Microscopic Foundations of the Meißner Effect – Thermodynamic Aspects

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We analyze the Meißner effect from first principles of quantum mechanics. We show in particular the existence of superconducting states minimizing the magnetic free–energy of a BCS–like model and carrying surface currents which annihilate the total magnetic induction inside the bulk in the thermodynamic limit. This study is a step towards a complete explanation of the Meißner effect from a microscopic model. It remains indeed to prove that those states are dynamically stable, i.e., quasi–stationary at low temperatures. Note that our analysis shows that the Meißner effect is not necessarily related to an effective magnetic susceptibility equal to -1.

I. INTRODUCTION

The so–called Meißner (or Meißner–Ochsenfeld) effect was discovered in 1933 by the physicists W. Meißner and R. Ochsenfeld, twenty two years after the discovery of mercury superconductivity in 1911. This represented an important experimental breakthrough and demonstrated, among other things, that superconductors cannot be seen as perfect classical conductors. This effect is well–described by phenomenological theories like the celebrated London equations. We observe however that its microscopic origin is far from being fully understood almost eighty years later. In other words, there is no rigorous microscopic foundation of the Meißner effect starting from first principles of quantum mechanics only.

In our papers [1, 2] we have recently showed, from a microscopic theory, a weak version of the Meißner effect defined by the absence of magnetization in presence of superconductivity, provided the (space–homogeneous) external magnetic induction does not reach a critical value. [3, Section 6.2] extends these results to space–inhomogeneous magnetic inductions. Nevertheless, the (full) Meißner effect also includes the existence of currents, concentrated near the surface of the bulk, which annihilate the total magnetic induction inside the superconductor. This phenomenon has not been analyzed in [1–3]. Such a study is the main subject of the present paper.

We base our microscopic theory on the strong–coupling BCS–Hubbard model with a *self–generated* magnetic induction, which is driven by a space–inhomogeneous external magnetic induction. Indeed, the strong–coupling BCS–Hubbard model at fixed magnetic induction shows qualitatively the same density dependency of the critical temperature observed in high–T_c superconductors [1, 2]. Depending on the choice of parameters, properties of conventional superconductors are also qualitatively well–described by such a model. These assertions [1, 2] directly result from the method described in [4], which gives access to domains of the phase diagram usually difficult to reach via other standard mathematical tools.

The analysis of the (full) Meißner effect from first principles of quantum mechanics is highly non–trivial and in the present paper we provide results concerning the free–energy, taking into account contributions of the magnetic energy due to currents. Note indeed that Gibbs states of the model under consideration do not manifest currents, at least for space–homogeneous external magnetic inductions. By adding a magnetic term to the usual free–energy density (similar to [5, Eq. (2.11)]), our results show that the minimizers of this new magnetic free–energy can create surface currents which annihilate the total magnetic induction inside the bulk, in the thermodynamic limit. The corresponding Euler–Lagrange equations for these minimizers seem to indicate that an effective magnetic susceptibility equal to -1 is not the mechanism behind the Meißner effect.

Note that such magnetic free–energy minimizing states should be, in some sense, stable with respect to dynamics to be named equilibrium states. Their existence is only a necessary condition to have the Meißner effect. Indeed, also for high temperatures, we can have minimizers of the magnetic free–energy density suppressing the magnetic induction within the bulk. This comes from the fact that the finite volume system can produce any current density by creating local superconducting patches within a negligible volume. Therefore, we conjecture that the quantum dynamics rapidly destroys all currents in the non–superconducting phase. In particular, the second step will be to show the dynamical instability/stability of such a phenomenon in the non–superconducting/superconducting phase. Such an analysis is not performed here because it requires an extension of [4] to include dynamics. We postpone it to a further paper.

Finally, note that thermodynamic studies of the Meißner effect have been performed in [5, 6] from an axiomatic point of view. They are based on assumptions, not proven for some concrete microscopic model, like the existence of equilibrium states with off–diagonal long range order. In the same spirit, we also discuss some model independent conditions for the existence of the Meißner effect in Section VI.

The present paper is organized as follows. In Section II we set up the quantum many–body problem at fixed magnetic induction and give one important result con-

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cerning the possibility of having currents with no energy cost in the thermodynamic limit. Section III explains the Biot–Savart operator used to define magnetic inductions from currents. The *self–generated* magnetic induction is then discussed in Section IV to obtain a proper definition of the magnetic free–energy density. The Meißner effect is finally discussed in Sections V and VI. Section VII explains in detail all technical proofs required to show the assertions of previous sections. Our main result is Theorem 12.

Notation 1 (Norms)
For any $x = (x_1, x_2, x_3) \in \mathbb{R}^3$, $|x| := \sqrt{x_1^2 + x_2^2 + x_3^2}$. For any subset $\Lambda \subset \mathbb{Z}$, $|\Lambda|$ is by definition the cardinality of Λ. For any $p \in \mathbb{N}$, $\|-\|_p$ stands for the L^p–norm, whereas $\|-\|$ is the operator norm. Additionally, $\|-\|_{\mathrm{Tr}}$ denotes the trace norm.

II. THERMODYNAMIC STABILITY OF CURRENTS

The host material for superconducting electrons is assumed to be a (perfect) cubic crystal. Other lattices could also be studied, but for simplicity we refrain from considering them. The unit of length is chosen so that the lattice spacing in this crystal is exactly 1. We thus use $\mathbb{Z}^3$ to represent the crystal. Our microscopic theory is based on the strong coupling BCS–Hubbard model studied in [1, 2].

In absence of magnetic induction, it is defined in the box $\Lambda_l := \{\mathbb{Z} \cap [-l, l-1]\}^3$ of side length $2l$ for $l \in \mathbb{N}$ by the Hamiltonian

$$T_l := -\mu \sum_{x \in \Lambda_l} (n_{x,\uparrow} + n_{x,\downarrow}) + 2\lambda \sum_{x \in \Lambda_l} n_{x,\uparrow} n_{x,\downarrow}$$
$$- \frac{\gamma}{|\Lambda_l|} \sum_{x,y \in \Lambda_l} a_{x,\uparrow}^* a_{x,\downarrow}^* a_{y,\downarrow} a_{y,\uparrow} \qquad (1)$$

with real parameters $\mu, \lambda \in \mathbb{R}$ and $\gamma \in \mathbb{R}^+$ (i.e., $\gamma > 0$). The operator $a_{x,s}^*$ resp. $a_{x,s}$ creates resp. annihilates a fermion with spin $s \in \{\uparrow, \downarrow\}$ at lattice position $x \in \mathbb{Z}^3$ whereas $n_{x,s} := a_{x,s}^* a_{x,s}$ is the particle number operator at position x and spin s.

The first term of the right hand side of (1) represents the strong coupling limit of the kinetic energy, with μ being the chemical potential of the system. Note that this "strong coupling limit" is also called "atomic limit" in the context of the Hubbard model. See, e.g., [7, 8]. For further discussions, we also strongly recommend Section VI (cf. **2.**).

The second term in the right hand side of (1) represents the (screened) Coulomb repulsion as in the celebrated Hubbard model. So, the parameter λ should be taken as a positive number but our results are also valid for any real Hubbard coupling $\lambda \in \mathbb{R}$.

The last term is the BCS interaction written in the

x–space since

$$\frac{\gamma}{|\Lambda_l|} \sum_{x,y \in \Lambda_l} a_{x,\uparrow}^* a_{x,\downarrow}^* a_{y,\downarrow} a_{y,\uparrow} = \frac{\gamma}{|\Lambda_l|} \sum_{k,q \in \Lambda_l^*} \tilde{a}_{k,\uparrow}^* \tilde{a}_{-k,\downarrow}^* \tilde{a}_{q,\downarrow} \tilde{a}_{-q,\uparrow}$$

with Λ_l^* being the reciprocal lattice of quasi–momenta and where $\tilde{a}_{q,s}$ is the corresponding annihilation operator for $s \in \{\uparrow, \downarrow\}$ and $q \in \Lambda_l^*$. Observe that the thermodynamics of the model for $\gamma = 0$ can easily be computed. Therefore, we restrict the analysis to the case $\gamma \in \mathbb{R}^+$. Note also that the BCS interaction can imply a superconducting phase. The mediator implying this effective interaction does not matter here, i.e., it could be a phonon, as in conventional type I superconductors, or anything else.

We now fix a magnetic induction $B \in L^2(\mathfrak{C}; \mathbb{R}^3)$, where $\mathfrak{C} := [-1/2, 1/2]^3$. The strong coupling BCS–Hubbard model with space–inhomogeneous magnetic induction is then defined by $H_l := T_l + \mathcal{M}_l$ with

$$\mathcal{M}_l := -\vartheta \sum_{x \in \Lambda_l} \left(a_{x,\uparrow}^* a_{x,\downarrow} + a_{x,\downarrow}^* a_{x,\uparrow} \right) \int_{\mathfrak{C}} b_1 \left(\frac{x+y}{2l} \right) d^3 y$$

$$+ i\vartheta \sum_{x \in \Lambda_l} \left(a_{x,\uparrow}^* a_{x,\downarrow} - a_{x,\downarrow}^* a_{x,\uparrow} \right) \int_{\mathfrak{C}} b_2 \left(\frac{x+y}{2l} \right) d^3 y$$

$$- \vartheta \sum_{x \in \Lambda_l} (n_{x,\uparrow} - n_{x,\downarrow}) \int_{\mathfrak{C}} b_3 \left(\frac{x+y}{2l} \right) d^3 y \qquad (2)$$

for any fixed parameter $\vartheta \in \mathbb{R}^+$ ($\vartheta > 0$) and

$$B(\mathfrak{t}) \equiv (b_1(\mathfrak{t}), b_2(\mathfrak{t}), b_3(\mathfrak{t})) \in \mathbb{R}^3$$

for $\mathfrak{t} \in \mathfrak{C}$ almost everywhere (a.e.). Indeed, the terms of $\mathcal{M}_l$ correspond to the interaction between spins and the total magnetic induction $B((x+y)/(2l))$ within a unit cell $\mathfrak{C}$ around $x \in \Lambda_l$.

Note that, for continuous fields $B \in C^0(\mathfrak{C}; \mathbb{R}^3)$ and in the thermodynamic limit $l \to \infty$,

$$\int_{\mathfrak{C}} B \left(\frac{x+y}{2l} \right) d^3 y = |\Lambda_l| \int_{(2l)^{-1}\mathfrak{C}} B \left(\frac{x}{2l} + \mathfrak{t} \right) d^3 \mathfrak{t}$$
$$= B \left(\frac{x}{2l} \right) + o(1) . \qquad (3)$$

If B is continuous, then we can equivalently take either (a) the integral of $B((x+y)/(2l))$ (with respect to y) in the unit cell $\mathfrak{C}$ or (b) the value $B(x/(2l))$ in the definition of $\mathcal{M}_l$. In fact, the thermodynamic limit of both systems (a)–(b) are identical for continuous magnetic inductions $B \in C^0(\mathfrak{C}; \mathbb{R}^3)$ (cf. Section VII), but an extension of our results to all $B \in L^2(\mathfrak{C}; \mathbb{R}^3)$ leads us to consider the definition (a) in (2) and not (b). Additionally, (a) is also more natural if one considers B as an effective field coming from a quantum magnetic induction. See, e.g., [4].

The scaling factor $(2l)^{-1}$ used in (2) means that the space fluctuations of the inhomogeneous magnetic induction involve a macroscopic number of lattice sites. This obviously does not prevent the space scale of these fluctuations from being extremely small as compared to the

side–length $2l$ of the box Λ_l. Similarly, we could also model mesoscopic fluctuations meaning that – in the thermodynamic limit – the space scale of inhomogeneities is infinitesimal with respect to the box side–length $2l$ whereas the lattice spacing is infinitesimal with respect to the space scale of inhomogeneities. See, e.g., [3, Sect. 5]. Microscopic fluctuations can also be handled provided they are periodic, see [3, Sect. 3]. Both situations (or any combination of them with the macroscopic one) are however omitted to simplify discussions and proofs.

We observe that T_l, $\mathcal{M}_l$ and $H_l = T_l + \mathcal{M}_l$ belong to the CAR C^*–algebra $\mathcal{U}_{\Lambda_l}$ with identity $\mathbf{1}$ and generators $\{a_{x,s}\}_{x \in \Lambda_l, s \in \{\uparrow,\downarrow\}}$ satisfying the canonical anti–commutation relations (CAR):

$$\begin{cases} a_{x,s} a_{x',s'} + a_{x',s'} a_{x,s} = 0 \ , \\ a_{x,s} a_{x',s'}^* + a_{x',s'}^* a_{x,s} = \delta_{x,x'} \delta_{s,s'} \mathbf{1} \ . \end{cases} \quad (4)$$

$\mathcal{U}_{\Lambda_l}$ is isomorphic to the C^*–algebra $\mathcal{L}(\bigwedge \mathcal{H}_{\Lambda_l})$ of all linear operators on the fermion Fock space $\bigwedge \mathcal{H}_{\Lambda_l}$, where

$$\mathcal{H}_{\Lambda_l} := \bigoplus_{x \in \Lambda_l} \mathcal{H}_x \ . \quad (5)$$

Here, for every $x \in \mathbb{Z}^3$, $\mathcal{H}_x$ is a copy of some fixed two dimensional Hilbert space $\mathcal{H}$ with orthonormal basis $\{|\uparrow\rangle, |\downarrow\rangle\}$. States on the C^*–algebra $\mathcal{U}_{\Lambda_l}$ are linear functionals $\rho \in \mathcal{U}_{\Lambda_l}^*$ which are positive, i.e., for all $A \in \mathcal{U}_{\Lambda_l}$, $\rho(A^*A) \geq 0$, and normalized, i.e., $\rho(\mathbf{1}) = 1$. We denote by $E_{\Lambda_l} \subset \mathcal{U}_{\Lambda_l}^*$ the set of all states on $\mathcal{U}_{\Lambda_l}$ for any $l \in \mathbb{N}$.

It is well–known that the physics of the system at thermodynamical equilibrium is given by the Gibbs state $\mathfrak{g}_l \in E_{\Lambda_l}$ defined by

$$\mathfrak{g}_l(A) := \mathrm{Trace}_{\wedge \mathcal{H}_{\Lambda_l}}(\mathrm{d}_{\mathfrak{g}_l} A) \ , \quad A \in \mathcal{U}_{\Lambda_l} \ , \quad (6)$$

with density matrix

$$\mathrm{d}_{\mathfrak{g}_l} := \frac{\mathrm{e}^{-\beta H_l}}{\mathrm{Trace}_{\wedge \mathcal{H}_{\Lambda_l}}(\mathrm{e}^{-\beta H_l})} \quad (7)$$

for any inverse temperature $\beta \in \mathbb{R}^+$ and $l \in \mathbb{N}$. Indeed, given any state $\rho \in E_{\Lambda_l}$ on $\mathcal{U}_{\Lambda_l}$, the energy observable $H_l = H_l^* \in \mathcal{U}_{\Lambda_l}$ fixes the finite volume free–energy density

$$f_l(\mathrm{B}, \rho) := |\Lambda_l|^{-1} \left\{ \rho(H_l) - \beta^{-1} S_l(\rho) \right\} \quad (8)$$

at fixed magnetic induction $\mathrm{B} \in L^2(\mathfrak{C}; \mathbb{R}^3)$ and inverse temperature $\beta \in \mathbb{R}^+$ for any $l \in \mathbb{N}$. If $\mathrm{B} \in L^2 \equiv L^2(\mathbb{R}^3; \mathbb{R}^3)$, then we set $f_l(\mathrm{B}, \rho) \equiv f_l(\mathrm{B}|_{\mathfrak{C}}, \rho)$. The first term in f_l is the mean energy per unit of volume of the physical system found in the state $\rho \in E_{\Lambda_l}$, whereas S_l is the von Neumann entropy defined, for all $\rho \in E_{\Lambda_l}$, by

$$S_l(\rho) := \mathrm{Trace}_{\wedge \mathcal{H}_{\Lambda_l}}(\eta(\mathrm{d}_\rho)) \geq 0 \ . \quad (9)$$

Here, $\eta(t) := -t \log(t)$ for $t \in \mathbb{R}^+$, $\eta(0) := 0$, and d_ρ is the density matrix of $\rho \in E_{\Lambda_l}$. The state of a system in thermal equilibrium and at fixed mean energy per

unit of volume maximizes the entropy, by the second law of thermodynamics. Therefore, it minimizes the free–energy density functional $\rho \mapsto f_l(\mathrm{B}, \rho)$. Such well–known arguments lead to the study of the variational problem $\inf f_l(\mathrm{B}, E_{\Lambda_l})$. The value of this variational problem is directly related to the so–called pressure $p_l(\mathrm{B})$ as

$$\begin{aligned} p_l(\mathrm{B}) := & (\beta |\Lambda_l|)^{-1} \ln \mathrm{Trace}_{\wedge \mathcal{H}_{\Lambda_l}}(\mathrm{e}^{-\beta H_l}) \\ = & -\inf_{\rho \in E_{\Lambda_l}} f_l(\mathrm{B}, \rho) \end{aligned} \quad (10)$$

for any magnetic induction $\mathrm{B} \in L^2(\mathfrak{C}; \mathbb{R}^3)$. (If $\mathrm{B} \in L^2$, then $p_l(\mathrm{B}) \equiv p_l(\mathrm{B}|_{\mathfrak{C}})$.) For any $\beta \in \mathbb{R}^+$ and $l \in \mathbb{N}$, the unique solution of this variational problem is precisely the Gibbs state $\mathfrak{g}_l \in E_{\Lambda_l}$ (6)–(7). This fact is named in the literature the passivity of Gibbs states and is a consequence of Jensen's inequality.

Our microscopic approach to the Meißner effect requires a definition of (charged) currents. Indeed, we would like to study the existence of currents near the surface of the bulk of the model. To this end, we note that, for all $x \in \Lambda_l$,

$$\begin{aligned} & \frac{\mathrm{d}}{\mathrm{d}t} \left\{ \mathrm{e}^{itH_l}(n_{x,\uparrow} + n_{x,\downarrow}) \mathrm{e}^{-itH_l} \right\} \\ & = \mathrm{e}^{itH_l} i[H_l, n_{x,\uparrow} + n_{x,\downarrow}] \mathrm{e}^{-itH_l} \end{aligned}$$

and

$$i[H_l, n_{x,\uparrow} + n_{x,\downarrow}] = \sum_{y \in \Lambda_l} \frac{4\gamma}{|\Lambda_l|} \mathrm{Im}\left(a_{y,\uparrow}^* a_{y,\downarrow}^* a_{x,\downarrow} a_{x,\uparrow}\right) \ .$$

The quantum observable describing the (charged) current from x to y is thus defined by

$$\mathrm{I}_l^{x,y} := \frac{4\gamma}{|\Lambda_l|} \mathrm{Im}\left(a_{x,\uparrow}^* a_{x,\downarrow}^* a_{y,\downarrow} a_{y,\uparrow}\right) \quad (11)$$

for any $x, y \in \Lambda_l$. These current observables naturally give rise to a magnetic induction functional which we define below by using the Biot–Savart law.

Indeed, given any state $\rho \in E_{\Lambda_l}$, we interpret the real number $\rho(\mathrm{I}_l^{x,y})$ as the current passing from x to y. We use this observation to define a current density induced by the system in the state ρ. One expects that the full current $\rho(\mathrm{I}_l^{x,y})$ between x and y is smoothly distributed in some region of size $|x - y|$ around $(x + y)/2$. The current profile is fixed by an arbitrary smooth, compactly supported, spherical symmetric and non–negative function $\xi \in C_0^\infty \equiv C_0^\infty(\mathbb{R}^3; \mathbb{R}^3)$ such that $\xi(0) > 0$,

$$\int_{\mathbb{R}^3} \xi(\mathfrak{t}) \, \mathrm{d}^3 \mathfrak{t} = 1 \quad \text{and} \quad \int_{\mathbb{R}^2} \xi(0, \mathfrak{t}_2, \mathfrak{t}_3) \, \mathrm{d}\mathfrak{t}_2 \mathrm{d}\mathfrak{t}_3 = 1 \ . \quad (12)$$

For any $l \in \mathbb{N}$, the current density induced by the system in the state $\rho \in E_{\Lambda_l}$ at $x \in \mathbb{R}^3$ is defined by

$$\rho \mapsto j_\rho(x) := \sum_{y,z \in \Lambda_l, \ y \neq z} \frac{z - y}{|z - y|^3} \xi\left(\frac{x - \frac{y+z}{2}}{|z - y|}\right) \rho(\mathrm{I}_l^{y,z}) \ . \quad (13)$$

It defines a map j from E_{Λ_l} to the real vector space C_0^∞ of compactly supported smooth fields. This map is named here the current density functional of the box Λ_l. Observe that the second condition of (12) ensures that the flow of the field

$$\frac{z-y}{|z-y|^3}\,\xi\left(\frac{x-\frac{y+z}{2}}{|z-y|}\right)\rho\,(\mathrm{I}_l^{y,z})$$

through the hyperplane perpendicular to $z-y$ at $(y+z)/2$ equals the full current $\rho\,(\mathrm{I}_l^{y,z})$ passing from y to z.

As we are interested in magnetic effects induced by the quantum system, for any state $\rho \in E_{\Lambda_l}$, we shall consider the smooth magnetic induction $B_\rho \in C^\infty \equiv C^\infty(\mathbb{R}^3;\mathbb{R}^3)$ created by the current density j_ρ together with some fixed external magnetic induction $\mathrm{B}_{\mathrm{ext}}$. Its definition uses the Biot–Savart operator $\mathcal{S}$ (Section III) and requires further explanations given in Section IV. We only note at this point that the smooth magnetic induction B_ρ has to be rescaled in order to be compared to B, see (2). We thus define the rescaled magnetic induction $B_\rho^{(l)} \in C^\infty$ by

$$B_\rho^{(l)}\,(\mathfrak{t}) := B_\rho\,(2l\mathfrak{t})\ ,\quad \mathfrak{t} \in \mathbb{R}^3\ , \tag{14}$$

for all $l \in \mathbb{N}$. Keeping in mind the Biot–Savart law (cf. (47)), we similarly need to define a rescaled current density $j_\rho^{(l)} \in C_0^\infty$ from the current density j_ρ as follows:

$$j_\rho^{(l)}(\mathfrak{t}) := 2l\, j_\rho(2l\mathfrak{t})\ ,\quad \mathfrak{t} \in \mathbb{R}^3\ , \tag{15}$$

for all $l \in \mathbb{N}$ and $\rho \in E_{\Lambda_l}$. Here, the support

$$\mathrm{supp}(j_\rho^{(l)}) := \{\mathfrak{t} \in \mathbb{R}^3\ :\ j_\rho^{(l)}(\mathfrak{t}) \neq 0\} \tag{16}$$

of $j_\rho^{(l)}$ is contained in a sufficiently large box $[-L,L]^3 \supset \mathfrak{C}$ which depends on the size of the support of the function ξ used in the definition of j_ρ but not on the length $l \in \mathbb{N}$. However, because of the prefactor $|\Lambda_l|^{-1}$ in the definition (11) of current observables, $j_\rho^{(l)}$ is strongly concentrated inside $\mathfrak{C}$, as $l \to \infty$.

The system also shows magnetization due to spinning charged particles, electrons in our case. The magnetization observables are seen as coordinates of an observable vector $\mathrm{M}^x := (\mathrm{m}_1^x, \mathrm{m}_2^x, \mathrm{m}_3^x)$ where, for all $x \in \mathbb{Z}^3$,

$$\begin{aligned}
\mathrm{m}_1^x &:= \vartheta\left(a_{x,\uparrow}^* a_{x,\downarrow} + a_{x,\downarrow}^* a_{x,\uparrow}\right)\ ,\\
\mathrm{m}_2^x &:= i\vartheta(a_{x,\downarrow}^* a_{x,\uparrow} - a_{x,\uparrow}^* a_{x,\downarrow})\ ,\\
\mathrm{m}_3^x &:= \vartheta\,(n_{x,\uparrow} - n_{x,\downarrow})\ .
\end{aligned} \tag{17}$$

For any strictly positive fixed parameter $\epsilon \in \mathbb{R}^+$, let

$$\xi_\epsilon(\mathfrak{t}) := \epsilon^{-3}\,\xi\left(\epsilon^{-1}\mathfrak{t}\right)\ ,\quad \mathfrak{t} \in \mathbb{R}^3\ . \tag{18}$$

Then, for any $l \in \mathbb{N}$ and $\rho \in E_{\Lambda_l}$, we define the coarse–grained magnetization density at $x \in \mathbb{R}^3$ by

$$\rho \mapsto m_\rho\,(x) := \frac{1}{|\Lambda_l|}\sum_{\substack{y \in \Lambda_l \\ (2l)^{-1}y+\mathrm{supp}(\Xi_\epsilon)\subset\mathfrak{C}}} \Xi_\epsilon\left(\frac{x-y}{2l}\right)\rho\,(\mathrm{M}^y)\ , \tag{19}$$

where

$$\Xi_\epsilon\,(\mathfrak{t}) := \int_{\mathfrak{C}}\xi_\epsilon\left(\mathfrak{t}-\frac{z}{2l}\right)\mathrm{d}z\ ,\quad \mathfrak{t} \in \mathbb{R}^3\ ,$$

for any $\epsilon \in \mathbb{R}^+$, whereas

$$\rho\,(\mathrm{M}^x) := \left(\rho(\mathrm{m}_1^x)\ ,\ \rho(\mathrm{m}_2^x),\ \rho(\mathrm{m}_3^x)\right) \in \mathbb{R}^3\ . \tag{20}$$

It is again a map from E_{Λ_l} to C_0^∞. Similar to the rescaled magnetic induction $B_\rho^{(l)} \in C^\infty$, the rescaled magnetization density is defined by

$$m_\rho^{(l)}\,(\mathfrak{t}) := m_\rho\,(2l\mathfrak{t})\ ,\quad \mathfrak{t} \in \mathbb{R}^3\ , \tag{21}$$

for any state $\rho \in E_{\Lambda_l}$.

The use of Ξ_ϵ in the definition of the magnetization density is technically convenient but not essential for our analysis. It is only a specific choice of a function with integral equal to 1 that implements the coarse–graining of the magnetization. The first condition of (12) ensures indeed that the full magnetization produced by one lattice site $x \in \Lambda_l$ equals $\rho\,(\mathrm{M}^x)$.

The restriction

$$(2l)^{-1}\,y + \mathrm{supp}\,(\Xi_\epsilon) \subset \mathfrak{C}$$

in the definition of m_ρ guarantees that $\mathrm{supp}(m_\rho^{(l)}) \subset \mathfrak{C}$. This is also technically convenient. We add that the scaling factor $(2l)^{-1}$ in (19) means that $m_\rho^{(l)} \in C_0^\infty$ is a *macroscopic* magnetization, used in Section IV to define the (also macroscopic) self–generated magnetic induction $B_\rho^{(l)}$ via the Maxwell equations in matter.

Remark 2 (Coarse–graining of the magnetization)
The coarse–grained magnetization density $m_\rho^{(l)}$ is defined for all $\epsilon \in \mathbb{R}^+$. However, we are only interested in the case where the space–scale of the coarse–graining of $m_\rho^{(l)}$ is very small as compared to the side length of the unit box $\mathfrak{C}$. This corresponds to take $\epsilon << \epsilon_\xi := 1/(2R_\xi)$ with

$$R_\xi := \sup\{|x| : \xi\,(x) \neq 0\} \in \mathbb{R}^+ \tag{22}$$

being the radius of the support of the function $\xi \in C_0^\infty$.

Remark 3 (Smooth from discrete)
The quantum many–body problem considered here uses discrete space coordinates. On the other hand, the Maxwell equations require differentiable fields. We have thus defined smooth magnetization and current densities m_ρ, j_ρ on $\mathbb{R}^3$. The latter is done without introducing any arbitrariness in our thermodynamic results since we take $\epsilon \to 0^+$ after the thermodynamic limit $l \to \infty$. The thermodynamics then becomes independent of the choice of $\xi \in C_0^\infty$.

We give now one of our main (technical) result about the creation of any smooth current density without energy costs:

Theorem 4 (Thermodynamic stability of currents)
For every $B \in L^2(\mathfrak{C}; \mathbb{R}^3)$ *and any smooth current density* $j \in C^\infty(\mathfrak{C}; \mathbb{R}^3)$, *there are states* $\rho_l \in E_{\Lambda_l}$ *for* $l \in \mathbb{N}$ *satisfying*

$$\lim_{l \to \infty} \left| f_l(B, \rho_l) - \inf_{\rho \in E_{\Lambda_l}} f_l(B, \rho) \right| = 0$$

(cf. (8)–(9)) as well as

$$\lim_{l \to \infty} \sup_{\mathfrak{t} \in \mathbb{R}^3} \left| j^{(l)}_{\rho_l}(\mathfrak{t}) - j(\mathfrak{t}) \right| = 0 .$$

Proof. The proof is a direct consequence of Lemmata 19 and 20 with, for instance, $\eta^\perp = 0.8$ and $\eta = 0.95$. Indeed, for any $j_1 \in C^\infty_0(\mathbb{R}^3; \mathbb{R})$, we construct in Lemma 20 a sequence $\{\rho_l\}_{l \in \mathbb{N}}$ of approximating minimizers which creates in the thermodynamic limit a current density $(j_1(\mathfrak{t}), 0, 0)$ at rescaled (macroscopic) position $\mathfrak{t} \in \mathfrak{C}$. In the same way, one constructs a sequence of approximating minimizers which creates in the thermodynamic limit a current density $(0, j_2(\mathfrak{t}), 0)$ or $(0, 0, j_3(\mathfrak{t}))$ at $\mathfrak{t} \in \mathfrak{C}$. Using the convexity of the free–energy density and the affinity of the current density functional $\rho \mapsto j_\rho$ together with a convex combination of such three approximating minimizers, one proves the assertion. $\square$

Remark 5 (Dynamical stability of currents)
The proof of Theorem 4 is based on the existence of mesoscopic superconducting domains in the (macroscopic) bulk, see (111)–(113). Therefore, we expect that the quantum dynamics rapidly destroys all currents j_{ρ_l} in the non–superconducting phase, even if this dynamics does not change the free–energy density.

Theorem 4 shows that the macroscopic system can create any smooth current density j by paying an infinitesimal energy price in the thermodynamic limit. Indeed, $\{\rho_l\}_{l \in \mathbb{N}}$ is a sequence of approximating minimizers of the free–energy density functional $\rho \mapsto f_l(B, \rho)$ in the thermodynamic limit $l \to \infty$. Therefore, one could use this phenomenon to create currents near the surface of the bulk which annihilate the magnetic induction inside the box Λ_l. This fact suggests the existence of a Meißner effect for the model under consideration.

Note however that the true minimizer of the free energy density, i.e., the Gibbs state (6)–(7), does *not* manifest any current, at least for space–homogeneous magnetic inductions B on $\mathfrak{C}$. This can be seen by using the symmetry properties of the model H_l. Therefore, one should also take into account the energy carried by the total magnetic induction, by adding a magnetic term to the free–energy density. Minimizers of this new magnetic free–energy density functional do carry currents, in general.

Such a magnetic term is introduced in Section IV, after the definition of the Biot–Savart operator $\mathcal{S}$ given in the next section.

III. THE BIOT–SAVART OPERATOR

The energy contained in the static configuration B of the total magnetic induction is given as usual by

$$E_{\mathrm{mag}}(B) := \frac{1}{2} \|B\|_2^2 := \frac{1}{2} \int_{\mathbb{R}^3} |B(\mathfrak{t})|^2 \, \mathrm{d}^3\mathfrak{t} . \tag{23}$$

See, e.g., [9, Chap. 5, 6] for an interesting derivation of the magnetic energy. As a consequence, we only consider magnetic inductions B which belong to the (real) Hilbert space $L^2 \equiv L^2(\mathbb{R}^3; \mathbb{R}^3)$ with L^2–norm $\| - \|_2$ and scalar product defined by

$$\langle B_1, B_2 \rangle_2 := \int_{\mathbb{R}^3} B_1(\mathfrak{t}) \cdot B_2(\mathfrak{t}) \, \mathrm{d}^3\mathfrak{t} , \quad B_1, B_2 \in L^2 . \tag{24}$$

A dense set of L^2 is of course given by the real vector space $C^\infty_0 \equiv C^\infty_0(\mathbb{R}^3; \mathbb{R}^3)$ of compactly supported smooth fields.

Units have here been chosen so that the magnetic permeability of free space equals 1, keeping in mind that the unit of length is already fixed to have a lattice spacing also equal to 1. Hence, a static magnetic induction B and a current density j satisfy in our units the Maxwell equation $\nabla \times B = j$.

Now therefore, the natural Hilbert space $\mathfrak{H}$ of current densities is defined as follows:

$$\langle j_1, j_2 \rangle_{\mathfrak{H}} := \frac{1}{4\pi} \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \frac{j_1(\mathfrak{t}) \cdot j_2(\mathfrak{s})}{|\mathfrak{t} - \mathfrak{s}|} \mathrm{d}^3\mathfrak{t} \, \mathrm{d}^3\mathfrak{s}, \quad j_1, j_2 \in C^\infty_0,$$

defines a (energy) scalar product in the real vector space C^∞_0 of compactly supported smooth current densities. This is easily seen by using the Fourier transform F. In particular, the (magnetic energy) norm

$$\|j\|_{\mathfrak{H}} := \langle j, j \rangle_{\mathfrak{H}}^{1/2} , \quad j \in C^\infty_0 , \tag{25}$$

clearly satisfies the parallelogram identity and we define the Hilbert space $\mathfrak{H} \equiv (\mathfrak{H}, \langle -, - \rangle_{\mathfrak{H}})$ to be the completion of $(C^\infty_0, \langle -, - \rangle_{\mathfrak{H}})$. The divergence–free subspaces of respectively $\mathfrak{H}$ and L^2 are isomorphic as Hilbert spaces. One natural isomorphism is given by the *Biot–Savart operator* $\mathcal{S}$ defined below.

Observe that the Fourier transform F defines a unitary map from $\mathfrak{H}$ to $L^2(\mathbb{R}^3, |k|^{-2} \mathrm{d}^3k; \mathbb{R}^3)$. Since

$$L^2(\mathbb{R}^3, |k|^{-2}\mathrm{d}^3k; \mathbb{R}^3) \hookrightarrow L^2(\mathbb{R}^3, (|k|^2 + 1)^{-1}\mathrm{d}^3k; \mathbb{R}^3) ,$$

$\mathfrak{H}$ can be seen as a subspace of distributions in $W^{-1,2}(\mathbb{R}^3; \mathbb{R}^3)$, where $W^{-1,2}(\mathbb{R}^3; \mathbb{R}^3)$ is the dual of the Sobolev space $W^{1,2}(\mathbb{R}^3; \mathbb{R}^3)$. The energy interpretation of the expression defining the norm $\| - \|_{\mathfrak{H}}$ above is well–known (see (38) below). Nevertheless, remark that, at least to our knowledge, the space $\mathfrak{H}$ does not seem to have been previously used in a similar context.

Note further that, for any arbitrary smooth compactly supported field $\Psi \in C^\infty_0$, there is a unique (Helmholtz)

decomposition $\Psi = \Psi^{\parallel} + \Psi^{\perp}$ with $\Psi^{\parallel}, \Psi^{\perp} \in C^{\infty}$ (not necessarily compactly supported), $\nabla \times \Psi^{\parallel} = 0$, $\nabla \cdot \Psi^{\perp} = 0$, and $\Psi^{\parallel}(\mathfrak{t}), \Psi^{\perp}(\mathfrak{t}) \to 0$, as $|\mathfrak{t}| \to \infty$. Moreover, $\Psi^{\perp}$ is the curl of some smooth field whereas $\Psi^{\parallel}$ is the gradient of a smooth function. This well–known result is the Helmholtz (decomposition) theorem. See, e.g., [10, Section 9.2, Theorem 3]. The fields $\Psi^{\parallel}$ and $\Psi^{\perp}$ are sometimes called the longitudinal and transverse components of Ψ, respectively.

Indeed, for any $j \in C_0^{\infty}$, $j^{\parallel} = P^{\parallel} j$ and $j^{\perp} = P^{\perp} j$, where $P^{\parallel}, P^{\perp}$ are the orthogonal projections respectively defined in Fourier space by

$$F[P^{\parallel} j](k) := \frac{k}{|k|^2} \, k \cdot F[j](k) \; , \tag{26}$$

$$F[P^{\perp} j](k) := F[j](k) - \frac{k}{|k|^2} \, k \cdot F[j](k) \; , \tag{27}$$

for $k \in \mathbb{R}^3$ and all current densities j in the dense subset $C_0^{\infty} \subset \mathfrak{H}$. Recall that F stands for the Fourier transform. Straightforward computations show that $P^{\parallel} j, P^{\perp} j \in C^{\infty}$ are smooth functions satisfying

$$\nabla \times [P^{\parallel} j] = 0 \; , \quad \nabla \cdot [P^{\perp} j] = 0 \; ,$$

and $[P^{\parallel} j](\mathfrak{t}), [P^{\perp} j](\mathfrak{t}) \to 0$, as $|\mathfrak{t}| \to \infty$.

We denote again by $P^{\parallel}$ and $P^{\perp}$ the unique orthogonal projections with ranges being respectively the closures of $P^{\parallel} C_0^{\infty}$ and $P^{\perp} C_0^{\infty}$ in $\mathfrak{H}$. The sets $P^{\parallel} \mathfrak{H}$ and $P^{\perp} \mathfrak{H}$ are clearly orthogonal. In fact, these projections can still be explicitly defined a.e. by (26)–(27) for any $j \in \mathfrak{H}$. The same construction can be carried out in L^2 and so, $P^{\parallel}$ and $P^{\perp}$ are also seen as mutually orthogonal projections acting on L^2. In fact,

$$\mathfrak{H} = P^{\parallel} \mathfrak{H} \oplus P^{\perp} \mathfrak{H} \; , \quad L^2 = P^{\parallel} L^2 \oplus P^{\perp} L^2 \; . \tag{28}$$

In other words, $P^{\perp} = 1 - P^{\parallel}$ and $P^{\perp} P^{\parallel} = P^{\parallel} P^{\perp} = 0$ as operators acting either on $\mathfrak{H}$ or L^2.

We define now the *restricted* Biot–Savart operator $\mathcal{S}_0$ on the dense set $C_0^{\infty} \subset \mathfrak{H}$ of smooth, compactly supported current densities $j \in C_0^{\infty}$ by

$$\mathcal{S}_0(j)(\mathfrak{t}) := \frac{1}{4\pi} \int_{\mathbb{R}^3} \frac{(\nabla \times j)(\mathfrak{s})}{|\mathfrak{t} - \mathfrak{s}|} \mathrm{d}^3 \mathfrak{s} \; , \quad \mathfrak{t} \in \mathbb{R}^3 \; . \tag{29}$$

In Fourier space, for any $j \in C_0^{\infty}$,

$$F[\mathcal{S}_0(j)](k) = \frac{ik}{|k|^2} \times F[j](k) \; , \quad k \in \mathbb{R}^3 \; . \tag{30}$$

Using the elementary equality

$$\Theta \times (\Psi \times \Phi) = (\Theta \cdot \Phi)\Psi - (\Theta \cdot \Psi)\Phi$$

together with (27), we remark that

$$ik \times F[\mathcal{S}_0(j)](k) = F[P^{\perp} j](k) \; , \quad k \in \mathbb{R}^3 \; . \tag{31}$$

In other words, $\mathcal{S}_0(j)$ and j satisfy the (generalized) Maxwell equation

$$\nabla \times \mathcal{S}_0(j) = j^{\perp} \; , \quad j \in C_0^{\infty} \; . \tag{32}$$

Additionally, we infer from (30) that

$$\nabla \cdot \mathcal{S}_0(j) = 0 \; , \quad j \in C_0^{\infty} \; . \tag{33}$$

The restricted Biot–Savart operator $\mathcal{S}_0$ also maps the dense set $C_0^{\infty} \subset \mathfrak{H}$ of current densities to the space L^2 of magnetic inductions. Indeed, for any $j_1, j_2 \in C_0^{\infty}$,

$$\langle \mathcal{S}_0(j_1), \mathcal{S}_0(j_2) \rangle_2 = \int_{\mathbb{R}^3} \mathcal{S}_0(j_1)(\mathfrak{t}) \cdot [\nabla \times \mathcal{A}(j_2)(\mathfrak{t})] \mathrm{d}^3 \mathfrak{t}$$

with the vector potential

$$\mathcal{A}(j_2)(\mathfrak{t}) := \frac{1}{4\pi} \int_{\mathbb{R}^3} \frac{j_2(\mathfrak{s})}{|\mathfrak{t} - \mathfrak{s}|} \mathrm{d}^3 \mathfrak{s} \; , \quad \mathfrak{t} \in \mathbb{R}^3 \; . \tag{34}$$

By using the well–known identity

$$\nabla \cdot (\Psi \times \Phi) = \Phi \cdot \nabla \times \Psi - \Psi \cdot \nabla \times \Phi \tag{35}$$

for smooth fields $\Psi, \Phi \in C^{\infty}$, the Maxwell equation (32), the Gauss Theorem, and decay of $\mathcal{S}_0(j_1)(\mathfrak{t}) \times \mathcal{A}(j_2)(\mathfrak{t})$ as $|\mathfrak{t}| \to \infty$, one gets

$$\langle \mathcal{S}_0(j_1), \mathcal{S}_0(j_2) \rangle_2 = \int_{\mathbb{R}^3} [\nabla \times \mathcal{S}_0(j_1)(\mathfrak{t})] \cdot \mathcal{A}(j_2)(\mathfrak{t}) \mathrm{d}^3 \mathfrak{t}$$

$$= \int_{\mathbb{R}^3} j_1^{\perp}(\mathfrak{t}) \cdot \mathcal{A}(j_2)(\mathfrak{t}) \mathrm{d}^3 \mathfrak{t}$$

$$= \langle j_1^{\perp}, j_2 \rangle_{\mathfrak{H}} \; .$$

The above computation is standard. Since $P^{\parallel}$, $P^{\perp}$ are mutually orthogonal projections acting on $\mathfrak{H}$, we infer from the last equality that

$$\langle \mathcal{S}_0(j_1), \mathcal{S}_0(j_2) \rangle_2 = \langle j_1^{\perp}, j_2^{\perp} \rangle_{\mathfrak{H}} , \quad j_1, j_2 \in C_0^{\infty} \; . \tag{36}$$

Therefore, we can extend $\mathcal{S}_0$ to a bounded operator $\mathcal{S}$ acting on $\mathfrak{H}$, named the *Biot–Savart operator*. By (36), the operator $\mathcal{S}$ restricted to $P^{\perp} \mathfrak{H}$ is an isometry:

$$\langle \mathcal{S}(j_1), \mathcal{S}(j_2) \rangle_2 = \langle j_1^{\perp}, j_2^{\perp} \rangle_{\mathfrak{H}} \; , \quad j_1, j_2 \in \mathfrak{H} \; . \tag{37}$$

In particular,

$$E_{\mathrm{mag}}(\mathcal{S}(j)) := \frac{1}{2} \|\mathcal{S}(j)\|_2^2 = \frac{1}{2} \|j^{\perp}\|_{\mathfrak{H}}^2 \; , \quad j \in \mathfrak{H} \; . \tag{38}$$

Clearly, $\ker \mathcal{S} = P^{\parallel} \mathfrak{H}$, i.e., $\mathcal{S}(j) = \mathcal{S}(j^{\perp})$.

Note that Equality (30) can be extended to all $j \in \mathfrak{H}$ because the Fourier transform F is a unitary map from $\mathfrak{H}$ to $L^2(\mathbb{R}^3, |k|^{-2} \mathrm{d}^3 k; \mathbb{R}^3)$. In other words, for all $j \in \mathfrak{H}$,

$$F[\mathcal{S}(j)](k) = \frac{ik}{|k|^2} \times F[j](k) \; , \quad k \in \mathbb{R}^3 \; \text{(a.e.)} \; . \tag{39}$$

Since (27) holds on the whole space $\mathfrak{H}$, we can also extend (32) to get

$$ik \times F[\mathcal{S}(j)](k) = F[P^{\perp} j](k) \; , \quad k \in \mathbb{R}^3 \; \text{(a.e.)} \; ,$$

for all $j \in \mathfrak{H}$, where

$$F[P^{\perp} j] \in L^2(\mathbb{R}^3, |k|^{-2} \mathrm{d}^3 k; \mathbb{R}^3) \; .$$

Consequently, the curl $\nabla \times$, which is seen here as an operator from the Hilbert space $P^\perp L^2$ to $P^\perp \mathfrak{H}$, defined in Fourier space by $ik \times$ is the left inverse of $\mathcal{S}$ on the subspace $P^\perp \mathfrak{H}$ of divergence–free currents. In particular, $\mathcal{S}(j)$ and j satisfy the (generalized) Maxwell equation

$$\nabla \times \mathcal{S}(j) = j^\perp , \qquad j \in \mathfrak{H} . \tag{40}$$

Analogously, one shows that $\nabla \times : P^\perp L^2 \to P^\perp \mathfrak{H}$ is the right inverse of $\mathcal{S}|_{P^\perp L^2}$:

$$\mathcal{S}(\nabla \times \mathrm{B}) = \mathrm{B} , \qquad \mathrm{B} \in P^\perp L^2 .$$

In particular, by (37), $\mathcal{S} : P^\perp \mathfrak{H} \to P^\perp L^2$ is an isomorphism of Hilbert spaces.

Similar to (33), one can use (39) to get also that

$$\nabla \cdot \mathcal{S}(j) = 0 , \qquad j \in \mathfrak{H} ,$$

i.e., $\mathcal{S}(\mathfrak{H}) \subseteq P^\perp L^2$. Here, $\nabla \cdot$ is defined in Fourier space by $ik \cdot$.

For further details on the Biot–Savart operator we recommend [11] where the latter is studied on the euclidean space $\mathbb{R}^3$. The results of [11] are also extended to the three–dimensional sphere in [12]. Note, however, that in [11, 12] the magnetic induction is restricted to bounded domains and the energy norm $\| - \|_{\mathfrak{H}}$ is not used. For example, in [11], the Biot–Savart operator is seen as a map from the Hilbert space $L^2(\Lambda; \mathbb{R}^3)$ to $L^2(\Lambda; \mathbb{R}^3)$ with $\Lambda \subset \mathbb{R}^3$ and $|\Lambda| < \infty$.

Remark 6 (Vector potentials)
To any $j \in C_0^\infty$ we associate a vector potential $\mathcal{A}(j) \in C^\infty$ as defined by (34). The definition of vector potentials $\mathcal{A}(j)$ for all $j \in \mathfrak{H}$ is given in Section VII A. In this case, $\mathcal{A}(j)$ is not anymore a function but a distribution, in general.

IV. MAGNETIC FREE–ENERGY DENSITY

We fix a smooth external magnetic induction $\mathrm{B}_{\mathrm{ext}} \in C^\infty \equiv C^\infty(\mathbb{R}^3; \mathbb{R}^3)$ that fulfills the Maxwell equation $\nabla \cdot \mathrm{B}_{\mathrm{ext}} = 0$ and has a finite magnetic energy:

$$E_{\mathrm{mag}}(\mathrm{B}_{\mathrm{ext}}) := \frac{1}{2} \| \mathrm{B}_{\mathrm{ext}} \|_2^2 := \frac{1}{2} \int_{\mathbb{R}^3} | \mathrm{B}_{\mathrm{ext}}(\mathfrak{t}) |^2 \, \mathrm{d}^3 \mathfrak{t} < \infty .$$

This field results from some fixed divergence–free current density $\mathrm{j}_{\mathrm{ext}}$, outside the electron (quantum) system.

More precisely, $\mathrm{B}_{\mathrm{ext}} = \mathcal{S}_0(\mathrm{j}_{\mathrm{ext}})$ for $\mathrm{j}_{\mathrm{ext}} \in C_0^\infty \cap P^\perp \mathfrak{H}$ with compact support

$$\mathrm{supp}(\mathrm{j}_{\mathrm{ext}}) := \{ \mathfrak{t} \in \mathbb{R}^3 \ : \ \mathrm{j}_{\mathrm{ext}}(\mathfrak{t}) \neq 0 \} \subset \mathbb{R} \backslash \mathfrak{C} . \tag{41}$$

Recall that $\mathcal{S}_0$ is the restricted Biot–Savart operator defined by (29), whereas $\mathfrak{C} := [-1/2, 1/2]^3$. Note that the assumption $\mathrm{B}_{\mathrm{ext}} \in C^\infty$ is clearly not necessary since the Biot–Savart operator $\mathcal{S}$ is defined for all $j \in \mathfrak{H}$. This stronger assumption is only used to simplify arguments using the usual Maxwell equations.

The external magnetic induction $\mathrm{B}_{\mathrm{ext}}$ can produce a magnetization density $m_\rho^{(l)}$ in the system. The latter can happen, for instance, if one assumes that $\mathrm{B} = \mathrm{B}_{\mathrm{ext}} \neq 0$ in (2) and $\rho = \mathfrak{g}_l$ is the corresponding Gibbs state. However, there is no reason for the magnetic induction $(\mathrm{B}_{\mathrm{ext}} + m_\rho^{(l)})$ to satisfy Gauss's law for magnetism $\nabla \cdot (\mathrm{B}_{\mathrm{ext}} + m_\rho^{(l)}) = 0$ when $m_\rho^{(l)} \neq 0$. On the other hand, in magnetostatics the total magnetic induction $B_\rho^{(l)}$ of the system must always satisfy $\nabla \cdot B_\rho^{(l)} = 0$. Observe moreover that spins of electrons interact with the total magnetic induction within the material and not only with the external magnetic induction $\mathrm{B}_{\mathrm{ext}}$. In other words, one must take $\mathrm{B} = B_\rho^{(l)}$ in (2) and not $\mathrm{B} = \mathrm{B}_{\mathrm{ext}}$. Therefore, it is crucial to properly define a total magnetic induction $B_\rho^{(l)}$ of the system satisfying $\nabla \cdot B_\rho^{(l)} = 0$ for any state $\rho \in E_{\Lambda_l}$.

It is well–known that the magnetization density $m_\rho^{(l)} \in C_0^\infty$ creates an effective current density, named bound current density, defined by

$$\mathrm{j}_{m_\rho}^{(l)} := \nabla \times m_\rho^{(l)} . \tag{42}$$

(Recall that units have here been chosen so that the magnetic permeability of free space equals 1.)

Remark 7 (Transverse projection of $m_\rho^{(l)}$)
As $\mathrm{j}_{m_\rho}^{(l)} \in C_0^\infty$, by (32) and [10, Section 9.2, Theorem 2], $\mathcal{S}_0(\mathrm{j}_{m_\rho}^{(l)})$ is the unique smooth divergence–free field satisfying $\nabla \times \mathcal{S}_0(\mathrm{j}_{m_\rho}^{(l)}) = \mathrm{j}_{m_\rho}^{(l)}$. By definition of the bound current density $\mathrm{j}_{m_\rho}^{(l)}$, $\mathcal{S}_0(\mathrm{j}_{m_\rho}^{(l)}) = (m_\rho^{(l)})^\perp$ must be the transverse component $(m_\rho^{(l)})^\perp$ of $m_\rho^{(l)} \in C_0^\infty$. In particular, the longitudinal component $(m_\rho^{(l)})^\parallel$ has no effect on the total magnetic induction.

Using Maxwell–Ampere's law, one then gets that the total magnetic induction $B_\rho^{(l)}$ must satisfy the equation

$$\nabla \times B_\rho^{(l)} = J_\rho^{(l)} + \frac{\partial D_\rho}{\partial t} \tag{43}$$

on $\mathbb{R}^3$, where

$$J_\rho^{(l)} := j_\rho^{(l)} + \mathrm{j}_{m_\rho}^{(l)} + \mathrm{j}_{\mathrm{ext}} \in C_0^\infty \tag{44}$$

is the total current density and $j_\rho^{(l)}$ is the rescaled internal (free) current density (15), whereas D_ρ is the electric induction produced by the system in the state $\rho \in E_{\Lambda_l}$.

If, at fixed time, $(J_\rho^{(l)} + \partial D_\rho / \partial t) \in C_0^\infty$ then there is a unique $B_\rho^{(l)} \in C^\infty$ satisfying the Maxwell equations $\nabla \cdot B_\rho^{(l)} = 0$ and (43) with $B_\rho^{(l)}(\mathfrak{t}) \to 0$, as $|\mathfrak{t}| \to \infty$. This unique magnetic induction is given, for any $\rho \in E_{\Lambda_l}$ and $l \in \mathbb{N}$, by

$$B_\rho^{(l)} := \mathcal{S}_0(J_\rho^{(l)} + \partial D_\rho / \partial t) . \tag{45}$$

For more details, we again recommend [10, Section 9.2, Theorem 2].

Note that $\nabla \cdot \mathrm{j}_{m_\rho}^{(l)} = 0$ because $\mathrm{j}_{m_\rho}^{(l)}$ is the curl of $m_\rho^{(l)}$, whereas $\nabla \cdot \mathrm{j}_{\mathrm{ext}} = 0$, by assumption. In other words, $(\mathrm{j}_{m_\rho}^{(l)}) = (\mathrm{j}_{m_\rho}^{(l)})^\perp$ and $\mathrm{j}_{\mathrm{ext}} = \mathrm{j}_{\mathrm{ext}}^\perp$. See the discussion in Section III about the Helmholtz theorem. The current density $j_\rho^{(l)} \in C_0^\infty$ (15) has generally a non–trivial decomposition: the rescaled longitudinal component $(j_\rho^{(l)})^\parallel$ could be non–zero, i.e., $\nabla \cdot j_\rho^{(l)} \neq 0$. Because of (43), one must have

$$0 = \left(\frac{\partial D_\rho}{\partial t} + J_\rho\right)^\parallel = \left(\frac{\partial D_\rho}{\partial t}\right)^\parallel + \left(j_\rho^{(l)}\right)^\parallel . \qquad (46)$$

In particular, only the divergence–free part $j_\rho^\perp$ of the current density functional $j_\rho \in C_0^\infty$ (13) is relevant with respect to the total magnetic induction $B_\rho^{(l)}$.

As we are interested in the system at equilibrium, it is natural to consider the stationary case. In particular, the electric induction D_ρ should be constant in time, i.e., we shall assume that $\partial D_\rho / \partial t = 0$. As a consequence, $(j_\rho^{(l)})^\parallel$ should vanish, to be consistent with (46). Because of the discrete nature of the system under consideration, $(j_\rho^{(l)})^\parallel$ is generally not exactly zero at fixed $l \in \mathbb{N}$, but the energy norm $\|(j_\rho^{(l)})^\parallel\|_{\mathfrak{H}}$ is taken below as arbitrarily small at large $l \in \mathbb{N}$, see (52).

In any case, the total magnetic induction functional is thus defined, for any $\rho \in E_{\Lambda_l}$ and $l \in \mathbb{N}$, by

$$B_\rho^{(l)} := \mathcal{S}_0(j_\rho^{(l)} + \mathrm{j}_{m_\rho}^{(l)} + \mathrm{j}_{\mathrm{ext}}) \in C^\infty , \qquad (47)$$

see (45) with $\partial D_\rho / \partial t = 0$. By definition of the operator $\mathcal{S}_0$, (47) corresponds to the Biot–Savart law which gives the total magnetic induction $B_\rho^{(l)}$ of the system from the total current density $J_\rho^{(l)}$. By construction, it is an affine map from E_{Λ_l} to C_0^∞ satisfying $\nabla \cdot B_\rho^{(l)} = 0$, cf. (33). Using the linearity of the (restricted) Biot–Savart operator $\mathcal{S}_0$, Remark 7, $\mathcal{S}_0 = \mathcal{S}_0 P^\perp$ and $\mathrm{B}_{\mathrm{ext}} = \mathcal{S}_0(\mathrm{j}_{\mathrm{ext}})$, we observe that

$$B_\rho^{(l)} = \mathcal{S}_0((j_\rho^{(l)})^\perp) + (m_\rho^{(l)})^\perp + \mathrm{B}_{\mathrm{ext}} \qquad (48)$$

and, by (32), its curl equals

$$\nabla \times B_\rho^{(l)} = (J_\rho^{(l)})^\perp = (j_\rho^{(l)})^\perp + \mathrm{j}_{m_\rho}^{(l)} + \mathrm{j}_{\mathrm{ext}}$$

for any $\rho \in E_{\Lambda_l}$ and $l \in \mathbb{N}$.

Using the magnetic energy E_{mag} defined by (23), we then finally introduce a magnetic, finite volume free–energy density functional $\mathcal{F}_l^{(\epsilon)}$ defined, for any $l \in \mathbb{N}$ and strictly positive parameter $\epsilon \in \mathbb{R}^+$, by

$$\rho \mapsto \mathcal{F}_l^{(\epsilon)}(\rho) := f_l(0,\rho) - \langle B_\rho^{(l)}, m_\rho^{(l)} \rangle_2 + E_{\mathrm{mag}}(B_\rho^{(l)}) \ (49)$$

on the set E_{Λ_l} of states. Here, $-\langle B_\rho^{(l)}, m_\rho^{(l)} \rangle_2$ is the magnetic interaction energy per unit of volume, whereas $E_{\mathrm{mag}}(B_\rho^{(l)})$ is the magnetic energy of B_ρ per unit of volume. Indeed, by (23)–(24),

$$\begin{aligned} E_{\mathrm{mag}}(B_\rho) &= |\Lambda_l| E_{\mathrm{mag}}(B_\rho^{(l)}) , \\ \langle B_\rho, m_\rho \rangle_2 &= |\Lambda_l| \langle B_\rho^{(l)}, m_\rho^{(l)} \rangle_2 \end{aligned} \qquad (50)$$

are respectively the magnetic energy of B_ρ (14) and, up to a minus sign, the magnetic interaction energy with the system in the state ρ. Note that $\mathcal{F}_l^{(\epsilon)}$ is defined for all $\epsilon \in \mathbb{R}^+$, but we are interested in the situation where $\epsilon << \epsilon_\xi$ is an arbitrarily small parameter, see Remark 2.

Our choice of the magnetic interaction energy is consistent with (42) and (48). This interaction energy can equivalently be seen as a quantum magnetic interaction energy with the divergence–free magnetic induction $B_\rho^{(l)} * \xi_\epsilon$ as

$$f_l(0,\rho) - \langle B_\rho^{(l)}, m_\rho^{(l)} \rangle_2 = f_l(B_\rho^{(l)} * \xi_\epsilon, \rho) \qquad (51)$$

for any $\epsilon \in \mathbb{R}^+$, $l \in \mathbb{N}$ and all states $\rho \in E_{\Lambda_l}$. Recall that $\xi_\epsilon \in C_0^\infty$ is defined by (18).

Note that a similar conceptual approach based on self–generated magnetic fields has been recently used in [13] to study electrons in atoms. See also [14, 15].

Remark 8 (The $E_\rho^{(l)} \equiv 0$ assumption)

Let $E_\rho^{(l)}$ be the rescaled static electric field induced by the system in the state ρ. Considering a purely magnetic energy implicitly corresponds to the situation where $E_\rho^{(l)}$ vanishes. One cannot expect this if the electron density is non–constant in space. One consequence of our analysis is that the electron density corresponding to minimizers of $\mathcal{F}_l^{(\epsilon)}$ is space–homogeneous within the superconducting regime, at large $l \in \mathbb{N}$ and small $\epsilon << \epsilon_\xi$ (cf. Remark 2). See discussions below Theorem 12 as well as Equation (89) with $h_{\mathfrak{t}} = 0$. Thus, the a priori assumption $E_\rho^{(l)} \equiv 0$ is justified within such a phase because of the electric neutrality of matter. In fact, minimizers of $\mathcal{F}_l^{(\epsilon)}$ with space–homogeneous electron density (as $\epsilon \to 0^+$) must also minimize the electromagnetic free–energy density functional

$$\rho \mapsto \mathcal{F}_l^{(\epsilon)}(\rho) + \frac{\epsilon_0}{2} \int_{\mathbb{R}^3} |E_\rho^{(l)}(\mathfrak{t})|^2 \ \mathrm{d}^3\mathfrak{t}$$

in the limit $\epsilon \to 0^+$. (Here, ϵ_0 is the relative permittivity of free space.) In the non–superconducting phase, this assumption is generally not consistent with the structure of minimizers of $\mathcal{F}_l^{(\epsilon)}$.

V. THERMODYNAMICS OF THE MEISSNER EFFECT

We analyze now the thermodynamics corresponding to the magnetic free–energy density functional $\mathcal{F}_l^{(\epsilon)}$ defined, for any $l \in \mathbb{N}$ and $\epsilon \in \mathbb{R}^+$, by (49) on the set E_{Λ_l} of states. In finite volume, equilibrium states $\omega_{\epsilon,l} \in E_{\Lambda_l}$ should minimize this functional.

We expect moreover their associated currents $j_{\omega_{\epsilon,l}}^{(l)}$ to be divergence–free in the limit $l \to \infty$. Indeed, the system should not be able, at thermodynamical equilibrium, to transmit energy in form of electromagnetic waves, i.e.,

the generated electromagnetic field should be static. The latter is consistent with (46) and the stationarity assumption $\partial D_\rho / \partial t = 0$. As a consequence, we only consider states creating quasi–divergence–free currents $j_\rho^{(l)}$, that is, states which belong to the set

$$E_{\Lambda_l}^\perp := \left\{ \rho \in E_{\Lambda_l} : \; \|(j_\rho^{(l)})^\| \|_{\mathfrak{H}} \leq l^{-\varkappa} \right\} \qquad (52)$$

for some small, but strictly positive parameter $\varkappa \in \mathbb{R}^+$. (For instance, take $\varkappa < 0.05$ with $\eta^\perp = 0.8$ and $\eta = 0.95$, see proof of Theorem 4.) It means in the thermodynamic limit $l \to \infty$ that, for all $\rho \in E_{\Lambda_l}^\perp$, the current density $j_\rho^{(l)}$ is divergence–free in the sense of the energy norm.

By (37)–(38) together with (42) and (47) note that, for any $\rho \in E_{\Lambda_l}$ and $l \in \mathbb{N}$,

$$\mathcal{F}_l^{(\epsilon)}(\rho) = f_l(0, \rho) - \langle J_\rho^{(l)}, j_{m_\rho}^{(l)} \rangle_{\mathfrak{H}} + \frac{1}{2} \|(J_\rho^{(l)})^\perp\|_{\mathfrak{H}}^2 \, ,$$

where $J_\rho^{(l)}$ is the total current density defined by (44) with transverse component $(J_\rho^{(l)})^\perp = P^\perp J_\rho^{(l)}$. If $\rho \notin E_{\Lambda_l} \backslash E_{\Lambda_l}^\perp$ then one should consider the energy of the induced electric field, but we refrain from considering this case in order to keep technical aspects as simple as possible. See also Remark 8.

Therefore, we shall consider the variational problem

$$\mathcal{P}_l^{(\epsilon)} := - \inf_{\rho \in E_{\Lambda_l}^\perp} \mathcal{F}_l^{(\epsilon)}(\rho) \, , \qquad l \in \mathbb{N} \, , \; \epsilon \in \mathbb{R}^+. \qquad (53)$$

The value $\mathcal{P}_l^{(\epsilon)}$ is named *finite volume magnetic pressure* by analogy with (10). Recall that the functionals $\rho \mapsto B_\rho^{(l)}$ and $\rho \mapsto m_\rho^{(l)}$ are affine. Hence, they are both continuous maps from $E_{\Lambda_l}^\perp$ to L^2, by finite dimensionality of $E_{\Lambda_l}^\perp$. The map $\rho \mapsto f_l(0, \rho)$ from $E_{\Lambda_l}^\perp$ to $\mathbb{R}$ is also continuous. Therefore, the functional $\mathcal{F}_l^{(\epsilon)}$ is continuous on $E_{\Lambda_l}^\perp$ for every $l \in \mathbb{N}$, $\epsilon \in \mathbb{R}^+$, and by compactness of $E_{\Lambda_l}^\perp$ and the Weierstraß theorem, the set

$$\Omega_l^{(\epsilon)} := \left\{ \omega_{\epsilon,l} \in E_{\Lambda_l}^\perp \; : \; \mathcal{F}_l^{(\epsilon)}(\omega_{\epsilon,l}) = \inf_{\rho \in E_{\Lambda_l}^\perp} \mathcal{F}_l^{(\epsilon)}(\rho) \right\} \qquad (54)$$

of finite volume minimizers is *non–empty* for any $l \in \mathbb{N}$ and $\epsilon \in \mathbb{R}^+$. In general, such minimizers are not Gibbs states $\mathfrak{g}_l \in E_{\Lambda_l}$ for $\mathrm{B} = B_\rho^{(l)}$, see (6)–(7). Moreover, $\mathcal{F}_l^{(\epsilon)}$ is a priori not a convex functional on $E_{\Lambda_l}^\perp$, i.e., its minimizer may not be unique.

Remark 9 (Quantum magnetic fields)
Considering a quantum (electro–) magnetic field interacting with the quantum system defined by H_l, we should obtain a convex free–energy density functional. In this context, $\mathcal{F}_l^{(\epsilon)}$ may be seen as an approximating free–energy density functional and elements of $\Omega_l^{(\epsilon)}$ as extreme states of the fully quantum system, as $l \to \infty$, $\epsilon \to 0^+$. An analogue situation is found in [4] where $\mathcal{F}_l^{(\epsilon)}$ would play the rôle of [4, Definition 2.6]. In particular, one shall instead consider the Γ–regularization of $\mathcal{F}_l^{(\epsilon)}$ on E_{Λ_l} and the new set of minimizers would then be the closed convex hull of $\Omega_l^{(\epsilon)}$, see [16, Theorem 1.4]. However, in order to keep mathematical aspects as simple as possible, we refrain from considering such a framework.

The aim of the present section is to analyze the thermodynamics of the quantum system under consideration with a self–generated magnetic induction in relation with the existence of the Meißner effect. To this end, we first observe that the thermodynamic pressure

$$\mathrm{B} \mapsto p_\infty(\mathrm{B}) := \lim_{l \to \infty} p_l(\mathrm{B}) < \infty \qquad (55)$$

is a well–defined continuous map from L^2 to $\mathbb{R}$. Its main properties are given by Theorem 15. Similarly, the thermodynamic limit

$$\mathcal{P}_\infty^{(\epsilon)} := \lim_{l \to \infty} \mathcal{P}_l^{(\epsilon)} < \infty$$

of the magnetic pressure $\mathcal{P}_l^{(\epsilon)}$ exists for all $\epsilon \in \mathbb{R}^+$, see Theorem 31. It is given by a variational problem over a closed subspace $\mathcal{B} \subset P^\perp L^2$ defined as follows: Consider the set

$$\mathcal{J} := C_0^\infty(\mathfrak{C}; \mathbb{R}^3) \cap P^\perp \mathfrak{H} \qquad (56)$$

of divergence–free smooth current densities supported in $\mathfrak{C}$. Recall that $\mathcal{S}_0$ is the restricted Biot–Savart operator defined by (29). Then, we denote by

$$\mathcal{B} := \overline{\mathcal{S}_0(\mathcal{J})} \subset P^\perp L^2 \qquad (57)$$

the closure of the set $\mathcal{S}_0(\mathcal{J}) \subset C^\infty$ in the weak topology of L^2.

We focus on the Meißner effect, that is, the existence of superconducting states $\omega_{\epsilon,l} \in \Omega_l^{(\epsilon)}$ with self–generated magnetic inductions $B_{\omega_{\epsilon,l}}^{(l)}$ which vanish inside the unit box $\mathfrak{C}$ while being created by currents supported on the boundary $\partial \mathfrak{C}$ of $\mathfrak{C}$, in the limit $\epsilon \to 0^+$ after $l \to \infty$. Indeed, we analyze in the limit $\epsilon \to 0^+$ the sets $\mathbb{B}_\epsilon^{(\pm)}$ defined by

$$\mathbb{B}_\epsilon^{(\pm)} := \bigcup_{\{\omega_{\epsilon,l}\}_{l \in \mathbb{N}} \subset \Omega^{(\epsilon)}} \mathbb{B}_\epsilon^{(\pm)}(\{\omega_{\epsilon,l}\}_{l \in \mathbb{N}}) \qquad (58)$$

for any $\epsilon \in \mathbb{R}^+$. Here, $\Omega^{(\epsilon)}$ is the set of all sequences $\{\omega_{\epsilon,l}\}_{l \in \mathbb{N}}$ with $\omega_{\epsilon,l} \in \Omega_l^{(\epsilon)}$, and $\mathbb{B}_\epsilon^{(\pm)}(\{\omega_{\epsilon,l}\}_{l \in \mathbb{N}})$ are the sets of all weak $(-)$ and norm $(+)$ cluster points of $\{B_{\omega_{\epsilon,l}}^{(l)}\}_{l \in \mathbb{N}}$.

Using Theorem 15 (ii) and $\mathcal{P}_\infty^{(\epsilon)} < \infty$, one verifies the existence of a radius $R \in \mathbb{R}^+$ such that $\|B_{\omega_{\epsilon,l}}^{(l)}\|_2 \leq R$ for all $\epsilon \in \mathbb{R}^+$, $l \in \mathbb{N}$ and $\omega_{\epsilon,l} \in \Omega_l^{(\epsilon)}$. Because of the Banach–Alaoglu theorem and the separability of L^2, the set $\{B_{\omega_{\epsilon,l}}^{(l)}\}_{l \in \mathbb{N}, \omega_{\epsilon,l} \in \Omega_l^{(\epsilon)}}$ is sequentially weak–precompact

and $\mathbb{B}_\epsilon^{(-)}$ is not empty. Therefore, one primary aim is to prove that elements of $\mathbb{B}_\epsilon^{(-)}$ can vanish inside the unit box $\mathfrak{C}$ while being created by currents supported on the boundary $\partial\mathfrak{C}$ of $\mathfrak{C}$, in the limit $\epsilon \to 0^+$.

As explained in Remark 3, we take the limit $\epsilon \to 0^+$ after $l \to \infty$ to avoid any arbitrariness. We thus define the infinite volume magnetic pressure by

$$\mathcal{P}_\infty := \lim_{\epsilon \to 0^+} \mathcal{P}_\infty^{(\epsilon)} < \infty . \qquad (59)$$

This pressure exists and equals:

Theorem 10 (Thermodynamics)
Let $\mathrm{B}_{\mathrm{ext}} = \mathcal{S}_0(\mathrm{j}_{\mathrm{ext}})$ *with* $\mathrm{j}_{\mathrm{ext}} \in C_0^\infty \cap P^\perp \mathfrak{H}$. *Then :*
(i) The infinite volume magnetic pressure equals

$$\mathcal{P}_\infty = \sup_{\mathrm{B}\in\mathcal{B}} \left\{ -\frac{1}{2} \|\mathrm{B} + \mathrm{B}_{\mathrm{ext}}\|_2^2 + p_\infty\left(\mathrm{B} + \mathrm{B}_{\mathrm{ext}}\right) \right\} .$$

(ii) For any $\epsilon \in \mathbb{R}^+$,

$$\mathbb{B}_\epsilon := \mathbb{B}_\epsilon^{(+)} = \mathbb{B}_\epsilon^{(-)} \neq \emptyset .$$

(iii) For any family $\{\mathrm{B}_\epsilon\}_{\epsilon\in\mathbb{R}^+}$ *with* $\mathrm{B}_\epsilon \in \mathbb{B}_\epsilon$,

$$\lim_{\epsilon\to 0^+}\left\{ -\frac{1}{2}\|\mathrm{B}_\epsilon + \mathrm{B}_{\mathrm{ext}}\|_2^2 + p_\infty(\mathrm{B}_\epsilon + \mathrm{B}_{\mathrm{ext}}) \right\} = \mathcal{P}_\infty .$$

Proof. (i) is Corollary 32. (ii)–(iii) result from Lemma 29 and Corollary 33. $\qquad\qquad\square$

Remark 11 (Existence of maximizer(s))
As explained in Section VII C, the variational problem $\mathcal{P}_\infty$ *could have no maximizer. Indeed, the map*

$$\mathrm{B} \mapsto \mathcal{G}(\mathrm{B}) := -\frac{1}{2} \|\mathrm{B} + \mathrm{B}_{\mathrm{ext}}\|_2^2 + p_\infty\left(\mathrm{B} + \mathrm{B}_{\mathrm{ext}}\right)$$

from $\mathcal{B}$ *to* $\mathbb{R}$ *is neither concave nor upper semi–continuous in the weak topology. However, under certain conditions, we show in Theorem 12 that* $\mathcal{G}$ *has a unique maximizer* $\mathrm{B}_{\mathrm{int}} \in \mathcal{B}$.

We prove now the Meißner effect at large enough inverse temperatures $\beta >> 1$ and large BCS couplings $\gamma >> 1$, i.e., in presence of a superconducting phase defined as follow: Consider the annihilation and creation operators

$$\mathfrak{c}_0 := \frac{1}{|\Lambda_l|^{1/2}} \sum_{x\in\Lambda_l} a_{x,\downarrow}a_{x,\uparrow} \text{ and } \mathfrak{c}_0^* := \frac{1}{|\Lambda_l|^{1/2}} \sum_{x\in\Lambda_l} a_{x,\uparrow}^* a_{x,\downarrow}^*$$

of Cooper pairs within the condensate, i.e., in the zero–mode for electron pairs. A *superconducting* phase is then characterized by a strictly positive (global) Cooper pair condensate density for all minimizers in the thermodynamic limit, that is,

$$\mathrm{r}_\beta := \lim_{\epsilon\to 0^+}\liminf_{l\to\infty}\left\{ \inf_{\omega_{\epsilon,l}\in\Omega_l^{(\epsilon)}} \omega_{\epsilon,l}\left(\frac{\mathfrak{c}_0^*\mathfrak{c}_0}{|\Lambda_l|}\right)\right\} > 0 . \quad (60)$$

This inequality corresponds to the existence of an off–diagonal long range order. The domain of parameters $(\beta, \mu, \lambda, \gamma, \mathrm{B}_{\mathrm{ext}})$ where r_β is strictly positive is non–empty. At sufficiently large inverse temperatures $\beta >> 1$, the latter holds for instance when $\mu < -\vartheta^2$ and $\gamma > |\mu - \lambda|\Gamma_0$ with

$$\Gamma_0 := \frac{4}{1 - \vartheta^2|\mu|^{-1}} > 4 .$$

See Theorem 40. The Meißner effect appears in this regime:

Theorem 12 (Meißner effect)
Let $\mu < -\vartheta^2$, $\gamma > |\mu - \lambda|\Gamma_0$ *and* $\mathrm{B}_{\mathrm{ext}} = \mathcal{S}_0(\mathrm{j}_{\mathrm{ext}})$ *with* $\mathrm{j}_{\mathrm{ext}} \in C_0^\infty \cap P^\perp \mathfrak{H}$. *Then, there is* $\beta_c \in \mathbb{R}^+$ *such that, for all* $\beta > \beta_c$:
(i) For any sequence of minimizers $\omega_{\epsilon,l} \in \Omega_l^{(\epsilon)}$,

$$\lim_{\epsilon\to 0^+}\liminf_{l\to\infty}\omega_{\epsilon,l}\left(\frac{\mathfrak{c}_0^*\mathfrak{c}_0}{|\Lambda_l|}\right) = \lim_{\epsilon\to 0^+}\limsup_{l\to\infty}\omega_{\epsilon,l}\left(\frac{\mathfrak{c}_0^*\mathfrak{c}_0}{|\Lambda_l|}\right) = \mathrm{r}_\beta(0)$$

with

$$\mathrm{r}_\beta(0) \geq \Gamma_0^{-2} - \gamma^{-2}(\mu - \lambda)^2 > 0$$

being the unique solution of (75) for $\mathrm{B} = 0$.
(ii) For any sequence of minimizers $\omega_{\epsilon,l} \in \Omega_l^{(\epsilon)}$,

$$\lim_{\epsilon\to 0^+}\limsup_{l\to\infty}\|B_{\omega_{\epsilon,l}}^{(l)} - \mathrm{B}_{\mathrm{int}}\|_2 = 0$$

with $\mathrm{B}_{\mathrm{int}} \in \mathcal{B}$ *being the unique maximizer of the variational problem* $\mathcal{P}_\infty$. *See Theorem 10 (i).*
(iii) The total magnetic induction vanishes inside the unit box $\mathfrak{C}$: $\mathrm{B}_{\mathrm{int}} + \mathrm{B}_{\mathrm{ext}} = 0$ *a.e. in* $\mathfrak{C}$.
(iv) If (41) also holds then the self–generated magnetic induction $\mathrm{B}_{\mathrm{int}} = \mathcal{S}(\mathrm{j}_{\mathrm{int}}^\perp)$ *is produced by some current* $\mathrm{j}_{\mathrm{int}}^\perp \in \overline{\mathcal{J}}$ *that is supported on the boundary* $\partial\mathfrak{C}$ *of* $\mathfrak{C}$.

Proof. (i) is Theorem 40 (iii). By Theorem 40 (ii), note that $\mathrm{B}_{\mathrm{int}} \in \mathcal{B}$ is the unique maximizer of the variational problem $\mathcal{P}_\infty$. By using Theorems 10 (ii)–(iii), 15 (ii), 40 (i) and a simple contradiction argument, we prove the second assertion (ii). Finally, (iii)–(iv) are consequences of Theorem 40 (i)–(ii) and Lemmata 34–35. $\qquad\square$

By Theorem 12, the electron density corresponding to minimizers of $\mathcal{F}_l^{(l)}$ is space–homogeneous within the superconducting regime, in the limit $\epsilon \to 0^+$ after $l \to \infty$. Indeed, the Meißner effect corresponds here to the absence of magnetic induction inside the unit box $\mathfrak{C}$, except within a ϵ–neighborhood of the boundary $\partial\mathfrak{C}$ of $\mathfrak{C}$. Therefore, in the limit $\epsilon \to 0^+$, the electron density d_β, defined for all $\mathfrak{t} \in \mathfrak{C}$ (a.e.) by (89) for a magnetic induction $(\mathrm{B}_{\mathrm{int}} + \mathrm{B}_{\mathrm{ext}})|_{\mathfrak{C}} = 0$, is constant in this case. This argument justifies a posteriori the use of a purely magnetic energy in (49). See Remark 8.

By Lemmata 34 and 36, observe finally that the Euler–Lagrange equations associated with $\mathcal{P}_\infty$ yield the equality

$$\mathrm{B}_0 + \mathrm{B}_{\mathrm{ext}} = \mathrm{M}_\beta^\perp(\mathrm{B}_0 + \mathrm{B}_{\mathrm{ext}}) \equiv \mathrm{M}_\beta^\perp \quad \text{a.e. in } \mathfrak{C} \quad (61)$$

for any maximizer $B_0 \in \mathcal{B}$ of the variational problem $\mathcal{P}_\infty$. Here, we denote as usual by $M_\beta^\perp = P^\perp M_\beta$ the transverse component of the (infinite volume) magnetization density $M_\beta \equiv M_\beta(B)$ defined a.e. on $\mathbb{R}^3$ for every $B \in L^2$ by

$$M_{\beta,\mathfrak{t}} \equiv M_{\beta,\mathfrak{t}}(B) := \frac{\mathbf{1}[\mathfrak{t} \in \mathfrak{C}] \, \vartheta \sinh(\beta h_\mathfrak{t})}{\cosh(\beta h_\mathfrak{t}) + \mathrm{e}^{-\beta\lambda}\cosh(\beta g_{r_\beta})} \frac{B(\mathfrak{t})}{|B(\mathfrak{t})|} \tag{62}$$

with $r_\beta \in [0, 1/4]$ being solution of the variational problem (75), $g_r := \{(\mu - \lambda)^2 + \gamma^2 r\}^{1/2}$ and $h_\mathfrak{t} := \vartheta |B(\mathfrak{t})|$ a.e. in the unit box $\mathfrak{C}$. By (84)–(85), $M_\beta(B)$ is indeed the magnetization density if one applies a fixed magnetic induction B on the system. See Section VII B for more details.

Recall that units have been chosen so that the magnetic permeability of free space equals 1. As a consequence, Equation (61) implies that the so–called magnetic field

$$H := B_0 + B_{\mathrm{ext}} - M_\beta^\perp \in L^2$$

is zero within the quantum system, i.e., $H|_{\mathfrak{C}} = 0$ a.e. in the unit cubic box $\mathfrak{C}$. The latter is satisfied for every maximizer B_0 and in the whole phase diagram (not only in the regime where the Meißner effect appears).

This suggests that $M_\beta^\perp = -H$ trivially holds in the superconducting phase because $M_\beta^\perp = 0 = H$, see Theorem 12. In other words, the Meißner effect is not necessarily related to an effective magnetic susceptibility equal to -1. A similar remark can be done about the magnetic permeability of the quantum system in the superconducting phase.

VI. UNIVERSALITY OF THE MEISSNER EFFECT

1. Conditions of Theorem 12 are only sufficient and clearly not necessary for the Meißner effect. See, e.g., Lemma 39. In particular, the inequalities $\mu < -\vartheta^2$ and $\gamma > |\mu - \lambda| \Gamma_0$ are far from being essential. The inequality $\gamma > 2|\mu - \lambda|$ is however necessary to get a superconducting phase. Therefore, Theorem 12 should be seen as an example where the thermodynamics of the Meißner effect is rigorously proven from first principles of quantum mechanics.

2. Of course, the model $H_l := T_l + \mathcal{M}_l$ under consideration is too simplified with respect to real superconductors, as explained for instance in [2]. However, from high–low temperature expansions, the more realistic model

$$H_{l,\mathfrak{e}} := H_l + \sum_{x,y \in \Lambda_l} \mathfrak{e}(x - y) \left(a_{x,\downarrow}^* a_{y,\downarrow} + a_{x,\uparrow}^* a_{y,\uparrow} \right) ,$$

which includes kinetic terms, should have essentially the same correlation functions as H_l at low temperatures, up to corrections of order $\|\mathfrak{e}\|_1$. Thus, the Hamiltonian H_l is a good model for certain kinds of superconductors or

ultra–cold Fermi gases in optical lattices for which the strong coupling regime is justified.

Additionally, by using [3, Section 6.1], a similar study could have been done for the usual (reduced) BCS model. In this model, the (screened) Coulomb repulsion is neglected, i.e., $\lambda = 0$, but the kinetic energy is taken into account.

3. Indeed, the Meißner effect is directly related to the existence of states minimizing the free–energy and having small magnetization densities at fixed magnetic induction B. It is the case in our model within the superconducting regime where the magnetization density M_β is generally exponentially small in the limit $\beta \to \infty$. See, e.g., (86). In fact, it is only necessary to verify the inequality $\|M_\beta(B)\|_2 \leq \mathbf{m}\|B\|_2$ with $\mathbf{m} < 1$ and that the system can produce currents without increasing the free–energy density in the thermodynamic limit. Then, assuming this phenomenon to happen in real superconductors, details of the model are not that important anymore. The self–generated magnetic induction $B_{\mathrm{int}} = \mathcal{S}(j_{\mathrm{int}}^\perp)$ is in this case the unique solution of the variational problem

$$\mathfrak{A} := \frac{1}{2} \inf_{B \in \mathcal{B}} \|B + B_{\mathrm{ext}}\|_2^2 , \tag{63}$$

whereas the corresponding divergence–free current density $j_{\mathrm{int}}^\perp$ is the unique minimizer of

$$\mathfrak{J} := \frac{1}{2} \inf_{j^\perp \in \overline{\mathcal{J}}} \|j^\perp + j_{\mathrm{ext}}\|_{\mathfrak{H}}^2 = \mathfrak{A} , \tag{64}$$

where $\overline{\mathcal{J}}$ is the (norm) closure of the set $\mathcal{J}$ defined by (56). See Equation (38) and Theorem 40. These variational problems are studied in Lemmata 34 and 35. See also the corresponding Euler–Lagrange equations (156) and (159).

Both variational problems can certainly be numerically studied in detail and the resulting (self–generated) magnetic induction B_{int} will correspond to the usual pictures found in textbooks on superconductors to illustrate the Meißner effect. The study of $\mathfrak{A}$ and $\mathfrak{J}$ may moreover be of relevance in completely different contexts, like in fluid dynamics where currents and magnetic inductions are respectively replaced by vortex lines and velocity fields.

Remark 13 (General superconducting domains)
Results of Lemmata 34 and 35 as well as the Euler–Lagrange equations (156) and (159) can be straightforwardly extended to all external magnetic inductions $B_{\mathrm{ext}} = \mathcal{S}(j_{\mathrm{ext}})$ with $j_{\mathrm{ext}} \in P^\perp \mathfrak{H}$, and all bounded domains $\mathfrak{C} \subset \mathbb{R}^3$ with (for example) piecewise smooth boundary $\partial\mathfrak{C}$.

4. Observe that a suppression of the magnetic induction in the box $\mathfrak{C}$ by minimizers of the magnetic free–energy density can also appear at small enough inverse temperatures $\beta < \vartheta^{-1}$, see Lemma 38. Indeed, for high temperatures, the pressure $p_\infty(B)$ mainly comes from its entropic part and so, it does not depend much on the magnetic

induction B in this regime. In particular, the magnetization density M_β becomes again small. On the other hand, as explained above, the minimizer of $\mathfrak{A}$ vanishes inside the box $\mathfrak{C}$. See again Lemmata 34–35.

From the physical point of view, the appearance of such a phenomenon at high temperature is however questionable. Indeed, our analysis is based on the possibility for the system to create any current density, see Theorem 4. It is proven by using patches of superconducting phases within negligible volume (with respect to $|\Lambda|$). As explained in Remark 5, such a configuration should be rapidly destroyed by the quantum dynamics of the system at (high) temperatures where no global superconducting phase exists. In other words, our results are physically well–founded for sufficiently low temperatures where we can ensure the existence of a (global) superconducting phase defined by $\mathbf{r}_\beta > 0$ and for sufficiently weak external magnetic inductions B_{ext}. Indeed, in order to suppress magnetic inductions B_{ext}, the quantum system has to produce currents via superconducting patches close to the boundary $\partial\mathfrak{C}$. Such patches are however rapidly destabilized by a too strong magnetic induction B_{ext} that succeeds to penetrate the region close to $\partial\mathfrak{C}$. The latter is suggested by Equation (81). Indeed, this equation shows that, for any B satisfying $|B_t| \geq \vartheta^{-1}h > \vartheta^{-1}h_c$ a.e. on a non–empty open set $\mathfrak{D} \subset \mathfrak{C}$, the local Cooper pair condensate density $r_{\beta,\mathfrak{D}} = \mathcal{O}(e^{-\beta(h-h_c)})$ in the region $\mathfrak{D}$ must be exponentially small, as $\beta \to \infty$. In particular, no superconducting current can be created within $\mathfrak{D}$.

VII. TECHNICAL PROOFS

A. Vector Potentials

We start this section by defining the vector potential $\mathcal{A}(j)$ for all $j \in \mathfrak{H}$. It will become important while proving the Meißner effect.

The vector potential $\mathcal{A}(j)$ associated with any current density $j \in \mathfrak{H}$ is the distribution defined by

$$\mathcal{A}(j)(\varphi) := \langle j, \varphi \rangle_{\mathfrak{H}} \in \mathbb{R}, \qquad \varphi \in C_0^\infty. \quad (65)$$

For any $j \in C_0^\infty$, $\mathcal{A}(j)$ can be seen as a C^∞–function, as usual. See (34). For convenience, we ignore this distinction and write

$$\mathcal{A}(j)(\mathfrak{t}) \equiv \frac{1}{4\pi} \int_{\mathbb{R}^3} \frac{j(\mathfrak{s})}{|\mathfrak{t} - \mathfrak{s}|} d^3\mathfrak{s}, \quad \mathfrak{t} \in \mathbb{R}^3, \quad (66)$$

for all $j \in C_0^\infty$ and

$$\mathcal{A}(j)(\varphi) \equiv \langle \mathcal{A}(j), \varphi \rangle_2, \qquad j, \varphi \in C_0^\infty. \quad (67)$$

Using Fourier transform, one verifies the following for the weak Laplace operator applied on $\mathcal{A}(j)$:

$$[-\Delta \mathcal{A}(j)](\varphi) := \mathcal{A}(j)(-\Delta \varphi) = j(\varphi) \quad (68)$$

for $j \in \mathfrak{H}$ and $\varphi \in C_0^\infty$. Indeed, recall that $\mathfrak{H}$ can be seen as a subspace of (tempered) distributions in $W^{-1,2}(\mathbb{R}^3; \mathbb{R}^3)$. Similarly, the curl of the vector potential distribution $\mathcal{A}(j)$ equals

$$\nabla \times \mathcal{A}(j) = \mathcal{S}(j^\perp), \qquad j \in \mathfrak{H},$$

in the weak sense. See, e.g., (39).

B. Thermodynamics at Fixed Magnetic Induction

For all inverse temperatures $\beta \in \mathbb{R}^+$, we associate to the Hamiltonian $H_l := T_l + \mathcal{M}_l \in \mathcal{U}_{\Lambda_l}$ the finite volume pressure $p_l(B)$ defined by (10) at fixed magnetic induction $B \in L^2(\mathfrak{C}; \mathbb{R}^3)$ with $\mathfrak{C} := [-1/2, 1/2]^3$. Its thermodynamic limit $p_\infty(B)$ is explicitly computed in two main steps.

The first step consists in assuming that $B \in C^0(\mathfrak{C}; \mathbb{R}^3)$ is a continuous magnetic induction in order to get the pressure $p_\infty(B)$ by using [3, Theorem 3]. The infinite volume pressure $p_\infty(B)$ is then given by the maximization of a functional $\mathfrak{F}$ defined on $\mathbb{R}_0^+$ by

$$\mathfrak{F}(r) \equiv \mathfrak{F}(r, B) := \mu + \beta^{-1} \ln 2 - \gamma r$$
$$+ \beta^{-1} \int_{\mathfrak{C}} \ln \text{Trace}_{\wedge \mathcal{H}_{\{0\}}} (e^{-\beta u(r, \mathfrak{t})}) d^3\mathfrak{t} \quad (69)$$

with the one–site Hamiltonian defined by

$$u(r, \mathfrak{t}) := -\mu(n_{0,\uparrow} + n_{0,\downarrow}) + 2\lambda n_{0,\uparrow} n_{0,\downarrow} \quad (70)$$
$$-\gamma\sqrt{r}(a_{0,\uparrow}^* a_{0,\downarrow}^* + a_{0,\downarrow} a_{0,\uparrow}) - B(\mathfrak{t}) \cdot M^0$$

for all $r \in \mathbb{R}_0^+$ (i.e., $r \geq 0$) and $\mathfrak{t} \in \mathfrak{C}$ (a.e.). Here, $M^0 := (m_1^0, m_2^0, m_3^0)$ is defined via (17). Indeed, one has:

Lemma 14 (Pressure for continuous fields)
For any $B \in C^0(\mathfrak{C}; \mathbb{R}^3)$,

$$p_\infty(B) := \lim_{l \to \infty} p_l(B) = \sup_{r \geq 0} \mathfrak{F}(r, B) < \infty.$$

Proof. If $B \in C^0(\mathfrak{C}; \mathbb{R}^3)$ is a continuous magnetic induction then one can replace in $\mathcal{M}_l$ (2) the mean value (3) of the magnetic induction B with $B(x/(2l))$, in order to compute the pressure $p_\infty(B)$. The latter results from [4, Eq. (3.11)] and straightforward estimates using the uniform continuity of B on the compact set $\mathfrak{C}$. Then, using the gauge symmetry of the model as well as a change of variable $r = \gamma\tilde{r}$ in the variational problem given by [3, Theorem 3], we arrive at the assertion. $\square$

The second step uses the density of the set $C^0(\mathfrak{C}; \mathbb{R}^3)$ in $L^2(\mathfrak{C}; \mathbb{R}^3)$ to compute $p_\infty(B)$ for all $B \in L^2(\mathfrak{C}; \mathbb{R}^3)$. Combined with Lemma 14 it leads to an explicit expression of $p_\infty(B)$ for all $B \in L^2(\mathfrak{C}; \mathbb{R}^3)$. This is resumed in the following theorem which serves as a springboard to the rest of the paper.

Theorem 15 (Infinite volume pressure − I)
(i) *For* $B \in L^2(\mathfrak{C}; \mathbb{R}^3)$*, the pressure* $p_l(B)$ *converges to*

$$p_\infty(B) := \lim_{l \to \infty} p_l(B) = \sup_{r \geq 0} \mathfrak{F}(r, B) < \infty$$

with $\mathfrak{F}$ *defined by (69)–(70). See also (74).*
(ii) *The family* $\{B \mapsto p_l(B)\}_{l \in \mathbb{N} \cup \{\infty\}}$ *of maps from* $L^2(\mathfrak{C}; \mathbb{R}^3)$ *to* $\mathbb{R}$ *is uniformly Lipschitz equicontinuous: For all* $l \in \mathbb{N} \cup \{\infty\}$,

$$|p_l(B) - p_l(C)| \leq 2\sqrt{3}\vartheta \|B - C\|_2 , \qquad B, C \in L^2(\mathfrak{C}; \mathbb{R}^3) . \tag{71}$$

Proof. The uniform Lipschitz equicontinuity (71) of the family $\{B \mapsto p_l(B)\}_{l \in \mathbb{N}}$ follows from (2) and (10) together with the Cauchy–Schwarz inequality, $|\mathfrak{C}| = 1$, $\|\rho\| = 1$ and $\|m_j^x\| \leq 2\vartheta$ for any $j \in \{1, 2, 3\}$ and all $x \in \mathbb{Z}^3$, see (17). As a consequence, by Lemma 14 and the density of $C^0(\mathfrak{C}; \mathbb{R}^3)$ in the separable Hilbert space $L^2(\mathfrak{C}; \mathbb{R}^3)$, the pressure $p_l(B)$ converges to some value $p_\infty(B) \in \mathbb{R}$ for all $B \in L^2(\mathfrak{C}; \mathbb{R}^3)$ and (71) is also satisfied for $l = \infty$. In particular, for any $B \in L^2(\mathfrak{C}; \mathbb{R}^3)$, there exists a sequence $\{B^{(n)}\}_{n \in \mathbb{N}} \subset C^0(\mathfrak{C}; \mathbb{R}^3)$ converging in norm to B such that

$$p_\infty(B) = \lim_{n \to \infty} p_\infty(B^{(n)}) = \lim_{n \to \infty} \left\{ \sup_{r \geq 0} \mathfrak{F}(r, B^{(n)}) \right\} . \tag{72}$$

On the other hand, by [4, Eq. (3.11)], one easily verifies that, for any $B, C \in L^2(\mathfrak{C}; \mathbb{R}^3)$ and $\mathfrak{t} \in \mathfrak{C}$ (a.e.),

$$\begin{aligned} | \ln \mathrm{Trace}_{\wedge \mathcal{H}_{\{0\}}} &(e^{-\beta u(r, \mathfrak{t}, B(\mathfrak{t}))}) \\ &- \ln \mathrm{Trace}_{\wedge \mathcal{H}_{\{0\}}} (e^{-\beta u(r, \mathfrak{t}, C(\mathfrak{t}))})| \\ &\leq 2\sqrt{3}\vartheta\beta \, |B(\mathfrak{t}) - C(\mathfrak{t})| \end{aligned}$$

with $u(r, \mathfrak{t}) \equiv u(r, \mathfrak{t}, B(\mathfrak{t}))$ being defined by (70). By (69) combined with the Cauchy–Schwarz inequality and $|\mathfrak{C}| = 1$, it follows that

$$|\mathfrak{F}(r, B) - \mathfrak{F}(r, C)| \leq 2\sqrt{3}\vartheta \|B - C\|_2 , \quad B, C \in L^2(\mathfrak{C}; \mathbb{R}^3) . \tag{73}$$

Combined with the limits (72), this last inequality in turn implies Lemma 14 extended to all $B \in L^2(\mathfrak{C}; \mathbb{R}^3)$, that is, (i) holds. $\square$

The functional $\mathfrak{F}$ can explicitly be computed and one gets

$$\mathfrak{F}(r) \equiv \mathfrak{F}(r, B) = \mu + \beta^{-1} \ln 2 - \gamma r \tag{74}$$

$$+ \beta^{-1} \int_{\mathfrak{C}} \ln \{\cosh(\beta h_\mathfrak{t}) + e^{-\lambda\beta} \cosh(\beta g_r)\} \, d^3\mathfrak{t}$$

with $g_r := \{(\mu - \lambda)^2 + \gamma^2 r\}^{1/2}$ for any $r \in \mathbb{R}_0^+$ and magnetic strength $h_\mathfrak{t} := \vartheta |B(\mathfrak{t})|$ a.e. for $\mathfrak{t} \in \mathfrak{C}$. Its properties can thus be studied in detail, exactly as in [1, Section 7].

In particular, for any $\beta, \gamma, \vartheta \in \mathbb{R}^+$, real numbers $\mu, \lambda \in \mathbb{R}$ and $B \in L^2(\mathfrak{C}; \mathbb{R}^3)$, it is clear that the supremum of

the variational problem in Theorem 15 (i) is reached for an order parameter $r \in \mathbb{R}_0^+$ in some bounded set. In particular, there is always $r_\beta \equiv r_\beta(B) \in \mathbb{R}_0^+$ such that

$$\sup_{r \geq 0} \mathfrak{F}(r, B) = \mathfrak{F}(r_\beta, B) , \quad B \in L^2(\mathfrak{C}; \mathbb{R}^3) . \tag{75}$$

Up to (special) points $(\beta, \mu, \lambda, \gamma, \vartheta, B)$ corresponding to a phase transition of first order, r_β should always be unique and continuous with respect to each parameter.

For small inverse temperatures $\beta << 1$, $r_\beta = 0$. See arguments of [1, Sections 2 and 7]. On the other hand, any non–zero solution r_β of the variational problem of Theorem 15 (i) has to be solution of the gap equation (or Euler–Lagrange equation):

$$\int_{\mathfrak{C}} \frac{\sinh(\beta g_{r_\beta})}{e^{\beta\lambda}\cosh(\beta h_\mathfrak{t}) + \cosh(\beta g_{r_\beta})} d^3\mathfrak{t} = \frac{2g_{r_\beta}}{\gamma} . \tag{76}$$

Because $\tanh(t) \leq 1$ for $t \in \mathbb{R}_0^+$, we then conclude that

$$r_\beta \leq \max\{0, r_{\max}\} \quad \text{with} \quad r_{\max} := \frac{1}{4} - \gamma^{-2}(\mu - \lambda)^2 . \tag{77}$$

In particular, if $\gamma \leq 2|\mu - \lambda|$ then $r_\beta = 0$ for any $\beta \in \mathbb{R}^+$. However, at fixed $\beta, \lambda, \mu, \vartheta, B$, there is $\gamma_c > 2|\lambda - \mu|$ such that $r_\beta > 0$ for any $\gamma \geq \gamma_c$. The latter can easily be seen like in [1, Section 7]. In other words, the domain of parameters $(\beta, \mu, \lambda, \gamma, \vartheta, B)$ where $r_\beta \in \mathbb{R}^+$ is non–empty.

To illustrate this, we give a regime where r_β becomes strictly positive for sufficiently low temperatures and large BCS couplings:

Lemma 16 (Superconducting phase − I)
Let $R \in \mathbb{R}^+$*,* $\mu < -R\vartheta$ *and* $\gamma > |\mu - \lambda| \Gamma_0$ *with*

$$\Gamma_0 := \frac{4}{1 - R\vartheta|\mu|^{-1}} > 4 .$$

Then, there is $\beta_c \in \mathbb{R}^+$ *such that, for all* $\beta > \beta_c$,

$$\inf_{B \in b_R(0)} r_\beta(B) \geq \Gamma_0^{-2} - \gamma^{-2}(\mu - \lambda)^2 > 0$$

with

$$b_R(0) := \{B \in L^2(\mathfrak{C}; \mathbb{R}^3) : \|B\|_2 \leq R\} .$$

Moreover, we can choose $\beta_c \equiv \beta_c(\gamma)$ *as a decreasing function of* γ.

Proof. For any $B \in b_R(0)$, note that $\|B\|_1 \leq R$, by the Cauchy–Schwarz inequality. Then, for any $\mu < 0$, the set

$$\mathfrak{D}_\mu := \{\mathfrak{t} \in \mathfrak{C} : |h_\mathfrak{t}| \geq |\mu|\}$$

satisfies $|\mathfrak{D}_\mu| \leq R\vartheta|\mu|^{-1}$ for all $B \in b_R(0)$. For every $\mu < -R\vartheta$ and $\varepsilon \in \mathbb{R}^+$, let

$$\Gamma_\varepsilon := (1 + \varepsilon)\Gamma_0 > 0 . \tag{78}$$

Since $\gamma > |\mu - \lambda| \Gamma_0$, we can choose $\varepsilon \in \mathbb{R}^+$ such that $\gamma > |\mu - \lambda| \Gamma_\varepsilon$. It follows that

$$\mathfrak{r}_\varepsilon := \Gamma_\varepsilon^{-2} - \gamma^{-2} (\mu - \lambda)^2 > 0 . \tag{79}$$

Then, by (78), there is $\beta_c \in \mathbb{R}^+$ such that, for all $\beta > \beta_c$,

$$\frac{\tanh(\beta g_{\mathfrak{r}_\varepsilon})}{g_{\mathfrak{r}_\varepsilon}} = \frac{\Gamma_\varepsilon}{\gamma} \tanh\left(\frac{\beta\gamma}{\Gamma_\varepsilon}\right) > \frac{4 + \varepsilon}{\gamma(1 - R\vartheta|\mu|^{-1})} .$$

Observe that $\beta_c \equiv \beta_c(\gamma)$ can be taken as a decreasing function of γ. The function $t^{-1}\tanh(\beta t)$ is decreasing on $\mathbb{R}_0^+$. Therefore, we deduce from the last inequality that

$$\left(1 - R\vartheta|\mu|^{-1}\right) \frac{\tanh(\beta g_r)}{g_r} > \frac{4 + \varepsilon}{\gamma} \tag{80}$$

for any $r \in [0, \mathfrak{r}_\varepsilon]$, all $\beta > \beta_c$ and fixed $\varepsilon \in \mathbb{R}^+$ such that $\gamma > |\mu - \lambda| \Gamma_\varepsilon$.

Now, we compute that $\partial_r \mathfrak{F}(r, \mathrm{B}) > 0$ is equivalent to

$$\int_{\mathfrak{C}} \frac{\sinh(\beta g_r)}{g_r (e^{\lambda\beta} \cosh(\beta h_{\mathfrak{t}}) + \cosh(\beta g_r))} d^3\mathfrak{t} > \frac{2}{\gamma} .$$

Using $|\mathfrak{D}_\mu| \leq R\vartheta|\mu|^{-1}$ and (80),

$$\int_{\mathfrak{C}} \frac{\sinh(\beta g_r)}{g_r (e^{\lambda\beta} \cosh(\beta h_{\mathfrak{t}}) + \cosh(\beta g_r))} d^3\mathfrak{t} > \frac{4 + \varepsilon}{2\gamma}$$

for any $\beta > \beta_c$, all $\mathrm{B} \in b_R(0)$, and $r \in [0, \mathfrak{r}_\varepsilon]$. In particular,

$$\partial_r \mathfrak{F}(r, \mathrm{B}) > 0 , \qquad r \in [0, \mathfrak{r}_\varepsilon] ,$$

which yields $\mathrm{r}_\beta(\mathrm{B}) \geq \mathfrak{r}_\varepsilon > 0$ for any $\beta > \beta_c$ and all magnetic inductions $\mathrm{B} \in b_R(0)$. □

By using Griffiths arguments (see, e.g., [1, Eq. (A.1)]) away from critical points (defined by the existence of a first order phase transition), one finds that, for any non–empty open region $\mathfrak{D} \subseteq \mathfrak{C}$, the Cooper pair condensate density

$$\mathrm{r}_{\beta,\mathfrak{D}} := \lim_{l \to \infty} \frac{1}{|\mathfrak{D}_l|^2} \sum_{x,y \in \mathfrak{D}_l} \mathfrak{g}_l \left(a_{x,\uparrow}^* a_{x,\downarrow}^* a_{y,\downarrow} a_{y,\uparrow}\right)$$

with $\mathfrak{D}_l := 2l\mathfrak{D} \cap \Lambda_l$ equals

$$\mathrm{r}_{\beta,\mathfrak{D}} = \frac{1}{|\mathfrak{D}|} \int_{\mathfrak{D}} \mathrm{r}_{\beta,\mathfrak{t}} \, d^3\mathfrak{t} \in [0, 1/4] . \tag{81}$$

(Note that $\mathfrak{D}_l$ is non–empty for sufficiently large $l \in \mathbb{N}$). Here, for all $\mathfrak{t} \in \mathfrak{C}$ (a.e.),

$$\mathrm{r}_{\beta,\mathfrak{t}} := \frac{\gamma \mathrm{r}_\beta \sinh(\beta g_{\mathrm{r}_\beta})}{2 g_{\mathrm{r}_\beta} (e^{\beta\lambda} \cosh(\beta h_{\mathfrak{t}}) + \cosh(\beta g_{\mathrm{r}_\beta}))} . \tag{82}$$

The inequality $\mathrm{r}_{\beta,\mathfrak{D}} \leq 1/4$ results from (76) and (77). In particular, for $\mathfrak{D} = \mathfrak{C}$,

$$\mathrm{r}_{\beta,\mathfrak{C}} \equiv \mathrm{r}_\beta \equiv \mathrm{r}_\beta(\mathrm{B}) \tag{83}$$

is the (global) Cooper pair condensate density, see (75)–(76). When $\mathrm{r}_\beta \in \mathbb{R}^+$ is the unique solution of the variational problem of (75), one obtains a s–wave superconducting phase with off–diagonal long range order. As an example, see [1, Theorems 3.1–3.3].

In a similar way, we compute the three components

$$\mathrm{M}_{\beta,\mathfrak{D}} := \lim_{l \to \infty} \frac{1}{|\mathfrak{D}_l|} \sum_{x \in \mathfrak{D}_l} \mathfrak{g}_l (\mathrm{M}^x) \in \mathbb{R}^3 \tag{84}$$

of the magnetization densities in the non–empty open region $\mathfrak{D} \subset \mathfrak{C}$. See (17) and (20). Away from critical points,

$$\mathrm{M}_{\beta,\mathfrak{D}} = \frac{1}{|\mathfrak{D}|} \int_{\mathfrak{D}} \mathrm{M}_{\beta,\mathfrak{t}} \, d^3\mathfrak{t} \in [-\vartheta, \vartheta]^3 \tag{85}$$

with $\vartheta \in \mathbb{R}^+$ and $\mathrm{M}_{\beta,\mathfrak{t}} \equiv \mathrm{M}_{\beta,\mathfrak{t}}(\mathrm{B})$ defined by (62).

In particular, in the limit $(\beta \to \infty)$ of low temperatures,

$$|\mathrm{M}_{\beta,\mathfrak{D}}| = \mathcal{O}(e^{-\beta(\mathrm{h_c}-\mathrm{h})}) \quad \text{and} \quad \mathrm{r}_{\beta,\mathfrak{D}} = \mathcal{O}(\mathrm{r}_\beta) \tag{86}$$

whenever, for all $\mathfrak{t} \in \mathfrak{D}$ (a.e.),

$$h_{\mathfrak{t}} \leq \mathrm{h} < \mathrm{h_c} \equiv \mathrm{h_c}(\mathrm{B}) := g_{\mathrm{r}_\beta} - \lambda \tag{87}$$

with $g_{\mathrm{r}_\beta} := \{(\mu - \lambda)^2 + \gamma^2 \mathrm{r}_\beta\}^{1/2}$. However, a strong and local magnetic induction such that $h_{\mathfrak{t}} \geq \mathrm{h} > \mathrm{h_c}$ (a.e.) on some non–empty open set $\mathfrak{D} \subseteq \mathfrak{C}$ implies a strong magnetization on $\mathfrak{D}_l := 2l\mathfrak{D} \cap \Lambda_l$, even if a global superconducting phase exists, that is, even if $\mathrm{r}_\beta \in \mathbb{R}^+$. In this case, $|\mathrm{M}_{\beta,\mathfrak{D}}| = \mathcal{O}(\vartheta)$, $\mathrm{r}_{\beta,\mathfrak{D}} = \mathcal{O}(e^{-\beta(\mathrm{h}-\mathrm{h_c})})$ and the magnetic induction expels the Cooper pair condensate from the (macroscopic) region $\mathfrak{D} \subseteq \mathfrak{C}$.

Meanwhile, away from critical points, the electron density

$$\mathrm{d}_{\beta,\mathfrak{D}} := \lim_{l \to \infty} \frac{1}{|\mathfrak{D}_l|} \sum_{x \in \mathfrak{D}_l} \mathfrak{g}_l (n_{x,\uparrow} + n_{x,\downarrow})$$

with $\mathfrak{D}_l := 2l\mathfrak{D} \cap \Lambda_l$ equals

$$\mathrm{d}_{\beta,\mathfrak{D}} := \frac{1}{|\mathfrak{D}|} \int_{\mathfrak{D}} \mathrm{d}_{\beta,\mathfrak{t}} \, d^3\mathfrak{t} \in [0, 2] \tag{88}$$

for any non–empty open region $\mathfrak{D} \subseteq \mathfrak{C}$, where, for all $\mathfrak{t} \in \mathfrak{C}$ (a.e.),

$$\mathrm{d}_{\beta,\mathfrak{t}} := 1 + \frac{(\mu - \lambda)\sinh(\beta g_{\mathrm{r}_\beta})}{g_{\mathrm{r}_\beta} (e^{\beta\lambda} \cosh(\beta h_{\mathfrak{t}}) + \cosh(\beta g_{\mathrm{r}_\beta}))} . \tag{89}$$

In particular, the electron density is space–homogeneous whenever the magnetic induction B is a.e. constant in space within the unit box $\mathfrak{C}$.

Apart from its physical interpretation (83) as the (global) Cooper pair condensate density, the solution r_β is extremely useful because it allows a construction of approximating minimizers of the free–energy in finite boxes. Indeed, let

$$\bar{u}_l(r, \mathfrak{t}) \equiv \bar{u}_l(r, \mathfrak{t}, \mathrm{B}) := \int_{\mathfrak{C}} u\left(r, \mathfrak{t} + \frac{y}{2l}\right) d^3 y$$

for all $r \in \mathbb{R}_0^+$ and $\mathfrak{t} \in \mathfrak{C}$. We define such approximating states by (well–defined, cf. [17, Theorem 11.2.]) product states of the form

$$\mathfrak{g}_{l,r} \equiv \mathfrak{g}_{l,r,\mathrm{B}} := \bigotimes_{x \in \Lambda_l} \omega_{(2l)^{-1}x,r} \in E_{\Lambda_l} \qquad (90)$$

for all $l \in \mathbb{N}$ and $r \in \mathbb{R}_0^+$, where $\omega_{(2l)^{-1}x,r} \equiv \omega_{(2l)^{-1}x,r,\mathrm{B}}$ is the (even) Gibbs state on $\mathcal{U}_{\{x\}}$ associated with the one–site Hamiltonian $\alpha_x(\bar{u}_l(r, (2l)^{-1}x))$ and thus defined by the density matrix

$$\frac{\mathrm{e}^{-\beta \alpha_x(\bar{u}_l(r,(2l)^{-1}x))}}{\mathrm{Trace}_{\wedge \mathcal{H}_{\{x\}}}(\mathrm{e}^{-\beta \alpha_x(\bar{u}_l(r,(2l)^{-1}x))})}$$

for all $x \in \mathbb{Z}^3$. Here, for every $x \in \mathbb{Z}^3$, α_x is the translation map from $\mathcal{U}_{\Lambda_l}$ to the C^*–algebra $\mathcal{U}_{\Lambda_l+x}$ with identity $\mathbf{1}$ and generators $\{a_{y+x,\mathrm{s}}\}_{y \in \Lambda_l, \mathrm{s} \in \{\uparrow,\downarrow\}}$. More precisely, for every $x \in \mathbb{Z}^3$, α_x is the isomorphism of C^*–algebras uniquely defined by the conditions

$$\alpha_x(a_{y,\mathrm{s}}) = a_{y+x,\mathrm{s}}, \qquad y \in \Lambda_l, \ \mathrm{s} \in \{\uparrow,\downarrow\} .$$

Then, as suggested by [3, Proposition 2], for any $\mathrm{B} \in L^2(\mathfrak{C}; \mathbb{R}^3)$, the product states $\{\mathfrak{g}_{l,\mathrm{r}_\beta}\}_{l \in \mathbb{N}}$ minimize the free–energy density functional f_l of the system in the thermodynamic limit $l \to \infty$. The proof is however more difficult than in [3, Proposition 2].

Similar to Lemma 14, we first consider continuous magnetic inductions $\mathrm{B} \in C^0(\mathfrak{C}; \mathbb{R}^3)$:

Lemma 17 (Approximating minimizers – I)
For any $\mathrm{B} \in C^0(\mathfrak{C}; \mathbb{R}^3)$ and any solution $\mathrm{r}_\beta = \mathrm{r}_\beta(\mathrm{B})$ of (75),

$$\lim_{l \to \infty} \left\{ f_l(\mathrm{B}, \mathfrak{g}_{l,\mathrm{r}_\beta}) - \inf_{\rho \in E_{\Lambda_l}} f_l(\mathrm{B}, \rho) \right\} = 0 .$$

Proof. For every $r \in \mathbb{R}_0^+$, define the continuous map $\mathfrak{p}_r$ from $\mathbb{R}$ to $\mathbb{R}$ by

$$x \mapsto \mathfrak{p}_r(x) := \frac{\gamma \sinh(\beta g_r)}{2g_r(\mathrm{e}^{\beta \lambda} \cosh(\beta x) + \cosh(\beta g_r))}$$

as well as $h_\mathfrak{t} \equiv h_\mathfrak{t}(\mathrm{B}) := \vartheta |\mathrm{B}(\mathfrak{t})|$ and

$$\bar{h}_{\mathfrak{t},l} \equiv \bar{h}_{\mathfrak{t},l}(\mathrm{B}) := \vartheta \left| \int_{\mathfrak{C}} \mathrm{B}\left(\mathfrak{t} + \frac{y}{2l}\right) \mathrm{d}^3 y \right|, \quad \mathfrak{t} \in \mathfrak{C}, \ l \in \mathbb{N} . \tag{91}$$

By explicit computations, for any $l \in \mathbb{N}$, $x \in \Lambda_l$, $\mathrm{B} \in C^0(\mathfrak{C}; \mathbb{R}^3)$ and $r \in \mathbb{R}_0^+$,

$$\omega_{(2l)^{-1}x,r}(a_{x,\downarrow} a_{x,\uparrow}) = \sqrt{r} \mathfrak{p}_r(\bar{h}_{(2l)^{-1}x,l}) \in \mathbb{R}, \qquad (92)$$

which, combined with the equicontinuity of $\mathfrak{p}_r$ and B on compact sets, yields

$$\lim_{l \to \infty} \left\{ \frac{1}{|\Lambda_l|} \sum_{x \in \Lambda_l} \omega_{(2l)^{-1}x,r}(a_{x,\downarrow} a_{x,\uparrow}) \right\} = \sqrt{r} \int_{\mathfrak{C}} \mathfrak{p}_r(h_\mathfrak{t}) \mathrm{d}^3 \mathfrak{t} . \tag{93}$$

In the same way, we show by explicit computations that

$$\lim_{l \to \infty} \left\{ \frac{1}{|\Lambda_l|} \sum_{x \in \Lambda_l} \ln \mathrm{Trace}_{\wedge \mathcal{H}_{\{x\}}}(\mathrm{e}^{-\beta \alpha_x(\bar{u}_l(r,(2l)^{-1}x))}) \right\}$$
$$= \mathfrak{F}(r, \mathrm{B}) + \gamma r \tag{94}$$

for any $\mathrm{B} \in C^0(\mathfrak{C}; \mathbb{R}^3)$ and $r \in \mathbb{R}_0^+$. Using the additivity of the von Neumann entropy of product states and the passivity of Gibbs states,

$$\sum_{x \in \Lambda_l} \mathfrak{g}_{l,r}\left(\alpha_x(\bar{u}_l(r,(2l)^{-1}x))\right) - \beta^{-1} S_l(\mathfrak{g}_{l,r}) \tag{95}$$
$$= -\beta^{-1} \sum_{x \in \Lambda_l} \ln \mathrm{Trace}_{\wedge \mathcal{H}_{\{x\}}}(\mathrm{e}^{-\beta \alpha_x(\bar{u}_l(r,(2l)^{-1}x))}) .$$

Since

$$H_l = \sum_{x \in \Lambda_l} \left(\alpha_x(\bar{u}_l(r,(2l)^{-1}x)) + \gamma\sqrt{r}(a_{x,\uparrow}^* a_{x,\downarrow}^* + a_{x,\downarrow} a_{x,\uparrow}) \right)$$
$$- \frac{\gamma}{|\Lambda_l|} \sum_{x,y \in \Lambda_l} a_{x,\uparrow}^* a_{x,\downarrow}^* a_{y,\downarrow} a_{y,\uparrow} ,$$

we infer from (8) and (93)–(95) that

$$\lim_{l \to \infty} f_l(\mathrm{B}, \mathfrak{g}_{l,r}) = -\mathfrak{F}(r, \mathrm{B}) - \gamma r \left(1 - \int_{\mathfrak{C}} \mathfrak{p}_r(h_\mathfrak{t}) \mathrm{d}^3 \mathfrak{t} \right)^2 \tag{96}$$

for any $\mathrm{B} \in C^0(\mathfrak{C}; \mathbb{R}^3)$ and $r \in \mathbb{R}_0^+$. In particular, by using the gap equation (76),

$$\lim_{l \to \infty} f_l(\mathrm{B}, \mathfrak{g}_{l,\mathrm{r}_\beta}) = -\mathfrak{F}(\mathrm{r}_\beta, \mathrm{B}) . \tag{97}$$

The latter yields the lemma because of (10), (75) and Theorem 15 (i). $\square$

Similar to Theorem 15 (i), we now extend Lemma 17 to all $\mathrm{B} \in L^2(\mathfrak{C}; \mathbb{R}^3)$ by using the density of $C^0(\mathfrak{C}; \mathbb{R}^3)$ in $L^2(\mathfrak{C}; \mathbb{R}^3)$:

Theorem 18 (Approximating minimizers – II)
For any $\mathrm{B} \in L^2(\mathfrak{C}; \mathbb{R}^3)$ and any solution $\mathrm{r}_\beta = \mathrm{r}_\beta(\mathrm{B})$ of (75),

$$\lim_{l \to \infty} \left\{ f_l(\mathrm{B}, \mathfrak{g}_{l,\mathrm{r}_\beta}) - \inf_{\rho \in E_{\Lambda_l}} f_l(\mathrm{B}, \rho) \right\} = 0 .$$

Proof. We start by proving the norm equicontinuity of the collection

$$\{\mathrm{B} \mapsto f_l(0, \mathfrak{g}_{l,r,\mathrm{B}})\}_{l \in \mathbb{N}} \tag{98}$$

of maps from $L^2(\mathfrak{C}; \mathbb{R}^3)$ to $\mathbb{R}$. To this end, we study, for all $\mathfrak{t} \in \mathfrak{C}$ and $l \in \mathbb{N}$, the maps

$$\mathrm{B} \mapsto \mathrm{d}_{\mathfrak{t},l}(\mathrm{B}) := \frac{\mathrm{e}^{-\beta \bar{u}_l(r,\mathfrak{t},\mathrm{B})}}{\mathrm{Trace}_{\wedge \mathcal{H}_{\{0\}}}(\mathrm{e}^{-\beta \bar{u}_l(r,\mathfrak{t},\mathrm{B})})}$$

from $L^2(\mathfrak{C}; \mathbb{R}^3)$ to the real space of self–adjoint elements of $\mathcal{U}_{\{0\}}$. Let $\|-\|_{\mathrm{Tr}}$ be the trace norm of $\mathcal{U}_{\{0\}}$. Observe that

$$\|\mathrm{d}_{\mathfrak{t},l}(\mathrm{B}) - \mathrm{d}_{\mathfrak{t},l}(\mathrm{C})\|_{\mathrm{Tr}} \leq \frac{2 \left\|\mathrm{e}^{-\beta \bar{u}_l(r,\mathfrak{t},\mathrm{B})} - \mathrm{e}^{-\beta \bar{u}_l(r,\mathfrak{t},\mathrm{C})}\right\|_{\mathrm{Tr}}}{\left\|\mathrm{e}^{-\beta \bar{u}_l(r,\mathfrak{t},\mathrm{B})}\right\|_{\mathrm{Tr}}}$$

for any $B, C \in L^2(\mathfrak{C}; \mathbb{R}^3)$. Using Duhamel's formula

$$e^{A_1} - e^{A_2} = \int_0^1 e^{\tau A_1} (A_1 - A_2) e^{(1-\tau)A_2} d^3\tau \,,$$

$\|A_1\| \leq \|A_1\|_{\mathrm{Tr}}$ and $\|A_1 A_2\|_{\mathrm{Tr}} \leq \|A_1\|\|A_2\|_{\mathrm{Tr}}$ for any $A_1, A_2 \in \mathcal{U}_{\{0\}}$, we then find that

$$\|\mathrm{d}_{\mathfrak{t},l}(B) - \mathrm{d}_{\mathfrak{t},l}(C)\|_{\mathrm{Tr}} \leq 2\beta \left\| \int_{\mathfrak{C}} (B - C) \left(\mathfrak{t} + \frac{y}{2l}\right) d^3 y \cdot M^0 \right\|_{\mathrm{Tr}}$$
$$\times \int_0^1 \mathfrak{n}(B, C, \tau) \, d^3\tau \qquad (99)$$

with

$$\mathfrak{n}(B, C, \tau) := \frac{\left\| e^{-\beta \tau \bar{u}_l(r,\mathfrak{t},B)} \right\| \left\| e^{-\beta(1-\tau)\bar{u}_l(r,\mathfrak{t},C)} \right\|}{\left\| e^{-\beta \bar{u}_l(r,\mathfrak{t},B)} \right\|} \,.$$

Straightforward computations show that

$$\left\| e^{-\beta \tau \bar{u}_l(r,\mathfrak{t},B)} \right\| = e^{\beta\tau\left(\mu + \max\{g_r - \lambda, \bar{h}_{\mathfrak{t},l}(B)\}\right)}$$

for any $B \in L^2(\mathfrak{C}; \mathbb{R}^3)$ and all $\tau \in [0, 1]$. Thus,

$$\mathfrak{n}(B, C, \tau) = e^{\beta(1-\tau)\left(\max\{g_r - \lambda, \bar{h}_{\mathfrak{t},l}(C)\} - \max\{g_r - \lambda, \bar{h}_{\mathfrak{t},l}(B)\}\right)}$$

and, by (99) and $\|\mathfrak{m}_j^x\| \leq 2\vartheta$ for any $j \in \{1, 2, 3\}$ and all $x \in \mathbb{Z}^3$,

$$\|\mathrm{d}_{\mathfrak{t},l}(B) - \mathrm{d}_{\mathfrak{t},l}(C)\|_{\mathrm{Tr}} \leq 4\sqrt{3}\beta\bar{h}_{\mathfrak{t},l}(B - C)$$
$$\times \int_0^1 \mathfrak{n}(B, C, \tau) \, d^3\tau$$

with $\bar{h}_{\mathfrak{t},l}$ defined by (91).

If $\bar{h}_{\mathfrak{t},l}(B - C) \leq 1$, then we deduce from the last two assertions that

$$\|\mathrm{d}_{\mathfrak{t},l}(B) - \mathrm{d}_{\mathfrak{t},l}(C)\|_{\mathrm{Tr}} \leq 4\sqrt{3}\left(e^\beta - 1\right)\bar{h}_{\mathfrak{t},l}(B - C) \,.$$

On the other hand, for any $B, C \in L^2(\mathfrak{C}; \mathbb{R}^3)$,

$$\|\mathrm{d}_{\mathfrak{t},l}(B) - \mathrm{d}_{\mathfrak{t},l}(C)\|_{\mathrm{Tr}} \leq \|\mathrm{d}_{\mathfrak{t},l}(B)\|_{\mathrm{Tr}} + \|\mathrm{d}_{\mathfrak{t},l}(C)\|_{\mathrm{Tr}} = 2 \,.$$

Therefore, for all $B, C \in L^2(\mathfrak{C}; \mathbb{R}^3)$,

$$\|\mathrm{d}_{\mathfrak{t},l}(B) - \mathrm{d}_{\mathfrak{t},l}(C)\|_{\mathrm{Tr}} \leq 4\sqrt{3}e^\beta \bar{h}_{\mathfrak{t},l}(B - C) \,.$$

We now use the product structure (90) of $\mathfrak{g}_{l,r}$, the uniform norm Lipschitz continuity of the von Neumann entropy and the last bound to deduce the existence of a finite constant $D \in \mathbb{R}^+$ such that

$$|f_l(0, \mathfrak{g}_{l,r,B}) - f_l(0, \mathfrak{g}_{l,r,C})| \leq \frac{D}{|\Lambda_l|} \sum_{x \in \Lambda_l} \bar{h}_{(2l)^{-1}x,l}(B - C)$$

for all $B, C \in L^2(\mathfrak{C}; \mathbb{R}^3)$ and all $l \in \mathbb{N}$. By definition of $\bar{h}_{\mathfrak{t},l}$ and the Cauchy–Schwarz inequality, we thus find that

$$|f_l(0, \mathfrak{g}_{l,r,B}) - f_l(0, \mathfrak{g}_{l,r,C})| \leq D\|B - C\|_2 \qquad (100)$$

for all $B, C \in L^2(\mathfrak{C}; \mathbb{R}^3)$ and all $l \in \mathbb{N}$. In other words, the collection (98) is norm equicontinuous.

We want now to prove from the last inequality that the collection

$$\{B \mapsto f_l(B, \mathfrak{g}_{l,r,B})\}_{l \in \mathbb{N}} \qquad (101)$$

of maps from $L^2(\mathfrak{C}; \mathbb{R}^3)$ to $\mathbb{R}$ is also norm equicontinuous. So, we need to show the norm equicontinuity of the family

$$\{B \mapsto \langle B, \mathfrak{m}_l(B)\rangle_2\}_{l \in \mathbb{N}} \qquad (102)$$

of maps from $L^2(\mathfrak{C}; \mathbb{R}^3)$ to $\mathbb{R}$, where

$$\mathfrak{m}_l(B)(\mathfrak{t}) := \sum_{x \in \Lambda_l} \mathbf{1}\left[2l\mathfrak{t} \in (\mathfrak{C} + x)\right] \mathfrak{g}_{l,r,B}(M^x) \,.$$

Indeed, for any $B, C \in L^2(\mathfrak{C}; \mathbb{R}^3)$,

$$|\langle B, \mathfrak{m}_l(B)\rangle_2 - \langle C, \mathfrak{m}_l(C)\rangle_2| \qquad (103)$$
$$\leq 2\sqrt{3}\vartheta\|B - C\|_2 + \|C\|_2\|\mathfrak{m}_l(B) - \mathfrak{m}_l(C)\|_2 \,,$$

using the Cauchy–Schwarz inequality and $\|\mathfrak{m}_j^x\| \leq 2\vartheta$. On the other hand, for any $B, C \in L^2(\mathfrak{C}; \mathbb{R}^3)$,

$$\|\mathfrak{m}_l(B) - \mathfrak{m}_l(C)\|_2^2 \qquad (104)$$
$$= \frac{1}{|\Lambda_l|} \sum_{x \in \Lambda_l} \left| \omega_{(2l)^{-1}x,r,B}(M^x) - \omega_{(2l)^{-1}x,r,C}(M^x) \right|^2 \,.$$

By explicit computations, for any $B \in L^2(\mathfrak{C}; \mathbb{R}^3)$,

$$\omega_{(2l)^{-1}x,r,B}(M^x) = \mathfrak{q}_r(\bar{h}_{(2l)^{-1}x,l}(B))\vartheta \int_{\mathfrak{C}} B\left(\frac{x+y}{2l}\right) d^3 y \,.$$
$$(105)$$

Here, for any $x \in \mathbb{R}^+$ and $r \in \mathbb{R}_0^+$,

$$\mathfrak{q}_r(x) := \frac{\vartheta \sinh(\beta x)}{x\left(\cosh(\beta x) + e^{-\beta\lambda}\cosh(\beta g_r)\right)} \,,$$

whereas at $x = 0$,

$$\mathfrak{q}_r(0) := \frac{\vartheta\beta}{1 + e^{-\beta\lambda}\cosh(\beta g_r)} \,.$$

Assume that $\bar{h}_{(2l)^{-1}x,l}(C) \leq 2$. Notice that there is a finite constant $D \in \mathbb{R}^+$ such that $|\mathfrak{q}_r(x)| \leq D$ for all $x \in \mathbb{R}_0^+$ and

$$|\mathfrak{q}_r(x) - \mathfrak{q}_r(y)| \leq D|x - y| \,, \quad x, y \in \mathbb{R}_0^+ \,,$$

by the mean value theorem. Then, by (105), for any $B \in L^2(\mathfrak{C}; \mathbb{R}^3)$,

$$\left| \omega_{(2l)^{-1}x,r,B}(M^x) - \omega_{(2l)^{-1}x,r,C}(M^x) \right|$$
$$\leq 2D \left| \bar{h}_{(2l)^{-1}x,l}(B) - \bar{h}_{(2l)^{-1}x,l}(C) \right|$$
$$+ D\vartheta \left| \int_{\mathfrak{C}} (B - C)\left(\frac{x+y}{2l}\right) d^3 y \right| \,.$$

Using Jensen's inequality, $(a+b)^2 \leq 2a^2 + 2b^2$ and $(|a| - |b|)^2 \leq |a-b|^2$, we then deduce from the last upper bound that

$$\left| \omega_{(2l)^{-1}x,r,\mathrm{B}}(\mathrm{M}^x) - \omega_{(2l)^{-1}x,r,\mathrm{C}}(\mathrm{M}^x) \right|^2$$
$$\leq 10D^2\vartheta^2 \int_{\mathfrak{C}} \left| (\mathrm{B}-\mathrm{C})\left(\frac{x+y}{2l}\right) \right|^2 \mathrm{d}^3 y \ , \qquad (106)$$

provided that $\bar{h}_{(2l)^{-1}x,l}(\mathrm{C}) \leq 2$.

Assume now that $\bar{h}_{(2l)^{-1}x,l}(\mathrm{C}) \geq 2$ and $\bar{h}_{\mathfrak{t},l}(\mathrm{B}-\mathrm{C}) \leq 1$. Remark that there is a finite constant $D \in \mathbb{R}^+$ such that, for all $x \geq 2$ and $|x - y| \leq 1$,

$$|\mathfrak{q}_r(x) - \mathfrak{q}_r(y)| \leq D \, |x|^{-1} \, |x - y| \ ,$$

again by the mean value theorem. Similar to (106), one then gets that

$$\left| \omega_{(2l)^{-1}x,r,\mathrm{B}}(\mathrm{M}^x) - \omega_{(2l)^{-1}x,r,\mathrm{C}}(\mathrm{M}^x) \right|^2$$
$$\leq 4D^2\vartheta^2 \int_{\mathfrak{C}} \left| (\mathrm{B}-\mathrm{C})\left(\frac{x+y}{2l}\right) \right|^2 \mathrm{d}^3 y \ , \qquad (107)$$

provided that $\bar{h}_{(2l)^{-1}x,l}(\mathrm{C}) \geq 2$ and $\bar{h}_{\mathfrak{t},l}(\mathrm{B}-\mathrm{C}) \leq 1$.

In the same way, we observe that

$$\left| \omega_{(2l)^{-1}x,r,\mathrm{B}}(\mathrm{M}^x) - \omega_{(2l)^{-1}x,r,\mathrm{C}}(\mathrm{M}^x) \right|^2$$
$$\leq 48\vartheta^2 \leq 48\vartheta^4 \int_{\mathfrak{C}} \left| (\mathrm{B}-\mathrm{C})\left(\frac{x+y}{2l}\right) \right|^2 \mathrm{d}^3 y \quad (108)$$

whenever $\bar{h}_{\mathfrak{t},l}(\mathrm{B}-\mathrm{C}) \geq 1$.

We then infer from (106)–(108) the existence of a finite constant $D \in \mathbb{R}^+$ so that, for any $\mathrm{B}, \mathrm{C} \in L^2(\mathfrak{C}; \mathbb{R}^3)$ and all $x \in \Lambda_l$,

$$\left| \omega_{(2l)^{-1}x,r,\mathrm{B}}(\mathrm{M}^x) - \omega_{(2l)^{-1}x,r,\mathrm{C}}(\mathrm{M}^x) \right|^2$$
$$\leq D \int_{\mathfrak{C}} \left| (\mathrm{B}-\mathrm{C})\left(\frac{x+y}{2l}\right) \right|^2 \mathrm{d}^3 y \ .$$

Hence, we deduce from (104) that

$$\|\mathfrak{m}_l(\mathrm{B}) - \mathfrak{m}_l(\mathrm{C})\|_2^2 \leq D\|\mathrm{B}-\mathrm{C}\|_2^2 \qquad (109)$$

for any $\mathrm{B}, \mathrm{C} \in L^2(\mathfrak{C}; \mathbb{R}^3)$.

By (100), (103) and (109), the families (98) and (102) are norm equicontinuous and so is the collection (101). Using this, (73) and the uniform Lipschitz continuity of the function $\mathfrak{p}_r$ together with the density of $C^0(\mathfrak{C}; \mathbb{R}^3) \subset L^2(\mathfrak{C}; \mathbb{R}^3)$, Equation (96) holds for all $\mathrm{B} \in L^2(\mathfrak{C}; \mathbb{R}^3)$ and $r \in \mathbb{R}_0^+$. By the gap equation (76), one gets (97) for all $\mathrm{B} \in L^2(\mathfrak{C}; \mathbb{R}^3)$ which implies the assertion because of (10), (75) and Theorem 15 (i). $\qquad\square$

Therefore, the sequence $\{\mathfrak{g}_{l,r_\beta}\}_{l \in \mathbb{N}}$ of approximating minimizers is a good starting point to construct the states $\rho_l \in E_{\Lambda_l}$ of Theorem 4. To this end, we define from $\{\mathfrak{g}_{l,r_\beta}\}_{l \in \mathbb{N}}$ states manifesting some current in subregions

of the box Λ_l with very small volumes with respect to the total volume $|\Lambda_l| = (2l)^3$. The latter is done as follows:

Take two positive real numbers $\eta^\perp, \eta \in \mathbb{R}^+$ such that

$$0 < \eta^\perp < \eta < 1 \qquad (110)$$

and define the small elementary box

$$\mathfrak{G}_l := \mathbb{Z}^3 \cap \{[-\ell^\eta, \ell^\eta] \times [-\ell^{\eta^\perp}, \ell^{\eta^\perp}]^2\} \subset \Lambda_l$$

with $\ell := l - 1$ for $l > 1$. (Note indeed that $\Lambda_l := \{\mathbb{Z} \cap [-l, l-1]\}^3$.) More conditions on the constants $\eta^\perp, \eta$ will be fixed later. We denote now by $[t]$ the integer part of $t \in \mathbb{R}_0^+$ to define the set

$$\mathcal{R}_l := \{(k_1, k_2, k_3) \in \mathbb{Z}^3 : |k_1| \leq [\ell^{1-\eta}] \, , \ |k_{2,3}| \leq [\ell^{1-\eta^\perp}]\} \ .$$

For any $k \in \mathcal{R}_l$, we define the translated elementary boxes

$$\mathfrak{G}_{l,k} := \mathfrak{G}_l + (k_1[\ell^\eta], k_2[\ell^{\eta^\perp}], k_3[\ell^{\eta^\perp}]) \subset \Lambda_l \ .$$

To create currents from $\mathrm{I}_l^{x,y}$ (11) we perform some gauge transformation inside these elementary boxes. Indeed, for any $k \in \mathcal{R}_l$ and $t \in \{0, 1\}$, we use the automorphism $U_{k,t}$ of the C^*–algebra $\mathcal{U}_{\Lambda_{l+[\ell^\eta]}}$ uniquely defined by

$$\forall x \notin \mathfrak{G}_{l,k} \, , \ s \in \{\uparrow, \downarrow\} \, , \qquad U_{k,t}(a_{x,s}) := \mathrm{e}^{it\pi/2} a_{x,s} \ ,$$
$$\forall x \in \mathfrak{G}_{l,k} \, , \ s \in \{\uparrow, \downarrow\} \, , \qquad U_{k,t}(a_{x,s}) := \mathrm{e}^{i\mathrm{C}_k x_1/(2l)} a_{x,s} \ .$$

The real parameter $\mathrm{C}_k \in \mathbb{R}$ will be chosen as a function of the current density j to be produced by the system.

Take now any solution $r_\beta = r_\beta(\mathrm{B})$ of (75) for $\mathrm{B} \in L^2(\mathfrak{C}; \mathbb{R}^3)$ and any even state $\varpi \in E_{\{0\}}$ satisfying

$$\varpi_0(a_{0,\downarrow} a_{0,\uparrow}) = 1 \ . \qquad (111)$$

Such a state exists because -1 and 1 both belong to the spectrum of $\mathrm{Re}(a_{0,\downarrow} a_{0,\uparrow})$. We denote by $\varpi_x := \varpi_0 \circ \alpha_{-x}$ the corresponding translated state on $\mathcal{U}_{\{x\}}$ for any $x \in \mathbb{Z}^d$. For any $k \in \mathcal{R}_l$, define the state

$$\nu_k := \frac{1}{2}\left\{\left(\underset{x \in \Lambda_{l+[\ell^\eta]}\backslash\mathfrak{G}_{l,k}}{\otimes} \omega_{(2l)^{-1}x,r_\beta}\right)\right. \qquad (112)$$
$$\left.\bigotimes\left(\underset{x \in \mathfrak{G}_{l,k}}{\otimes} \varpi_x\right)\right\} \circ (U_{k,0} + U_{k,1})$$

satisfying $\nu_k(\mathrm{I}_l^{x,y}) = 0$ whenever $\{x, y\} \cap (\Lambda_l \backslash \mathfrak{G}_{l,k}) \neq \emptyset$. Note indeed that, for any $x \in \Lambda_{l+[\ell^\eta]}$,

$$\omega_{(2l)^{-1}x,r_\beta}(a_{x,\downarrow} a_{x,\uparrow}) \in \mathbb{R} \ ,$$

see (92) which clearly holds for all $\mathrm{B} \in L^2(\mathfrak{C}; \mathbb{R}^3)$. Finally, for any $l \in \mathbb{N}$, we set

$$\rho_l := \frac{1}{|\mathcal{R}_l| \, |\mathfrak{G}_l|} \sum_{k \in \mathcal{R}_l, \ x \in \mathfrak{G}_l} \nu_k \circ \alpha_x|_{E_{\Lambda_l}} \in E_{\Lambda_l} \ . \qquad (113)$$

This state implies currents, in general. Moreover, observe that the term

$$\underset{x\in\Lambda_l\backslash\mathfrak{G}_{l,k}}{\otimes}\omega_{(2l)^{-1}x,r_\beta}=\mathfrak{g}_{l,r_\beta}|_{\mathcal{U}_{\Lambda_l\backslash\mathfrak{G}_{l,k}}}\qquad(114)$$

in the definition (112) of ν_k is the restriction of the approximating minimizer $\mathfrak{g}_{l,r_\beta}$ (90) on $\mathcal{U}_{\Lambda_l\backslash\mathfrak{G}_{l,k}}$. Therefore, observing that H_l is gauge invariant, one can infer from Theorem 18 that $\{\rho_l\}_{l\in\mathbb{N}}$ is also a sequence of approximating minimizers:

Lemma 19 (Approximating minimizers – III)
For any $\mathrm{B}\in L^2(\mathfrak{C};\mathbb{R}^3)$ and any set $\{\mathrm{C}_k\}_{k\in\mathcal{R}_l}\subset\mathbb{R}$,

$$\lim_{l\to\infty}\left\{f_l(\mathrm{B},\rho_l)-\inf_{\rho\in E_{\Lambda_l}}f_l(\mathrm{B},\rho)\right\}=0\ .$$

Proof. Note that (110) implies that, for all $k\in\mathcal{R}_l$,

$$|\mathfrak{G}_{l,k}|=|\mathfrak{G}_l|=o(|\Lambda_l|)\ .$$

Therefore, we deduce from (8), Theorem 18, (112) and (114) that

$$\lim_{l\to\infty}\left\{f_l(\mathrm{B},\nu_k\circ\alpha_x|_{E_{\Lambda_l}})-\inf_{\rho\in E_{\Lambda_l}}f_l(\mathrm{B},\rho)\right\}=0\ ,$$

uniformly for all $k\in\mathcal{R}_l$ and $x\in\mathfrak{G}_l$. Since the state ρ_l is a convex combination of states $\{\nu_k\circ\alpha_x|_{E_{\Lambda_l}}\}_{k\in\mathcal{R}_l,x\in\mathfrak{G}_l}$, the assertion then follows from the convexity of the free–energy density f_l. $\qquad\square$

We use now the sequence $\{\rho_l\}_{l\in\mathbb{N}}$ of approximating minimizers to create a one–component current arbitrarily close to any prescribed smooth function $\mathrm{j}_1\in C_0^\infty(\mathfrak{C};\mathbb{R})$ at large $l\in\mathbb{N}$. With respect to Theorem 4, the function j_1 is the first coordinate of some smooth current density $\mathrm{j}\in C^\infty(\mathfrak{C};\mathbb{R}^3)$.

We now fix the real parameters $\{\mathrm{C}_k\}_{k\in\mathcal{R}_l}$ as follows: Observing that

$$\int_{-l^\eta}^{l^\eta}\int_{-l^\eta}^{l^\eta}(z_1-y_1)^2\,\mathrm{d}z_1\,\mathrm{d}y_1=\frac{8l^{4\eta}}{3}\ ,$$

we remark that

$$\mathrm{K}:=\lim_{l\to\infty}\left\{l^{-4\eta-4\eta^\perp}\sum_{y,z\in\mathfrak{G}_l}(z_1-y_1)^2\right\}=\frac{2^7}{3}\ .\qquad(115)$$

Then, we define the constant C_k by

$$\mathrm{C}_k:=\frac{8}{\mathrm{K}\gamma}l^{6-4\eta-4\eta^\perp}\mathrm{j}_1\left(\frac{k_1[\ell^\eta]}{2l},\frac{k_2[\ell^{\eta^\perp}]}{2l},\frac{k_3[\ell^{\eta^\perp}]}{2l}\right)\qquad(116)$$

for all $k\in\mathcal{R}_l$ and any $\mathrm{j}_1\in C_0^\infty(\mathfrak{C};\mathbb{R})$. With this choice of parameters, as $l\to\infty$, one produces indeed the current density $(\mathrm{j}_1(\mathrm{t}),0,0)$ for $\mathrm{t}\in\mathbb{R}^3$:

Lemma 20 (One–component currents)
Assume that (110) holds with $3\eta+4\eta^\perp>6$ and $\mathrm{j}_1\in C_0^\infty(\mathfrak{C};\mathbb{R})$. Then, for any $\varkappa<\min\{1-\eta,\eta-\eta^\perp\}$,

$$\sup_{\mathrm{t}\in\mathbb{R}^3}\left|j_{\rho_l}^{(l)}(\mathrm{t})-(\mathrm{j}_1(\mathrm{t}),0,0)\right|=o(l^{-\varkappa})$$

with the rescaled current density $j_{\rho_l}^{(l)}$ defined by (15).

Proof. By (112), observe that, for any $k\in\mathcal{R}_l$ and $x_0\in\mathfrak{G}_l$, the state $\nu_{k,x_0}:=\nu_k\circ\alpha_{x_0}|_{E_{\Lambda_l}}\in E_{\Lambda_l}$ creates a current density $j_{\nu_{k,x_0}}(x)$ at $x\in\mathbb{R}^3$ which is equal to

$$j_{\nu_{k,x_0}}(x)=\frac{4\gamma}{|\Lambda_l|}\sum_{y,z\in\mathfrak{G}_{l,k},\ y\neq z}\frac{y-z}{|y-z|^3}\xi\left(\frac{x+x_0-\frac{y+z}{2}}{|y-z|}\right)$$
$$\times\sin\left[\mathrm{C}_k(y_1-z_1)l^{-1}\right]\ .\qquad(117)$$

See Equation (13). Because of (116), note that the condition

$$\delta:=3\eta+4\eta^\perp-6>0$$

implies that, for any $k\in\mathcal{R}_l$,

$$l^\eta\mathrm{C}_k=\mathcal{O}(l^{-\delta})\qquad(118)$$

vanishes in the thermodynamic limit $l\to\infty$. As a consequence, we can deduce from (117) that

$$2l\,j_{\nu_{k,x_0}}(x)$$
$$=\frac{4\gamma}{|\Lambda_l|}\sum_{y,z\in\mathfrak{G}_{l,k},\ y\neq z}\frac{y-z}{|y-z|^3}\xi\left(\frac{x+x_0-\frac{y+z}{2}}{|y-z|}\right)$$
$$\times\left[2\mathrm{C}_k(y_1-z_1)+\mathcal{O}(l^{-2}|\mathrm{C}_k(y_1-z_1)|^3)\right],\ (119)$$

because $|z_1-y_1|\leq l^\eta$ for all $y,z\in\mathfrak{G}_{l,k}$. Note that the factor $2l$ above is related to the definition of the rescaled current density (15). The current density functional $\rho\mapsto j_\rho$ of the box Λ_l defined by (13) is affine. It follows that the current density induced by the approximating minimizer ρ_l (113) at $x\in\mathbb{R}^3$ equals

$$j_{\rho_l}(x)=\frac{1}{|\mathcal{R}_l|\,|\mathfrak{G}_l|}\sum_{k\in\mathcal{R}_l\ ,\ x_0\in\mathfrak{G}_l}j_{\nu_{k,x_0}}(x)\ .\qquad(120)$$

Since $\xi\in C_0^\infty(\mathbb{R}^3;\mathbb{R})$ is a smooth and compactly supported function, we infer from (118) that the norm of the vector

$$\frac{4\gamma}{l^2\,|\Lambda_l|\,|\mathcal{R}_l|\,|\mathfrak{G}_l|}\sum_{k\in\mathcal{R}_l\ ,\ x_0\in\mathfrak{G}_l}\sum_{y,z\in\mathfrak{G}_{l,k},\ y\neq z}\frac{y-z}{|y-z|^3}$$
$$\times\xi\left(\frac{x+x_0-\frac{y+z}{2}}{|y-z|}\right)|\mathrm{C}_k(y_1-z_1)|^3$$

converges to zero as $l\to\infty$ faster than l^{-1}, uniformly for $x\in\mathbb{R}^3$. Using the explicit parameters (116), the smoothness of j_1 as well as (119), we can rewrite (120) as

$$j_{\rho_l}^{(l)}(\mathrm{t})=\frac{1}{|\mathcal{R}_l|\,|\mathfrak{G}_l|}\sum_{k\in\mathcal{R}_l}\sum_{2l\mathfrak{s}_k\in\mathfrak{G}_{l,k}}\chi_l(\mathrm{t}-\mathfrak{s}_k)$$
$$\times\left(\mathrm{j}_1(\mathfrak{s}_k)+\mathcal{O}\left(l^{\eta-1}\right)\right)+o(l^{\eta-1})\ ,\quad(121)$$

uniformly for all $\mathfrak{t} \in \mathbb{R}^3$. Here, the function $\chi_{\scriptscriptstyle l} \in C_0^\infty$ is defined, for all $\mathfrak{t} \in \mathbb{R}^3$, by

$$\chi_{\scriptscriptstyle l}(\mathfrak{t}) := \frac{8}{\mathrm{K}} l^{3-4\eta-4\eta^\perp} \sum_{y,z \in \mathfrak{G}_l} \frac{y-z}{|y-z|^3}(y_1-z_1)$$
$$\times \xi\left(\frac{2l\mathfrak{t} - \frac{y+z}{2}}{|y-z|}\right) .$$

By (12), note that

$$\int_{\mathbb{R}^3} \chi_{\scriptscriptstyle l}(\mathfrak{t})\,\mathrm{d}^3\mathfrak{t} = \frac{1}{\mathrm{K}} l^{-4\eta-4\eta^\perp} \sum_{y,z \in \mathfrak{G}_l}(y-z)(y_1-z_1) .$$

Observe also that we have chosen the constant K (115) in the definition of C_k to have exactly the limit

$$\lim_{l \to \infty} \int_{\mathbb{R}^3} \chi_{\scriptscriptstyle l}(\mathfrak{t})\,\mathrm{d}^3\mathfrak{t} = (1,0,0) .$$

More precisely,

$$\int_{\mathbb{R}^3} \chi_{\scriptscriptstyle l}(\mathfrak{t})\,\mathrm{d}^3\mathfrak{t} - (1,0,0) = \mathcal{O}(l^{\eta^\perp - \eta}) .$$

Since $\xi \in C_0^\infty(\mathbb{R}^3;\mathbb{R})$ is compactly supported, the support $\mathrm{supp}(\chi_{\scriptscriptstyle l})$ of $\chi_{\scriptscriptstyle l} \in C_0^\infty$ has radius

$$\sup\{|x| : \chi_{\scriptscriptstyle l}(x) \neq 0\} = \mathcal{O}(l^{\eta-1}) \qquad (122)$$

and belongs to a sufficiently large box $[-L, L]^3$, $L > 0$, for all $l \in \mathbb{N}$:

$$[-L, L]^3 \supset \bigcup_{l \in \mathbb{N}} \mathrm{supp}(\chi_{\scriptscriptstyle l}) . \qquad (123)$$

In fact, the sequence $\{\chi_{\scriptscriptstyle l}\}_{l \in \mathbb{N}}$, seen as a family of distributions, converges to the delta function, as $l \to \infty$.

Now therefore, the right hand side of (121) approximates the convolution $\chi_{\scriptscriptstyle l} * \mathrm{j}_1(\mathfrak{t})$ since the sum is a Riemann sum. By (121), it is then straightforward to verify the existence of a constant D not depending on (a sufficiently large) $l \in \mathbb{N}$ and $\mathfrak{t} \in \mathfrak{C}$ such that

$$\left| j_{\rho_l}^{(l)}(\mathfrak{t}) - (\mathrm{j}_1(\mathfrak{t}), 0, 0) \right| \qquad (124)$$
$$\leq D \sup_{\mathfrak{s} \in \mathrm{supp}(\chi_{\scriptscriptstyle l})} \left| \mathrm{j}_1(\mathfrak{s} + \mathfrak{t}) - \mathrm{j}_1(\mathfrak{t}) \right| + D(l^{\eta-1} + l^{\eta^\perp-\eta}) .$$

The continuity of $\mathrm{j}_1 \in C_0^\infty(\mathfrak{C};\mathbb{R})$ implies its equicontinuity on any compact set. Hence, by (122)–(123),

$$\sup_{\mathfrak{t} \in \mathfrak{C}} \sup_{\mathfrak{s} \in \mathrm{supp}(\chi_{\scriptscriptstyle l})} \left| \mathrm{j}_1(\mathfrak{s} + \mathfrak{t}) - \mathrm{j}_1(\mathfrak{t}) \right| = \mathcal{O}(l^{\eta-1}) . \qquad (125)$$

The lemma is then a consequence of (124)–(125). $\qquad \square$

Lemma 20 can be extended to all current densities $\mathrm{j} \in C^\infty(\mathfrak{C};\mathbb{R}^3)$. See Theorem 4. Meanwhile, as explained in Section III, for every approximating minimizer ρ_l (112)–(113), the current density $j_{\rho_l}^{(l)} \in C_0^\infty$ can be decomposed into longitudinal and transverse components $(j_{\rho_l}^{(l)})^\| = P^\| j_{\rho_l}^{(l)}$ and $(j_{\rho_l}^{(l)})^\perp = P^\perp j_{\rho_l}^{(l)}$, respectively. So, we conclude this section by showing that the energy norm of $(j_{\rho_l}^{(l)})^\|$ is negligible as $l \to \infty$ whenever $\nabla \cdot \mathrm{j} = 0$:

Lemma 21 (Energy norm estimates)
Assume that (110) holds with $3\eta + 4\eta^\perp > 6$ and $\mathrm{j}_1 \in C_0^\infty(\mathfrak{C};\mathbb{R})$. Then, for any $\varkappa < \min\{1-\eta, \eta-\eta^\perp\}$,

$$\|\mathrm{j}^\perp - (j_{\rho_l}^{(l)})^\perp\|_\mathfrak{H} \leq \|\mathrm{j} - j_{\rho_l}^{(l)}\|_\mathfrak{H} = o(l^{-\varkappa}) .$$

In particular, if j is divergence–free (i.e., $\mathrm{j} = \mathrm{j}^\perp$), then

$$\|(j_{\rho_l}^{(l)})^\|\|_\mathfrak{H} = o(l^{-\varkappa}) .$$

Proof. First, observe that

$$\|j\|_\mathfrak{H}^2 \leq \|j\|_2^2 + \int_{\{k \in \mathbb{C} \,:\, |k|^2 \leq 1\}} \frac{|F[j](k)|^2}{|k|^2}\mathrm{d}^3 k \qquad (126)$$

for all $j \in L^2 \cap \mathfrak{H}$, where $F[j]$ is the Fourier transform of j. Therefore, the assertion follows from Theorem 4 together with the fact that $P^\|$, $P^\perp$ are mutually orthogonal projections. Recall that, for some sufficiently large $L \in \mathbb{R}^+$ and all $l \in \mathbb{N}$, the support (16) of $j_{\rho_l}^{(l)}$ is contained in the box $[-L, L]^3$. $\qquad \square$

C. Thermodynamics with Self–Generated Magnetic Inductions

We analyze now the thermodynamics corresponding to the magnetic free–energy density functionals $\mathcal{F}_l^{(\epsilon)}$ defined by (49) on the sets E_{Λ_l} of states for all $l \in \mathbb{N}$ and $\epsilon \in \mathbb{R}^+$. By contrast with the previous section, the magnetic induction $\mathrm{B} = B_\rho^{(l)}$ (47) is now self–generated by the system in the state $\rho \in E_{\Lambda_l}$.

We first need to compute the thermodynamic limit $\mathcal{P}_\infty^{(\epsilon)}$ of the magnetic pressure (53), that is,

$$\mathcal{P}_l^{(\epsilon)} := -\inf_{\rho \in E_{\Lambda_l}^\perp} \mathcal{F}_l^{(\epsilon)}(\rho) , \qquad l \in \mathbb{N} , \; \epsilon \in \mathbb{R}^+,$$

where $E_{\Lambda_l}^\perp$ is defined by (52). This requires various arguments and we present them in several lemmata.

Recall Equation (51) which is actually satisfied for all $\mathrm{B} \in L^2$:

$$\langle \mathrm{B}, m_\rho^{(l)} \rangle_2 = f_l(0,\rho) - f_l(\mathfrak{T}_\epsilon \mathrm{B}, \rho) , \quad \mathrm{B} \in L^2, \; \rho \in E_{\Lambda_l} . \qquad (127)$$

Here, $\mathfrak{T}_\epsilon$ is the Hilbert–Schmidt operator defined, for any $\epsilon \in \mathbb{R}^+$, by

$$\mathfrak{T}_\epsilon \mathrm{B} := \mathbf{1}[\mathfrak{t} \in \mathfrak{C}](\xi_\epsilon * \mathrm{B}) , \qquad \mathrm{B} \in L^2 . \qquad (128)$$

See also (18). In particular, $\mathfrak{T}_\epsilon$ has Hilbert–Schmidt norm equal to $\epsilon^{-3/2}\|\xi\|_2$ but its operator norm satisfies $\|\mathfrak{T}_\epsilon\| \leq 1$, because of $\|\xi_\epsilon\|_1 = 1$ and Young's inequality. We also add that

$$\lim_{\epsilon \to 0^+} \|(\mathfrak{T}_\epsilon - \mathbf{1}[\mathfrak{t} \in \mathfrak{C}])\mathrm{B}\|_2 = 0 , \qquad \mathrm{B} \in L^2 . \qquad (129)$$

The latter can easily be proven for all $\mathrm{B} \in C_0^\infty$ by direct estimates. Then, one uses the density of C_0^∞ in L^2 as well

as $\|\mathfrak{T}_\epsilon\| \leq 1$ for any $\epsilon \in \mathbb{R}^+$ to get (129), i.e., the strong convergence of $\mathfrak{T}_\epsilon$ as $\epsilon \to 0^+$ towards the (non–compact) operator $\mathfrak{T}_0$ defined by

$$\mathfrak{T}_0 B := \mathbf{1}\left[t \in \mathfrak{C}\right] B , \qquad B \in L^2 . \qquad (130)$$

Obviously, $\|\mathfrak{T}_0\| \leq 1$.

The first step is to study the collection $\{B \mapsto p_l(\mathfrak{T}_\epsilon B)\}_{l \in \mathbb{N} \cup \{\infty\}}$ of maps from L^2 to $\mathbb{R}$ at any fixed $\epsilon \in \mathbb{R}^+$. Indeed, since $\mathfrak{T}_\epsilon$ is a compact operator for every $\epsilon \in \mathbb{R}^+$, such maps have much stronger continuity properties than the maps $B \mapsto p_l(B)$ analyzed for all $l \in \mathbb{N} \cup \{\infty\}$ in Theorem 15. An important additional feature at any $\epsilon \in \mathbb{R}^+$ is the weak equicontinuity of the collection $\{B \mapsto p_l(\mathfrak{T}_\epsilon B)\}_{l \in \mathbb{N}}$ of maps on any ball

$$b_R(0) := \left\{B \in L^2 : \|B\|_2 \leq R\right\} \qquad (131)$$

of radius $R \in \mathbb{R}^+$ centered at 0. The latter is a consequence of the following lemma:

Lemma 22 (Magnetic interaction energy)
The family $\{B \mapsto \langle B, m_\rho^{(l)}\rangle_2\}_{l \in \mathbb{N}, \rho \in E_{\Lambda_l}}$ of maps from $b_R(0)$ to $\mathbb{R}$ is equicontinuous in the weak topology.

Proof. Since $\mathfrak{T}_\epsilon$ is a Hilbert–Schmidt operator satisfying $\|\mathfrak{T}_\epsilon\| \leq 1$ for every $\epsilon \in \mathbb{R}^+$, it is compact and its singular value decomposition is

$$\mathfrak{T}_\epsilon = \sum_{n=1}^\infty \lambda_n |v_n\rangle\langle w_n| ,$$

where $\{v_n\}_{n=1}^\infty, \{w_n\}_{n=1}^\infty$ are orthonormal bases of L^2 and $\{\lambda_n\}_{n=1}^\infty \subset [0,1]$ is a set of real numbers satisfying

$$\sum_{n=1}^\infty \lambda_n^2 < \infty .$$

Take any $\varepsilon \in \mathbb{R}^+$. Then, there is $N \in \mathbb{N}$ such that

$$\left\| \mathfrak{T}_\epsilon - \sum_{n=1}^N \lambda_n |v_n\rangle\langle w_n| \right\| \leq \frac{\varepsilon}{8R\sqrt{3}\vartheta} .$$

Choose now $B \in L^2$ and $\delta := \varepsilon/(4N\sqrt{3}\vartheta)$. Meanwhile, remark that $\|\rho\| = 1$ and $\|m_j^x\| \leq 2\vartheta$ for any $j \in \{1,2,3\}$ and all $x \in \mathbb{Z}^3$, see (17). Therefore, by (127) and the Cauchy–Schwarz inequality,

$$\left| \langle C - B, m_\rho^{(l)}\rangle_2 \right| \leq \varepsilon , \quad B \in b_R(0) , \ C \in \mathcal{V}_\delta(B) ,$$

where $\mathcal{V}_\delta(B)$ is the weak neighborhood

$$\mathcal{V}_\delta(B) := \left\{ C \in b_R(0) : \sup_{n \in \{1,\ldots,N\}} |\langle C - B, w_n\rangle_2| \leq \delta \right\} .$$

In other words, the maps $B \mapsto \langle B, m_\rho^{(l)}\rangle_2$ from $b_R(0)$ to $\mathbb{R}$ are equicontinuous in the weak topology for all $l \in \mathbb{N}$ and $\rho \in E_{\Lambda_l}$. $\qquad \square$

We now use Lemma 22 to prove a stronger version of Ascoli's theorem [18, Theorem A.5] for the weak equicontinuous family $\{p_l(\mathfrak{T}_\epsilon B)\}_{l \in \mathbb{N}}$ at fixed $\epsilon \in \mathbb{R}^+$: $p_l(\mathfrak{T}_\epsilon B)$ converges to $p_\infty(\mathfrak{T}_\epsilon B)$ as $l \to \infty$ (and not only along a subsequence), uniformly for any $B \in b_R(0)$.

Lemma 23 (Uniform convergence of pressures)
For any $\epsilon \in \mathbb{R}^+$, the sequence $\{p_l(\mathfrak{T}_\epsilon B)\}_{l \in \mathbb{N}}$ is a uniformly Cauchy sequence on any ball $b_R(0) \subset L^2$ of arbitrary radius $R \in \mathbb{R}^+$ centered at 0.

Proof. For any $R \in \mathbb{R}^+$, the ball $b_R(0)$ is weakly compact in L^2 (Banach–Alaoglu theorem) and the weak topology is metrizable on $b_R(0)$, see, e.g., [4, Theorem 10.10]. Denote by d_R any metric on $b_R(0)$ generating the weak topology. Define also by

$$\mathfrak{b}_\delta(B) := \{C \in b_R(0) : d_R(B,C) < \delta\}$$

the weak ball of radius $\delta \in \mathbb{R}^+$ centered at $B \in L^2$. Balls $\mathfrak{b}_\delta(B)$ are clearly weakly open sets in $b_R(0)$. Thus, using the weak compactness of $b_R(0)$ as well as the weak density of C^0 in L^2, for any $\delta \in \mathbb{R}^+$, there is a finite number $N_\delta \in \mathbb{N}$ of continuous centers $\{B^{(n)}\}_{n=1}^{N_\delta} \subset b_R(0) \cap C^0$ such that

$$b_R(0) = \bigcup_{n=1}^{N_\delta} \mathfrak{b}_\delta(B^{(n)}) . \qquad (132)$$

Fix $\epsilon \in \mathbb{R}^+$. From Lemma 22, the collection $\{B \mapsto p_l(\mathfrak{T}_\epsilon B)\}_{l \in \mathbb{N}}$ of maps from $b_R(0)$ to $\mathbb{R}$ is equicontinuous in the weak topology. See also (10) and (127). By the weak compactness and metrizability of the ball $b_R(0)$, the family $\{B \mapsto p_l(\mathfrak{T}_\epsilon B)\}_{l \in \mathbb{N}}$ is uniformly equicontinuous in the weak topology: For any $\varepsilon \in \mathbb{R}^+$ there is $\delta \in \mathbb{R}^+$ such that, for all $l \in \mathbb{N} \cup \{\infty\}$, $B \in b_R(0)$ and $C \in \mathfrak{b}_\delta(B)$,

$$|p_l(\mathfrak{T}_\epsilon B) - p_l(\mathfrak{T}_\epsilon C)| \leq \frac{\varepsilon}{3} . \qquad (133)$$

By Lemma 14, for any $\varepsilon \in \mathbb{R}^+$, there is $L \in \mathbb{R}^+$ such that, for any $n \in \{1,\ldots,N_\delta\}$ and integers $l_1, l_2 > L$,

$$\left| p_{l_1}(\mathfrak{T}_\epsilon B^{(n)}) - p_{l_2}(\mathfrak{T}_\epsilon B^{(n)}) \right| \leq \frac{\varepsilon}{3} . \qquad (134)$$

By (132), (133) and (134), for any $\varepsilon \in \mathbb{R}^+$, there is $L \in \mathbb{R}^+$ such that, for all $B \in b_R(0)$ and integers $l_1, l_2 > L$,

$$|p_{l_1}(\mathfrak{T}_\epsilon B) - p_{l_2}(\mathfrak{T}_\epsilon B)| \leq \varepsilon .$$

$\qquad \square$

We now use Lemmata 22 and 23 to deduce a stronger version of Theorem 15:

Theorem 24 (Infinite volume pressure – II)
Let $b_R(0) \subset L^2$ be any ball of radius $R \in \mathbb{R}^+$ centered at 0, see (131). Then, for any $\epsilon \in \mathbb{R}^+$, one has:
(i) The pressure $p_l(\mathfrak{T}_\epsilon B)$ converges to $p_\infty(\mathfrak{T}_\epsilon B)$ uniformly on $b_R(0)$, as $l \to \infty$. See Theorem 15 (i).
(ii) The family $\{B \mapsto p_l(\mathfrak{T}_\epsilon B)\}_{l \in \mathbb{N} \cup \{\infty\}}$ of maps from $b_R(0)$ to $\mathbb{R}$ is equicontinuous in the weak topology.

Proof. Both assertions (i) and (ii) are direct consequences of Lemmata 22 and 23 combined with (10) and (127). □

Recall that $\mathfrak{T}_\epsilon$ is defined, for any $\epsilon \in \mathbb{R}_0^+$, by (128) and (130) and always satisfy $\|\mathfrak{T}_\epsilon\| \leq 1$. We now study the variational problems defined, for all $\epsilon \in \mathbb{R}_0^+$ and $l \in \mathbb{N} \cup \{\infty\}$, by

$$\mathfrak{B}_l^{(\epsilon)} := \inf_{B \in \mathcal{B}} \left\{ \frac{1}{2} \|B + B_{\mathrm{ext}}\|_2^2 - p_l \left(\mathfrak{T}_\epsilon B + \mathfrak{T}_\epsilon B_{\mathrm{ext}} \right) \right\}$$
(135)

with $\mathcal{B}$ defined by (57). We break this further preliminary analysis in three short Lemmata. Note that Lemmata 26 and 27 both exclude the case $\epsilon = 0$.

Lemma 25 (Variational problems $\mathfrak{B}_l^{(\epsilon)}$ – I)
For any $B_{\mathrm{ext}} \in L^2$, there is $R \in \mathbb{R}^+$ such that, for all $\epsilon \in \mathbb{R}_0^+$ and $l \in \mathbb{N} \cup \{\infty\}$,

$$\mathfrak{B}_l^{(\epsilon)} = \inf_{B \in \mathcal{B} \cap b_R(0)} \left\{ \frac{1}{2} \|B + B_{\mathrm{ext}}\|_2^2 - p_l \left(\mathfrak{T}_\epsilon B + \mathfrak{T}_\epsilon B_{\mathrm{ext}} \right) \right\} ,$$

where $b_R(0) \subset L^2$ is the ball (131) of radius $R \in \mathbb{R}^+$ centered at 0.

Proof. The assertion is a direct consequence of $\|\mathfrak{T}_\epsilon\| \leq 1$ together with the uniform Lipschitz continuity of the collection $\{B \mapsto p_l(B)\}_{l \in \mathbb{N} \cup \{\infty\}}$ of maps from $L^2(\mathfrak{C}; \mathbb{R}^3)$ to $\mathbb{R}$, see Theorem 15 (ii). We omit the details. □

Lemma 26 (Variational problems $\mathfrak{B}_l^{(\epsilon)}$ – II)
For any $B_{\mathrm{ext}} \in L^2$, all $\epsilon \in \mathbb{R}^+$ and $l \in \mathbb{N} \cup \{\infty\}$, there is a sequence $\{B_\epsilon^{(l,n)}\}_{n \in \mathbb{N}} \subset \mathcal{S}_0(\mathcal{J})$ converging in norm to $B_\epsilon^{(l)} \in \mathcal{B}$ as $n \to \infty$ such that

$$\lim_{n \to \infty} \left\{ \frac{1}{2} \|B_\epsilon^{(l,n)} + B_{\mathrm{ext}}\|_2^2 - p_l(\mathfrak{T}_\epsilon(B_\epsilon^{(l,n)} + B_{\mathrm{ext}})) \right\}$$
$$= \frac{1}{2} \|B_\epsilon^{(l)} + B_{\mathrm{ext}}\|_2^2 - p_l(\mathfrak{T}_\epsilon(B_\epsilon^{(l)} + B_{\mathrm{ext}})) = \mathfrak{B}_l^{(\epsilon)}.$$

Proof. The map $B \mapsto \|B\|_2$ is lower semi–continuous in the weak topology, whereas $B \mapsto p_l(\mathfrak{T}_\epsilon B)$ is weakly continuous on any ball $b_R(0)$ for all $\epsilon \in \mathbb{R}^+$ and $l \in \mathbb{N} \cup \{\infty\}$, see Theorem 24 (ii). Using these properties together with Lemma 25, the weak closure $\mathcal{B}$ of $\mathcal{S}_0(\mathcal{J})$ and the weak compactness of $b_R(0)$, we deduce the existence of (a possibly non–unique minimizer) $B_\epsilon^{(l)} \in \mathcal{B}$ such that

$$\mathfrak{B}_l^{(\epsilon)} = \frac{1}{2} \|B_\epsilon^{(l)} + B_{\mathrm{ext}}\|_2^2 - p_l(\mathfrak{T}_\epsilon B_\epsilon^{(l)} + \mathfrak{T}_\epsilon B_{\mathrm{ext}})$$

for any $\epsilon \in \mathbb{R}^+$ and $l \in \mathbb{N} \cup \{\infty\}$.

Meanwhile, the set $\mathcal{S}_0(\mathcal{J})$ with $\mathcal{J}$ defined by (56) is a convex subset of the Hilbert space L^2. Therefore, we infer from [18, Theorem 3.12] that its weak closure $\mathcal{B}$ (57) coincides with the norm closure of $\mathcal{S}_0(\mathcal{J})$. In particular, for any $\epsilon \in \mathbb{R}^+$ and $l \in \mathbb{N} \cup \{\infty\}$, there is a sequence $\{B_\epsilon^{(l,n)}\}_{n \in \mathbb{N}} \subset \mathcal{S}_0(\mathcal{J})$ converging in norm to $B_\epsilon^{(l)} \in \mathcal{B}$, as $n \to \infty$. Since, by Theorem 15 (ii), the maps $B \mapsto \|B\|_2$

and $B \mapsto p_l(\mathfrak{T}_\epsilon B)$ are both norm continuous, we deduce that

$$\lim_{n \to \infty} \left\{ \frac{1}{2} \|B_\epsilon^{(l,n)} + B_{\mathrm{ext}}\|_2^2 - p_l(\mathfrak{T}_\epsilon B_\epsilon^{(l,n)} + \mathfrak{T}_\epsilon B_{\mathrm{ext}}) \right\}$$
$$= \frac{1}{2} \|B_\epsilon^{(l)} + B_{\mathrm{ext}}\|_2^2 - p_l(\mathfrak{T}_\epsilon B_\epsilon^{(l)} + \mathfrak{T}_\epsilon B_{\mathrm{ext}})$$

for any $\epsilon \in \mathbb{R}^+$ and $l \in \mathbb{N} \cup \{\infty\}$. □

Lemma 27 (Variational problems $\mathfrak{B}_l^{(\epsilon)}$ – III)
For any $B_{\mathrm{ext}} \in L^2$ and $\epsilon \in \mathbb{R}^+$, $\lim_{l \to \infty} \mathfrak{B}_l^{(\epsilon)} = \mathfrak{B}_\infty^{(\epsilon)}$.

Proof. By Lemmata 25–26, there are $R \in \mathbb{R}^+$ and minimizers $B_\epsilon^{(l)} \in \mathcal{B}$ of $\mathfrak{B}_l^{(\epsilon)}$ satisfying $B_\epsilon^{(l)} \in b_R(0)$ for all $\epsilon \in \mathbb{R}^+$ and $l \in \mathbb{N} \cup \{\infty\}$. Therefore, the lemma follows from the uniform convergence of $p_l(\mathfrak{T}_\epsilon B)$ towards $p_\infty(\mathfrak{T}_\epsilon B)$ on $b_R(0)$, see Theorem 24 (i). □

Even if the map $B \mapsto \|B\|_2$ is lower semi–continuous in the weak topology and although Lemma 25 also holds for $\epsilon = 0$ and $l = \infty$, the existence of minimizer(s) of the variational problem $\mathfrak{B}_\infty^{(0)}$ is far from being clear. Indeed, one can check that the map $B \mapsto p_\infty(B + B_{\mathrm{ext}})$ is not upper semi–continuous in the weak topology. Nevertheless, $\mathfrak{B}_\infty^{(0)}$ can be obtained from $\mathfrak{B}_\infty^{(\epsilon)}$ by taking the limit $\epsilon \to 0^+$:

Lemma 28 (Variational problems $\mathfrak{B}_\infty^{(\epsilon)}$ – I)
For any $B_{\mathrm{ext}} \in L^2$, $\lim_{\epsilon \to 0^+} \mathfrak{B}_\infty^{(\epsilon)} = \mathfrak{B}_\infty^{(0)}$.

Proof. Take any sequence $\{B_{0,n}\}_{n \in \mathbb{N}} \subset \mathcal{B}$ of approximating minimizers of $\mathfrak{B}_\infty^{(0)}$, that is,

$$\mathfrak{B}_\infty^{(0)} = \lim_{n \to \infty} \left\{ \frac{1}{2} \|B_{0,n} + B_{\mathrm{ext}}\|_2^2 - p_\infty(B_{0,n} + B_{\mathrm{ext}}) \right\} .$$
(136)

Then, for any $\epsilon \in \mathbb{R}^+$ and every $n \in \mathbb{N}$, $\mathfrak{B}_l^{(\epsilon)}$ is by definition bounded from above by

$$\mathfrak{B}_\infty^{(\epsilon)} \leq \frac{1}{2} \|B_{0,n} + B_{\mathrm{ext}}\|_2^2 - p_\infty(\mathfrak{T}_\epsilon(B_{0,n} + B_{\mathrm{ext}})) .$$
(137)

The operator $\mathfrak{T}_\epsilon$ converges in the strong topology to $\mathfrak{T}_0$, as $\epsilon \to 0^+$. See (129) and (130). Moreover, the map $B \mapsto p_\infty(B)$ is uniformly Lipschitz continuous, by Theorem 15 (ii). It follows that

$$\lim_{\epsilon \to 0^+} p_\infty(\mathfrak{T}_\epsilon(B_{0,n} + B_{\mathrm{ext}})) = p_\infty(B_{0,n} + B_{\mathrm{ext}}) .$$

Combining this with (136) and (137), we then obtain the upper bound

$$\limsup_{\epsilon \to 0^+} \mathfrak{B}_\infty^{(\epsilon)} \leq \mathfrak{B}_\infty^{(0)}$$
(138)

for any $B_{\mathrm{ext}} \in L^2$.

On the other hand, we note that the map

$$x \mapsto \mathfrak{h}_r(x) := \beta^{-1} \ln \left\{ \cosh(\beta \vartheta |x|) + \mathrm{e}^{-\lambda \beta} \cosh(\beta g_r) \right\}$$
(139)

from $\mathbb{R}^3$ to $\mathbb{R}^+$ is a convex function at any fixed $(\beta, \mu, \lambda, \gamma, \vartheta, r)$. Using this together with $\|\xi_\epsilon\|_1 = 1$ and Jensen's inequality, we find that, for any $B \in L^2$ and $\mathfrak{t} \in \mathfrak{C}$ (a.e),

$$\mathfrak{h}_r \left((\xi_\epsilon * B)(\mathfrak{t}) \right) \leq \xi_\epsilon * (\mathfrak{h}_r \circ B)(\mathfrak{t}) \ ,$$

which in turn implies

$$\mathfrak{F}(r, \xi_\epsilon * B) \leq \mu + \beta^{-1} \ln 2 - \gamma r + \int_{\mathfrak{C}} \xi_\epsilon * (\mathfrak{h}_r \circ B)(\mathfrak{t}) \, d^3\mathfrak{t}$$
$$(140)$$

for any $\epsilon \in \mathbb{R}^+$, $r \in \mathbb{R}_0^+$ and $B \in L^2$, see (74). Using Fubini's theorem, for any $\epsilon \in (0, \epsilon_\xi)$, $r \in \mathbb{R}_0^+$ and $B \in L^2$, we get the equality

$$\int_{\mathfrak{C}} \xi_\epsilon * (\mathfrak{h}_r \circ B)(\mathfrak{t}) \, d^3\mathfrak{t}$$
$$= \int_{\mathbb{R}^3} \xi_\epsilon(\mathfrak{s}) \int_{\mathfrak{C} \backslash \mathfrak{C}_\epsilon} \mathfrak{h}_r(B(\mathfrak{t} - \mathfrak{s})) \, d^3\mathfrak{t} \, d^3\mathfrak{s}$$
$$+ \int_{\mathbb{R}^3} \xi_\epsilon(\mathfrak{s}) \int_{\mathfrak{C}_\epsilon} \mathfrak{h}_r(B(\mathfrak{t} - \mathfrak{s})) \, d^3\mathfrak{t} \, d^3\mathfrak{s} \ , \quad (141)$$

where, for any $\epsilon < \epsilon_\xi$,

$$\mathfrak{C}_\epsilon := \left\{ \mathfrak{t} \in \mathfrak{C} : \inf \{ |\mathfrak{t} - \mathfrak{s}| : \mathfrak{s} \in \partial \mathfrak{C} \} > \epsilon R_\xi \right\} \ . \quad (142)$$

Here, $\partial \mathfrak{C}$ is the boundary of $\mathfrak{C}$ and $\epsilon_\xi := 1/(2R_\xi)$ with R_ξ being the radius of the support of the function $\xi \in C_0^\infty$, see (18) and (22). By $\|\xi_\epsilon\|_1 = 1$ together with the Cauchy–Schwarz inequality and

$$|\mathfrak{h}_r(x)| \leq D(|x| + 1) \ , \qquad x \in \mathbb{R}^3,$$

for some finite constant $D \in \mathbb{R}^+$, the absolute value of the first integral in the right hand side of (141) is bounded by

$$D(|\mathfrak{C} \backslash \mathfrak{C}_\epsilon|^{1/2} \|B\|_2 + |\mathfrak{C} \backslash \mathfrak{C}_\epsilon|) \qquad (143)$$

for any $\epsilon \in (0, \epsilon_\xi)$, $r \in \mathbb{R}_0^+$ and $B \in L^2$. Meanwhile, using similar arguments,

$$\left| \int_{\mathbb{R}^3} \xi_\epsilon(\mathfrak{s}) \int_{\mathfrak{C}_\epsilon} \mathfrak{h}_r(B(\mathfrak{t} - \mathfrak{s})) \, d^3\mathfrak{t} \, d^3\mathfrak{s} - \int_{\mathfrak{C}} \mathfrak{h}_r(B(\mathfrak{t})) \, d^3\mathfrak{t} \right|$$
$$\leq D(|\mathfrak{C} \backslash \mathfrak{C}_\epsilon|^{1/2} \|B\|_2 + |\mathfrak{C} \backslash \mathfrak{C}_\epsilon|) \ . \qquad (144)$$

From (140)–(144) and Theorem 15 (i), we thus deduce that

$$p_\infty(\mathfrak{T}_\epsilon(B + B_{\text{ext}})) \leq p_\infty(B + B_{\text{ext}}) \qquad (145)$$
$$+ 2D(|\mathfrak{C} \backslash \mathfrak{C}_\epsilon|^{1/2} \|B\|_2 + |\mathfrak{C} \backslash \mathfrak{C}_\epsilon|)$$

for any $\epsilon \in (0, \epsilon_\xi)$, $B_{\text{ext}} \in L^2$ and all $B \in b_R(0)$ with $R \in \mathbb{R}^+$ being any fixed radius. As a consequence, for any $B_{\text{ext}} \in L^2$, there is $R \in \mathbb{R}^+$ such that, for all $\epsilon \in (0, \epsilon_\xi)$,

$$\mathfrak{B}_\infty^{(\epsilon)} \geq \mathfrak{B}_\infty^{(0)} - 2D(R|\mathfrak{C} \backslash \mathfrak{C}_\epsilon|^{1/2} + |\mathfrak{C} \backslash \mathfrak{C}_\epsilon|) \ , \qquad (146)$$

because of Lemma 25.

Since $|\mathfrak{C} \backslash \mathfrak{C}_\epsilon| = \mathcal{O}(\epsilon)$, we therefore combine the lower bound (146) in the limit $\epsilon \to 0^+$ with the upper bound (138) to arrive at the assertion. $\qquad \square$

We can now deduce that minimizers of $\mathfrak{B}_\infty^{(\epsilon)}$ are approximating minimizers of the variational problem $\mathfrak{B}_\infty^{(0)}$:

Lemma 29 (Variational problems $\mathfrak{B}_\infty^{(\epsilon)} -$ II)
Let $B_{\text{ext}} \in L^2$. Then, any family $\{B_\epsilon\}_{\epsilon \in \mathbb{R}^+} \subset \mathcal{B}$ of minimizers B_ϵ of $\mathfrak{B}_\infty^{(\epsilon)}$ minimizes $\mathfrak{B}_\infty^{(0)}$ in the limit $\epsilon \to 0^+$.

Proof. Take any family $\{B_\epsilon\}_{\epsilon \in \mathbb{R}^+} \subset \mathcal{B} \cap b_R(0)$ of minimizers B_ϵ of $\mathfrak{B}_\infty^{(\epsilon)}$, see Lemmata 25–26. By Lemma 28 and (145),

$$\lim_{\epsilon \to 0^+} \left\{ \frac{1}{2} \|B_\epsilon + B_{\text{ext}}\|_2^2 - p_\infty(B_\epsilon + B_{\text{ext}}) \right\} = \mathfrak{B}_\infty^{(0)} \ .$$
$$(147)$$

In other words, $\{B_\epsilon\}_{\epsilon \in \mathbb{R}^+}$ is a family of approximating minimizers of $\mathfrak{B}_\infty^{(0)}$. $\qquad \square$

Remark 30
By Lemma 25, the Banach–Alaoglu theorem and the separability of L^2, any family $\{B_\epsilon\}_{\epsilon \in \mathbb{R}^+}$ of minimizers of $\mathfrak{B}_\infty^{(\epsilon)}$ converges in the weak topology and along a subsequence to some $B_0 \in \mathcal{B} \cap b_R(0)$, as $\epsilon \to 0^+$. In general, B_0 may not be a minimizer of $\mathfrak{B}_\infty^{(0)}$. Sufficient conditions to ensure that B_0 is a minimizer of $\mathfrak{B}_\infty^{(0)}$ are given in Theorem 40.

We are now in position to obtain the magnetic pressures $\mathcal{P}_\infty^{(\epsilon)}$ as the variational problems $-\mathfrak{B}_\infty^{(\epsilon)}$ for all $\epsilon \in \mathbb{R}_0^+$. We start by considering the case $\epsilon \in \mathbb{R}^+$. The case $\epsilon = 0$ will then be a direct consequence of Lemma 28.

Theorem 31 (Infinite volume magnetic pressure)
Let $B_{\text{ext}} = \mathcal{S}_0(j_{\text{ext}})$ with $j_{\text{ext}} \in C_0^\infty \cap P^\perp \mathfrak{H}$. Then,

$$\mathcal{P}_\infty^{(\epsilon)} := \lim_{l \to \infty} \mathcal{P}_l^{(\epsilon)} = -\mathfrak{B}_\infty^{(\epsilon)} \ , \qquad \epsilon \in \mathbb{R}^+.$$

Proof. By (52), note first that

$$\mathcal{P}_l^{(\epsilon)} \leq - \inf_{j \in \mathcal{J}_l} \left\{ \frac{1}{2} \|\mathcal{S}_0(j) + B_{\text{ext}}\|_2^2 - p_l(\mathfrak{T}_\epsilon(\mathcal{S}_0(j) + B_{\text{ext}})) \right\}$$
$$(148)$$

with $\mathcal{J}_l$ being the set defined by

$$\mathcal{J}_l := \{ j \in \mathfrak{H} : \|j^\|\|_{\mathfrak{H}} \leq l^{-\varkappa} \} \cap C_0^\infty(\mathfrak{C}; \mathbb{R}^3) \ . \qquad (149)$$

Since $\ker \mathcal{S} = P^\| \mathfrak{H}$, i.e., $\mathcal{S}(j) = \mathcal{S}(j^\perp)$, we thus infer from (57), (135) and (148) that $\mathcal{P}_l^{(\epsilon)} \leq -\mathfrak{B}_l^{(\epsilon)}$ for any $\epsilon \in \mathbb{R}^+$ and $l \in \mathbb{N}$. In particular, in the limit $l \to \infty$ one gets $\mathcal{P}_\infty^{(\epsilon)} \leq -\mathfrak{B}_\infty^{(\epsilon)}$ for any $\epsilon \in \mathbb{R}^+$, using Lemma 27. It remains to show that, for any $\epsilon \in \mathbb{R}^+$, $-\mathfrak{B}_\infty^{(\epsilon)}$ is a lower bound of the magnetic pressure $\mathcal{P}_\infty^{(\epsilon)}$.

By Lemma 26, there is a norm convergent sequence $\{B_\epsilon^{(n)}\}_{n\in\mathbb{N}} \subset \mathcal{S}_0(\mathcal{J})$ such that

$$\mathfrak{B}_\infty^{(\epsilon)} = \lim_{n\to\infty} \left\{ \frac{1}{2}\|B_\epsilon^{(n)} + B_{\text{ext}}\|_2^2 - p_\infty(\mathfrak{T}_\epsilon(B_\epsilon^{(n)} + B_{\text{ext}})) \right\} .$$
(150)

Moreover, for any $n \in \mathbb{N}$, there is by definition a current density $j_\epsilon^{(n)} \in \mathcal{J}$ generating the magnetic induction $B_\epsilon^{(n)} = \mathcal{S}_0(j_\epsilon^{(n)})$. Therefore, by Lemmata 19, 21 and 22 together with (28) and (38), for any fixed $n \in \mathbb{N}$, there is a sequence $\{\rho_l\}_{l\in\mathbb{N}}$ of quasi–divergence–free states $\rho_l \in E_{\Lambda_l}^\perp$ satisfying

$$\lim_{l\to\infty} \|B_{\rho_l}^{(l)} - B_\epsilon^{(n)}\|_2 = 0 , \qquad n \in \mathbb{N} , \qquad (151)$$

and

$$\lim_{l\to\infty} \left\{ f_l(\mathfrak{T}_\epsilon B_{\rho_l}^{(l)}, \rho_l) - \inf_{\rho\in E_{\Lambda_l}} f_l(\mathfrak{T}_\epsilon(B_\epsilon^{(n)} + B_{\text{ext}}), \rho) \right\} = 0 .$$
(152)

Hence, the lower bound $\lim_{l\to\infty} \mathcal{P}_l^{(\epsilon)} \geq -\mathfrak{B}_\infty^{(\epsilon)}$ for any $\epsilon \in \mathbb{R}^+$ is a direct consequence of (51), (53), (150), (151) and (152) together with Theorem 15 (i). $\qquad\square$

Corollary 32 (Magnetic pressure for $\epsilon = 0$)
Let $B_{\text{ext}} = \mathcal{S}_0(j_{\text{ext}})$ with $j_{\text{ext}} \in C_0^\infty \cap P^\perp \mathfrak{H}$. Then,

$$\mathcal{P}_\infty := \lim_{\epsilon\to0^+} \mathcal{P}_\infty^{(\epsilon)} = -\mathfrak{B}_\infty^{(0)} .$$

Proof. See Lemma 28 and Theorem 31. $\qquad\square$

It remains to establish the relation between the solutions of the variational problem $\mathfrak{B}_\infty^{(\epsilon)}$ for $\epsilon \in \mathbb{R}^+$ and the sets $\mathbb{B}_\epsilon^{(\pm)}$ of all weak $(-)$ and norm $(+)$ cluster points of self–generated magnetic inductions $B_{\omega_{\epsilon,l}}^{(l)}$, see (58). This result is a relatively direct corollary of Theorems 24 and 31.

Corollary 33 (Magnetic inductions)
Let $\epsilon \in \mathbb{R}^+$ and $B_{\text{ext}} = \mathcal{S}_0(j_{\text{ext}})$ with $j_{\text{ext}} \in C_0^\infty \cap P^\perp \mathfrak{H}$. Then, $\mathbb{B}_\epsilon^{(+)} = \mathbb{B}_\epsilon^{(-)} \subset \mathcal{B}$ is a set of minimizers of $\mathfrak{B}_\infty^{(\epsilon)}$.

Proof. The inclusion $\mathbb{B}_\epsilon^{(+)} \subseteq \mathbb{B}_\epsilon^{(-)}$ is clear and $\mathbb{B}_\epsilon^{(-)} \neq \emptyset$, by weak compactness of balls. Take any $B_\epsilon \in \mathbb{B}_\epsilon^{(-)}$. By definition of $\mathbb{B}_\epsilon^{(-)}$, there is a subsequence $\{l_n\}_{n\in\mathbb{N}}$ such that $B_{\omega_{\epsilon,l_n}}^{(l_n)}$ converges in the weak topology to $B_\epsilon \in \mathbb{B}_\epsilon$, as $n \to \infty$. Note that $\{B_{\omega_{\epsilon,l_n}}^{(l_n)}\}_{n\in\mathbb{N}} \subset \mathcal{S}_0(\mathcal{J}_l)$ with $\mathcal{J}_l$ being the set defined by (149). Since $\ker \mathcal{S} = P^\| \mathfrak{H}$, we have $\{B_{\omega_{\epsilon,l_n}}^{(l_n)}\}_{n\in\mathbb{N}} \subset \mathcal{S}_0(\mathcal{J})$. Therefore,

$$\mathbb{B}_\epsilon^{(-)} \subset \mathcal{B} := \overline{\mathcal{S}_0(\mathcal{J})} ,$$

see (57). Using Theorem 24 (ii), the weak lower semi–continuity of the map $B \mapsto \|B\|_2$ as well as Theorem 31, we also deduce that $B_\epsilon \in \mathbb{B}_\epsilon^{(-)}$ must be a solution of the variational problem $\mathfrak{B}_\infty^{(\epsilon)}$ and

$$\lim_{n\to\infty} \|B_{\omega_{\epsilon,l_n}}^{(l_n)}\|_2 = \|B_\epsilon\|_2 . \qquad (153)$$

To prove the latter, use the equality

$$\|B_{\omega_{\epsilon,l_n}}^{(l_n)} + B\|_2^2 = \|B_{\omega_{\epsilon,l_n}}^{(l_n)}\|_2^2 + \|B\|_2^2 + 2\langle B_{\omega_{\epsilon,l_n}}^{(l_n)}, B\rangle_2 \quad (154)$$

for $B = B_{\text{ext}}$, as well as the weak continuity of the map

$$B \mapsto 2\langle B, B_{\text{ext}}\rangle_2 + p_\infty(\mathfrak{T}_\epsilon(B + B_{\text{ext}})) ,$$

see Theorem 24 (ii). It follows that $\mathbb{B}_\epsilon^{(-)}$ is a set of minimizers of $\mathfrak{B}_\infty^{(\epsilon)}$. Using (153) and (154) with $B = -B_\epsilon$, we deduce that $B_{\omega_{\epsilon,l_n}}^{(l_n)}$ converges in norm to B_ϵ, as $l \to \infty$. In other words, $\mathbb{B}_\epsilon^{(-)} \subseteq \mathbb{B}_\epsilon^{(+)}$. $\qquad\square$

By Lemma 29, this corollary also links in the limit $\epsilon \to 0^+$ the sets $\{\mathbb{B}_\epsilon^{(\pm)}\}_{\epsilon\in\mathbb{R}^+}$ to the approximating minimizers of the variational problem $\mathfrak{B}_\infty^{(0)}$.

We analyze now in detail the variational $\mathfrak{B}_\infty^{(\epsilon)}$ for all $\epsilon \in \mathbb{R}_0^+$. In the limit $\beta \to \infty$ of low temperatures, recall (86), that is, $|M_{\beta,\mathfrak{D}}| = \mathcal{O}(e^{-\beta(h_c-h)})$ whenever (87) is satisfied. It means that, as $\beta \to \infty$, the pressure p_∞ (B) does not depend much on magnetic inductions $B \in \mathcal{B}$ that satisfy (87) on $\mathfrak{C}$. Therefore, we first study the variational problem (63), that is,

$$\mathfrak{A} := \frac{1}{2} \inf_{B\in\mathcal{B}} \|B + B_{\text{ext}}\|_2^2 . \qquad (155)$$

Lemma 34 (Variational problem $\mathfrak{A}$)
Let $B_{\text{ext}} = \mathcal{S}_0(j_{\text{ext}})$ with $j_{\text{ext}} \in C_0^\infty \cap P^\perp \mathfrak{H}$. Then, there is a unique minimizer $B_{\text{int}} \in \mathcal{B}$ of $\mathfrak{A}$. The latter fulfills $B_{\text{int}} = -B_{\text{ext}}$ a.e. in $\mathfrak{C}$.

Proof. By [18, Theorem 3.12], recall that the weak closure $\mathcal{B}$ (57) coincides with the norm closure of $\mathcal{S}_0(\mathcal{J})$. By linearity of the Biot–Savart operator $\mathcal{S}$, we then conclude that $\mathcal{B} \subset P^\perp L^2$, equipped with the L^2–scalar product, is a sub–Hilbert space of L^2. As a consequence, by strict convexity and weak lower semi–continuity of the map $B \mapsto \|B\|_2$, there is a unique minimizer $B_{\text{int}} \in \mathcal{B}$ satisfying the Euler–Lagrange equations

$$\langle B_{\text{int}} + B_{\text{ext}}, B\rangle_2 = 0 , \qquad B \in \mathcal{B} . \qquad (156)$$

Now, since the space $C_0^\infty(\mathfrak{C};\mathbb{R}^3)$ is dense in $L^2(\mathfrak{C};\mathbb{R}^3)$, it suffices to prove (156) for all $B \in C_0^\infty(\mathfrak{C};\mathbb{R}^3)$ (instead of $B \in \mathcal{B}$). Take

$$j^\perp := \nabla \times B \in C_0^\infty(\mathfrak{C};\mathbb{R}^3)$$

for any $B \in C_0^\infty(\mathfrak{C};\mathbb{R}^3)$. Then, clearly, $\nabla \cdot j^\perp = 0$ and thus $j^\perp \in \mathcal{J}$. Moreover, as explained in Remark 7, $\mathcal{S}_0(j^\perp) = P^\perp B$ and $P^\perp B \in \mathcal{B}$. Since, by definition of the Biot–Savart operator, $B_{\text{int}} \in P^\perp L^2$ and, by assumption, $B_{\text{ext}} \in P^\perp L^2$, we infer from (156) at $B = P^\perp B$ that

$$\langle B_{\text{int}} + B_{\text{ext}}, P^\perp B\rangle_2 = \langle B_{\text{int}} + B_{\text{ext}}, B\rangle_2 = 0 \quad (157)$$

for any $B \in C_0^\infty(\mathfrak{C};\mathbb{R}^3)$. Indeed, $P^\|$ and $P^\perp$ are mutually orthogonal projections. See, e.g., (28). $\qquad\square$

Since $\mathcal{B}$ (57) is a closed space with respect to the L^2–norm (cf. [18, Theorem 3.12]), we use (38) to observe that $\mathcal{B} = \mathcal{S}(\overline{\mathcal{J}})$, where

$$\overline{\mathcal{J}} \subseteq P^\perp \mathfrak{H} \cap \overline{C_0^\infty(\mathfrak{C}; \mathbb{R}^3)}^{\|-\|_{\mathfrak{H}}}$$

is the (norm) closure of the set $\mathcal{J}$. As a consequence, Equations (38) and (155) yield (64), that is,

$$\mathfrak{A} = \mathfrak{J} := \frac{1}{2} \inf_{j^\perp \in \overline{\mathcal{J}}} \left\| j^\perp + j_{\text{ext}} \right\|_{\mathfrak{H}}^2 \ . \tag{158}$$

In particular, there is a one–to–one map from minimizers of (155) and minimizers of (158). By (37) and (156), the unique minimizer $j_{\text{int}}^\perp \in \overline{\mathcal{J}}$ satisfies the Euler–Lagrange equations

$$\langle j_{\text{int}}^\perp + j_{\text{ext}}, j^\perp \rangle_{\mathfrak{H}} = 0 \ , \qquad j^\perp \in \overline{\mathcal{J}} \ . \tag{159}$$

The latter implies that $j_{\text{int}}^\perp$ is a distribution supported on the boundary $\partial\mathfrak{C}$ of $\mathfrak{C}$, provided (41) holds:

Lemma 35 (Variational problem $\mathfrak{J}$)
Let $B_{\text{ext}} = \mathcal{S}_0(j_{\text{ext}})$ with $j_{\text{ext}} \in C_0^\infty \cap P^\perp \mathfrak{H}$. Assume additionally (41), that is, $\text{supp}(j_{\text{ext}}) \subset \mathbb{R}\backslash\mathfrak{C}$. Then, there is a unique minimizer $j_{\text{int}}^\perp \in \overline{\mathcal{J}}$ of $\mathfrak{J}$ which, as a distribution, is supported on the boundary $\partial\mathfrak{C}$ of $\mathfrak{C}$.

Proof. Uniqueness and existence is a direct consequence of Lemma 34, as explained after (158). Now, similar to (157), we deduce from (159) that

$$\langle j_{\text{int}}^\perp + j_{\text{ext}}, j \rangle_{\mathfrak{H}} = 0 \ , \qquad j \in C_0^\infty(\mathfrak{C}; \mathbb{R}^3) \ . \tag{160}$$

The current density j_{ext} and the minimizer $j_{\text{int}}^\perp$ both create vector potentials respectively equal to $\mathcal{A}(j_{\text{ext}})$ and $\mathcal{A}(j_{\text{int}}^\perp)$, see (65)–(67). By (160), the distributions $\mathcal{A}(j_{\text{ext}})$ and $-\mathcal{A}(j_{\text{int}}^\perp)$ are equal to each other within the unit box $\mathfrak{C}$:

$$\mathcal{A}(j_{\text{int}}^\perp) = -\mathcal{A}(j_{\text{ext}}) \ , \qquad \text{on } C_0^\infty(\mathfrak{C}; \mathbb{R}^3) \ . \tag{161}$$

Since $j_{\text{ext}} \in C_0^\infty \cap P^\perp \mathfrak{H}$ is by assumption supported on $\mathbb{R}\backslash\mathfrak{C}$, $-\Delta\mathcal{A}(j_{\text{ext}}) = 0$ (in the strong sense) on the unit box $\mathfrak{C}$, which, together with (161), implies that

$$-\Delta\mathcal{A}(j_{\text{int}}^\perp) = 0 \ , \qquad \text{on } C_0^\infty(\mathfrak{C}; \mathbb{R}^3) \ . \tag{162}$$

Combining this equality with (68) and $j_{\text{int}}^\perp \in \overline{\mathcal{J}}$, we arrive at the assertion. $\square$

Therefore, we deduce from Lemmata 34 and 35 that the solution $B_{\text{int}} = \mathcal{S}(j_{\text{int}}^\perp)$ of the variational problem $\mathfrak{A}$ (155) comes from surface currents $j_{\text{int}}^\perp \in \overline{\mathcal{J}}$ which annihilate a.e. all the total magnetic induction inside the bulk $\mathfrak{C}$. We take advantage of this property to analyze the full variational problem $\mathfrak{B}_\infty^{(\epsilon)}$ (135).

Like in Lemma 34, we start with a first consequence of the Euler–Lagrange equations associated with $\mathfrak{B}_\infty^{(\epsilon)}$ for any $\epsilon \in \mathbb{R}_0^+$. Recall that $\mathfrak{B}_\infty^{(\epsilon)}$ has minimizer(s) $B_\epsilon \in \mathcal{B}$ for all $\epsilon \in \mathbb{R}^+$ (cf. Lemma 26), but the existence of minimizer(s) of the variational problem $\mathfrak{B}_\infty^{(0)}$ is unclear.

Lemma 36 (Variational problems $\mathfrak{B}_\infty^{(\epsilon)}$ – III)
Let $\epsilon \in \mathbb{R}_0^+$ and $B_{\text{ext}} = \mathcal{S}_0(j_{\text{ext}})$ with $j_{\text{ext}} \in C_0^\infty \cap P^\perp \mathfrak{H}$. Assume that B_ϵ is a minimizer of $\mathfrak{B}_\infty^{(\epsilon)}$. Then,

$$B_\epsilon - B_{\text{int}} = M_\beta^\perp(\mathfrak{T}_\epsilon B_\epsilon + \mathfrak{T}_\epsilon B_{\text{ext}}) \quad \text{a.e. in } \mathfrak{C} \ .$$

Here, B_{int} is the unique minimizer of $\mathfrak{A}$ (Lemma 34), whereas $M_\beta^\perp = P^\perp M_\beta \in C^\infty$ is the transverse component of the magnetization density $M_\beta \equiv M_\beta(B)$ defined on $\mathbb{R}^3$ by (62) for all $B \in L^2$.

Proof. For any $r \in \mathbb{R}_0^+$ and all $B, C \in L^2$, the map $t \mapsto \mathfrak{F}(r, B + tC)$ from $\mathbb{R}$ to $\mathbb{R}$ is differentiable. Explicit computations show that

$$\partial_t \mathfrak{F}(r, B + tC)|_{t=0} \tag{163}$$
$$= \int_{\mathbb{R}^3} \frac{\mathbf{1}[t \in \mathfrak{C}]\,\vartheta \sinh(\beta h_t)}{\cosh(\beta h_t) + e^{-\beta\lambda}\cosh(\beta g_r)} \frac{B(t)}{|B(t)|} \cdot C(t) \ d^3t \ .$$

On the other hand, by (135) combined with Theorem 15 (i),

$$\mathfrak{B}_\infty^{(\epsilon)} = \inf_{r \geq 0} \inf_{B \in \mathcal{B}} \left\{ \frac{1}{2} \|B + B_{\text{ext}}\|_2^2 - \mathfrak{F}(r, \mathfrak{T}_\epsilon B + \mathfrak{T}_\epsilon B_{\text{ext}}) \right\} \tag{164}$$

for any $\epsilon \in \mathbb{R}_0^+$. Therefore, by (62), (75), (163) and (164), the corresponding Euler–Lagrange equations associated with $\mathfrak{B}_\infty^{(\epsilon)}$ read: For all $B \in \mathcal{B}$, all $r_\beta \in \mathbb{R}_0^+$ solution of (75), and any minimizer B_ϵ of $\mathfrak{B}_\infty^{(\epsilon)}$ (provided it exists when $\epsilon = 0$),

$$\langle B_\epsilon + B_{\text{ext}}, B \rangle_2 = \langle M_\beta(\mathfrak{T}_\epsilon B_\epsilon + \mathfrak{T}_\epsilon B_{\text{ext}}), B \rangle_2 \ . \tag{165}$$

(Note that M_β depends on r_β.) Using (156) and (165), we arrive at the equality

$$\langle B_\epsilon - B_{\text{int}}, B \rangle_2 = \langle M_\beta(\mathfrak{T}_\epsilon B_\epsilon + \mathfrak{T}_\epsilon B_{\text{ext}}), B \rangle_2 \ , \quad B \in \mathcal{B} \ , \tag{166}$$

from which one easily shows the assertion. See proof of Lemma 34 for more details. $\square$

Lemma 36 or Equation (166) yields a general estimate on the L^2–norm of solutions of $\mathfrak{B}_\infty^{(\epsilon)}$ and $\mathfrak{A}$:

Lemma 37 (Variational problems $\mathfrak{B}_\infty^{(\epsilon)}$ – IV)
Let $\epsilon \in \mathbb{R}_0^+$ and $B_{\text{ext}} = \mathcal{S}_0(j_{\text{ext}})$ with $j_{\text{ext}} \in C_0^\infty \cap P^\perp \mathfrak{H}$. Assume that B_ϵ is a minimizer of $\mathfrak{B}_\infty^{(\epsilon)}$. Then, $B_\epsilon - B_{\text{int}} \in b_\vartheta(0)$. Moreover, if $\epsilon < \epsilon_\xi$ (cf. (22)) then

$$\|B_\epsilon - B_{\text{int}}\|_2 \leq \|M_\beta(\mathfrak{T}_\epsilon(B_\epsilon - B_{\text{int}}))\|_2 + \vartheta|\mathfrak{C}\backslash\mathfrak{C}_\epsilon|^{1/2}$$

with $\vartheta|\mathfrak{C}\backslash\mathfrak{C}_\epsilon|^{1/2} = \mathcal{O}(\sqrt{\epsilon})$, see (142).

Proof. Assume that B_ϵ is a minimizer of $\mathfrak{B}_\infty^{(\epsilon)}$ for some $\epsilon \in \mathbb{R}_0^+$. We already know that such a minimizer exists for all $\epsilon \in \mathbb{R}^+$ (cf. Lemma 26). Since $B_\epsilon, B_{\text{int}} \in \mathcal{B}$, it follows from (166) applied to

$$B = \tilde{B}_\epsilon := B_\epsilon - B_{\text{int}} \in \mathcal{B}$$

that

$$\|\tilde{\mathrm{B}}_\epsilon\|_2^2 = \int_{\mathfrak{C}} \mathrm{M}_{\beta,\mathfrak{t}}(\mathfrak{T}_\epsilon(\mathrm{B}_\epsilon + \mathrm{B}_{\mathrm{ext}})) \cdot \tilde{\mathrm{B}}_\epsilon \ \mathrm{d}^3\mathfrak{t} \ . \qquad (167)$$

In particular, by (62) and the Cauchy–Schwarz inequality, we obtain $\|\tilde{\mathrm{B}}_\epsilon\|_2 \leq \vartheta$. In other words, $\mathrm{B}_\epsilon - \mathrm{B}_{\mathrm{int}} \in b_\vartheta(0)$, see (131).

Take now $\epsilon < \epsilon_\xi$. Then, $\mathfrak{C}_\epsilon \subseteq \mathfrak{C}$ is a non–empty set, see (142). By Lemma 34, $\mathfrak{T}_\epsilon \mathrm{B}_{\mathrm{int}} = -\mathfrak{T}_\epsilon \mathrm{B}_{\mathrm{ext}}$ a.e. in $\mathfrak{C}_\epsilon$. We thus rewrite (167) as

$$\|\tilde{\mathrm{B}}_\epsilon\|_2^2 = \int_{\mathfrak{C}\backslash\mathfrak{C}_\epsilon} \mathrm{M}_{\beta,\mathfrak{t}}(\mathfrak{T}_\epsilon(\mathrm{B}_\epsilon + \mathrm{B}_{\mathrm{ext}})) \cdot \tilde{\mathrm{B}}_\epsilon \ \mathrm{d}^3\mathfrak{t}$$
$$+ \int_{\mathfrak{C}_\epsilon} \mathrm{M}_\beta(\mathfrak{T}_\epsilon\tilde{\mathrm{B}}_\epsilon) \cdot \tilde{\mathrm{B}}_\epsilon \ \mathrm{d}^3\mathfrak{t} \ . \qquad (168)$$

By (62) and the Cauchy–Schwarz inequality, we deduce from (168) that

$$\|\tilde{\mathrm{B}}_\epsilon\|_2 \leq \vartheta|\mathfrak{C}\backslash\mathfrak{C}_\epsilon|^{1/2} + \|\mathrm{M}_\beta(\mathfrak{T}_\epsilon\tilde{\mathrm{B}}_\epsilon)\|_2 \ .$$

(This bound is proven in the same way as (143).) $\qquad\square$

Lemma 37 directly yields the suppression of the total magnetic induction within $\mathfrak{C}$ (a.e.) for sufficiently high temperatures because in this case the map $\mathrm{B} \mapsto \mathrm{M}_\beta(\mathrm{B})$ from L^2 to $L^2(\mathfrak{C};\mathbb{R}^3)$ satisfies $\|\mathrm{M}_\beta(\mathrm{B})\|_2 \leq \mathbf{m}\|\mathrm{B}\|_2$ with $\mathbf{m} < 1$:

Lemma 38 (Variational problems $\mathfrak{B}_\infty^{(\epsilon)}$ – V)
Let $\epsilon \in [0, \epsilon_\xi)$ and $\mathrm{B}_{\mathrm{ext}} = \mathcal{S}_0(\mathrm{j}_{\mathrm{ext}})$ with $\mathrm{j}_{\mathrm{ext}} \in C_0^\infty \cap P^\perp\mathfrak{H}$. Assume that B_ϵ is a minimizer of $\mathfrak{B}_\infty^{(\epsilon)}$. If $\beta < \vartheta^{-1}$ then

$$\|\mathrm{B}_\epsilon - \mathrm{B}_{\mathrm{int}}\|_2 \leq (\vartheta^{-1} - \beta)^{-1} |\mathfrak{C}\backslash\mathfrak{C}_\epsilon|^{1/2}$$

with $\vartheta|\mathfrak{C}\backslash\mathfrak{C}_\epsilon|^{1/2} = \mathcal{O}(\sqrt{\epsilon})$, see (142).

Proof. For all $\mathrm{B} \in L^2(\mathfrak{C};\mathbb{R}^3)$ and $\beta \in \mathbb{R}^+$, note that

$$\|\mathrm{M}_\beta(\mathrm{B})\|_2^2 \leq \int_{\mathfrak{C}} \vartheta^2\beta^2 h_\mathfrak{t}^2 \frac{\tanh^2(\beta h_\mathfrak{t})}{\beta^2 h_\mathfrak{t}^2} \ \mathrm{d}^3\mathfrak{t} \leq \vartheta^2\beta^2\|\mathrm{B}\|_2^2 \ ,$$

using (62) and $\tanh(t) \leq t$ for all $t \in \mathbb{R}_0^+$. Since $\|\mathfrak{T}_\epsilon\| \leq 1$ for any $\epsilon \in \mathbb{R}_0^+$, we thus arrive at the assertion by combining the last upper bound with $\vartheta\beta < 1$ and Lemma 37. $\qquad\square$

This last situation, i.e., the high temperature regime, is of course not the main case of interest. Moreover, it is questionable from the physical point of view, see discussions in Section VI (cf. **4.**). We are instead interested in showing the Meißner effect at large enough inverse temperatures $\beta \gg 1$ and large BCS couplings $\gamma \gg 1$ to ensure the presence of a superconducting phase.

Now therefore, we pursue our analysis of the variational problem $\mathfrak{B}_\infty^{(\epsilon)}$ by using the asymptotics $|\mathrm{M}_{\beta,\mathfrak{D}}| = \mathcal{O}(\mathrm{e}^{-\beta(\mathrm{h}_c - \mathrm{h})})$ (cf. (86)) whenever (87) is a.e. satisfied on some open subset $\mathfrak{D} \subseteq \mathfrak{C}$. We give below a sufficient (but *not* necessary) condition to prove in Theorem 40 the Meißner effect at low temperatures.

Lemma 39 (Variational problems $\mathfrak{B}_\infty^{(\epsilon)}$ – VI)
Let $\epsilon \in [0, \epsilon_\xi)$ and $\mathrm{B}_{\mathrm{ext}} = \mathcal{S}_0(\mathrm{j}_{\mathrm{ext}})$ with $\mathrm{j}_{\mathrm{ext}} \in C_0^\infty \cap P^\perp\mathfrak{H}$. Assume that B_ϵ is a minimizer of $\mathfrak{B}_\infty^{(\epsilon)}$ such that

$$\mathfrak{z} := \frac{\beta \cosh(\beta\mathrm{h})}{\mathrm{e}^{-\beta\lambda}\cosh(\beta g_{\mathrm{r}_\beta(\mathfrak{T}_\epsilon(\mathrm{B}_\epsilon - \mathrm{B}_{\mathrm{int}}))})} + \vartheta\mathrm{h}^{-1} < \vartheta^{-1}$$

for some $\mathrm{h} \in \mathbb{R}^+$. Here, $\mathrm{r}_\beta(\mathfrak{T}_\epsilon(\mathrm{B}_\epsilon - \mathrm{B}_{\mathrm{int}}))$ stands for any arbitrary solution of (75) with $\mathrm{B} = \mathfrak{T}_\epsilon(\mathrm{B}_\epsilon - \mathrm{B}_{\mathrm{int}})$. Then,

$$\|\mathrm{B}_\epsilon - \mathrm{B}_{\mathrm{int}}\|_2 \leq (\vartheta^{-1} - \mathfrak{z})^{-1} |\mathfrak{C}\backslash\mathfrak{C}_\epsilon|^{1/2}$$

with $\vartheta|\mathfrak{C}\backslash\mathfrak{C}_\epsilon|^{1/2} = \mathcal{O}(\sqrt{\epsilon})$, see (142).

Proof. Take any constant $\mathrm{h} \in \mathbb{R}^+$. The magnetization density $\mathrm{M}_\beta \equiv \mathrm{M}_\beta(\mathrm{B}) \in C_0^\infty$, defined by (62) for all $\mathrm{B} \in L^2(\mathfrak{C};\mathbb{R}^3)$, trivially satisfies

$$\|\mathrm{M}_\beta\|_2 \leq \|\mathbf{1}[h_\mathfrak{t} \leq \mathrm{h}]\mathrm{M}_\beta\|_2 + \|\mathbf{1}[h_\mathfrak{t} \geq \mathrm{h}]\mathrm{M}_\beta\|_2 \qquad (169)$$

with $h_\mathfrak{t} := \vartheta|\mathrm{B}(\mathfrak{t})|$ a.e. for $\mathfrak{t} \in \mathfrak{C}$. Using (62) and the mean value theorem, one gets that

$$\|\mathbf{1}[h_\mathfrak{t} \leq \mathrm{h}]\mathrm{M}_\beta\|_2 \leq \frac{\beta\vartheta\cosh(\beta\mathrm{h})}{\mathrm{e}^{-\beta\lambda}\cosh(\beta g_{\mathrm{r}_\beta})} \|\mathrm{B}\|_2 \ . \qquad (170)$$

On the other hand,

$$\|\mathbf{1}[h_\mathfrak{t} \geq \mathrm{h}]\mathrm{M}_\beta\|_2 \leq \|\mathrm{M}_\beta\|_2 \|\mathbf{1}[h_\mathfrak{t} \geq \mathrm{h}]\|_2 \leq \vartheta^2\mathrm{h}^{-1}\|\mathrm{B}\|_2 \ .$$

We combine this with (169)–(170) to deduce that

$$\|\mathrm{M}_\beta\|_2 \leq \left(\frac{\beta\vartheta\cosh(\beta\mathrm{h})}{\mathrm{e}^{-\beta\lambda}\cosh(\beta g_{\mathrm{r}_\beta})} + \vartheta^2\mathrm{h}^{-1} \right) \|\mathrm{B}\|_2 \ .$$

Applying this inequality to $\mathrm{B} = \mathfrak{T}_\epsilon(\mathrm{B}_\epsilon - \mathrm{B}_{\mathrm{int}})$ and using Lemma 37 and $\|\mathfrak{T}_\epsilon\| \leq 1$, we finally find that

$$\|\mathrm{B}_\epsilon - \mathrm{B}_{\mathrm{int}}\|_2 \leq \mathfrak{z}\vartheta\|\mathrm{B}_\epsilon - \mathrm{B}_{\mathrm{int}}\|_2 + \vartheta|\mathfrak{C}\backslash\mathfrak{C}_\epsilon|^{1/2}$$

from which we deduce the lemma $\qquad\square$

Sufficient conditions to ensure that assumptions of the last lemma hold at large $\beta > 0$ are given by $\mu < -\vartheta^2$ and $\gamma > |\mu - \lambda|\Gamma_0$ with

$$\Gamma_0 := \frac{4}{1 - \vartheta^2|\mu|^{-1}} > 4 \ .$$

See Lemmata 16 and 37. Indeed, the Meißner effect is directly related with the existence of a superconducting phase, which is characterized by a strictly positive Cooper pair condensate density for all minimizers $\omega_{\epsilon,l} \in \Omega_l^{(\epsilon)}$ in the limit $\epsilon \to 0^+$, see (60).

Theorem 40 (Superconducting phase – II)
Let $\mu < -\vartheta^2$, $\gamma > |\mu - \lambda|\Gamma_0$ and $\mathrm{B}_{\mathrm{ext}} = \mathcal{S}_0(\mathrm{j}_{\mathrm{ext}})$ with $\mathrm{j}_{\mathrm{ext}} \in C_0^\infty \cap P^\perp\mathfrak{H}$. Then, there is $\beta_c \in \mathbb{R}^+$ such that, for all $\beta > \beta_c$:
(i) Any family $\{\mathrm{B}_\epsilon\}_{\epsilon \in \mathbb{R}^+} \subset \mathcal{B}$ of minimizers of $\mathfrak{B}_\infty^{(\epsilon)}$ converges in norm to the unique minimizer $\mathrm{B}_{\mathrm{int}}$ of $\mathfrak{A}$ (cf.

Lemma 34).

(ii) B_{int} *is also the unique minimizer of* $\mathfrak{B}_\infty^{(0)}$.

(iii) *For any sequence of minimizers* $\omega_{\epsilon,l} \in \Omega_l^{(\epsilon)}$,

$$\lim_{\epsilon \to 0^+} \liminf_{l \to \infty} \omega_{\epsilon,l}\left(\frac{\mathfrak{c}_0^* \mathfrak{c}_0}{|\Lambda_l|}\right) = \lim_{\epsilon \to 0^+} \limsup_{l \to \infty} \omega_{\epsilon,l}\left(\frac{\mathfrak{c}_0^* \mathfrak{c}_0}{|\Lambda_l|}\right) = r_\beta(0)$$

with

$$r_\beta(0) \geq \Gamma_0^{-2} - \gamma^{-2}(\mu - \lambda)^2 > 0$$

being the unique solution of (75) for $B = 0$.

Proof. (i) For any $B \in b_\vartheta(0)$, $\mu < -\vartheta^2$ and $\gamma > |\mu - \lambda|\Gamma_0$, by Lemma 16, there is $\beta_c \in \mathbb{R}^+$ such that, for all $\beta > \beta_c$,

$$h_c(B) := g_{r_\beta} - \lambda > h := g_{r_0} - \lambda > \vartheta^2$$

with r_0 defined by (79) for $\varepsilon = 0$ and $R = \vartheta$. See also (87). By Lemma 37, it follows that the conditions of Lemma 39 are satisfied for $\epsilon \in \mathbb{R}^+$. The latter yields

$$\|B_\epsilon - B_{int}\|_2 = \mathcal{O}(\sqrt{\epsilon}) .$$

(ii) We combine (i) and (147) with Theorem 15 (ii) to check that B_{int} is a minimizer of $\mathfrak{B}_\infty^{(0)}$. On the other hand, recall that, for any $B \in b_\vartheta(0)$, the conditions of Lemma 39 are also satisfied for $\epsilon = 0$. Hence, B_{int} is the unique minimizer of $\mathfrak{B}_\infty^{(0)}$.

(iii) Since, by definition, $\omega_{\epsilon,l} \in \Omega_l^{(\epsilon)}$ minimizes the magnetic free–energy density functional $\mathcal{F}_l^{(\epsilon)}$ (49), every $\omega_{\epsilon,l} \in \Omega_l^{(\epsilon)}$ can be seen as a tangent functional to the magnetic pressure $\mathcal{P}_l^{(\epsilon)}$ (53). See [4, Section 2.6] for further details. In particular,

$$\lim_{\delta \to 0^+} \partial_\gamma \mathcal{P}_l^{(\epsilon)}(\gamma - \delta) \leq \omega_{\epsilon,l}\left(\frac{\mathfrak{c}_0^* \mathfrak{c}_0}{|\Lambda_l|}\right) \leq \lim_{\delta \to 0^+} \partial_\gamma \mathcal{P}_l^{(\epsilon)}(\gamma + \delta) \tag{171}$$

for any $l \in \mathbb{N}$. Observe now that the finite volume magnetic pressure $\mathcal{P}_l^{(\epsilon)} \equiv \mathcal{P}_l^{(\epsilon)}(\gamma)$ is a continuous convex function of $\gamma \in \mathbb{R}^+$. Indeed, $\mathcal{P}_l^{(\epsilon)}(\gamma)$ is the supremum over a family of affine functions of $\gamma \in \mathbb{R}^+$. Therefore, by Griffiths arguments (see, e.g., [1, Eq. (A.1)])

together with (171), the point–wise convergence of functions $\mathcal{P}_l^{(\epsilon)} \equiv \mathcal{P}_l^{(\epsilon)}(\gamma)$ towards the continuous convex function $\mathcal{P}_\infty^{(\epsilon)} \equiv \mathcal{P}_\infty^{(\epsilon)}(\gamma)$ yields

$$\lim_{\delta \to 0^+} \partial_\gamma \mathcal{P}_\infty^{(\epsilon)}(\gamma - \delta) \leq r_\epsilon^{(-)} \leq r_\epsilon^{(+)} \leq \lim_{\delta \to 0^+} \partial_\gamma \mathcal{P}_\infty^{(\epsilon)}(\gamma + \delta) \tag{172}$$

for any $\epsilon \in \mathbb{R}^+$, where

$$r_\epsilon^{(-)} := \liminf_{l \to \infty} \omega_{\epsilon,l}\left(\frac{\mathfrak{c}_0^* \mathfrak{c}_0}{|\Lambda_l|}\right) , \quad r_\epsilon^{(+)} := \limsup_{l \to \infty} \omega_{\epsilon,l}\left(\frac{\mathfrak{c}_0^* \mathfrak{c}_0}{|\Lambda_l|}\right) . \tag{173}$$

By applying once again [1, Eq. (A.1)] to the family $\{\mathcal{P}_\infty^{(\epsilon)}(\gamma)\}_{\epsilon \in \mathbb{R}^+}$ of continuous convex functions, we deduce from (59) and (172)–(173) that the limits

$$r_0^{(-)} := \liminf_{\epsilon \to 0^+} r_\epsilon^{(-)} \quad \text{and} \quad r_0^{(+)} := \limsup_{\epsilon \to 0^+} r_\epsilon^{(+)} \tag{174}$$

must obey:

$$\lim_{\delta \to 0^+} \partial_\gamma \mathcal{P}_\infty(\gamma - \delta) \leq r_0^{(-)} \leq r_0^{(+)} \leq \lim_{\delta \to 0^+} \partial_\gamma \mathcal{P}_\infty(\gamma + \delta) . \tag{175}$$

Note that $\mathcal{P}_\infty(\gamma)$ is of course well–defined for all $\gamma \in \mathbb{R}^+$.

In fact, we combine (ii) with Theorem 15 (i), Corollary 32 and Lemma 34 to obtain that, for any $\gamma > |\mu - \lambda|\Gamma_0$,

$$\mathcal{P}_\infty \equiv \mathcal{P}_\infty(\gamma) = -\frac{1}{2}\|\tilde{B}_{int}\|_2^2 + \mathfrak{F}(r_\beta(0), 0) \tag{176}$$

with $\tilde{B}_{int} := B_{int} + B_{ext}$ and $r_\beta(0)$ being a solution of (75) for $B = 0$. By Lemmata 16 together with [1, Lemma 7.1], the solution of (75) is unique. Using (176) while keeping in mind that B_{int} is the minimizer of $\mathfrak{B}_\infty^{(0)}$ for all $\gamma \geq \gamma' > |\mu - \lambda|\Gamma_0$ (cf. (ii)) and $\beta > \beta_c = \beta_c(\gamma')$ (cf. Lemma 16), we then conclude that

$$\lim_{\delta \to 0^+} \partial_\gamma \mathcal{P}_\infty(\gamma - \delta) = \lim_{\delta \to 0^+} \partial_\gamma \mathcal{P}_\infty(\gamma + \delta) = r_\beta(0) .$$

Because of (174)–(175), the latter yields (iii). $\square$

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