \pm 1\}$ and consider the autonomous system

$$\omega^k(\mathbf{x}) = \rho_k\, x_k^{(1+\sigma_k)} \prod_{j \neq k} x_j^{\sigma_j},\ 1 \leq k \leq n,\ \rho_j \in \mathbb{R} - \{0\}$$

where the conditions $\left(4 + \sum_{j=1}^{n} \sigma_j\right) \neq 0$ and $4 + \sum_{j=1}^{n} \sigma_j - \sigma_k \neq 0$ are satisfied for all k. As a consequence of proposition 2, the system does not admit symmetries of type $X_{ij} = x_i \frac{\partial}{\partial x_j}$ and $Y_j = \psi_j(t) \frac{\partial}{\partial x_j}$. A routine verification further shows that linear combinations of these vector fields do neither satisfy the symmetry condition. Proposition 6 implies that $V_0 = \frac{\partial}{\partial t}$ is the only symmetry of type (42). From the homogeneity of the $\omega^k(\mathbf{x})$ we obtain that

$$x_k \frac{\partial \omega^k(\mathbf{x})}{\partial x_k} = (1 + \sigma_k)\, \omega^k(\mathbf{x}),\ x_l \frac{\partial \omega^k(\mathbf{x})}{\partial x_l} = \sigma_l\, \omega^k(\mathbf{x}),\ l \neq k.$$

Therefore, the equations of motion satisfy a linear equation (54) if

$$\alpha t \frac{\partial \omega^k(t, \mathbf{x})}{\partial t} + \sum_{j=1}^{n} \beta_j x_j \frac{\partial \omega^k(t, \mathbf{x})}{\partial x_j} + (2\alpha - \beta_k)\, \omega^k(t, \mathbf{x}) = \left(\sum_{j=1}^{n} \sigma_j \beta_j + 2\alpha\right) \omega^k(t, \mathbf{x}) = 0$$

holds for any $1 \leq k \leq n$. Now, chosing $\alpha = 2$ and $\beta_j = 1$ for $j \neq k$, the preceding condition reduces to

$$\sum_{j \neq k} \sigma_j + \sigma_k \beta_k + 4 = 0,$$

thus taking $\beta_k = -\left(\sum_{j \neq k} \sigma_j + 4\right)/\sigma_k$ shows that the system admits the scaling symmetry

$$V_k = 2t \frac{\partial}{\partial t} - \left(\frac{4 + \sum_{j \neq k} \sigma_j}{\sigma_k}\right) x_k \frac{\partial}{\partial x_k} + \sum_{j \neq k} x_j \frac{\partial}{\partial x_j},\ 1 \leq k \leq n. \tag{55}$$

It is not difficult to justify that $\{V_0, \cdots, V_n\}$ exhaust all possible symmetries of the system. Computing the brackets gives

$$[V_0, V_k] = \frac{2\sigma_k}{4 + \sum_{j \neq k} \sigma_j} V_k,\ [V_k, V_l] = 0, 1 \leq k, l \leq n \tag{56}$$

showing that the Lie algebra $\mathfrak{L}$ of point symmetries is solvable of dimension $n+1$.

4 Conclusions

In this work we have analyzed some types of generic symmetries of a system of n second order ordinary differential equations $\ddot{\mathbf{x}} = \omega^{\alpha}(t, \mathbf{x})$. Having discarded those symmetries that ensure that the system is (locally) linearizable to the free particle system, the symmetry condition (3) has been formulated for non-maximally symmetric systems. Basing on this simplification, some specific types of symmetries have been considered in detail. The first case considered corresponds to symmetries satisfying the constraint $\xi(t, \mathbf{x}) = 0$. Analyzing their commutators, a procedure to generate semisimple (actually reductive) subalgebras of type $\mathfrak{sl}(k, \mathbb{R})$ has been obtained. Enlarging the argument to linear combinations of these symmetries, the case of orthogonal subalgebras $\mathfrak{so}(k)$ has also been obtained. Certainly, not all possibilities of symmetries generating semisimple Lie algebras have been covered, but a detailed description for all semisimple real Lie algebras forms would require an extended analysis of their representation theory, as well as their realizations in terms of differential operators [27,28]. Such an analysis will be performed elsewhere.

A second type of symmetries analyzed corresponds to those fulfilling the condition $\xi(t, \mathbf{x}) \neq 0$. It has been shown that such symmetries are scarcer, and depending on the non-autonomous/ autonomous character of the system, they give rise to either Abelian subalgebras or to subalgebras isomorphic to $\mathfrak{sl}(2, \mathbb{R})$. In particular, a procedure for the determination of systems with minimal semisimple Lie algebra of symmetries isomorphic to $\mathfrak{sl}(2, \mathbb{R})$ can be easily deduced from proposition 6. This shows the existence of large classes of non-maximally symmetry systems with (minimal) semisimple symmetry algebras. Another special case within symmetries satisfying $\xi(t, \mathbf{x}) \neq 0$ is given by scaling symmetries, which have played a certain role in the study of integrals of motion and Noether symmetries [10,12]. A somewhat unexpected result is that a highly nonlinear system can have a number of independent scaling symmetries as high as $n + 1$, a property usually related to linearizable and linear systems [23].

Summarizing, the results are potentially useful for the explicit construction of systems possessing a prescribed semisimple or reductive subalgebra of symmetries, and could be applied to classify systems according to a fixed Levi factor.
There still remain several open questions concerning the generic structure of symmetries for arbitrary systems, like the exact coupling of the analyzed symmetry types. Another important problem is whether for systems $\ddot{\mathbf{x}} = \omega^{\alpha}(t, \dot{\mathbf{x}}, \mathbf{x})$ explicitly depending on the derivatives, similar statements on the structure of subalgebras of the symmetry algebra $\mathfrak{L}$ can be made. The essential difficulty in this case is that the equations resulting from the symmetry condition (3) cannot be further separated. In this context, it is worthy to be inspected whether the imposition of fixed subalgebras generated by symmetries of certain types allow to derive general properties of the equations of motion and further symmetries. Work in this direction is currently in progress.

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