Absence of absolutely continuous spectrum for the Kirchhoff Laplacian on radial trees

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Abstract

In this paper we prove that the existence of absolutely continuous spectrum of the Kirchhoff Laplacian on a radial metric tree graph together with a finite complexity of the geometry of the tree implies that the tree is in fact eventually periodic. This complements the results by Breuer and Frank in [3] in the discrete case as well as for sparse trees in the metric case.

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1 Introduction

Quantum graphs and their discrete counterparts have been a subject of intense interest recently – see [1] for the bibliography – both as a source of practically important models and an object of inspiring mathematical complexity. One of the important question concerns transport on such graphs, in particular, the presence or absence of the absolutely continuous spectrum of the corresponding Hamiltonian.

It was demonstrated in [3] for the Laplacian Δ on a discrete radially symmetric rooted tree graph, that if the sequence of branching numbers (b_n) is bounded and Δ has nonempty absolutely continuous spectrum then (b_n) is eventually periodic, in other words, we may think of the geometry of the tree being eventually periodic. This result makes it possible to make a similar conclusion for metric graphs as long as they are equilateral using the known duality – cf. [8] and references therein. This tells us nothing, however, about the spectrum in the metric setting beyond the equilateral case when the geometry of the tree encoded in the edge lengths should come into play.

The aim of this note is address this question and to prove the analogous result for the Laplacian on radial metric trees with Kirchhoff boundary conditions. The proof will combine three facts:

(a) a unitary equivalence of the Kirchhoff Laplacian on the tree with a direct sum of self-adjoint operators on halflines – see [13, Theorem 3.5],

(b) an adapted version of Remling's Oracle Theorem [9, Theorem 2] for such halfline operators,

(c) an adapted version of absence of absolutely continuous spectrum for "finite local complexity" [6, Theorem 4.1].

Sections 2 to 4 below are devoted respectively to these three facts; after discussing them we state and prove in Section 5 our main theorem and comment on its possible extensions.

2 Radial tree graphs and unitary equivalence

Let $\Gamma = (V, E)$ be a rooted radially symmetric metric tree graph with vertex set V and edge set E. Let $O \in V$ be the root, and for a vertex $v \in V$ of the nth generation of vertices let b_n be the branching number of v, i.e., the number of forward neighbors, and $t_n > 0$ be the length of the path connecting the vertex v with the root O. We set $t_0 := 0$ and $b_0 := 1$. In order to obtain a well-defined operator we will assume

$$\inf_{n\in\mathbb{N}} (t_{n+1} - t_n) > 0, \tag{1}$$

i.e., the the edge lengths are bounded away from 0. Without loss of generality, we may also assume that

$$\inf_{n\in\mathbb{N}} b_n > 1, \tag{2}$$

i.e., each vertex except the root will have at least two forward neighbors (otherwise, by Kirchhoff boundary conditions, we can delete such vertices).

Let H_Γ be the Laplacian on Γ with Kirchhoff boundary conditions at the vertices except O, and Dirichlet boundary conditions at the root, i.e., H_Γ is the unique self-adjoint operator associated with the form τ,

$$
\begin{aligned}
D(\tau) &:= \{u \in L_2(\Gamma);\ u' \in L_2(\Gamma),\, u(O) = 0,\, u \,\text{continuous on}\, \Gamma\}\,,\\
\tau(u,v) &:= \int_\Gamma u(x)\overline{v(x)}\,dx.
\end{aligned}
$$

The following result was shown in [13, Theorem 3.5], see also [3, Proposition 5].

Proposition 2.1 ([13, Theorem 3.5], [3, Proposition 5]). *H_Γ is unitarily equivalent to*

$$
A_0^+ \oplus \bigoplus_{k=1}^{\infty} (A_k^+ \otimes I_{\mathbb{C}^{b_1\cdots b_{k-1}(b_k-1)}}),
$$

where, for $k \geq 0$, A_k^+ is a linear operator in $L_2(t_k,\infty)$ defined by

$$
\begin{aligned}
D(A_k^+) := \Big\{ & u \in L_2(t_k,\infty);\ u \in W_2^2\Big(\bigcup_{n\geq k}(t_n,t_{n+1})\Big),\, u(t_k) = 0,\\
& u(t_n+) = \sqrt{b_n}\,u(t_n-),\, u'(t_n+) = \frac{1}{\sqrt{b_n}}u'(t_n-) \quad (n > k)\Big\}\\
(A_k^+u)(t) := {} & -u''(t) \quad \Big(t \in \bigcup_{n\geq k}(t_n,t_{n+1})\Big).
\end{aligned}
$$

The preceding proposition reduces the the study of (the spectrum of) H_Γ to the study of (the spectra of) the operators A_k^+. These are operators on halflines. We will describe such operators by means of measures in the following way.

Definition. A measure μ on $\mathbb{R}$ is called atomic, if $\operatorname{spt}\mu$ is countable, i.e., if there exists $J \subseteq \mathbb{N}$ such that $\mu = \sum_{n\in J}\beta_n\delta_{t_n}$ with suitable $(\beta_n), (t_n)$ in $\mathbb{R}$.

For sequences (b_n) in $(1,\infty)$, (t_n) in $\mathbb{R}$ satisfying (1) and (2), we associate a measure $\mu = \sum_{n=1}^{\infty}\beta_n\delta_{t_n}$, where $\beta_n := \frac{\sqrt{b_n}+1}{\sqrt{b_n}-1}$ $(n \in \mathbb{N})$. Then define H_μ in

$L_2(\mathbb{R})$ by

$$D(H_\mu) := \Big\{ u \in L_2(\mathbb{R});\ u \in W_2^2\Big(\bigcup_{n\in\mathbb{N}_0} (t_n, t_{n+1}) \Big),$$
$$u(t_n+) = \sqrt{b_n} u(t_n-),\ u'(t_n+) = \frac{1}{\sqrt{b_n}} u'(t_n-) \quad (n \in \mathbb{N}) \Big\}$$
$$(H_\mu u)(t) := -u''(t) \quad \Big(t \in \bigcup_{n\in\mathbb{N}_0} (t_n, t_{n+1}) \Big).$$

Then H_μ is self-adjoint and non-negative. In case (t_n) in $(0,\infty)$ we can also consider the corresponding halfline operators with Dirichlet boundary conditions at 0, then denoted by H_μ^+. Note that each A_k^+ in Proposition 2.1 can be associated to some atomic measure μ such that A_k^+ unitarily equivalent (by translation) to H_μ^+.

3 The Oracle Theorem

A signed Radon measure μ on $\mathbb{R}$ is said to be translation bounded, if

$$\|\mu\|_{\text{loc}} := \sup_{x\in\mathbb{R}} |\mu| ([x, x+1]) < \infty.$$

Let $\mathcal{M}_{\text{loc,unif}}(\mathbb{R})$ be the space of all translation bounded measures.

For an interval $I \subseteq \mathbb{R}$ and $C > 0$ let

$$\mathcal{M}^C(I) := \{ \mathbb{1}_I \mu;\ \mu \in \mathcal{M}_{\text{loc,unif}}(\mathbb{R}),\ \|\mu\|_{\text{loc}} \le C \}.$$

Equipped with the topology of vague convergence for measures $\mathcal{M}^C(I)$ is compact and hence metrizable (see, e.g., [11, Proposition 4.1.2]).

For $\gamma > 0$ let

$$\mathcal{M}_a^\gamma(\mathbb{R}) := \big\{ \mu \in \mathcal{M}_{\text{loc,unif}}(\mathbb{R});\ \mu \text{ nonegative and atomic},$$
$$|s - t| \ge \gamma \quad (s, t \in \operatorname{spt}\mu, s \ne t) \big\}.$$

Moreover, let

$$\mathcal{M}_a^{\gamma,+}(\mathbb{R}) := \{ \mu \in \mathcal{M}_a^\gamma(\mathbb{R});\ \operatorname{spt}\mu \subseteq [\gamma, \infty) \}.$$

We will also need the subsets

$$\mathcal{M}_a^{C,\gamma}(\mathbb{R}) := \mathcal{M}^C(\mathbb{R}) \cap \mathcal{M}_a^\gamma(\mathbb{R}), \quad \mathcal{M}_a^{C,\gamma,+}(\mathbb{R}) := \mathcal{M}^C(\mathbb{R}) \cap \mathcal{M}_a^{\gamma,+}(\mathbb{R}).$$

Remark 3.1. Let $\mu = \sum_{n\in\mathbb{N}} \beta_n \delta_{t_n} \in \mathcal{M}_a^{C,\gamma}(\mathbb{R})$, where $\beta_n = \frac{\sqrt{b_n}+1}{\sqrt{b_n}-1}$.

Let $s, t \in \mathbb{R} \setminus \mathrm{spt}\mu$, $s < t$. Let $u, v \in W_2^2((s,t) \setminus \mathrm{spt}\mu)$ satisfying

$$u(t_n+) = \sqrt{b_n} u(t_n-), \quad u'(t_n+) = \frac{1}{\sqrt{b_n}} u'(t_n-) \quad (t_n \in (s,t)),$$

and similarly for v. Then Green's formula holds in this case as well:

$$\int_s^t (-u'')(r)\overline{v}(r)\, dr - \int_s^t u(r)\overline{(-v'')}(r)\, dr = W(u,\overline{v})(t) - W(u,\overline{v})(s),$$

where $W(u,v)(t) = u'(t)v(t) - u(t)v'(t)$ is the Wronskian of u and v at t.

Remark 3.2. Let $\mu \in \mathcal{M}_a^{C,\gamma}(\mathbb{R})$, $z \in \mathbb{C}^+ := \{z \in \mathbb{C};\ \mathrm{Im}\, z > 0\}$.

(a) Let $t \in \mathbb{R} \setminus \mathrm{spt}\mu$. Let $u_N(z,\cdot)$, $u_D(z,\cdot)$ be the two formal solutions of $H_\mu u = zu$ satisfying Neumann and Dirichlet conditions at t, i.e.

$$\begin{aligned} u_N(z,t) &= 1 & \qquad u_D(z,t) &= 0 \\ u_N'(z,t) &= 0 & \qquad u_D'(z,t) &= 1. \end{aligned}$$

Let $b > t$. Consider the formal solution $u(z,\cdot)$ of $H_\mu u = zu$ satisfying a Robin condition with angle β at b, i.e.

$$\cos\beta u(z,b) + \sin\beta u'(z,b) = 0.$$

We can write $u(z,\cdot) = u_N(z,\cdot) + m(z,t,b,\mu)u_D(z,\cdot)$ Then

$$m(z,t,b,\mu) = -\frac{u_N(z,b)\cot\beta + u_N'(z,b)}{u_D(z,b)\cot\beta + u_D'(z,b)}.$$

Thus, $m(z,t,b,\mu)$ lies on the circle with center

$$M(z,b,\mu) := \frac{W(u_N(z,\cdot), \overline{u_D(z,\cdot)})(b)}{W(u_D(z,\cdot), \overline{u_D(z,\cdot)})(b)}.$$

and radius

$$r(z,b,\mu) := \left| \frac{W(u_N(z,\cdot), u_D(z,\cdot))(b)}{W(u_D(z,\cdot), \overline{u_D(z,\cdot)})(b)} \right| = \frac{1}{\left| W(u_D(z,\cdot), \overline{u_D(z,\cdot)})(b) \right|}.$$

By Green's formula we deduce that increasing the value b yields smaller and smaller circles, where the smaller one is contained in the larger one. As in the Schrödinger case (see, e.g., [4, Lemma III.1.4 and Corollary III.1.5]) one can show that $r(z,b,\mu) \to 0$ as $b \to \infty$, i.e., we are in the so-called limit point case.

Taking $b \to \infty$, one therefore obtains $m(z,t,b,\mu) \to m_+(z,t,\mu)$ which lies in the interior of the circle to the value b, for all $b > 0$. Analogous reasoning for $b \to -\infty$ yields a limit point $m_-(z,t,\mu)$. The limits are called the m-functions of H_μ.

(b) Similarly as in the case of Schrödinger operators one can show that there exist (unique up to multiplication by constants) formal solutions $u_\pm(z,\cdot)$ of $H_\mu u = zu$ lying in L_2 at $\pm\infty$. Then

$$m_\pm(z,t,\mu) = \pm\frac{u'_\pm(z,t)}{u_\pm(z,t)} \quad (t \in \mathbb{R} \setminus \operatorname{spt}\mu).$$

The m-functions are Herglotz functions, so its boundary values on the real line exist a.e. If $\mu \in \mathcal{M}_a^{C,\gamma,+}(\mathbb{R})$ then $\mathbb{C}^+ \ni z \mapsto m_+(z,0,\mu)$ depends only on the restriction of μ to $[0,\infty)$. Note that the m-functions contain spectral information of the operator in the following way: the set

$$\Sigma_{ac}(H_\mu^+) := \{E \in \mathbb{R};\ 0 < \operatorname{Im} m_+(E+i0,0,\mu) < \infty\}$$

is an essential support of the absolutely continuous spectrum of H_μ^+. Hence, H_μ^+ has absolutely continuous spectrum if and only if $\Sigma_{ac}(H_\mu^+)$ has positive measure.

Lemma 3.3. *Let $C,\gamma > 0$, (μ_n) in $\mathcal{M}_a^{C,\gamma,+}(\mathbb{R})$, $\mu \in \mathcal{M}_a^{C,\gamma}(\mathbb{R})$, $\mu_n \to \mu$ vaguely. Then, for $t \in \mathbb{R} \setminus \operatorname{spt}\mu$, we have $m_\pm(\cdot,t,\mu_n) \to m_\pm(\cdot,t,\mu)$ uniformly on compact subsets of $\mathbb{C}^+$.*

Proof. Since $\mu_n \to \mu$, we conclude that for $s \in \mathbb{R}$ there exists (s_n) such that $s_n \to s$ and $\mu_n(\{s_n\}) \to \mu(\{s\})$. Let $b \notin \bigcup_{n\in\mathbb{N}} \operatorname{spt}\mu_n \cup \operatorname{spt}\mu$ and consider the formal solutions of $H_{\mu_n} u = zu$ satisfying Neumann and Dirichlet solutions at t. Then these solutions and their derivatives at b converge locally uniformly in z to the corresponding Neumann and Dirichlet solutions of $H_\mu u = zu$. Thus, also the circle center's and radii as is in Remark 3.2 converge locally uniformly in z.

Now, let $\varepsilon > 0$ and $K \subseteq \mathbb{C}^+$ be compact. There exists $b > 0$ such that

$$\sup_{z\in K} r(z,b,\mu) < \varepsilon.$$

There exists $N \in \mathbb{N}$ such that for all $n \geq N$ we have

$$\sup_{z\in K} |M(z,b,\mu_n) - M(z,b,\mu)| < \varepsilon, \quad \sup_{z\in K} |r(z,b,\mu_n) - r(z,b,\mu)| < \varepsilon.$$

Thus,

$$\sup_{z\in K} |m_+(z,t,\mu_n) - m(z,t,\mu)| \leq 3\varepsilon \quad (n \geq N).$$

□

Remark 3.4. Let $\mathcal{H} := \{F\colon \mathbb{C}^+ \to \mathbb{C}^+;\ F \,\text{holomorphic}\}$ be the set of Herglotz functions. Let (F_n) be a sequence in $\mathcal{H}$, $F \in \mathcal{H}$.

(a) We say that $F_n \to F$ in *value distribution*, if

$$\lim_{n\to\infty} \int\limits_A \omega_{F_n(t)}(S)\, dt = \int\limits_A \omega_{F(t)}(S)\, dt$$

for all Borel sets $A, S \subseteq \mathbb{R}$ with $\lambda(A) < \infty$. Here, for $z = x + iy \in \mathbb{C}^+$ we have

$$\omega_z(S) := \frac{1}{\pi} \int\limits_S \frac{y}{(t-x)^2 + y^2}\, dt$$

and for $G \in \mathcal{H}$ and for a.a. $t \in \mathbb{R}$ we have

$$\omega_{G(t)}(S) := \lim_{y\to 0+} \omega_{G(t+iy)}(S).$$

Note that $G(t) := \lim_{y\to 0+} G(t+iy)$ exists for a.a. $t \in \mathbb{R}$. For these t we can define $\omega_{G(t)}(S)$ directly (which coincides with the definition via the limit).

(b) As shown in [10, Theorem 2.1] the convergence $F_n \to F$ in value distribution holds if and only if $F_n \to F$ uniformly on compact subsets of $\mathbb{C}^+$.

(c) Let $F, G \in \mathcal{H}$. Assume that $\omega_{F(t)}(S) = \omega_{G(t)}(S)$ for a.a. $t \in A$ and for all $S \subseteq \mathbb{R}$. Then $F = G$ a.e. on A, see [9, paragraph before Theorem 2.1].

Definition. Let $C, \gamma > 0$, $\mu \in \mathcal{M}_a^{C,\gamma}(\mathbb{R})$, $\Lambda \subseteq \mathbb{R}$ measurable. Then μ is called *reflectionless* on Λ, if

$$m_+(E + i0, t, \mu) = -\overline{m_-(E + i0, t, \mu)} \quad (\text{a.e. } E \in \Lambda) \tag{3}$$

for some $t \in \mathbb{R} \setminus \operatorname{spt}\mu$. Let $\mathcal{R}^{C,\gamma}(\Lambda) := \big\{\mu \in \mathcal{M}_a^{C,\gamma}(\mathbb{R});\ \mu\,\text{reflectionless on}\,\Lambda\big\}$ be the set of reflectionless (atomic) measures on Λ with parameters C and γ.

Note that if (3) holds for some $t \in \mathbb{R} \setminus \operatorname{spt}\mu$ then it automatically holds for all $t \in \mathbb{R} \setminus \operatorname{spt}\mu$.

Lemma 3.5. *Let $\mu \in \mathcal{M}_a^{C,\gamma}(\mathbb{R})$, $\mu(\{0\}) = 0$, $\Lambda \subseteq \mathbb{R}$ a Borel set. Then $\mu \in \mathcal{R}^{C,\gamma}(\Lambda)$ if and only if*

$$\int_B \omega_{m_-(E,0,\mu)}(-S)\, dE = \int_B \omega_{m_+(E,0,\mu))}(S)\, dE \tag{4}$$

for all Borel sets $B \subseteq \Lambda$, $\lambda(B) < \infty$ and $S \subseteq \mathbb{R}$.

Proof. Assume that $\mu \in \mathcal{R}^{C,\gamma}(\Lambda)$. Then $\operatorname{Im} m_\pm(E, 0, \mu) > 0$ for a.a. $E \in \Lambda$. Since μ is reflectionless on Λ, we have

$$m_+(E, 0, \mu) = -\overline{m_-(E, 0, \mu)}$$

for a.a. $E \in \Lambda$. Since $\omega_z(-S) = \omega_{-\overline{z}}(S)$ for all $z \in \mathbb{C}^+$, we obtain

$$\int_B \omega_{m_-(E,0,\mu)}(-S)\, dE = \int_B \omega_{m_+(E,0,\mu)}(S)\, dE.$$

On the other hand, assume (4). Lebesgue's differentiating theorem yields

$$\omega_{m_-(E,0,\mu)(-S)} = \omega_{m_+(E,0,\mu)}(S) \quad \text{(a.e. } E \in \Lambda).$$

Since $\omega_z(-S) = \omega_{-\overline{z}}(S)$ for all $z \in \mathbb{C}^+$ we obtain

$$\omega_{-\overline{m_-(E,0,\mu)}}(S) = \omega_{m_+(E,0,\mu)}(S) \quad \text{(a.e. } E \in \Lambda).$$

Thus, $m_+(E, 0, \mu) = -\overline{m_-(E, 0, \mu)}$ for a.a. $E \in \Lambda$. □

Next, we again consider measures on halflines:

$$\mathcal{M}_+^C := \left\{ \mathbb{1}_{(0,\infty)}\mu;\ \mu \in \mathcal{M}^C(\mathbb{R}) \right\},$$
$$\mathcal{M}_-^C := \left\{ \mathbb{1}_{(-\infty,0)}\mu;\ \mu \in \mathcal{M}^C(\mathbb{R}) \right\}.$$

Note that $\mathcal{M}_+^C$ can be identified with $\mathcal{M}^C((0, \infty))$, and we use the topology and metric d_+ from this space. Similar identifications apply to $\mathcal{M}_-^C$. Then the restriction maps $\mathcal{M}^C(\mathbb{R}) \to \mathcal{M}^C_\pm$ are continuous and thus $(\mathcal{M}^C_\pm, d_\pm)$ are compact. Furthermore, these restriction maps are injective. Let

$$\mathcal{R}^{C,\gamma}_+(\Lambda) := \left\{\mathbb{1}_{(0,\infty)}\mu;\ \mu \in \mathcal{R}^{C,\gamma}(\Lambda)\right\} \subseteq \mathcal{M}^C_+,$$
$$\mathcal{R}^{C,\gamma}_-(\Lambda) := \left\{\mathbb{1}_{(-\infty,0)}\mu;\ \mu \in \mathcal{R}^{C,\gamma}(\Lambda)\right\} \subseteq \mathcal{M}^C_-,$$

equipped with the metrics $d_\pm$.

We can now prove the analogon of [9, Proposition 2] (which was proven for the Schrödinger case) in our setting.

Proposition 3.6. *Let $\Lambda \subseteq \mathbb{R}$ be measurable and $C, \gamma > 0$. Then $(\mathcal{R}^{C,\gamma}(\Lambda), d)$ and $(\mathcal{R}^{C,\gamma}_\pm(\Lambda), d_\pm)$ are compact and the restriction maps $\mathcal{R}^{C,\gamma}(\Lambda) \to \mathcal{R}^{C,\gamma}_\pm(\Lambda)$ are homeomorphisms.*

Proof. It suffices to show that $\mathcal{R}^{C,\gamma}(\Lambda)$ is closed. Let (μ_n) in $\mathcal{R}^{C,\gamma}(\Lambda)$, $\mu \in \mathcal{M}^C(\mathbb{R})$ an atomic measure, $\mu_n \to \mu$. Then Lemma 3.3 yields $m_\pm(\cdot, 0, \mu_n) \to m_\pm(\cdot, 0, \mu)$ uniformly on compact subsets of $\mathbb{C}^+$. Thus, also $m_\pm(\cdot, 0, \mu_n) \to m_\pm(\cdot, 0, \mu)$ in value distribution. By Lemma 3.5, for $n \in \mathbb{N}$ we have

$$\int_B \omega_{m_-(E,0,\mu_n)}(-S)\, dE = \int_B \omega_{m_+(E,0,\mu_n)}(S)\, dE$$

for all Borel sets $B \subseteq \Lambda$, $\lambda(B) < \infty$ and $S \subseteq \mathbb{R}$. Taking the limit $n \to \infty$ yields

$$\int_B \omega_{m_-(E,0,\mu)}(-S)\, dE = \int_B \omega_{m_+(E,0,\mu)}(S)\, dE$$

for all Borel sets $B \subseteq \Lambda$, $\lambda(B) < \infty$ and $S \subseteq \mathbb{R}$. Again applying Lemma 3.5 we obtain $\mu \in \mathcal{R}^{C,\gamma}(\Lambda)$.

Since the restriction maps are continuous, $(\mathcal{R}^{C,\gamma}_\pm(\Lambda), d_\pm)$ are compact as continuous images of a compact space. Since the restriction maps are bijective and continuous between these compact metric spaces, their inverses are continuous as well. □

For $\mu \in \mathcal{M}^C(\mathbb{R})$ we write $S_x\mu := \mu(\cdot + x)$ for the translate by x. Then we can define the ω limit set of μ as

$$\omega(\mu) := \left\{\nu \in \mathcal{M}^C(\mathbb{R});\ \text{there exists } (x_n) \text{ in } \mathbb{R} : x_n \to \infty,\ d(S_{x_n}\mu, \nu) \to 0\right\}.$$

Note that if $\mu \in \mathcal{M}^{C,\gamma}_a(\mathbb{R})$, then $\omega(\mu) \subseteq \mathcal{M}^{C,\gamma}_a(\mathbb{R})$.

The following result was proven in [3, Theorem 16].

Proposition 3.7 ([3, Theorem 16])**.** *Let $C, \gamma > 0$, $\mu \in \mathcal{M}_a^{C,\gamma,+}(\mathbb{R})$. Then $\omega(\mu) \subseteq \mathcal{R}^{C,\gamma}(\Sigma_{ac}(H_\mu^+))$.*

We can now prove a version of Remling's Oracle Theorem ([9, Theorem 2]) for our setting.

Theorem 3.8. *Let $\Lambda \subseteq \mathbb{R}$ be a Borel set of positive Lebesgue measure, $\varepsilon > 0$, $a, b, \in \mathbb{R}$, $a < b$, $C > 0$. Then there exist $L > 0$ and a continuous function*

$$\triangle \colon \mathcal{M}^C(-L, 0) \to \mathcal{M}^C(a, b)$$

such that $\triangle$ maps atomic measures to atomic measures so that the following holds. If there is $\gamma > 0$ and $\mu \in \mathcal{M}_a^{C,\gamma,+}(\mathbb{R})$ such that $\Sigma_{ac}(H_\mu^+) \supseteq \Lambda$, then there exists $x_0 > 0$ so that for all $x \geq x_0$ we have

$$d(\triangle(\mathbb{1}_{(-L,0)} S_x \mu), \mathbb{1}_{(a,b)} S_x \mu) < \varepsilon.$$

Proof. We follow the proof of [9, Theorem 2], but replace the application of Theorem 3 in there (in step 4) by Proposition 3.7.

(i) By compactness it suffices to prove the statement for *some* metric d that generates the topology of vague convergence.

Let $J_- := (-L, 0)$ and $J_+ := (a, b)$. Let $d_\pm$ be the metric on $M^C(J_\pm)$. For $\mu \in \mathcal{M}^C(\mathbb{R})$ let $\mu_\pm$ be the restriction of μ to $J_\pm$.

Fix a metric d such that d dominates $d_\pm$: if $\mu, \nu \in \mathcal{M}^C(\mathbb{R})$ then

$$d_-(\mu_-, \nu_-) \leq d(\mu, \nu), \quad d_+(\mu_+, \nu_+) \leq d(\mu, \nu).$$

(ii) By Proposition 3.6 the mapping $\mathcal{R}_-^{C,\gamma}(\Lambda) \to \mathcal{R}_+^{C,\gamma}(\Lambda)$ is uniformly continuous. Hence, by the definition of the topologies we may find $L > 0$ and $0 < \delta < \varepsilon < 1$ such that if $\nu, \tilde{\nu} \in \mathcal{R}^{C,\gamma}(\Lambda)$, then

$$d_-(\nu_-, \tilde{\nu}_-) < 5\delta \Longrightarrow d_+(\nu_+, \tilde{\nu}_+) < \varepsilon^2.$$

(iii) The set

$$\mathcal{R}_{J_-}^{C,\gamma}(\Lambda) := \{\mu_-;\ \mu \in \mathcal{R}^{C,\gamma}(\Lambda)\}$$

is compact by Proposition 3.6. Since $\mathcal{M}^{C,\gamma}(J_-)$ is compact, the closed δ-neighborhood

$$\overline{U}_\delta = \{\mu_- \in \mathcal{M}^{C,\gamma}(J_-);\ \exists\, \nu \in \mathcal{R}^{C,\gamma}(\Lambda) : d_-(\mu_-, \nu_-) \leq \delta\}$$

is compact, too. Hence, there exist $\mathcal{F} \subseteq \mathcal{R}^{C,\gamma}(\Lambda)$ finite so that the balls of radius 2δ around $\mathcal{F}$ cover $\overline{U}_\delta$. Define $\triangle(\nu_-) := \nu_+$ for $\nu \in \mathcal{F}$.

(iv) For $\sigma \in \overline{U}_\delta$ define

$$\triangle(\sigma) := \frac{\sum_{\nu\in\mathcal{F}}(3\delta - d_-(\sigma,\nu_-))^+ \triangle(\nu_-)}{\sum_{\nu\in\mathcal{F}}(3\delta - d_-(\sigma,\nu_-))^+}.$$

Then $\triangle(\sigma)$ is atomic for $\sigma \in \overline{U}_\delta$ and $\triangle\colon \overline{U}_\delta \to \mathcal{M}^C(J_+)$ is continuous. Moreover, for all $\tilde\nu \in \mathcal{F}$ with $d_-(\sigma,\tilde\nu_-) < 2\delta$ we have $d_-(\nu_-,\tilde\nu_-) < 5\delta$ for all $\nu \in \mathcal{F}$ contributing to the sum. Thus, (ii) implies $d_+(\nu_+,\tilde\nu_+) < \varepsilon^2$ for these ν and by [9, Lemma 2] we obtain

$$d_+(\triangle(\sigma),\tilde\nu_+) < \varepsilon$$

for sufficiently small ε. Note that for every $\sigma \in \overline{U}_\delta$ there exists $\tilde\nu \in \mathcal{F}$ such that $d_-(\sigma,\tilde\nu_-) < 2\delta$.

So, $\triangle$ is defined on $\overline{U}_\delta$ and maps every $\sigma \in \overline{U}_\delta$ to some nonegative atomic measure $\triangle(\sigma)$. The extension theorem of Dugundji and Borsuk [2, Chapter II, Theorem 3.1] yields a continuous extension of $\triangle$ to $\mathcal{M}^C(J_-)$. Since this extension is obtained by convex combinations, $\triangle$ maps atomic measures to atomic measures.

(v) Now, choose $\mu \in \mathcal{M}_a^{C,\gamma,+}(\mathbb{R})$ with $\Sigma_{ac}(H_\mu^+) \supseteq \Lambda$. Then there exists $x_0 > 0$ such that

$$d(S_x\mu,\omega(\mu)) < \delta \quad (x \geq x_0),$$

i.e., for fixed $x \geq x_0$ there exists $\nu \in \omega(\mu)$ such that $d(S_x\mu,\nu) < \delta$. By Proposition 3.7 we have $\nu \in \mathcal{R}^{C,\gamma}(\Lambda)$. We thus obtain

$$d_\pm((S_x\mu)_\pm,\nu_\pm) < \delta.$$

Hence, $(S_x\mu)_- \in \overline{U}_\delta$, so there exists $\tilde\nu \in \mathcal{F}$ such that

$$d_\pm((S_x\mu)_-,\tilde\nu_-) < 2\delta.$$

By (iv) we obtain

$$d_+(\triangle((S_x\mu)_-),\tilde\nu_+) < \varepsilon.$$

Since also $d_-(\nu_-,\tilde\nu_-) < 3\delta$, (ii) implies $d_+(\nu_+,\tilde\nu_+) < \varepsilon^2$. Therefore, we finally conclude

$$d_-(\triangle((S_x\mu)_-),(S_x\mu)_+) < \delta + \varepsilon + \varepsilon^2 < 3\varepsilon.$$

□

4 Finite local complexity

Let us recall some definitions from [6].

Definition. A *piece* is a pair (ν, I) consisting of a left-closed right-open interval $I \subseteq \mathbb{R}$ with positive length $\lambda(I) > 0$ (which is then called the *length* of the piece) and a $\nu \in \mathcal{M}_{\text{loc,unif}}(\mathbb{R})$ supported on I. We abbreviate pieces by ν^I. A *finite piece* is a piece of finite length. We say ν^I *occurs* in a measure μ at $x \in \mathbb{R}$, if $\mathbb{1}_{x+I}\mu$ is a translate of ν.

The *concatenation* $\nu^I = \nu_1^{I_1} \mid \nu_2^{I_2} \mid \ldots$ of a finite or countable family $(\nu_j^{I_j})_{j\in N}$, with $N \subseteq \mathbb{N}$, of finite pieces is defined by

$$I = \left[\min I_1, \min I_1 + \sum_{j\in N} \lambda(I_j)\right),$$

$$\nu = \nu_1 + \sum_{j\in N,\, j\geq 2} \nu_j\Big(\,\cdot - \Big(\min I_1 + \sum_{k=1}^{j-1} \lambda(I_k) - \min I_j\Big)\Big).$$

We also say that ν^I is *decomposed* by $(\nu_j^{I_j})_{j\in N}$.

Definition. Let μ be a measure on $\mathbb{R}$. We say that μ has the *finite decomposition property* (f.d.p.), if there exist a finite set $\mathcal{P}$ of finite pieces (called the *local pieces*) and $x_0 \in \mathbb{R}$, such that $\mathbb{1}_{[x_0,\infty)}\mu^{[x_0,\infty)}$ is a translate of a concatenation $v_1^{I_1} \mid \nu_2^{I_2} \mid \ldots$ with $\nu_j^{I_j} \in \mathcal{P}$ for all $j \in \mathbb{N}$. Without restriction, we may assume that $\min I = 0$ for all $\nu^I \in \mathcal{P}$.

A measure μ has the *simple finite decomposition property* (s.f.d.p.), if it has the f.d.p. with a decomposition such that there is $\ell > 0$ with the following property: Assume that the two pieces

$$\nu_{-m}^{I_{-m}} \mid \ldots \mid \nu_0^{I_0} \mid \nu_1^{I_1} \mid \ldots \mid \nu_{m_1}^{I_{m_1}} \quad \text{and} \quad \nu_{-m}^{I_{-m}} \mid \ldots \mid \nu_0^{I_0} \mid \mu_1^{J_1} \mid \ldots \mid \mu_{m_2}^{J_{m_2}}$$

occur in the decomposition of μ with a common first part $\nu_{-m}^{I_{-m}} \mid \ldots \mid \nu_0^{I_0}$ of length at least ℓ and such that

$$\mathbb{1}_{[0,\ell)}(\nu_1^{I_1} \mid \ldots \mid \nu_{m_1}^{I_{m_1}}) = \mathbb{1}_{[0,\ell)}(\mu_1^{J_1} \mid \ldots \mid \mu_{m_2}^{J_{m_2}}),$$

where $\nu_j^{I_j}$, $\mu_k^{J_k}$ are pieces from the decomposition (in particular, all belong to $\mathcal{P}$ and start at 0) and the latter two concatenations are of lengths at least ℓ. Then

$$\nu_1^{I_1} = \mu_1^{J_1}.$$

Lemma 4.1. *Let (t_n) in $\mathbb{R}$, (b_n) in $(1,\infty)$, $\mu = \sum_{n=1}^{\infty} \beta_n \delta_{t_n}$ be an atomic measure, where $\beta_n = \frac{\sqrt{b_n}+1}{\sqrt{b_n}-1}$ $(n \in \mathbb{N})$. Then:*

(a) If (t_n) and (b_n) satisfy (1) and (2) then $\mu \in \mathcal{M}_{\mathrm{loc,unif}}(\mathbb{R})$.

(b) μ is eventually periodic if and only if $((t_{n+1} - t_n, b_n))$ is eventually periodic.

(c) μ has the s.f.d.p. if $\{t_{n+1} - t_n;\ n \in \mathbb{N}\}$ and $\{b_n;\ n \in \mathbb{N}\}$ are finite.

Proof. In order to prove (a) we remark that (β_n) is bounded by (2) and that

$$\|\mu\|_{\mathrm{loc}} \leq \frac{\sup_{n\in\mathbb{N}} \beta_n}{\inf_{n\in\mathbb{N}}(t_{n+1} - t_n)}.$$

Part (b) is clear by definition of μ. To prove (c) note that μ can be decomposed by $\mathcal{P} := \left\{(\mathbb{1}_{[t_n,t_{n+1})}\beta_n\delta_{t_n})(\cdot + t_n);\ n \in \mathbb{N}\right\}$ and this set is, by assumption, finite. To show that the decomposition is simple note that

$$\mathbb{1}_{[0,\ell)}(\nu_1^{I_1} \mid \ldots \mid \nu_{m_1}^{I_{m_1}}) = \mathbb{1}_{[0,\ell)}(\mu_1^{J_1} \mid \ldots \mid \mu_{m_2}^{J_{m_2}}),$$

where all the ν_j's and μ_j's are elements from $\mathcal{P}$ directly implies that $\nu_1^{I_1} = \mu_1^{J_1}$. □

Theorem 4.2. *Let (t_n) in $(0,\infty)$ and (b_n) satisfy (1) and (2). Assume that the corresponding measure μ has the s.f.d.p. Then, if H_μ^+ has nonempty absolutely continuous spectrum the measure μ is eventually periodic.*

Proof. Assume that $\sigma_{ac}(H_\mu^+)$ is nonempty. Then $\Sigma_{ac}(H_\mu^+)$ has positive measure. Let ℓ and $\mathcal{L}$ be the minimum and maximum of the length of the finitely many pieces of μ according to s.f.d.p., respectively, and $G := \{x_k;\ k \in \mathbb{N}_0\}$ be the grid. Let

$$\mathcal{D} := \left\{\mathbb{1}_{(-1,\ell)}(S_x\mu);\ x \in G\right\}.$$

Then $\mathcal{D}$ is finite and hence there exists $\varepsilon > 0$ such that $d(\nu_1,\nu_2) > 2\varepsilon$ for $\nu_1,\nu_2 \in \mathcal{D}$ with $\nu_1 \neq \nu_2$. Let $a := -1$ and $b := \ell$ and choose $L > \ell$ according to Theorem 3.8.

We construct a coarser grid $G_L \subseteq G$ by $y_0 := x_0$, $y_{k+1} - y_k \in [L, L+\mathcal{L}]$. Since $G \cap [y_k, y_{k+1}]$ is finite, also

$$\mathcal{P}_L := \left\{\left(S_{y_k}(\mathbb{1}_{[y_k,y_{k+1})}\mu), S_{y_k}([y_k,y_{k+1}))\right);\ k \in \mathbb{N}\right\}$$

is finite. Hence, there exists $k \in \mathbb{N}$ such that infinitely many translates of $\mathbb{1}_{[y_k, y_{k+1})}\mu$ occur in μ, so that the corresponding parts of G are translates of $G \cap [y_k, y_{k+1})$.

Let $z^1 := y_{m+1}$, where y_m is one of the corresponding points in G_L, $m > k$ and denote $y^1 := y_{k+1}$. By Theorem 3.8 applied with $x = y^1$ and $x = z^1$ we obtain

$$d(\mathbb{1}_{(-1,\ell)}(S_{y^1}\mu), \mathbb{1}_{(-1,\ell)}(S_{z^1}\mu)) \leq 2\varepsilon.$$

Hence, $\mathbb{1}_{(-1,\ell)}(S_{y^1}\mu) = \mathbb{1}_{(-1,\ell)}(S_{z^1}\mu)$ by the choice of ε. Moreover, we know that the pieces of μ starting at y_k and y_m, respectively, are decomposed in the same way. The s.f.d.p yields that the pieces starting at y^1 and z^1 are translates from each other, i.e., for $y^2 := \min G \cap (y^1, \infty)$ and $z^2 := \min G \cap (z^1, \infty)$ we have

$$z^2 - z^1 = y^2 - y^1, \quad \text{and} \quad S_{y^1}(\mathbb{1}_{[y^1,y^2)}\mu) = S_{z^1}(\mathbb{1}_{[z^1,z^2)}\mu).$$

Thus, $\mathbb{1}_{[y_k,y^2)}\mu$ is a translate of $\mathbb{1}_{[y_m,z^2)}\mu$. Iterating, we obtain sequences (y^n) and (z^n) in G such that $\mathbb{1}_{[y_k,y^n)}\mu$ is a translate of $\mathbb{1}_{[y_m,z^n)}\mu$ for all $n \in \mathbb{N}$. Since $y^{n+1} - y^n = z^{n+1} - z^n \geq \ell$ for all $n \in \mathbb{N}$ we obtain that $\mathbb{1}_{[y_k,\infty)}\mu$ is a translate of $\mathbb{1}_{[y_m,\infty)}\mu$, i.e., μ is eventually periodic. □

5 Absence of absolutely continuous spectrum on trees

We can now state our main theorem, which is the analogue of [3, Theorem 1] for radially symmetric metric tree graphs.

Theorem 5.1. *Let (t_n) in $(0,\infty)$ and (b_n) in $(1,\infty)$ satisfy (1) and (2). Assume that $\{t_{n+1} - t_n;\ n \in \mathbb{N}\}$ and $\{b_n;\ n \in \mathbb{N}\}$ are finite. Then, if H_Γ has nonempty absolutely continuous spectrum the sequence $((t_{n+1} - t_n, b_n))$ is eventually periodic.*

Proof. By Proposition 2.1 it suffices to prove the statement for all A_k^+, $k \geq 0$.

Note that, for $k \geq 0$, A_k^+ is unitarily equivalent to $H_{\mu_k}^+$, where $\mu_k := S_{t_k}(\mathbb{1}_{(t_k,\infty)}\mu)$ and μ is associated to $((t_n, b_n))$. Indeed, the unitary transformation is just the shift by t_k. Since μ has the s.f.d.p. by Lemma 4.1, clearly also μ_k has the s.f.d.p. for all $k \geq 0$. Theorem 4.2 proves that μ_k is eventually periodic for all $k \geq 0$. Thus, also μ is eventually periodic. By Lemma 4.1 we conclude that $((t_{n+1} - t_n, b_n))$ is eventually periodic. □

Remark 5.2. Kirchhoff conditions are the simplest but by far not the only way to couple the tree edges in a self-adjoint way. In particular, a wide class of boundary conditions on rooted radially symmetric metric tree graphs was considered in [5] and it was shown, in analogy with the corresponding result in [3], that if such a tree is sparse the absolutely continuous spectrum is absent. One can treat such boundary conditions also in the present context. With the help of [5, Theorem 6.4] one can prove an Oracle Theorem similar to Theorem 3.8 for these more general boundary conditions, and furthermore, one can associate with them two or three atomic measures [5, Section VI]; if all of them are s.f.d.p. and at least one is nontrivial, then one can derive the result on absence of absolutely continuous spectrum analogous to Theorem 5.1. We stress the nontriviality requirement: there is a subset of boundary conditions [5, Example 7.1] which are mapped by the unitary equivalence of Section 2 to free motion on the family of halflines, hence the absolutely continuous spectrum is present in such cases, in fact covering the whole positive halfline, irrespective of the edge lengths.

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