A PARTIAL JUSTIFICATION OF GREENE'S CRITERION FOR CONFORMALLY SYMPLECTIC SYSTEMS

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Abstract. Greene's criterion for twist mappings asserts the existence of smooth invariant circles with preassigned rotation number if and only if the periodic trajectories with frequency approaching that of the quasi-periodic orbit are linearly stable.

We formulate an extension of this criterion for conformally symplectic systems in any dimension and prove one direction of the implication, namely that if there is a smooth invariant attractor, we can predict the eigenvalues of the periodic orbits whose frequencies approximate that of the tori. The proof of this result is very different from the proof in the area preserving case, since in the conformally symplectic case the existence of periodic orbits requires adjusting parameters. Also, as shown in [13], in the conformally symplectic case there are no Birkhoff invariants giving obstructions to linearization near an invariant torus.

As a byproduct of the techniques developed here, we obtain quantitative information on the existence of periodic orbits in the neighborhood of quasi-periodic tori and we provide upper and lower bounds on the width of the Arnold tongues in n-degrees of freedom conformally symplectic systems.

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1. Introduction

The goal of this paper is to present some results (see Theorem 4) on a partial justification of Greene's criterion for conformally symplectic mappings (i.e., mappings that transform a symplectic form into a multiple of itself, see [17] and (2.1)).

Conformally symplectic mappings appear in several applications, including mechanical systems with friction proportional to the velocity (an emblematic example is the spin-orbit problem in celestial mechanics, [19]), in the study of critical points of discounted functionals (very common in finance, [9]) and in non-equilibrium statistical mechanics (Gaussian thermostats, [29, 67]).

In this section, we will describe briefly the motivation for the results we obtained, present them informally and outline the organization of the paper.

1.1. **The Greene's criterion.**

The paper [39] studied the breakdown of smooth quasi-periodic solutions (KAM circles) in twist mappings. The main result of [39] was the assertion (verified by careful numerics) that a smooth invariant circle exists if and only if the periodic orbits approximating it are stable. This assertion is called *"The Greene's criterion"*.

The idea of [39], which generalizes to other contexts, is that the existence or not of a smooth KAM circle can be investigated by studying the stability properties of orbits which are close to the KAM circle.

In this paper, we will be concerned with conformally symplectic systems in any number of degrees of freedom. Hence, one can expect relevant differences in details with the symplectic case in one degree of freedom, which was investigated in [39]. Notably, adjusting parameters will play a significant role and, as shown in [17], for conformally symplectic systems there are no Birkhoff invariants.

The work developed in [39] also contained a number of other results on the behavior of the distance of the eigenvalues at the critical point, as well as scaling properties, which we will not consider here (see however [56, 16]). The study of the breakdown for twist mappings can be investigated by a variety of other methods; a comparative survey of these methods can be found in an appendix of [15].

A partial justification of Greene's criterion in the symplectic case was obtained in [32, 46], by showing that if there is an invariant circle, then the bounds on the residue (a measure of stability of the periodic orbits) hold. The work presented in [32] provides a version for complex values of the parameter, which was implemented in [31].

The idea of the proof of the above partial justification for symplectic systems is conceptually simple. By the theory of Birkhoff normal forms, one can perform a change of variables that makes the system almost integrable in a neighborhood of the torus. The error in the approximate integrability can be bounded (in a smooth norm) by an arbitrarily high power of the distance to the torus.

The residue of a periodic orbit, being a function of the eigenvalues, is invariant under changes of variables, hence it can be computed in any system of coordinates, in particular in the Birkhoff normal form coordinates. Therefore using perturbation theories one can show that the orbits that lie at a small distance from the torus have a small residue. Using the twist properties, one can also show that the orbits with a rotation number close to that of the torus, indeed lie in a small neighborhood.

A justification of the other direction of Greene's criterion (that if all the approximating orbits are stable, there is a smooth invariant circle) is more difficult, but there has been a partial progress. The paper [24] shows that, if the dynamics of the renormalization group satisfies certain geometric hypotheses (for which [24] supplies numerical evidence), then Greene's criterion would hold in the domain of universality. That is, under the geometric hypothesis on the dynamics of the renormalization group, if there is no smooth invariant circle, the residue will grow for families in the domain of universality. Furthermore, [24] shows that in such a case the rate of growth of the residue satisfies several scaling properties, which are verified numerically (see also [2]).

Greene's criterion has been generalized to several other geometric contexts. The case of symplectic mappings in higher dimensions has been considered numerically in [64, 21] and some partial justication was obtained in [63]. Non-twist maps were studied in [26] and [38]. The case of volume preserving maps has been studied numerically in [48, 49] and rigorously in [37].

1.2. **Greene's criterion for conformally symplectic systems.** The goal of this paper is to present a partial justification of Greene's criterion in yet another context, namely conformally symplectic mappings (the case of conformally symplectic flows is also done in a similar way, see Section 9), that is systems that transform a symplectic structure into a multiple of itself. More precisely, we consider a mapping f such that $f^*\Omega = \lambda\Omega$, where Ω is a symplectic form and $0 < \lambda < 1$ is a number. The case $\lambda = 1$ corresponds to symplectic mappings, which we will not consider here and which is indeed very different. The case $\lambda > 1$ can be reduced to the previous one by considering the inverse of the mapping.

Since conformally symplectic mappings may be dissipative, one needs to adjust parameters to obtain that there are quasi-periodic orbits with a fixed rotation number. Recently, there have been several geometric studies of quasi-periodic orbits in conformally symplectic mappings. The paper [19] obtained a KAM theory for the spin-orbit problem, while a general KAM theory for conformally symplectic systems in any dimension was obtained in [17]. The KAM theory developed in [17] involves adjusting parameters, so that the unknowns are the embedding of the torus with the given rotational frequency and the parameters.

More relevantly for the present purposes, in [13] one can find a study of Poincaré-Birkhoff theory of normal forms in a neighborhood of an invariant Lagrangian torus. The somewhat surprising result of [13] is that the theory of normal forms around invariant tori for conformally symplectic systems has much stronger results than Poincaré-Birkhoff theory in Hamiltonian systems. In [13] it is shown that, for dissipative conformally symplectic systems, in a neighborhood of a KAM torus, there is a smooth change of variables which reduces the motion to a rotation and a linear contraction (by the conformal factor λ). In particular, for dissipative conformally symplectic systems, there is no analogue of Birkhoff invariants. Furthermore, the transformations to normal form are convergent.

In view of the results of [13], and also in agreement with the KAM theory developed in [17], it is clear that Greeene's criterion and its possible justifications should treat parameters as unknowns. In contrast to what happens for area preserving or symplectic mappings, there are no periodic orbits in a neighborhood of the tori for the same parameter values. For these parameter values, the torus attracts all the orbits in a neighborhood and there are no other recurrent orbits in a neighborhood (a very strong form of this phenomenon is presented in [13]). The best that we can expect for conformally symplectic systems is that if there is one parameter for which there is an invariant circle, then there is a sequence of parameters converging to the parameter of the torus and that for each of them there is a periodic orbit. These sequence of periodic orbits (for the maps with parameters in the sequences) approximates the invariant torus in the sense of the Hausdorff distance of sets. Since adjusting parameters plays a role similar to changing initial conditions in the symplectic case, it seems clear that we will need some non-degeneracy on parameters as an analogue of the twist mapping assumption.

In our case, since there are several eigenvalues for derivatives of periodic orbits, we will define a measure of stability as the distance of the spectrum of a periodic orbit of period L to the set $\{1, \lambda^L\}$, which is the spectrum of the integrable case. Similar definitions were presented in [63, 49], which also considered higher dimensional systems. Of course, the distance of the spectrum to this set can be related to the coefficients of the characteristic polynomial in the spirit of [39] (see Remark 5).

Since we want to deal with approximation of tori by periodic orbits, we quickly run into the problem of approximating irrational numbers by rational ones. In one dimension, this is clearly solved by the continued fraction expansion. In higher dimensions, it is easy to see that there is no analogue of continued fractions. A recent discussion on rational approximations well suited for Greene's criterion and with references to the original sources can be found in [49].

Hence, following the strategies of previous papers studying Greene's criterion in higher dimensions, we have divided the results into two types:

(1) Showing that if there is a KAM torus for a certain value of the parameter, then for small nearby values of the parameter, the eigenvalues of a periodic orbit (in a neighborhood of the torus) with frequency $\rho = \frac{1}{L}(a_1, ..., a_n)$ with $a_j \in \mathbb{Z}$, $L \in \mathbb{N}$, are close to $\{1, \lambda^L\}$ (see Theorem 4).

(2) Showing that, given appropriate non-degeneracy conditions, for small values of the parameter we can obtain periodic orbits of frequency approximating that of the torus (see Section 7).

We conclude this section calling attention to the following observations.

i) The above results (1) and (2) are proved by very different methods and have different hypotheses. The estimate of the eigenvalues is done using some versions of averaging theory. Theorem 4 makes statements about all periodic orbits satisfying some proximity conditions, but does not consider whether such orbits exist. The existence of periodic orbits (with good quantitative estimates) is considered in Section 7. Note that Proposition 24 includes some non-degeneracy hypotheses that are not needed in Theorem 4; it seems that these non-degeneracy assumptions could be weakened.

Notice that this is very different from the situation for twist mappings considered in [39, 32], since in that case we have the non-degeneracy assumptions from the start. As discovered in [13], the Birkhoff normal form theory is much stronger in the dissipative conformally symplectic case than in the symplectic case.

ii) The results we obtain estimating the spectral numbers are valid not only for periodic orbits, but also for the spectrum of the linearization of other invariant objects.

iii) From the numerical point of view, it might be advantageous to extend the notion of the residue in terms of the coefficients of the characteristic polynomial, which depends in a differentiable way on the values of the matrix entry (see Remark 5). On the other hand – because the spectrum is very degenerate – the eigenvalues may depend in a Hölder fashion on the coefficients of the characteristic polynomials, so that they are more sensitive to changes in parameters.

iv) Notice that we only study one of the directions of the Greene's criterion. In particular, we show that if there is a KAM torus, then the linearization around neighboring periodic orbits is controlled. We do not have any rigorous statement in the opposite direction, namely showing that if we have control on the Lyapunov exponents of periodic orbits, there will be an invariant torus.

v) Greene's method for dissipative systems has already been implemented numerically in the literature. In [12] both Greene's method and the method of growth of Sobolev norms ([14]) are implemented and agree within the precision achieved in the ranges of parameters where both of them are implementable (Greene's criterion depends on computing periodic orbits, which may be hard for some values of the parameter, while the Sobolev growth has a wider range of practical applicability). Since the method of [14] was justified rigorously in [15], we think that a full justification of Greene's method could be true, at least in some regions where the renormalization group is available (we note that a renormalization theory for conformally symplectic maps was obtained in [56, 57, 58]).

vi) Using topological methods (which apply to area-preserving maps as well as to dissipative maps), Hall ([41]) proved that a twist map of the annulus having a periodic orbit with frequency p/q must have a Birkhoff periodic orbit with the same rotation number, where *Birkhoff* means to have periodic orbits whose radial order is preserved by the map. In Section 7 we show that quasi-periodic (non contractible) KAM tori can be approximated by periodic orbits. In view of the result given in [41] we have the existence of Birkhoff periodic orbits and therefore we obtain the existence of a quasi-periodic Birkhoff limit set of periodic orbits, which can be a torus or a Cantor set.

1.3. **Organization of the paper.**
In Section 2 we present the notation on conformally symplectic systems and analyze the local behavior in a neighborhood of a rotational, Lagrangian, invariant torus (we just reproduce the results of [13] that we need). In Section 3 we introduce Diophantine numbers, we recall the pairing rule property of conformally symplectic matrices and we state the main result (see Theorem 4); the result for flows is given in Section 9. A sketch of the proof of the main theorem is given in Section 4. Two different proofs are presented in Sections 5 and 6: the first proof is based on deformation theory (which consists in developing a suitable normal form), while the second proof combines the theory of normally hyperbolic invariant manifolds and averaging theory. As a byproduct we obtain informations about the width of the Arnold tongues as shown in Section 7, which contains also a proof of the existence of the approximating periodic orbits.

2. Local behavior near rotational Lagrangian tori in conformally symplectic systems

2.1. **Conformally symplectic systems.**
We consider a symplectic manifold $\mathcal{M}$ of dimension $2n$, endowed with the standard scalar product and a symplectic form Ω. The diffeomorphism $f : \mathcal{M} \to \mathcal{M}$ is conformally symplectic, whenever the following condition is satisfied:

$$f^*\Omega = \lambda\,\Omega \tag{2.1}$$

for some λ real (see [6, 7, 42, 47, 65, 67] for other studies of conformally symplectic geometry).

Remark 1. *When $n = 1$, taking as Ω the area form, any mapping admits a non-constant function λ. In this work we will only consider the case of a constant λ; as it is well known* ([7])*, when $n \geq 2$ and $\mathcal{M}$ is connected, then λ is a constant function.*

Given $\omega \in \mathbb{R}^n$, we denote by $T_\omega : \mathbb{T}^n \to \mathbb{T}^n$ the shift

$$T_\omega(\varphi) = \varphi + \omega\ .$$

We say that an embedding $K : \mathbb{T}^n \to \mathcal{M}$ defines a rotational invariant torus for f, whenever the following invariance equation is satisfied:

$$f \circ K = K \circ T_\omega\ . \tag{2.2}$$

We say that a parameterized torus of dimension n is Lagrangian if

$$K^*\Omega = 0\ . \tag{2.3}$$

In coordinates the relation (2.3) is equivalent to

$$DK^{\mathsf{T}}(\varphi)\ J \circ K(\varphi)\ DK(\varphi)\ =\ 0\ , \tag{2.4}$$

where the superscript T denotes transposition and where $J = J(x)$ is the matrix representing Ω at x, namely if u, $v \in T_x\mathcal{M}$, then one has

$$\Omega(u, v) = (J(x)u, v)\ ,$$

where $(\cdot,\cdot)$ denotes the Euclidean scalar product. We remark that tori that satisfy (2.3) are called *isotropic* in symplectic geometry. When the dimension of the torus is half of the dimension of the phase space, they are called *Lagrangian*.

For a family f_μ of conformally symplectic mappings depending on a parameter vector $\mu \in \mathbb{R}^\ell$, $\ell \leq n$, we require that the following invariance equation (extending (2.2)) is satisfied:

$$f_\mu \circ K = K \circ T_\omega \ . \tag{2.5}$$

We note that in (2.5) both the embedding K and the parameter μ are unknowns. The reason to adjust parameters is that, if we fix the parameter, it could well happen that there are no embeddings satisfying (2.2). For example it could happen that all orbits are attracted to a periodic orbit. It is remarkable that, when the rotation vector is Diophantine and some non-degeneracy conditions are satisfied, we can get the existence of the tori just by adjusting an n dimensional parameter.

Following [13], in a neighborhood of an invariant torus, we can assume, without loss of generality, that the manifold $\mathcal{M}$ coincides with the manifold $\mathcal{M}_0$ defined as

$$\mathcal{M}_0 \equiv \mathbb{T}^n \times \mathcal{B} \ ,$$

where $\mathcal{B}$ denotes a ball of $\mathbb{R}^n$ around zero.

We note that in $\mathcal{M}_0$ there is the standard symplectic form

$$\Omega_0 = \sum_{i=1}^{n} dI_i \wedge d\varphi_i \ , \tag{2.6}$$

where I_i are the coordinates in $\mathcal{B} \subset \mathbb{R}^n$ and φ_i are the coordinates on $\mathbb{T}^n$.
We denote by f_0 the conformally symplectic mapping with respect to Ω_0 defined as

$$f_0(\varphi, I) = (\varphi + \omega, \lambda I) \ . \tag{2.7}$$

Notice that (2.7) provides a rotation by ω in the angles and a constant contraction (for $\lambda < 1$) in the actions.

For $r > 0$ integer we consider a C^r-family of functions $f_\mu : \mathcal{M} \to \mathcal{M}$ with finite regularity, where C^r is the space of differentiable functions whose derivatives, both with respect to coordinates or to the parameter of the family, of order smaller or equal than r are continuous. We endow the C^r-space with the usual norm of the supremum of all the derivatives. For later use, we define $\|\mu\| = (\mu_1^2 + ... + \mu_\ell^2)^{\frac{1}{2}}$ for any $\mu \in \mathbb{R}^\ell$.

2.2. **Local behavior in a neighborhood of a KAM torus.** We now recall the result presented in [13] concerning the conjugation to a rotation and a contraction near a rotational, Lagrangian invariant torus.

Theorem 2. *([13]) Let $(\mathcal{M}, \Omega)$ be a $2n$-dimensional symplectic manifold with symplectic form Ω; for $r > 0$ integer, let f be a C^r-conformally symplectic contractive diffeomorphism (see (2.1)). Let $\omega \in \mathbb{R}^n$ and assume that there exists an embedding $K : \mathbb{T}^n \to \mathcal{M}$, such that (2.2) and (2.4) are satisfied. Then, there exists a C^r-diffeomorphism g from $\mathcal{M}$ to a neighborhood of the torus, such that*

$$\begin{aligned} g^{-1} \circ f \circ g &= f_0 \\ g^* \Omega_0 &= \Omega \\ g(\varphi, 0) &= K(\varphi) \ , \end{aligned}$$

where f_0 is given in (2.7) *and Ω_0 is given in* (2.6).

Notice that in Theorem 2 we do not impose any non-resonance condition on ω. It even applies to $\omega \in \mathbb{Q}^n$. However, when ω is not Diophantine, there are no theorems that guarantee the existence of rotational invariant tori for perturbations of the systems. Indeed when ω is resonant, there are arguments that show that the Lagrangian tori do not remain rotational (of course, they persist as normally hyperbolic invariant manifolds).

3. Definitions and statement of the main result

3.1. **Diophantine property.** In order to state the main result, we start by considering a family f_μ of conformally symplectic mappings with $f_\mu \in C^r$, $r > 0$, depending on a parameter vector $\mu \in \mathbb{R}^\ell$, $\ell \leq n$. We assume that for $\mu = 0$ the mapping f_0 admits a KAM rotational invariant circle with rotation number ω satisfying the Diophantine condition

$$|\omega \cdot q - p| \ \geq \ \nu |q|^{-\delta} \ , \qquad p \in \mathbb{Z} \ , \quad q \in \mathbb{Z}^n \backslash \{0\} \ , \tag{3.1}$$

for some positive constants ν, $\delta \in \mathbb{R}$. We remark that due to Theorem 2, close to the invariant curve we can write f_0 as in (2.7).

3.2. **The pairing rule.** We recall the important particular case of the results presented in [28, 29, 67], which shows that the eigenvalues of a conformally symplectic matrix can be paired.

Lemma 3. *Assume that M is a λ-conformally symplectic matrix in a $2n$-dimensional space, that is:*

$$M^\mathsf{T} J M = \lambda J \ .$$

Then, the spectrum of M has the form:

$$\mathrm{Spec}(M) = \{(\gamma_i, \lambda\gamma_i^{-1}) \ : \quad i = 1, \cdots, n\} \ .$$

Proof. The result follows from the observation that if γ_i is an eigenvalue of M with right eigenvector v, then $\lambda\gamma_i^{-1}$ is an eigenvalue of M with left eigenvector $(Jv)^\mathsf{T}$, as it follows from $M^\mathsf{T} Jv = \gamma_i^{-1} M^\mathsf{T} JMv = \gamma_i^{-1}\lambda Jv$. □

3.3. **Main result.** The main result of this paper is the following Theorem 4, which is related to Theorem 2.1 of [32] and shows that, when the invariant curve exists, the spectral numbers of the return map associated to the nearby periodic orbits are close to those of the KAM torus. Of course, as explained in Section 1, in conformally symplectic systems the Birkhoff normal form around the torus is trivial and we have to include extra parameters.

Theorem 4. *For $r > 0$ integer, let f_μ, $\mu \in \mathbb{R}^\ell$ ($\ell \in \mathbb{N}$, $\ell \leq n$), be a C^r-family of conformally symplectic diffeomorphisms, such that f_0 admits a Lagrangian invariant torus with Diophantine rotation vector ω. Then, there is a neighborhood $\mathcal{U}$ of the torus with the following property. Let*

$$\rho = \frac{1}{L}\ (a_1, ..., a_n) \tag{3.2}$$

with $(a_1, ..., a_n) \in \mathbb{Z}^n$ and $L \in \mathbb{N}$. Assume that the orbit of a point x has rotation number ρ and is contained in $\mathcal{U}$. Denote by γ_i the spectral numbers of the return map of the periodic orbit which, according to Lemma 3, we define as $\mathrm{Spec}(Df^L(x)) = \{\gamma_i, \lambda^L\gamma_i^{-1}\}$.

Then, there exists $N_0 \in \mathbb{N}$ depending on r, such that for every $N \leq N_0$ there exists a constant $C_N > 0$, depending also on r and on the Diophantine exponent of ω, so that

$$|\gamma_i - 1| \leq L C_N \|\mu\|^N \ . \tag{3.3}$$

Note that, of course, for C^∞ or analytic mappings $N_0 = \infty$ and then (3.3) holds for all N.

If one had explicit forms of the C_N, for a fixed $\|\mu\|$ it would be possible to obtain the N that gives the sharpest bound (this is a standard procedure in Nekhoroshev's theory, [54]). In this paper, for the sake of brevity, we will not undertake obtaining the explicit form of the C_N. We note that since we present two methods of proof, it would be advantageous to obtain explicit estimates for both methods and take the best one.

Remark 5. *The results provided in* [39] *were stated in terms of a quantity, called the "residue", which provides information on the stability of the periodic orbits. In the case of a 1-dimensional, area-preserving map, the residue is defined in terms of the trace of the matrix Df^q computed along a full cycle of the periodic orbit with frequency p/q. Even though we will not use the definition of residue, due to the fact that our results will be stated in terms of spectral numbers, we find it useful for computations to generalize as follows the definition of residue to n-dimensional, conformally symplectic maps. Let*

$$c(x) \equiv x^{2n} + c_{2n-1}x^{2n-1} + \dots + c_1 x + \lambda^{nL} \tag{3.4}$$

be the characteristic polynomial of the derivative over a full cycle of the periodic orbit with frequency ρ as in (3.2) *for some real coefficients c_j, $j = 1, \dots, 2n-1$. Let the characteristic polynomial associated to f_0 in* (2.7) *be written as the distance between the coefficients of the characteristic polynomials* (3.4), (3.5)*:*

$$(x-1)^n (x - \lambda^L)^n \equiv x^{2n} + c^0_{2n-1}x^{2n-1} + \dots + c^0_1 x + \lambda^{nL} \tag{3.5}$$

for some real coefficients c^0_j, $j = 1, \dots, 2n-1$. We can define the residue R for example as

$$R \equiv \sum_{j=1}^{2n-1} |c_j - c^0_j| \ .$$

Notice that, due to the pairing rule, the $2n-1$ coefficients c_j are parameterized by the n spectral numbers γ_i.

Remark 6. *Note that the statement of Theorem 4 is to provide bounds on the spectral numbers of an orbit of type $\rho = a/L$ in terms of L and of $\|\omega - \rho\|$ (which, under the non-degeneracy assumptions, we will make comparable to $\|\mu\|$, see Section 6). The bounds on the distance of the spectral numbers from unity, see* (3.3)*, get better when $\|\omega - \rho\|$ decreases and get worse when L increases.*

Of course, to obtain sharp results requires some compromise since to get $\|\omega - \rho\|$ small, we need to get L large. The Diophantine property (3.1) *precisely quantifies that L has to be at least $\nu^{\frac{1}{\delta+1}} \|\omega - \rho\|^{-\frac{1}{\delta+1}}$.*

If we assume that the exponents in the Diophantine property are optimal, we know that there is a sequence of L that almost saturates the Diophantine bounds and for which $L \approx \|\omega - \rho\|^{-1/(\delta+1)}$. Hence we see that for N sufficiently large, the quantity $|\gamma_i - 1|$ has to go to zero.

We notice that by Liouville theorem we know that there are other sequences for which $L \approx \|\omega - \rho\|^{-\frac{1}{n}}$*. Therefore, especially in the* $n \geq 2$ *case, the behavior of approximations with respect to* L *can have a variety of asymptotic rates and this is a motivation for us to leave the results in the theorem without choosing a particular one. The problem of how to approximate vectors by rational vectors with denominators as small as possible have been much studied (*multidimensional Diophantine approximation*). A recent practical review close to the problems we are dealing with and with references to the previous literature is* [49]*. A detailed study of the approximation of quasi-periodic tori by periodic orbits in the dissipative standard mapping can be found in* [18, 20]*.*

4. The idea of the proof of Theorem 4

In sections 5 and 6 we present two different proofs of Theorem 4. The first proof is based on deformation theory (see Section 5) and the second proof is based on the theory of normally hyperbolic manifolds and on the averaging theory of rotations (see Section 6). We present both proofs in the hope that they lead to a better insight and that they can have applications to other problems. In particular, in this paper we will use the theory of normally hyperbolic invariant manifolds and averaging theory developed in the second proof to obtain information on the location of the Arnold tongues.

Let us sketch briefly the arguments which will be developed in complete detail in the rest of the paper.

The proof presented in Section 5 is based on developing a normal form, i.e. a smooth change of variables that reduces the system to a particularly simple form up to an error. The spectrum is invariant under smooth changes of variables. For the systems given by the normal form neglecting the reminder, the spectral numbers are identically one. Hence, the spectrum can be estimated by bounding the error of the normal form, which is computed in Theorem 7 by using the theory of deformations (see [8, 23]). Indeed, such theory gives bounds on the derivatives of the map at every step of the orbit. To put them together and obtain better bounds, in Section 5.4 we use some techniques which are standard in the theory of the so-called exponential dichotomies ([61, 22, 51]).

A different method of proof of Theorem 4 is presented in Section 6. We first appeal to the theory of normally hyperbolic manifolds (see [34, 45, 55]) to show that we obtain a smooth family $\mathcal{T}_\mu$ of n-dimensional tori invariant under f_μ. We note that any periodic orbit close enough to the torus has to be in the normally hyperbolic invariant manifold, which moreover is Lagrangian.

We denote by $R_\mu(\varphi)$ the dynamics of f_μ restricted to $\mathcal{T}_\mu$; in particular, R_0 is a Diophantine rotation. Using the averaging theory of torus maps C^r-close to a Diophantine rotation, for every $N \leq N_0(r)$ as in Theorem 4 we can find a family B_μ^N of diffeomorphisms of the n-dimensional torus and a family of rotations $T_{\omega_\mu^N}$ in such a way that

$$(B_\mu^N)^{-1} \circ R_\mu \circ B_\mu^N = T_{\omega_\mu^N} + O(\|\mu\|^{N+1}) \ .$$

If we find a periodic orbit (in the normally hyperbolic invariant manifold), we see that n of its Lyapunov exponents have to be very close to 1. Using the pairing rule of exponents for conformally symplectic maps (see [28, 29, 67] and Lemma 3) and using that the normally hyperbolic manifold is Lagrangian, we obtain that the remaining n-Lyapunov exponents associated to a periodic orbit with frequency $\rho = a/L$ have to be close to the conformal factor λ^L.

5. Proof of Theorem 4 using deformation theory

In this section we prove the following theorem that shows that if $f_0(\varphi, I) = (\varphi + \omega, \lambda I)$ has a KAM torus, we can find a change of variables V_μ^N for every $N \in \mathbb{N}$, $N \leq N_0(r)$, that reduces the system to a particularly simple form up to an error which is small in μ (see equation (5.1)).

Note that for the main part of the normal form (5.1), the spectrum is equal to that of f_0. Then, the proof of Theorem 4 is obtained by a perturbation argument, treating the remainder of the normal form as a perturbation.

Theorem 7. *For $r > 0$ integer, let f_μ be a C^r-family of conformally symplectic systems, where $\mu \in \mathbb{R}^\ell$, $\ell \leq n$. Assume that $f_0(\varphi, I) = (\varphi + \omega, \lambda I)$ for some $\lambda < 1$ and let ω be a Diophantine vector. We can find functions $A(N), B(N, r)$ – depending on the Diophantine exponent of ω – with the following property.*

For any $N > 0$ integer, $N \leq N_0(r)$, we can find a $C^{r-A(N)}$ change of variables V_μ^N and a polynomial function S_μ taking values in $\mathbb{R}^n$, such that

$$(V_\mu^N)^{-1} \circ f_\mu \circ V_\mu^N(\varphi, I) = (\varphi + S_\mu, \lambda I) + R_N(\varphi, I) \ , \tag{5.1}$$

where the remainder R_N can be bounded as

$$\|R_N\|_{C^{r-B(N,r)}} \leq C \ \|\mu\|^{N+1} \ ,$$

for a positive real constant C depending on N, r, on the Diophantine exponent of ω and on the C^r-norm of f_μ.

The proof of Theorem 7 will be accomplished by implementing a suitable normal form and we will show that Theorem 7 implies Theorem 4.

Remark 8. *A byproduct of Theorem 7 is that it provides information on the Lyapunov exponents (and indeed on any quantity invariant under smooth changes of variables which admits a good perturbation theory) of any invariant object in a neighborhood of the torus.*

If we evaluate these quantities in the system of coordinates produced in Theorem 7, we see that these Lyapunov exponents will be those of the normal form up to a small error.

5.1. **The deformation method.**

The deformation method (also known as the *method of the paths*) is a powerful tool to discuss equivalence of dynamical systems. It was introduced in singularity theory ([62]), but it was soon realized that it can be used to study not only a differentiable equivalence, but also an equivalence that preserves other geometric structures (see [52] for volume preserving systems, [66] for symplectic systems and [23, 8] for more systematic developments closer to our goals in the theory of normal forms and perturbation theory).

The main idea is that one considers not only a problem, but interpolates by a smooth family of problems having a trivial solution, beside the solution of the desired problem. Then, one studies the derivatives with respect to parameters of the solutions of the problems in the family. As we will see, the equations for the derivatives are linear (heuristically, think of derivatives as infinitesimal transformations and observe that the only equations one can form among infinitesimal quantities are linear, since any nonlinear effect is an infinitesimal of higher order). Furthermore, the geometric constraints, which are nonlinear and non-local for diffeomorphisms, become linear and local constraints on the derivatives.

In Section 5.1.1 we summarize some geometric considerations for conformally symplectic families. In Section 5.1.2 we present a geometric formulation of the deformation method for conformally symplectic mappings; in Lemma 13 we show how one can transform our problem into a rigid rotation and a contraction to any order in the perturbation. It will be important to note that the regularity of the conjugation that achieves this transformation can be estimated by the regularity of the original family (see the estimate (5.15)).

The traditional formulation of the deformation method considers one-dimensional families, but for our problem we also need n-dimensional families. In Section 5.3 we present a remark showing that results for one-parameter families that have some uniformity (see (5.15)) immediately imply results for n-dimensional parameters. These results are enough to prove the upper bounds on the spectral numbers appearing in Theorem 4. Nevertheless, we point out that there is a more geometric treatment of the multivariable parameter deformations, quite analogous to the theory of *symplectic actions* and to the *momentum map* developed in [5, 40] (see also [38] for a version very close to our needs). We outline this theory in Section 5.3, providing an alternative derivation, but also leading to a smooth dependence on parameters, which is useful for the theory of existence of periodic orbits developed in Section 7. The end of the proof of Theorem 4 is provided in Section 5.4, using the theory of dichotomies of mappings.

5.1.1. *Exact conformally symplectic mappings.* We first consider the case of a family of diffeomorphisms $f_\tau : \mathcal{M} \to \mathcal{M}$ depending on a parameter $\tau \in \mathbb{R}$; as already remarked, the extension to a family depending on a parameter vector $\mu \in \mathbb{R}^\ell$ with $\ell \le n$ will be treated in Section 5.3.

Let f be a λ-conformally symplectic diffeomorphism defined as in (2.1) on a symplectic manifold $\mathcal{M}$ and let $\Omega = d\alpha$. Then, we have

$$f^* d\alpha - \lambda d\alpha = 0 \ ,$$

namely

$$d(f^*\alpha - \lambda\alpha) = 0 \ .$$

We say that f is exact conformally symplectic if the following holds:

$$f^*\alpha - \lambda\alpha = dP_f \ ,$$

where P_f is called the primitive function of f.

The following Lemma shows that the set of (exact) conformally symplectic mappings is a graded group under composition, the grading being the conformal factor.

Lemma 9. *Let f be a λ-conformally symplectic diffeomorphism and let g be an η-conformally symplectic diffeomorphism.*

Then, $f \circ g$ is a $\lambda\eta$-conformally symplectic diffeomorphism. Furthermore, if f and g are exact, so is $f \circ g$ and the primitive function of $f \circ g$ is given by

$$P_{f\circ g} = \lambda P_g + g^* P_f \ . \tag{5.2}$$

Proof. Notice that

$$(f \circ g)^* \Omega = g^* f^* \Omega = g^* \lambda \Omega = \lambda \eta \Omega \ .$$

If f and g are exact, we obtain

$$\begin{aligned} (f \circ g)^* \alpha &= g^* f^* \alpha \\ &= g^* (\lambda \ \alpha + dP_f) \\ &= \lambda \eta \ \alpha + \ \lambda dP_g + d(g^* P_f) \\ &= \lambda \eta \ \alpha + d(\lambda P_g + g^* P_f) \ , \end{aligned}$$

which proves (5.2). □

5.1.2. *Generators and Hamiltonians of a deformation.* Given a differentiable family of diffeomorphisms $f_\tau : \mathcal{M} \to \mathcal{M}$ depending on the parameter $\tau \in \mathbb{R}$, we say that f_τ is an isotopy if

$$\frac{d}{d\tau} f_\tau = \mathcal{F}_\tau \circ f_\tau \ , \tag{5.3}$$

where $\mathcal{F}_\tau$ is called the family of generator vector fields. Our goal now is to translate the geometric properties of f_τ (f_τ being conformally symplectic or exact conformally symplectic) to the isotopy properties of the generator. We start by considering a family of λ-conformally symplectic maps, such that

$$f_\tau^* \Omega = \lambda \Omega \ . \tag{5.4}$$

We observe that (5.4) is equivalent to

$$\frac{d}{d\tau} (f_\tau^* \Omega) = 0 \quad \text{and} \quad f_0^* \Omega = \lambda \Omega \ .$$

Indeed, using Cartan's *magic formula* for the derivatives, we obtain:

$$\frac{d}{d\tau} (f_\tau^* \Omega) = f_\tau^* (i_{\mathcal{F}_\tau} d\Omega + d(i_{\mathcal{F}_\tau} \Omega)) = f_\tau^* (d(i_{\mathcal{F}_\tau} \Omega)) = 0 \ .$$

Therefore, we see that the the family f_τ consists of conformally symplectic mappings, if and only if

$$d(i_{\mathcal{F}_\tau} \Omega) = 0 \quad \text{and} \quad f_0^* \Omega = \lambda \Omega \ . \tag{5.5}$$

We say that the family of diffeomorphisms f_τ is a λ-exact conformally symplectic isotopy (for short λ-ECSI), whenever

$$f_\tau^* \alpha = \lambda \alpha + dP_\tau \ , \tag{5.6}$$

where P_τ is the primitive of f_τ. Proceeding as before, setting $\Omega = d\alpha$ and taking derivatives we obtain:

$$\frac{d}{d\tau} (f_\tau^* \ \alpha) = f_\tau^* \ (i_{\mathcal{F}_\tau} d\alpha + d(i_{\mathcal{F}_\tau} \alpha)) = f_\tau^* \ (i_{\mathcal{F}_\tau} \Omega + d(i_{\mathcal{F}_\tau} \alpha)) \ .$$

Hence, we see that f_τ is a λ-ECSI, if and only if

$$i_{\mathcal{F}_\tau} \Omega = dF_\tau \quad \text{and} \quad f_0^* \Omega = \lambda \Omega \ , \tag{5.7}$$

where the function F_τ is called the Hamiltonian associated to f_τ.

If we prescribe F_τ (and hence dF_τ), because the symplectic form is non-degenerate, we can obtain $\mathcal{F}_\tau$ and, hence, if we prescribe f_0, we can obtain the deformation by solving the differential equation (5.3) with the initial condition f_0, of course provided that the vector field $\mathcal{F}_\tau$ is differentiable enough and that the differential equations have unique solutions.

In other words, for sufficiently smooth isotopies, it is equivalent to prescribe f_τ than to prescribe (f_0, F_τ). As we will see, the preservation of geometric structures and the calculations of the functional equations are much simpler in the (f_0, F_τ) representation.

Remark 10. *When the deformation f_τ is exact conformal, we can compute the change in the primitive function P_τ in* (5.6) *as follows:*

$$\frac{d}{d\tau}(dP_\tau) = \frac{d}{d\tau}(f_\tau^*\ \alpha) = d[f_\tau^*(F_\tau + i_{\mathcal{F}_\tau}\alpha)]\ .$$

Hence, we can choose the constant so that

$$\frac{d}{d\tau}P_\tau = f_\tau^*(F_\tau + i_{\mathcal{F}_\tau}\alpha)\ .$$

Remark 11. *The characterization of* (5.7) *is a complete analogue as Hamilton's equations for an exact symplectic deformation. Similarly, we observe that the characterization in* (5.5) *is the same as the characterization of symplectic deformations by locally Hamiltonian flows.*

This is clear if we observe that in a family f_τ of λ-conformally symplectic diffeomorphisms, the change from one element of the family to another is symplectic, namely

$$(f_{\tau'}^{-1} \circ f_\tau)^*\ \Omega = \Omega\ ,$$

for any τ, $\tau' \in \mathbb{R}$ (see also Lemma 9).

This shows that a conformally symplectic deformation is the same as a symplectic deformation, except that the initial point is conformally symplectic.

5.1.3. *Calculus of deformations.* The key result of the deformation method is the following Lemma 12, which transforms the non-linear and non-local composition problems into linear and local problems among the generators and their Hamiltonians.

Of course, as mentioned before the isotopies are determined by their initial value and their Hamiltonians. The main point of Lemma 12 is that the functional equations involving compositions become much easier when studied in terms of Hamiltonians.

Lemma 12. *Assume that f_τ is a λ-ECSI, g_τ is an η-ECSI, then*

- *$h_\tau = f_\tau \circ g_\tau$ is a $\lambda\eta$-ECSI*
- *$k_\tau = f_\tau^{-1}$ is a $1/\lambda$-ECSI.*
- *$e_\tau = (g_\tau)^{-1} \circ f_\tau \circ g_\tau$ is a λ-ECSI.*

Their generators are given respectively by

$$\mathcal{H}_\tau = \mathcal{F}_\tau + f_{\tau*}\mathcal{G}_\tau \tag{5.8}$$
$$\mathcal{K}_\tau = -({f_\tau}^{-1})_*\mathcal{F}_\tau \tag{5.9}$$
$$\mathcal{E}_\tau = -({g_\tau}^{-1})_*\mathcal{G}_\tau + ({g_\tau}^{-1})_*(\mathcal{F}_\tau + f_{\tau*}\mathcal{G}_\tau)\ , \tag{5.10}$$

while their Hamiltonians are respectively:

$$H_\tau = F_\tau + \lambda f_{\tau *} G_\tau = F_\tau + \lambda G_\tau \circ f_\tau^{-1} \tag{5.11}$$

$$K_\tau = -\frac{1}{\lambda} (f_\tau^{-1})_* F_\tau = -\frac{1}{\lambda} F_\tau \circ f_\tau \tag{5.12}$$

$$\begin{aligned} E_\tau &= -\frac{1}{\eta} (g_\tau^{-1})_* G_\tau + \frac{1}{\eta} (g_\tau^{-1})_* (F_\tau + \lambda f_{\tau *} G_\tau) \\ &= -\frac{1}{\eta} G_\tau \circ g_\tau + \frac{1}{\eta}(F_\tau \circ g_\tau + \lambda G_\tau \circ f_\tau^{-1} \circ g_\tau) . \end{aligned} \tag{5.13}$$

Proof. Formula (5.8) is very easy to obtain, since one just needs to take derivatives; it does not use any geometric structure, but just the chain rule

$$\begin{aligned} \frac{d}{d\tau} h_\tau &= \mathcal{F}_\tau \circ f_\tau \circ g_\tau + Df_\tau \circ g_\tau \, \mathcal{G}_\tau \circ g_\tau \\ &= \mathcal{F}_\tau \circ f_\tau \circ g_\tau + Df_\tau \circ (f_\tau)^{-1} \circ f_\tau \circ g_\tau \, \mathcal{G}_\tau \circ (f_\tau)^{-1} \circ f_\tau \circ g_\tau \\ &= \mathcal{F}_\tau \circ h_\tau + (f_{\tau *}\mathcal{G}_\tau) \circ h_\tau . \end{aligned}$$

Since $\mathrm{Id} = k_\tau \circ f_\tau$, applying (5.8) we obtain

$$0 = \mathcal{K}_\tau + k_{\tau *}\mathcal{F}_\tau ,$$

which is equivalent to (5.9) through algebraic manipulations. To obtain (5.10), we just apply (5.8) twice.

Taking contractions with Ω in (5.8), we obtain (using [1, Prop. 2.4.14] and that $f_\tau^*\Omega = \lambda\Omega$):

$$\begin{aligned} i(\mathcal{H}_\tau)\Omega &= i(\mathcal{F}_\tau)\Omega + i(f_{\tau *}\mathcal{G}_\tau)\Omega = i(\mathcal{F}_\tau)\Omega + i((f_\tau^{-1})^*\mathcal{G}_\tau)\lambda(f_\tau^{-1})^*\Omega \\ &= i(\mathcal{F}_\tau)\Omega + (f_\tau^{-1})^*(\lambda dG_\tau) = i(\mathcal{F}_\tau)\Omega + \lambda d[(f_\tau^{-1})^* G_\tau] \\ &= i(\mathcal{F}_\tau)\Omega + \lambda dG_\tau \circ f_\tau^{-1} , \end{aligned}$$

which leads to

$$dH_\tau = dF_\tau + \lambda \, f_{\tau *} \, dG_\tau .$$

Finally, we obtain (5.11) which, of course, is unique up to an additive constant.

Again, we obtain (5.12) by observing that $\mathrm{Id} = k_\tau \circ f_\tau$ and applying (5.11) to the right hand side, while of course the Hamiltonian of the left hand side is zero. We obtain (5.13) by applying (5.11) twice. ☐

5.2. **The deformation Lemma.** For $r > 0$ let us consider a C^r-family $f_\tau : \mathcal{M} \to \mathcal{M}$ depending on a parameter $\tau \in \mathbb{R}$. We define the set $B_\eta \equiv \{\tau \in \mathbb{R} : \ |\tau| \leq \eta\}$. Let $\|\cdot\|_{C^r,\eta}$ denote the supremum over B_η of the C^r-norm.

Lemma 13. *Let $f_\tau : \mathcal{M} \to \mathcal{M}$ be a C^r-differentiable family of λ-ECSI with $f_0(\varphi, I) = (\varphi + \omega, \lambda I)$. Assume that ω is Diophantine. We can find an integer $N_0(r)$ – depending on the Diophantine exponent of ω – such that for every $N \in \mathbb{N}$, $N \leq N_0(r)$, there exists a function $\tilde{A}(n)$ and a $C^{r-\tilde{A}(N)}$-family of symplectic mappings v_τ^N with $v_0^N = \mathrm{Id}$ and a* *family of vectors* S_τ^N *with* $S_0^N = \omega$*, such that* $g_\tau^N = (v_\tau^N)^{-1} \circ f_\tau \circ v_\tau^N$ *and its Hamiltonian* G_τ^N *satisfy the following relations*

$$\begin{aligned} g_\tau^N(\varphi, I) =& (\varphi + S_\tau^N, \lambda I) + e_\tau^N(\varphi, I)\tau^{N+1} \\ G_\tau^N(\varphi, I) =& I\ S_\tau^N + E_\tau^N(\varphi, I)\tau^{N+1}\ , \end{aligned} \tag{5.14}$$

for some error function $e_\tau^N(\varphi, I)$ *and associated Hamiltonian* $E_\tau^N(\varphi, I)$ *(see Proposition 14).*

Furthermore, S_τ^N *is a polynomial in* τ*, which can be considered as the partial sum of a formal power series of the form*

$$S_\tau^N = \sum_{i=0}^{N} S^{(i)} \tau^i$$

for some real coefficients $S^{(i)}$*,* $i = 1, ..., N$*. Finally, we can bound* G_τ^N *in terms of the norm of* f_τ *for some* $\tilde\eta < \eta$*,* $\tilde r < r$ *and for some constant* $C_N > 0$ *depending on the Diophantine exponent of* ω*, as*

$$\|G_\tau^N\|_{C^{\tilde r}, \tilde\eta} \le C_N \|f_\tau\|_{C^r, \eta}\ . \tag{5.15}$$

Proof of Lemma 13. Using (5.13), the desired equation (5.14) is obtained by requiring that V_τ^N and G_τ^N satisfy

$$- V_\tau^N \circ v_\tau^N + F_\tau \circ v_\tau^N + \lambda V_\tau^N \circ f_\tau^{-1} \circ v_\tau^N = G_\tau^N\ . \tag{5.16}$$

To derive the order by order equations, we use the (very well known) observation provided by the following result.

Proposition 14. *Let us consider a family of* C^r *functions* A_τ *with* $A_0 = 0$ *and let us expand* A_τ *as* $A_\tau = \sum_{i=0}^N A^{(i)} \tau^i + O(|\tau|^{N+1})$ *for suitable functions* $A^{(i)}$*. Let* h_τ *be a* C^r*, exact, conformally symplectic isotopy. Then, there is a unique expansion*

$$B_\tau \equiv A_\tau \circ h_\tau = \sum_{i=0}^{r} B^{(i)} \tau^i\ ,$$

where the functions $B^{(i)}$ *are such that* $B^{(0)} = 0$ *and, for* $j > 0$*,*

$$B^{(j)} = A^{(j)} \circ h_0 + R_j\ , \tag{5.17}$$

where R_j *is a polynomial expression involving* $A^{(1)}, \dots, A^{(j-1)}$ *and their derivatives up to the order* j*, involving also the Hamiltonians* $H^{(i)}$ *corresponding to the terms* $h_\tau^{(i)}$ *of the expansion of* h_τ *and their derivatives up to the order* n*.*

Note that the main content of Proposition 14 is that we have identified in (5.17) the order j term of the expansion and that $A^{(j)}$ appears in the simple way indicated above, while $H^{(j)}$ does not appear. This allows to derive order by order equations.

Proof. The desired expansion (5.17) is, of course, just the Taylor expansion of B_τ in τ. The coefficients can be obtained by computing derivatives of B_τ with respect to τ and evaluating them at $\tau = 0$. The derivative of the composition can be computed using Faa-di-Bruno formula ([30, 3]), as well as using derivatives of A_τ and h_τ. Of course, the derivatives of h_τ can be referred to derivatives of the Hamiltonian. □

Even if we will not use it in this paper, we note that there are computationally efficient ways to determine the $S^{(i)}$ in Lemma 13; in particular, one can compute the coefficients rather efficiently using the so called *"Poisson brackets"* (see, for example, [23, Appendix C] for the Hamiltonian case and [47] for the conformally symplectic case).

Applying Proposition 14, equating the coefficients of order j, we obtain:

$$-V^{(j)}(\varphi, I) + F^{(j)}(\varphi, I) + \lambda V^{(j)} \circ f_0^{-1}(\varphi, I) = I\, S^{(j)} + T_j(\varphi, I)\ , \tag{5.18}$$

where $F^{(j)}$ are the coefficients of the Hamiltonian F_τ; we stress that in (5.18) T_j is, according to the results of Proposition 14, a polynomial whose terms involve $S^{(1)}, \dots, S^{(j-1)}$ and the derivatives of $V^{(1)}, \dots, V^{(j-1)}$ up to order j.

We will consider (5.18) as an equation for $V^{(j)}$, $S^{(j)}$, assuming known $S^{(1)}, \dots, S^{(j-1)}$, $V^{(1)}, \dots, V^{(j-1)}$. In the next Lemma 16 we will show that these equations can always be solved. Hence, (5.18) is an inductive procedure that allows us to match recursively all the coefficients of powers of τ. The initial case, $j = 0$, is trivial. Furthermore, in Lemma 16 we will obtain estimates that establish the bounds claimed in Lemma 13.

Note that, composing with f_0 on the right and multiplying by λ^{-1}, equation (5.18) is equivalent to

$$-\lambda^{-1} V^{(j)} \circ f_0(\varphi, I) + V^{(j)}(\varphi, I) - \lambda^{-1}\, I\ S^{(j)} = B^{(j)}(\varphi, I)\ , \tag{5.19}$$

where we have denoted by $B^{(j)}$ a function which is already known. We emphasize that we should consider (5.19) as an equation for $V^{(j)}, S^{(j)}$, given $B^{(j)}$.

Remark 15. *We remark that the method proposed here relies on solving the order-by-order equation* (5.18) *(or, equivalently,* (5.19)*). In* [17] *one can find a quadratically convergent method, which provides the solution of* (5.18)*.*

Lemma 16. *Given ω satisfying the Diophantine condition* (3.1) *and given a C^r-differentiable function B, with r sufficiently large, defined on a manifold $\mathcal{M}$, we can find a unique $C^{r-\delta}$ function V and a vector $S \in \mathbb{R}^n$ such that*

$$-\lambda^{-1} V \circ f_0(\varphi, I) + V(\varphi, I) - \lambda^{-1}\, I\ S = B(\varphi, I)\ . \tag{5.20}$$

Furthermore, for $\tilde r < r - \delta$ we have the bounds:

$$\begin{aligned} \|V\|_{C^{\tilde r}} &\le C\|B\|_{C^r} \quad (r \ge 3)\ , \\ |S| &\le |\lambda|\ \|B\|_{C^1} \end{aligned} \tag{5.21}$$

for a suitable positive constant C depending on the Diophantine exponent of ω.

Proof. We start by writing B and V as

$$B(\varphi, I) = B_0(\varphi) + B_1(\varphi) I + B^{>}(\varphi, I)$$

and

$$V(\varphi, I) = V_0(\varphi) + V_1(\varphi) I + V^{>}(\varphi, I)\ , \tag{5.22}$$

with $B^{>}(\varphi, 0) = \partial_I B^{>}(\varphi, 0) = V^{>}(\varphi, 0) = \partial_I V^{>}(\varphi, 0) = 0$. Note that B_0, B_1, V_0, and V_1 are uniquely determined by either B or V.

Then, we can solve equation (5.20) in three steps, precisely we solve the three equations below:

$$V_0(\varphi) - \lambda^{-1} V_0(\varphi + \omega) = B_0(\varphi) \tag{5.23}$$

$$V_1(\varphi) - V_1(\varphi + \omega) = B_1(\varphi) + \lambda^{-1}\, S \tag{5.24}$$

$$V^>(\varphi, I) - \lambda^{-1} V^>(\varphi + \omega, \lambda I) = B^>(\varphi, I) \ . \tag{5.25}$$

The solution of equation (5.23) is obtained by observing that (5.23) can be rewritten as $-\lambda V_0(\varphi - \omega) + V_0(\varphi) = -\lambda B_0(\varphi - \omega)$. Hence, iterating, we obtain the solution:

$$V_0(\varphi) = -\sum_{i=1}^{\infty} B_0(\varphi - i\omega)\lambda^i \ . \tag{5.26}$$

Note that the sum in (5.26) is absolutely convergent, if B_0 is a continuous function. The equation (5.24) is the standard small divisor equation that appears often in KAM theory. Note that, since the left hand side of (5.24) must have zero average, we obtain that

$$S = -\frac{\lambda}{(2\pi)^n} \int_{\mathbb{T}^n} B_1(\varphi)\, d\varphi \ . \tag{5.27}$$

A convenient way to express the Fourier coefficients $(\hat{V}_1)_k$ of V_1 for $|k| > 0$ is the following:

$$(\hat{V}_1)_k = \frac{(\hat{B}_1)_k}{1 - e^{2\pi i k\cdot\omega}} \ .$$

If ω is Diophantine, the equation can be solved for all sufficiently differentiable functions (see Section 8) and the result can be estimated in spaces of less differentiable functions. Optimal regularity results were obtained in [59, 60]. The solution of (5.25) is given by the expression

$$V^>(\varphi, I) = \sum_{i=0}^{\infty} \lambda^{-i} \ B^> \circ f_0^i(\varphi, I) \ . \tag{5.28}$$

The sum in (5.28) converges uniformly because the function $B^>$ satisfies $\|B^>\| \le C|I|^2$ for some constant $C > 0$. Hence, the general term in (5.28) is bounded by $C\lambda^i$.

In conclusion, the solution V of the problem (5.20) is obtained as in (5.22).

Now, we turn to the problem of obtaining estimates. We start by estimating V_0 and V_1 as

$$\|V_0\|_{C^r} \le \|B_0\|_{C^r} \le \|B\|_{C^r}$$
$$\|V_1\|_{C^{r-\delta-1}} \le C\nu^{-1}\|B_1\|_{C^{r-1}} \le C\nu^{-1}\|B\|_{C^r}$$

(compare with Lemma 25 of Section 8). Moreover, by (5.28) we can estimate the derivatives of $V^>$ in terms of the derivatives of $B^> \circ f_0^i$, where

$$\partial_\varphi^k \partial_I^j (B^> \circ f_0^i)(\varphi, I) = ((\partial_\varphi^k \partial_I^j) B^>) \circ f_0^i(\varphi, I)\lambda^{ji} \ .$$

Hence, we obtain that the series of derivatives

$$\partial_\varphi^k \partial_I^j (B^> \circ f_0^i)$$

converges uniformly for $j \ge 2$. For $j = 0$ we know that

$$|\partial_\varphi^k (B^> \circ f_0^i)| \le C\, \lambda^{2i} \ ,$$

because $|\partial_\varphi^k B^>(\varphi, I)| \le C|I|^2$. For $j = 1$ we have

$$\partial_\varphi^k \partial_I (B^> \circ f_0^i)(\varphi, I) = \lambda^i \ (\partial_\varphi^k \partial_I B^>) \circ f_0^i(\varphi, I) \ ,$$

so that

$$|\partial_\varphi^k \partial_I (B^> \circ f_0^i)| \le C\lambda^i \ .$$

Finally, the last in (5.21) comes from (5.27). □

5.3. **Higher dimensional parameters.** The results we have obtained so far show that one can get the case $\ell = 1$ in Theorem 7. The goal of this section is to present a very simple argument to show that one can deduce the version for arbitrary ℓ from the case $\ell = 1$.

Let us consider a family f_μ with $\mu \in \mathbb{R}^\ell$ of exact conformally symplectic mappings as in the hypotheses of Theorem 7. For a fixed μ_0 we consider the one dimensional family $\tilde{f}_\tau = f_{\tau \frac{\mu_0}{|\mu_0|}}$, Then, applying the case $\ell = 1$ of Theorem 7, we obtain a transformation $\tilde{v}_\tau^N$ corresponding to f_τ, satisfying (5.16).

We note that, if the family is uniformly C^r, the one dimensional families $\tilde{f}_\tau$ obtained for different μ_0 in a small neighborhood are also uniformly C^r. Hence, we conclude that all the families $\tilde{v}_\mu^N$ are uniformly $C^{\tilde{r}}$ for $\tilde{r} < r$ and that we can bound the remainders also uniformly in $\|\mu\|^{N+1}$, independently of the direction.

5.3.1. *The momentum mappings.* A geometrically more natural treatment of the problem of higher dimensional parameters is to borrow the strategy of [5, 40]. We just outline the small modifications to the calculus given in Lemma 12.

We consider a family of conformally symplectic diffeomorphisms f_μ, $\mu \in \mathbb{R}^\ell$, such that

$$f_\mu^* \Omega = \lambda \Omega \ .$$

We denote by ∂_v the directional derivative in the direction v in the μ-space. Then we have by the chain rule

$$\partial_v f_\mu = (\mathcal{F}_\mu \cdot v) \circ f_\mu \ ,$$

where $\mathcal{F}_\mu$ is a linear function from $\mathbb{R}^\ell$ to the space of vector fields X, that is,

$$\mathcal{F}_\mu : \mathbb{R}^\ell \to X \ .$$

Again, we introduce the notion of Hamiltonian F_μ. The calculations can be done in every direction and, as was argued before (see (5.10)), this has the consequence that for each direction v, we have

$$i_{\mathcal{F}_\mu \cdot v} \Omega = d(F_\mu \cdot v) \ .$$

Because contraction and exterior derivatives are linear, we can assume (by choosing appropriately the constants on each Hamiltonian) that F_μ is a linear function from the space of directions to the space of C^r-functions.

As we will see, all computations that we have carried out remain valid in this context. Notice that since the main equations we consider, e.g. (5.19), are linear equations among functions, they remain identical, when the symbols stand for linear functionals producing functions. In fact, to compute the generator of the composition of two families of ECSI, we use the ideas above together with the chain rule. Consider f_μ a λ-ECSI and g_μ an η-ECSI; we compute the generator of $h_\mu = f_\mu \circ g_\mu$ by applying the chain rule to directional derivatives. More precisely, let us use the notation ∂_v to indicate as above the directional derivative in the direction v in the μ parameter, while D stands for the derivative in the geometric variables; then, we have:

$$\begin{aligned}
\partial_v h_\mu &= \partial_v f_\mu \circ g_\mu + D f_\mu \circ g_\mu \partial_v g_\mu \\
&= (\mathcal{F}_\mu \cdot v) \circ f_\mu \circ g_\mu + D f_\mu \circ g_\mu (\mathcal{G}_\mu \cdot v) \circ g_\mu \\
&= (\mathcal{F}_\mu \cdot v) \circ f_\mu \circ g_\mu + D f_\mu \circ f_\mu^{-1} \circ f_\mu \circ g_\mu (\mathcal{G}_\mu \cdot v) \circ f_\mu^{-1} \circ f_\mu \circ g_\mu \\
&= (\mathcal{F}_\mu \cdot v) \circ h_\mu + \big(f_{\mu *} (\mathcal{G}_\mu \cdot v) \big) \circ h_\mu \ .
\end{aligned}$$

Therefore, we obtain

$$\mathcal{H}_\mu \cdot v = \mathcal{F}_\mu \cdot v + f_{\mu *} (\mathcal{G}_\mu \cdot v)$$

or, more concisely, we see that we can suppress the argument v from both sides, to obtain the identity

$$\mathcal{H}_\mu = \mathcal{F}_\mu + f_{\mu *} \mathcal{G}_\mu \ , \tag{5.29}$$

which is typographically identical to (5.8). The only difference is that now the meaning of $\mathcal{H}_\mu, \mathcal{F}_\mu, \mathcal{G}_\mu$ is that of linear functions producing vector fields, when given $\mathbb{R}^\ell$ arguments, rather than vector fields.

Once we have justified (5.29), we see that all the other identities of Lemma 12 are obtained by repeated applications of (5.29) or contractions with Ω. Since the contraction with Ω is a linear operation, we get that all the formulas for the Hamiltonians in Lemma 12 make sense, when we interpret the Hamiltonians as linear functionals from $\mathbb{R}^\ell$ to the space of functions.

5.4. **End of the proof of Theorem 4.** In the previous subsection, we have obtained that, if μ is small enough, then we can change variables to a system with easy derivatives (up to a small error). The goal of this section is to obtain bounds on the spectrum of the periodic orbit that have a moderate dependence on the length of the orbit.

Such questions have been considered in the theory of dichotomies of mappings (see, for example, [61, 22, 51]). The results we present, e.g. Lemma 17, are slightly sharper than the results obtained applying the general theory of dichotomies, because we take advantage of the fact that the system is conformal.

We start with some preliminaries. If the orbit $\{x_j = f_\mu^j(x_0)\}_{j=0}^{L-1}$ is contained in the neighborhood $\mathcal{U}$ claimed in Theorem 4, we have that

$$D f_\mu(x_i) = \Lambda + E_i \ ,$$

where $\Lambda = \begin{pmatrix} \lambda\,\mathrm{Id} & 0 \\ 0 & \mathrm{Id} \end{pmatrix}$ and $\|E_i\| \le C_N \|\mu\|^N$. By the chain rule, we have

$$D f_\mu^L(x_0) = (\Lambda + E_{L-1}) \cdots (\Lambda + E_0) \ . \tag{5.30}$$

The claim (3.3) in Theorem 4 is just an estimate on the spectrum of the product in (5.30), given estimates on the norms of E_j which we have been already obtained.

One can get rather straightforward estimates for $C_N \|\mu\|^N$ small enough depending on L:

$$\begin{aligned}
||D f_\mu^L(x_0) - \Lambda^L|| &\le C_L C_N \ \lambda^{L-1} \|\mu\|^N \ , \qquad && \lambda > 1 \\
||D f_\mu^L(x_0) - \Lambda^L|| &\le C_L C_N \|\mu\|^N \ , && \lambda < 1 \ ,
\end{aligned} \tag{5.31}$$

where C_L is a combinatorial constant.

The estimates (5.31), even if interesting, are not enough to obtain the better estimates (3.3) we claimed, so in the rest of the section we will use some methods inspired by the theory of dichotomies ([61]) that allow us to improve the estimates. We note that we take advantage of the fact that the transformations are conformal to obtain sharper results than those obtained by the general theory of dichotomies; we present it in full details.

We start by observing that for any sequence $U_0, \ldots, U_{L-1}$ of invertible matrices we have

$$\mathrm{Spec}(Df_\mu^L(x_0)) = \mathrm{Spec}(U_0^{-1} Df_\mu^L(x_0) U_0)$$

and, using (5.30):

$$U_0^{-1} Df_\mu^L(x_0) U_0 = U_0^{-1} Df_\mu(x_{L-1}) U_{L-1}\, U_{L-1}^{-1} Df_\mu(x_{L-2}) U_{L-2} \cdots U_1^{-1} Df_\mu(x_0) U_0 \ . \quad (5.32)$$

We will show in Proposition 17 that it is possible to choose appropriate U_i in such a way that the product in (5.32) becomes simpler and therefore leads to a more accurate estimate than those in (5.31).

Proposition 17. *With the notations above, it is possible to find matrices*

$$A_i = \begin{pmatrix} 0 & A_i^{12} \\ A_i^{21} & 0 \end{pmatrix} \ , \qquad Q_i = \begin{pmatrix} Q_i^{11} & 0 \\ 0 & Q_i^{22} \end{pmatrix}$$

in such a way that, defining $U_i = \mathrm{Id}_{2n} + A_i$*, one has*

$$(\Lambda + E_i) U_i = U_{i+1} (\Lambda + Q_i) \quad (5.33)$$

(here we use arithmetic modulus L *for the index* i *so that* $U_L = U_0$*).*

Moreover, there exists a constant C *independent of* L *such that*

$$\sup_{0 \le i \le L-1} ||A_i||, \sup_{0 \le i \le L-1} ||Q_i|| \le C \sup_{0 \le i \le L-1} ||E_i|| \ .$$

The geometric meaning of the U's is that we are choosing a system of coordinates in each of the tangent spaces $T_{x_i}\mathcal{M}$ in which the mapping is block diagonal, equivalently we are finding some invariant subspaces. Such construction is very typical in the theory of invariant dichotomies, but in our case it is even simpler than in the general case, since the systems are conformal.

Proof. We can rewrite (5.33) as

$$E_i + \Lambda A_i = A_{i+1} \Lambda + Q_i + R(A_i, E_i, A_{i+1}, Q_i) \ , \quad (5.34)$$

where R is a quadratic homogeneous polynomial in its arguments.

We should think of the sequence of equations (5.34) as an equation acting on sequences of matrices of length L. The key point is that we will obtain estimates independent of L. We denote by E, A, Q the sequences whose components are E_i, A_i, Q_i. We consider the space of sequences endowed with the supremum norm

$$||E|| = \sup_{0 \le i \le L-1} ||E_i|| \ .$$

We can write (5.34) more concisely as

$$\mathcal{L}(A, Q) = -E + \mathcal{R}(E, A, Q) \quad (5.35)$$

where $\mathcal{L}$ is the operator acting on sequences of matrices by

$$\mathcal{L}(A, Q)_i \equiv \Lambda A_i - A_{i+1} \Lambda - Q_i$$

and

$$\mathcal{R}(E,A,Q)_i = R(A_i,E_i,A_{i+1},Q_i) \ .$$

We note that $\mathcal{R}(0,0,0)=0$, $D\mathcal{R}(0,0,0)=0$ and that

$$\sup_{||E||,||Q||,||A||\le 1} ||D^2\mathcal{R}(E,A,Q)|| \le C \ ,$$

where C is a constant independent of L. We now show that $\mathcal{L}$ is invertible with bounds independent of L. Then, the desired result, Proposition 17, follows by rewriting (5.35) as

$$(A,Q) = \mathcal{L}^{-1}(-E+\mathcal{R}(E,A,Q)) \equiv \mathcal{T}_E(A,Q)$$

and applying the standard contraction mapping principle.

To show that indeed $\mathcal{L}$ is invertible, we see that the equation $\mathcal{L}(A,Q)=\eta$ for A,Q given η can be written more explicitly, taking components and blocks as

$$\begin{aligned}
&\lambda A_i^{12} - A_{i+1}^{12} = \eta_i^{12}\\
&A_i^{21} - \lambda A_{i+1}^{21} = \eta_i^{21}\\
&\lambda A_i^{11} - \lambda A_{i+1}^{11} - Q_i^{11} = \eta_i^{11}\\
&A_i^{22} - A_{i+1}^{22} - Q_i^{22} = \eta_i^{22} \ .
\end{aligned} \tag{5.36}$$

Clearly, the equations (5.36) can be solved with bounds that are independent of L. □

Once we have Proposition 17, we see that the spectrum of $Df_\mu^L(x_0)$ is the union of the spectra of

$$(\lambda\,\mathrm{Id}_n + Q_{L-1}^{11})(\lambda\,\mathrm{Id}_n + Q_{L-2}^{11})\cdots(\lambda\,\mathrm{Id}_n + Q_0^{11})$$

and of

$$(\mathrm{Id}_n + Q_{L-1}^{22})(\mathrm{Id}_n + Q_{L-2}^{22})\cdots(\mathrm{Id}_n + Q_0^{22}) \ .$$

Because of the pairing rule (see Lemma 3) it is clear that the γ_i are the spectra of

$$(\lambda\,\mathrm{Id}_n + Q_{L-1}^{11})(\lambda\,\mathrm{Id}_n + Q_{L-2}^{11})\cdots(\lambda\,\mathrm{Id}_n + Q_0^{11}) \ ,$$

but it is standard that

$$||(\lambda\,\mathrm{Id}_n + Q_{L-1}^{11})(\lambda\,\mathrm{Id}_n + Q_{L-2}^{11})\cdots(\lambda\,\mathrm{Id}_n + Q_0^{11}) - \lambda^L\,\mathrm{Id}_n\,|| \le CL||Q|| \ ,$$

provided that $L||Q||$ is small. In particular, since $||Q|| \le C||E|| \le C_N\|\mu\|^N$, we obtain the claim (3.3) and thus we finish the proof of Theorem 4.

□

6. Proof of Theorem 4 using normally hyperbolic manifolds and averaging

In this section, we give a different proof of Theorem 4, using two standard techniques in dynamical systems: the theory of persistence of normally hyperbolic manifolds and the theory of averaging. This method also leads to some developments and insights, in particular in order to obtain information on the Arnold tongues (see Section 7).

6.1. **Overview of the method.** In the following we indicate the main steps of the proof of Theorem 4 developed in this section. Of course, fleshing out the outline in this section will require making precise and quantitative the steps, in particular providing the norms used to specify the meaning of *small*, the orders of perturbation theory, etc. Since some of the techniques are rather standard, we hope that the experts in each of the techniques will be able to develop some of the steps without consulting our exposition.

(1) We observe that the invariant torus for f_0 is a normally hyperbolic manifold in the sense of [33, 34, 45, 55].
(2) We apply the theory of normally hyperbolic manifolds to conclude that for μ sufficiently small there is a smooth family of manifolds $\mathcal{T}_\mu$ invariant by f_μ and normally hyperbolic.
(3) We can write these manifolds as the image of a torus under a family of maps K_μ from the torus to the phase space.
(4) We note that the parameterization K_μ satisfies

$$f_\mu \circ K_\mu = K_\mu \circ R_\mu \ ,$$

where R_μ is a family of maps from the torus to itself with $R_0(\varphi) = \varphi + \omega$ (see Theorem 19).
(5) Using that ω is Diophantine, we can develop an averaging theory for the map R_μ, showing that there are smooth families of smooth changes of variables B_μ that transform R_μ into a rigid rotation up to an error which is of high order in μ. This averaging procedure is very standard (see for example [11, 50, 4]).
(6) Since $B_\mu^{-1} \circ R_\mu \circ B_\mu$ is close to a rotation (in a smooth topology) for μ small enough, we conclude that for μ small enough, R_μ has Lyapunov multipliers close to 1.
(7) Hence, under a suitable non-degeneracy condition all the nearby periodic orbits have Lyapunov multipliers close to 1 in the restriction to the invariant manifold.
(8) Using the fact that the map is conformally symplectic we show that the Lyapunov multipliers in the directions complementary to the invariant manifold are close to λ.

Note that most of the steps in the above strategy (the normally hyperbolic theory and the averaging theory) are standard. The most subtle step of the above strategy is the averaging theorem, which requires using the Diophantine properties and delicate estimates.

We remark that we use the averaging theory only for maps on the torus which are close to a Diophantine rotation (i.e., the restrictions to the invariant manifold). Nevertheless, we can use the geometry to obtain bounds on all the Lyapunov multipliers of the original map. Informally, we use the geometry to transfer the very accurate results obtained in the slow degrees of freedom case to the whole map. This strategy has also appeared recently in other contexts ([27, 10]).

6.1.1. *A remark on the persistence of KAM tori based on normally hyperbolic manifolds.* The strategy outlined in Section 6.1 gives also a proof of the persistence of KAM tori (i.e., invariant tori with a motion which is topologically conjugate to a rotation) in families of conformally symplectic systems. We will not present a full theorem.

Theorem 18. *Let f_μ be a smooth family of smooth conformally symplectic systems. Assume that g is a conformally symplectic system admitting a smooth KAM torus with a Diophantine rotation ω. Assume that f_0 is sufficiently close to g in a smooth topology. Assume furthermore that f_μ satisfies some appropriate non-degeneracy conditions. Then, we can find μ_* close to zero such that f_{μ_*} admits a KAM torus of frequency ω.*

As we will see, the non-degeneracy condition is expressed as the fact that the average of an explicit matrix involving derivatives of f_μ – both with respect to space variables and parameters – has rank n (see below for details on how to compute the matrix).

Proof. By the theory of normally hyperbolic manifolds, we obtain that there is a smooth family of manifolds which are normally hyperbolic invariant manifolds for f_μ. By the implicit function theorem, since these manifolds are close to the KAM torus for g, we can choose a coordinate system in which the motion on the manifolds is given by a family R_μ in such a way that R_0 is close to a rotation of frequency ω (notice that R_μ is obtained through the implicit function theorem and will depend also on derivatives of f_μ with respect to the space variables). The non-degeneracy condition assumed in the theorem is just that $\int_{\mathbb{T}^n} \partial_\mu R_0(\varphi)\, d\varphi$ has full rank.

Under these conditions, the result of [53] shows that we can find a value of the parameter μ_* close to zero, such that the dynamics is smoothly conjugated to the rotation by ω. Clearly, the invariant manifold for μ_* is a KAM torus for f_{μ_*}. □

Comparing Theorem 18 with the results in [17], we note that the above approach has some disadvantages with respect to the proof provided in [17]: it does not establish the existence of C^∞ or analytic invariant tori, since normally hyperbolic manifolds are not C^∞, it does not have an a-posteriori format, it does not allow to discuss the zero dissipation limit or Lindstedt series and it does not lead to efficient algorithms.

6.2. **The theory of normally hyperbolic manifolds.** The main result presented in this section is the following well known result.

Theorem 19. *In the assumptions of Theorem 4 for any $r \in \mathbb{N}$ we can find $a > 0$ and a C^r-family K_μ of embeddings of the torus indexed by $\mu \in B_a \equiv \{\mu \in \mathbb{R}^\ell : \ |\mu| \leq a\} \subset \mathbb{R}^\ell$, $\ell \leq n$, say $K_\mu : \mathbb{T}^n \times B_a \to \mathcal{M}$, and a C^r-family of maps R_μ of the torus to itself, such that*

$$f_\mu \circ K_\mu = K_\mu \circ R_\mu \ . \tag{6.1}$$

Furthermore, the family R_μ is C^r-arbitrarily close to the family $R_0(\varphi) = \varphi + \omega$.

The Theorem 19 is an easy consequence of the standard theory of normally hyperbolic manifolds (see [33, 45, 55]). Below we will indicate how to deduce Theorem 19 from more customary statements in the literature. The study of the functional equation (6.1) can also be used as the basis of the theory of normally hyperbolic manifolds and leads to very efficient algorithms ([44, 43]).

Note that the range of parameters for which we can claim C^r regularity depends on r and may have an empty intersection, when we consider all values of r sufficiently large. It is known that there are examples of analytic normally hyperbolic manifolds in analytic families of maps, which are not C^∞ for any neighborhood of parameters (see [34, 35]); this is even true when the dynamics of the unperturbed map is a rigid rotation.

The way that the theory of normally hyperbolic manifolds is presented is based on the persistence of overflowing manifolds ([33]). To obtain smooth dependence on parameters, we just observe that if Λ_0 is an invariant manifold for f_0, then $\tilde{\Lambda} \equiv \Lambda_0 \times B_a$ is an invariant manifold (with boundary) for the extended map $\tilde{f}(x,\mu) = (f_0(x),\mu)$. If Λ_0 is normally hyperbolic for f_0, $\tilde{\Lambda}$ is normally hyperbolic for $\tilde{f}$. Notice also that the rates of growth of the derivatives in the tangent direction of $\tilde{\Lambda}$ are the same as those for Λ_0 and that the stable and unstable directions for $\tilde{\Lambda}$ are the same as those for Λ_0.

We also observe that for a small enough, the map $\hat{f}(x,\mu) = (f_\mu(x),\mu)$ is C^r-close to $\tilde{f}$. Applying the standard theory of persistence of normally hyperbolic invariant manifolds (with boundary), we conclude that there exists a manifold $\hat{\Lambda}$ locally invariant for $\hat{f}$. By the structure of $\hat{f}$, this locally invariant manifold has to be of the form $\cup_\mu \Lambda_\mu \times \{\mu\}$, where Λ_μ is a manifold invariant under f_μ.

Note that the above argument concludes as much regularity for the families of manifolds Λ_μ as the regularity of the invariant manifolds for the extended mapping. As it is well known ([34, 35, 45]), this regularity is determined by the regularity of the map and by some ratios between logarithms of rates of contraction in the stable/unstable directions and the rates of contraction on the mappings restricted to the manifold. In our case, the mappings are smooth, the dynamics of the unperturbed map restricted to the manifold is smoothly conjugate to a rigid rotation (because we are assuming it is a KAM torus) and the normal directions are a contraction. In this circumstances, given any r, we can obtain that the manifold $\hat{\Lambda}$ is C^r, restricting the domain of the μ considered.

Since the manifold invariant for f_0 is a KAM torus, we know that there is a parameterization K_0 in such a way that $f_0 \circ K_0(\varphi) = K_0(\varphi+\omega)$. Applying the implicit function theorem, we can obtain smooth parameterizations for Λ_μ depending on μ. The R_μ is just the dynamics restricted to any of the manifolds Λ_μ.

6.3. **Averaging theory.** With reference to the conclusions of the previous subsection, we denote by $R_\mu = R_\mu(\varphi)$ the family describing the dynamics restricted to Λ_μ as

$$R_\mu(\varphi) \equiv \varphi + R_\mu^{\leq N}(\varphi) + R_{N+1}(\varphi) \ , \tag{6.2}$$

for some $N > 0$, where we can expand $R_\mu^{\leq N}$ as

$$R_\mu^{\leq N}(\varphi) = \omega + R^1(\varphi)\mu + ... + R^N(\varphi)\mu^{\otimes N} \ . \tag{6.3}$$

The functions R^1, ..., R^N can be computed explicitly, while the remainder R_{N+1} can be bounded as

$$\|R_{N+1}\|_{C^{r-N}(\mathbb{T}^n)} \leq C\|\mu\|^{N+1} \ ,$$

for some real constant $C > 0$ and for $r \geq N$.

Proposition 20. *Let f_μ be a C^r-family of conformally symplectic diffeomorphisms for $\mu \in \mathbb{R}^\ell$, $\ell \leq n$. Assume that $f_0(\varphi, I) = (\varphi+\omega, \lambda I)$ for some $\lambda < 1$ with ω a Diophantine vector. For any $N > 0$ integer, $N \leq N_0(r)$, there exists a transformation of coordinates B_μ^N and a frequency ω_μ^N, such that*

$$(B_\mu^N)^{-1} \circ R_\mu \circ B_\mu^N(\varphi, I) = \varphi + \omega_\mu^N + S_N(\varphi,\mu) \ , \tag{6.4}$$

where S_N can be bounded for some $\tilde{r} < r$ as

$$\|S_N\|_{C^{r-\tilde{r}}} \leq C\|\mu\|^{N+1} \ , \tag{6.5}$$

for some positive constant C depending on N, on the Diophantine exponent of ω and on the C^r-norm of f_μ.

Furthermore, let $\rho = (a_1, ..., a_n)/L$ for $(a_1, ..., a_n) \in \mathbb{Z}^n$ and $L > 0$ integer. Assume that the non-degeneracy condition $\bar{R}^1 \neq 0$ is satisfied with $\bar{R}^1$ denoting the average of R^1 introduced in (6.3)*; then, there exists a value $\tilde{\mu} = \tilde{\mu}(\rho)$ such that* (6.4) *becomes*

$$(B^N_{\tilde{\mu}})^{-1} \circ R_{\tilde{\mu}} \circ B^N_{\tilde{\mu}}(\varphi, I) = \varphi + \rho + S_N(\varphi, \mu)$$

with S_N bounded as before.

Before giving the proof of Proposition 20, we state the following Lemma, which provides a normal form near the rotation curve, such that it transforms the conjugation to a rigid rotation with frequency ω^N_μ.

Lemma 21. *Given R_μ as in* (6.2) *for some $N > 0$ integer, there exists a change of coordinates B^N_μ and a frequency $\omega^N_\mu \in \mathbb{R}^n$, such that*

$$R_\mu \circ B^N_\mu = B^N_\mu \circ T_{\omega^N_\mu} \ . \tag{6.6}$$

Proof. We expand the frequency as

$$\omega^N_\mu = \omega + \omega_1 \mu + \omega_2 \mu^{\otimes 2} + ... + \omega_N \mu^{\otimes N} \tag{6.7}$$

and we also expand B^N_μ as

$$B^N_\mu(\varphi) \equiv \varphi + B^1(\varphi)\mu + B^2(\varphi)\mu^{\otimes 2} + ... + B^N(\varphi)\mu^{\otimes N} \ .$$

Equation (6.6) implies that we need to solve equations of the form

$$R^i(\varphi) + B^i(\varphi) - \omega_i = B^i(\varphi + \omega) + W^i(\varphi) \ , \qquad 1 \leq i \leq N \ , \tag{6.8}$$

where $W^1(\varphi) = 0$ and, supposing to have solved the equation up to the order $i-1$ for $i > 1$, then W^i becomes an expression involving the known terms $\omega_1, ..., \omega_{i-1}$, as well as the known functions $R^1, ..., R^{i-1}, B^1, ..., B^{i-1}$ and their derivatives up to the order $i-1$.

Indeed, equation (6.8) can be written in the form

$$B^i(\varphi + \omega) - B^i(\varphi) = -\omega_i + R^i(\varphi) - W^i(\varphi) \ . \tag{6.9}$$

As in Section 5.1 we first compute the average of (6.8), which provides ω_i leaving the average of B^i undetermined:

$$\omega_i = \bar{R}^i - \bar{W}^i \qquad \text{for } 1 \leq i \leq N \ ,$$

where the bar denotes again the average. In particular, we have that $\omega_1 = \bar{R}^1$. Then, we compute the remaining part of B^i as the solution of

$$B^i(\varphi + \omega) - B^i(\varphi) = \tilde{R}^i(\varphi) - \tilde{W}^i(\varphi) \ ,$$

where $\tilde{R}^i(\varphi) = R^i(\varphi) - \bar{R}^i$, $\tilde{W}^i(\varphi) = W^i(\varphi) - \bar{W}^i$. Thus we obtain the solution of (6.9), which yields at each order the terms B^i and ω^i satisfying (6.6). ☐

Proof. (Proposition 20) To obtain the bounds in (6.5), we notice that we can bound the solution of (6.9) for $\tilde{r} < r$ as (compare with Lemma 25 of Section 8)

$$\|B^i\|_{C^{\tilde{r}}} \leq C\nu^{-1} \, \|\tilde{R}^i - \tilde{W}^i\|_{C^r} \ .$$

Assuming that equation (6.9) is solved up to the order N with ω^N_μ as in (6.7) determined up to the same order, then the remainder S_N is bounded as in (6.5).

Assume now to have a periodic orbit as in Theorem 4 with frequency $\rho = (a_1, ..., a_n)/L$ with $(a_1, ..., a_n) \in \mathbb{Z}^n$ and $L \in \mathbb{N}$; assume that its trajectory lies in a neighborhood $\mathcal{U}$ of the torus. Since ω satisfies the Diophantine condition (3.1), we obtain that

$$\|\omega - \rho\| = L^{-1}\|\omega L - a\| \geq \nu L^{-\delta-1} \ ,$$

namely $L \geq \nu^{\frac{1}{\delta+1}} \|\omega - \rho\|^{-\frac{1}{\delta+1}}$.

In the new coordinate system determined in Lemma 21, the frequency of motion is given by ω_μ^N as in (6.7). Therefore, we proceed to find $\mu = \tilde{\mu}$, such that

$$\omega_{\tilde{\mu}}^N = \rho \ ;$$

if the non-degeneracy condition $\omega_1 \neq 0$ is satisfied, then we can determine $\tilde{\mu}$ as

$$\tilde{\mu}(\rho) = \omega_1^{-1}(\rho - \omega) + O(\rho - \omega)^2 \ . \tag{6.10}$$

This ends the proof of the Proposition. □

The conclusion of the proof of Theorem 4 relies on the fact that if there exists a periodic orbit close to the invariant curve, then μ must be small according to (6.10), provided that the non-degeneracy condition is satisfied (the existence of such periodic orbits is deferred to Section 7). Due to Lemma 21, the dynamics is a small perturbation of order of $\|\mu\|^{N+1}$ of a rotation in the angles with frequency ω_μ and a contraction in the actions. Let us denote by γ_i^{tg}, γ_i^{nor}, $i = 1, ..., n$, the eigenvalues in the tangent and normal directions, respectively. The eigenvalues along the manifold are approximately one; using the pairing rule for exponents of conformally symplectic systems (see Lemma 3), we have that $\gamma_i^{tg}\gamma_i^{nor} = \lambda$, which shows that the eigenvalues in the normal direction are close to λ. Due to the fact that in the angles the dynamics is close to a rigid rotation with frequency ρ up to orders of $\|\mu\|^{N+1}$ as in (6.4), we have that the corresponding eigenvalues, say γ_i for $i = 1, ..., n$, satisfy the inequality

$$|\gamma_i - 1| \leq C_{N,L} \, \|\mu\|^{N+1}$$

for some constant $C_{N,L}$. This ends the proof of Theorem 4 using the averaging theory.

7. Arnold tongues

In this section we study the range of parameters for which we can find periodic orbits of rotation number $\rho = a/L$, $a \in \mathbb{Z}^n$, $L \in \mathbb{N}$. We will assume that ρ is close to ω and, hence that μ will be small. The goal is to quantify how small μ should be and, hence, to obtain bounds for the residue that depend only on $\|\omega - \rho\|$ and L. We will also show that these periodic orbits exist.

Since the periodic orbits have to be contained in the normally hyperbolic manifold discussed in Section 6.2 and that the existence of periodic orbits is invariant under changes of variables, it suffices to study the existence of periodic orbits for maps of the torus $\mathbb{T}^n$ of the form in (6.4). Indeed, all the results that we present in this section are valid for families of torus maps close to a Diophantine rotation, independently of the theory of conformally symplectic mappings.

When $n = 1$, the theory of existence or not of periodic maps of the circle is well developed thanks to the theory of rotation number ([25]). In this one-dimensional case, the sets of parameter values for which the periodic orbits exist are called the *phase* *locking intervals* and, when we consider two or more parameter families, the sets of two dimensional parameters for which the periodic orbits exist are called *Arnold tongues*.

We use similar notations for arbitrary n and we call the *phase locking set* of type ρ, the set of parameter values for which there is a periodic orbit of type ρ or, if the number of parameters is larger than n, we again refer to them as *Arnold tongues*. One important difference between the case $n \geq 2$ and the case $n = 1$ is that, when $n \geq 2$ the rotation number may depend on the orbits: a map may have several rotations (one speaks about the rotation set of the map), and the phase locking sets for different ρ may have intersections.

The main observation is that the L-th iteration of the map in (6.4) is

$$f_\mu^L(\varphi) = \varphi + L\omega_\mu^N + E_{L,N}(\mu, \varphi) \tag{7.1}$$

where $\|E_{L,N}\| \leq LC_N \|\mu\|^N$ and, as we will see later, derivatives of $E_{L,N}(\mu, \varphi)$ with respect to μ and to φ are also small. Equation (7.1) and the estimates on $E_{L,N}$ will allow us to treat the domains of existence of periodic solutions as a perturbation of the study of the existence of periodic rotations of the map

$$g_\mu^{N,L}(\varphi) = \varphi + L\omega_\mu^N \ .$$

7.1. **Upper bounds on the phase locking set.** Obtaining upper bounds on the phase locking sets is rather trivial.

Starting from equation (7.1) and applying the implicit function theorem to the function ω_μ^N, we have the following result.

Proposition 22. *With the notations above, if* $\|L\omega_\mu^N - a\| > LC_N\|\mu\|^N$*, the equation* (7.1) *for* φ *does not have any solution. Hence, a necessary condition for the existence of periodic orbits of rotation* ρ *is that*

$$\|\omega_\mu^N - \rho\| \leq C_N \|\mu\|^N \ .$$

An upper bound on the phase locking set is provided by the following result, which is an easy consequence of Proposition 22.

Corollary 23. *If* ω_1 *is an invertible matrix, for any* ρ *given by the Implicit Function Theorem we can obtain a unique* μ^* *so that* $\omega_{\mu^*}^N = \rho$ *and the phase locking set corresponding to* ρ *is contained in the set*

$$\{\|\mu - \mu^*\| \leq C_N \|\mu\|^N\} \ .$$

7.2. **Existence of periodic orbits.** In this section, we just study the existence of solutions of the periodic point equation. The tool we will use is the implicit function theorem. The only difficulty is that we have to make sure that indeed the derivative with respect to μ of the function $E_{L,N}(\mu, \varphi)$ is small, so that we obtain the following proposition.

Proposition 24. *Under the assumptions of Proposition 20, assume that there exists a* $\mu = \mu^*$*, so that* $f_{\mu^*}(\varphi)$ *has an orbit of rotation* ρ *and that the matrix* $\partial_\mu \omega_{\mu^*}^N$ *has full rank.*

Then, there exists an open set of parameters Λ *containing* μ^**, such that for any* $\mu \in \Lambda$ *the map of the torus* $f_\mu(\varphi)$ *has an orbit of rotation* ρ*.*

Proof. We start by considering the L-th iterate of $f_\mu(\varphi)$ given by formula (7.1). We want to obtain an open set of $\beta = \mu - \mu^*$ for which the iterate, $f_\mu^L(\varphi)$, has a rotation number equal to $a = L\,\rho$, i.e. we look for parameter values μ such that the following relation is satisfied:

$$f_\mu^L(\varphi) = \varphi + a \ . \tag{7.2}$$

We notice that since $a = L\omega_{\mu^*}^N$, we can rewrite (7.2) as a fixed point problem:

$$0 = L(\omega_\mu^N - \omega_{\mu^*}^N) + E_{L,N}(\mu, \varphi) \ . \tag{7.3}$$

By assumption the partial derivative of ω_μ^N with respect to μ at μ^* is a non-singular matrix. Then, if we denote by $\omega_{1*}^N \equiv \partial_\mu \omega_{\mu^*}^N$, we have that

$$\omega_\mu^N - \omega_{\mu^*}^N = \omega_{1*}^N(\mu - \mu^*) + R(\mu - \mu^*) \ ,$$

where $R = R(\beta)$ is a function satisfying

$$R(0) = 0 \ , \qquad D_\beta R(0) = 0 \ .$$

We now rewrite the problem (7.3) as the following fixed point problem:

$$\beta = \mathcal{T}(\beta) \equiv -(\omega_{1*}^N)^{-1} \left[\frac{E_{L,N}(\beta + \mu^*, \varphi)}{L} + R(\beta) \right] \ . \tag{7.4}$$

The proposition follows by applying a standard contraction mapping principle. In order to verify that $\mathcal{T}(\beta)$ is a contraction, we prove that the derivative of $\mathcal{T}(\beta)$ with respect to β is less than 1/2. The bound on the partial derivative comes from estimates on the partial derivatives of $E_{L,N}$ with respect to the parameter μ. We obtain the estimates for $\partial_\mu E_{L,N}(\mu, \varphi)$ inductively through (6.4) and by noticing that

$$f_\mu^j(\varphi) = f_\mu \circ f_\mu^{j-1}(\varphi) = f_\mu^{j-1}(\varphi) + \omega_\mu^N + S_N(f_\mu^{j-1}(\varphi), \mu) \ ,$$

which provides a recursive formula for $E_{j,N}(\mu, \varphi)$, namely

$$E_{j,N}(\mu, \varphi) = E_{j-1,N}(\mu, \varphi) + S_N(f_\mu^{j-1}(\varphi), \mu) \ .$$

If we denote by $\mathcal{E}_{j,N} = \sup_{\varphi \in \mathbb{T}^n} |\partial_\mu E_{j,N}(\mu, \varphi)|$, the estimate (6.5) implies that for every j, $\mathcal{E}_{j,N} \leq \mathcal{E}_{j-1,N} + C_N \|\mu\|^{N+1}$. Therefore, $\mathcal{E}_{j,N}$ is bounded by $\|\mu\|^{N+1}$ times a constant depending on N and j, say

$$\mathcal{E}_{j,N} \leq j C_N \|\mu\|^{N+1} \ . \tag{7.5}$$

We now show that $\mathcal{T}(\beta)$ is a contraction for β small enough, since (7.4) and (7.5) imply that

$$\|D_\beta \mathcal{T}(\beta)\| \leq \|(\omega_{1*}^N)^{-1}\| \left[\|D_\beta R(\beta)\| + \frac{\mathcal{E}_{L,N}}{L} \right] \leq \|(\omega_{1*}^N)^{-1}\| \left[\|D_\beta R(\beta)\| + C_N \|\mu\|^{N+1} \right] \ . \tag{7.6}$$

Taking into account that $\|\mu\|$ is small, we now choose $\delta^* > 0$ small enough, so that the right hand side of (7.6) is strictly less than 1/2 when $\|\beta\| < \delta^*$. The argument proves the existence of an open set of parameters μ for which there are periodic orbits with frequency ρ and Proposition 24 follows. □

8. Appendix: solutions of the cohomological equation

We quote two Lemmas which solve a cohomological equation of the form

$$w(\varphi + \omega) - \lambda w(\varphi) = \eta(\varphi) \tag{8.1}$$

for some functions w and η; the proof of the Lemmas con be found for example in [17].

Lemma 25. *For $\lambda \in [A_0, A_0^{-1}]$, $0 < A_0 < 1$ and for ω Diophantine as in (3.1), consider the equation (8.1). Assume that η is a C^r-differentiable function, $r > 0$, with zero average. Then, there is one and only one solution of (8.1) with zero average. Furthermore, if w is $C^{r-\delta}$-differentiable for $\delta > 0$, we have*

$$\|w\|_{r-\delta} \leq C\nu^{-1} \ \|\eta\|_r \ ,$$

where C is a constant that depends on A_0 and n, but it is uniform in λ and it is independent of the Diophantine constant ν.

Lemma 26. *Let $|\lambda| \neq 1$ and $\omega \in \mathbb{R}^n$. Given any Lebesgue measurable function η, there is one Lebesgue measurable function w satisfying*

$$w(\varphi + \omega) - \lambda w(\varphi) = \eta(\varphi) \ .$$

Furthermore, for $r > 0$ the following estimates hold:

$$\|w\|_r \leq \big| |\lambda| - 1\big|^{-1} \|\eta\|_r \ .$$

Finally, one can bound the derivatives of w with respect to λ as

$$\|D^j_\lambda w\|_r \leq \frac{j!}{\big| |\lambda| - 1\big|^{j+1}} \ \|\eta\|_r \ , \qquad j \geq 1 \ .$$

9. Appendix: the extension to vector fields

In this section we provide the extension of Theorem 4 to the case of conformally symplectic flows, which are defined as follows.

Let $\mathcal{M}$ be a symplectic manifold of dimension $2n$, endowed with a symplectic form Ω. Let X be a vector field with associated flow $f_t : \mathcal{M} \to \mathcal{M}$ at time t.

Definition 27. *We say that the flow f_t is a conformally symplectic flow, if*

$$f_t^*\Omega = e^{\eta t}\Omega \tag{9.1}$$

for some η real.

Notice that the above definition is equivalent to require that

$$L_X\Omega = \eta\Omega \ ,$$

where L_X denotes the Lie derivative.

In the case that $\Omega = d\alpha$, denoting by i_X the contraction with the vector field X, one has

$$d(\eta \ \alpha) = \eta \, d\alpha = L_X\Omega = i_X d\Omega + d(i_X\Omega) = d(i_X\Omega) \ ,$$

which shows that $\eta\alpha$ and $i_X\Omega$ differ by a closed form. This leads to the definition of an *exact conformally symplectic* vector field X, whenever there exists a function H such that

$$i_X\Omega = \eta\alpha + dH \ .$$

All exact conformally symplectic fields are conformally symplectic, but the converse is not necessarily true.

Like for maps we introduce an embedding K defining a rotational invariant torus, provided that the following invariance equation is satisfied:

$$f_t \circ K = K \circ T_{\omega t} \ . \tag{9.2}$$

Note that if K is an embedding, then $K(\mathbb{T}^n)$ is a diffeomorphic copy of the torus.

We say that an n-dimensional torus is Lagrangian if

$$K^*\Omega = 0 \ ,$$

which can be expressed in coordinates as

$$DK^{\mathsf{T}}(\varphi)\ J \circ K(\varphi)\ DK(\varphi) = 0 \ . \tag{9.3}$$

In the neighborhood of an invariant torus, we can identify $\mathcal{M}$ with the manifold $\mathcal{M}_0$ defined as

$$\mathcal{M}_0 \equiv \mathbb{T}^n \times B \ ,$$

where $B \subset \mathbb{R}^n$ is a ball around zero; the manifold $\mathcal{M}_0$ is endowed with the standard symplectic form

$$\Omega_0 \equiv \sum_{j=1}^{n} dI_j \wedge d\varphi_j \ .$$

An example of a conformally symplectic flow with respect to Ω_0 is given by

$$f_t^0(\varphi, I) = (\varphi + \omega t, e^{\eta t} I) \ , \tag{9.4}$$

which provides a rotation by ω in the angles and a constant contraction in the actions for $\eta < 0$.

We now state the equivalent of Theorem 2 for flows.

Theorem 28. *([13]) Let $(\mathcal{M}, \Omega)$ be a $2n$-dimensional analytic symplectic manifold and for $r > 0$ integer let f_t be a C^r-conformally symplectic flow as in (9.1). Let $\omega \in \mathbb{R}^n$ and assume that there exists an embedding $K : \mathbb{T}^n \to \mathcal{M}$, such that (9.2) and (9.3) are satisfied. Then, there exists $g \in C^r$ from $\mathcal{M}$ to a neighborhood of the torus, such that*

$$\begin{aligned} g^{-1} \circ f_t \circ g =& f_t^0 \\ g^*\Omega_0 =& \Omega \\ g(\varphi, 0) =& K(\varphi) \ , \end{aligned}$$

where f_t^0, η are given in (9.4).

In analogy to (3.1) we consider a frequency $\omega \in \mathbb{R}^n$ satisfying the Diophantine condition

$$|\omega \cdot k| \geq \nu |k|^{-\delta} \ , \qquad k \in \mathbb{Z}^n \backslash \{0\} \ ,$$

for suitable positive real constants ν, δ.

We introduce a frequency vector

$$\rho = \frac{1}{T}\ (a_1, ..., a_n)$$

with $(a_1, ..., a_n) \in \mathbb{Z}^n$ and $T > 0$. Let us denote by $\Phi = \Phi(t)$ the principal fundamental matrix around the periodic orbit with frequency ρ. Let γ_1, ..., γ_{2n} be the *Floquet multipliers* associated to $\Phi(T)$ which, by Floquet theory ([36]), can be written as $\Phi(T) = e^{PT}$ for some constant matrix P. We denote by η_1, ..., η_{2n} the *characteristic exponents* associated to γ_j by

$$\gamma_j = e^{\eta_j T} \ . \tag{9.5}$$

In analogy to Lemma 3 we state the following result.

Lemma 29. *Let f_t be a C^r-conformally symplectic flow and let γ_1, ..., γ_{2n} be the eigenvalues of the principal fundamental matrix $\Phi(T)$; then, one has that the spectrum of $\Phi(T)$ has the form*

$$Spec(\Phi(T)) = \{(\gamma_i, e^{\eta T}\gamma_i^{-1}) : \ i = 1, ..., n\} \ .$$

This result has been proven in [28] (see Section V), as a consequence of the conformally symplectic condition (9.1), which can be written as

$$Df_t(x)^{\mathsf{T}} \ J \circ f_t(x) \ Df_t(x) \ = \ e^{\eta t} \ J(x) \ .$$

Remark 30. *Due to* (9.5)*, the pairing rule of Lemma 29 implies that the characteristic exponents sum in pairs and that the sum is equal to the conformal factor.*

The extension of Theorem 4 for flows is stated below. We shall consider a C^r-family f_t^μ of conformally symplectic flows, depending on a parameter $\mu \in \mathbb{R}^\ell$, $\ell \le n$; let $\Phi_\mu = \Phi_\mu(t)$ be the associated principal fundamental matrix.

Theorem 31. *For $r > 0$ integer, let f_μ^t, $\mu \in \mathbb{R}^\ell$ ($\ell \in \mathbb{N}$, $\ell \le n$), be a C^r-family of conformally symplectic flows, such that f_t^0 admits a Lagrangian invariant torus with Diophantine rotation vector ω. Then, there is a neighborhood $\mathcal{U}$ of the torus with the following property. Let*

$$\rho = \frac{1}{T} \ (a_1, ..., a_n)$$

with $(a_1, ..., a_n) \in \mathbb{Z}^n$ and $T > 0$. Assume that the orbit of a point x has rotation number ρ and is contained in $\mathcal{U}$. Denote by γ_i the Floquet multipliers which, according to Lemma 29, we define as $Spec(\Phi_\mu(T)) = \{(\gamma_i, e^{\eta T}\gamma_i^{-1}) : \ i = 1, ..., n\}$, where $\Phi_\mu(T)$ is the principal fundamental matrix at period T. Then, there exists $N_0 \in \mathbb{N}$ depending on r, such that for every $N \le N_0$ there exists a constant $C_N > 0$, depending also on r and on the Diophantine exponent of ω, so that

$$|\gamma_i - 1| \le T C_N \|\mu\|^N \ .$$

The proof can be done as for maps using deformation theory (see Section 5.1) or using the theory of normally hyperbolic invariant manifolds and the averaging theory of rotations (see Section 6), provided that suitable modifications are implemented to take into account that f_0 (see equation (2.7)) is replaced by f_t^0 as in (9.4) or that the invariance equation (2.2) is replaced by (9.2).

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