DOUBLE EXPONENTIAL GROWTH OF THE VORTICITY GRADIENT FOR THE TWO-DIMENSIONAL EULER EQUATION

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Abstract. For two-dimensional Euler equation on the torus, we prove that the L^∞–norm of the vorticity gradient can grow as double exponential over arbitrarily long but finite time. The method is based on the perturbative analysis around the singular stationary solution studied by Bahouri and Chemin in [1].

1. Upper bounds

Consider the two-dimensional Euler equation for the vorticity

$$\dot\theta = \nabla\theta\cdot\nabla^\perp u, \quad u=\Delta^{-1}\theta, \quad \theta(x,y,0)=\theta_0(x,y) \tag{1}$$

and θ is 2π–periodic in both x and y (e.g., the equation is considered on the torus $\mathbb{T}^2$). We assume that θ_0 has zero average over $\mathbb{T}^2$ and then Δ^{-1} is well-defined since the Euler flow is area-preserving and the average of $\theta(\cdot,t)$ is zero as well.

The global existence of the smooth solution for smooth initial data is well-known [9]. The estimate on the possible growth of the Sobolev norms, however, is double exponential. We sketch the proof of this bound for H^2–norm. The estimates for $H^s, s>2$ can be obtained similarly. More general results on regularity can be found in [4]. Let

$$j_k(t)=\|\theta(t)\|_{H^k}$$

Lemma 1.1. *If θ is the smooth solution of (1), then*

$$j_2(t)\le \exp\Big((1+\log j_2(0))\exp(C\|\theta(0)\|_\infty t)-1\Big) \tag{2}$$

Proof. Acting on (1) with Laplacian we get

$$\Delta\dot\theta=\Delta\theta_x u_y+2\nabla\theta_x\cdot\nabla u_y-\Delta\theta_y u_x-2\nabla\theta_y\cdot\nabla u_x$$

Multiply by $\Delta\theta$ and integrate to get

$$\partial_t\|\theta\|^2_{H^2}\le 16\|H(u)\|_\infty\|\theta\|^2_{H^2} \tag{3}$$

where $H(u)$ denotes the Hessian of u. The next inequality follows from the Littlewood-Payley decomposition ([4], proposition 1.4)

$$\|H(u)\|_\infty < C(\sigma)\|\theta\|_\infty(1+\log\|\theta\|_{H^\sigma}) \tag{4}$$

for any $\sigma>1$. Notice that $\|\theta\|_\infty$ is invariant under the motion so combine (3) and (4) to get (2). □

Keywords: Two-dimensional Euler equation, growth of the vorticity gradient
2000 AMS Subject classification: primary 76B99, secondary 76F99.

Remark. In the same way one can prove bounds for higher Sobolev norms, e.g.,

$$j_4(t) \le \exp\Big((1+\log j_4(0))\exp(C\|\theta(0)\|_\infty t) - 1\Big) \tag{5}$$

Similar estimates can be proved for different norms, e.g., the uniform norm of the vorticity gradient. For example, the Sobolev embedding and the estimates on j_4 already yield the double exponential growth. In this paper, we will be working with that norm as many arguments are easier in that setting.

In the regime of large $j_k(0)$, we will be able to show that, given arbitrarily large λ, the estimate $\max_{t\in[0,T]}\|\nabla\theta(t,\cdot)\|_\infty > \lambda^{e^{0,001T}}\|\nabla\theta_0\|_\infty$ can hold for some infinitely smooth initial data. This is far from showing that (2) or (5) are sharp however this does prove the possibility of double exponential growth and rules out the estimates like

$$j_k(t) \le j_k(0)\exp(Ct) \tag{6}$$

under the normalization $\|\theta\|_\infty = 1$. The question of whether $j_k(0)$ can be taken ~ 1 is left wide open, see the discussion in last section.

Our results rigorously confirm the following: if the 2D incompressible dynamics gets into a certain "instability mode" then the Sobolev norms can grow very fast in local time (i.e. counting from the time the "instability regime was reached"). Can the Sobolev norms grow at all infinitely in time assuming that initially they are small? The answer to this question is yes, see [6] and [8, 10, 11, 12]. The important questions of linear and nonlinear instabilities were addressed before (see, e.g. [7] and references there).

It must be mentioned here that 2D Euler allows rescaling which provides the tradeoff between how large the T and $\theta(0)$ are, i.e. if $\theta(x,y,t)$ is a solution then $\lambda\theta(x,y,\lambda t)$ is also a solution for any $\lambda > 0$. However, in our construction we will always have $\|\theta\|_p \sim 1, \quad \forall p\in[1,\infty]$. The bounds like (6) can be easily obtained for Euler-type equations with Δ^{-1} replaced by Δ^{-1-}. In our opinion, the double exponential growth is the first indication of the nonlinear nature of the problem.

2. The singular stationary solution and dynamics on the torus

The following singular stationary solutions was studied before (see, e.g., [1, 3] in the context of $\mathbb{R}^2$). We consider the following function

$$\theta_0^s(x,y) = sgn(x)\cdot sgn(y), \quad |x|\le\pi, |y|\le\pi$$

This is a steady state as can be easily seen by checking directly. Indeed, the function $u_0 = \Delta^{-1}\theta_0^s$ is odd with respect to each variable as can be verified on the Fourier side. That, in particular, implies that u_0 is zero on the axis and then its gradient is orthogonal to them. This steady state, of course, is a weak solution, a vortex-type steady state. Another consequence of u_0 being odd is that the origin is a stationary point of the dynamics.

By the Poisson summation formula, we have

$$\sum_{n\in\mathbb{Z}^2, n\neq(0,0)} |n|^{-2}e^{in\cdot x} = C\ln|x| + \phi(x), \quad x\sim 0$$

where $\phi(x)$ is smooth and even.

Therefore, around the origin we have

$$\nabla u_0(x,y) \sim \int\int_{B_\delta(0)} \frac{(x-\xi_1, y-\xi_2)}{(x-\xi_1)^2+(y-\xi_2)^2} sgn(\xi_1)sgn(\xi_2)d\xi_1 d\xi_2 + \dots$$

Due to symmetry, it is sufficient to consider the domain $D = \{0 < x < y < 0,001\}$. Then, taking the integrals, we see that

$$\mu(x,y) = (\mu_1, \mu_2) = \left(\nabla^{\perp} u_0\right)(x,y) = \tag{7}$$

$$= C\left(-\int_0^x \ln(y^2+\xi^2)d\xi + xr_1(x,y), \int_0^y \ln(x^2+\xi^2)d\xi + yr_2(x,y)\right)$$
$$= C(-x\log y + xO(1), y\log y + yO(1)) \quad \text{if } (x,y)\in D$$

The correction terms $r_{1(2)}$ are smooth. Without loss of generality we will later assume that $C = 1$ in the last formula. Notice also that the flow given by the vector-field μ is area-preserving.

Thus, the dynamics of the point $(x_0, \delta) \in D, x_0 \ll \delta$ is

$$(C_1\delta)^{e^t} \lesssim y(t) \lesssim (C_2\delta)^{e^t}, \quad x_0(C_1\delta)^{-e^t+1} \lesssim x(t) \lesssim x_0(C_2\delta)^{-e^t+1},\ t\in[0,t_0], \tag{8}$$

where t_0 is the time the trajectory leaves the domain D. These estimates therefore give a bound on t_0. The attraction to the origin, the stationary point, is double exponential along the vertical axis and the repulsion along the horizontal axis is also double exponential.

3. The idea

The idea of constructing the smooth initial data for a double exponential scenario is quite simple and roughly can be summarized as follows: given any $T > 0$, we will smooth out the singular steady state such that the dynamics is double exponential over $[0,T]$ in a certain domain away from the coordinate axes. Then we will place a small but steep bump in the area of double exponential behavior and will let it evolve hoping that the vector field generated by this bump itself is not going to ruin the double exponential contraction in OY direction. The rest of the paper verifies that this indeed is the case.

4. The Model Equation

We start this section with the study of the following system

$$\begin{cases} \dot{x} = \mu_1(x,y) + \nu_1(x,y,t), & x(0,\alpha,\beta) = \alpha \\ \dot{y} = \mu_2(x,y) + \nu_2(x,y,t), & y(0,\alpha,\beta) = \beta \end{cases} \tag{9}$$

Here we assume the following

$$|\nu_{1(2)}| < 0,0001\upsilon r, \quad r = \sqrt{x^2+y^2} \tag{10}$$

and

$$|\nabla \nu_{1(2)}| < 0,0001\upsilon \tag{11}$$

with small υ (to be specified later) and these estimates are valid in the area of interest

$$\aleph = \{y > \sqrt{x}\} \cap \{y < \epsilon_2\} \cap \{x > \epsilon_1\}$$

where

$$\upsilon \ll \epsilon_1 \ll \epsilon_2$$

The functions $\nu_{1(2)}$ are infinitely smooth in all variables in $\aleph$ but we have no control over higher derivatives. We also assume that this flow is area preserving. Our goal is to study the behavior of trajectories within the time interval $[0,T]$. In this section, the parameters will eventually be chosen in the following order

$$T \longrightarrow \epsilon_2 \longrightarrow \epsilon_1 \longrightarrow \upsilon$$

Here are some obvious observations:

1. If $\epsilon_{1(2)}$ are small and

$$\alpha \gtrsim \upsilon \left|\frac{\beta}{\log\beta}\right| \tag{12}$$

then $x(t)$ always increases and $y(t)$ decreases. This monotonicity persists as long as the trajectory stays within $\aleph$. Assuming that $\epsilon_{1(2)}$ are fixed, (12) can always be satisfied by taking υ small enough, i.e.,

$$\upsilon \lesssim \frac{\epsilon_1|\log\epsilon_2|}{\epsilon_2} \tag{13}$$

2. We have estimates

$$x(-\log y+C)+\upsilon y > \dot{x} > x(-\log y-C)-\upsilon y, \quad -y(|\log y|+C) < \dot{y} < -y(|\log y|-C) \tag{14}$$

The second estimate yields

$$e^{e^t(\log\beta+C)} \gtrsim y(t) \gtrsim e^{e^T(\log\beta-C)} = \kappa(T,\beta) \tag{15}$$

and so for $x(t)$ we have

$$x(t) \le \alpha\exp\left(Ct-\int_0^t \log y(\tau)d\tau\right)+\upsilon\int_0^t y(\tau)\exp\left(C(t-\tau)-\int_\tau^t \log y(s)ds\right)d\tau$$

$$x(T) < (\alpha+\upsilon\beta T)\exp\Big(T(C+|\log\kappa(T,\beta)|)\Big)$$

Thus, the trajectory will stay inside $\aleph$ for any $t\in[0,T]$ as long as

$$\alpha < \kappa\exp(-(T+2)|\log\kappa|)-\upsilon\epsilon_2 T$$

and if we make

$$\upsilon < \epsilon_2^{e^{2T}} \tag{16}$$

then condition

$$\alpha < \beta^{e^{2T}} \tag{17}$$

is sufficient for the trajectory to stay inside $\aleph$ for $t\in[0,T]$. Thus, we are taking

$$\epsilon_1 < \epsilon_2^{e^{2T}}$$

and focus on the domain

$$\Omega_0 = \{(\alpha,\beta) : \epsilon_1 < \alpha < \beta^{2e^T}, \beta < \epsilon_2\}$$

Then, any point from this set stays inside $\aleph$ over $[0,T]$, $x(t)$ grows monotonically and $y(t)$ monotonically decays with the double-exponential rate given in (15).

Now, we will prove that the derivative in α of $x(t,\alpha,\beta)$ grows with the double-exponential rate and this will be the key calculation. For any $t\in[0,T]$, (7) yields

$$\left\{\begin{array}{l} \dot{x}_\alpha = -x_\alpha\log(x^2+y^2)+x_\alpha r_1+ \\ \qquad\qquad +xx_\alpha r_{1x}+xy_\alpha r_{1y}+\nu_{1x}x_\alpha+\nu_{1y}y_\alpha-2y_\alpha\arctan(xy^{-1}) \\ \dot{y}_\alpha = y_\alpha\log(x^2+y^2)+y_\alpha r_2+yx_\alpha r_{2x}+ \\ \qquad\qquad +yy_\alpha r_{2y}+\nu_{2x}x_\alpha+\nu_{2y}y_\alpha+2x_\alpha\arctan(yx^{-1}) \end{array}\right. \tag{18}$$

and $x_\alpha(0,\alpha,\beta)=1$, $y_\alpha(0,\alpha,\beta)=0$. Let

$$f_{11}(t)=\nu_{1x}-\log(x^2+y^2)+r_1+xr_{1x}+\nu_{1x}$$
$$f_{12}(t)=xr_{1y}+\nu_{1y}-2\arctan(xy^{-1})$$
$$f_{21}(t)=yr_{2x}+\nu_{2x}+2\arctan(yx^{-1})$$
$$f_{22}(t)=\log(x^2+y^2)+r_2+yr_{2y}+\nu_{2y}$$

$$x_\alpha=\exp\left(\int_0^t f_{11}(\tau)d\tau\right)\psi,\quad y_\alpha=\exp\left(\int_0^t f_{22}(\tau)d\tau\right)\phi$$

If

$$g=f_{11}-f_{22}$$

then

$$\psi(t)=1+\int_0^t\psi(s)f_{21}(s)\int_s^t f_{12}(\tau)\exp\left(-\int_s^\tau g(\xi)d\xi\right)d\tau ds$$

Since the trajectory is inside $\aleph$, we have $y>\sqrt{x}$ and so

$$|f_{12}|\lesssim y+\upsilon,\quad |f_{21}|\lesssim 1,\quad g(t)>1$$

From (15), we get

$$|\psi(t)-1|\lesssim \upsilon\int_0^t|\psi(\tau)|d\tau+\int_0^t|\psi(s)|\int_s^t e^{e^\tau(\log\beta+C)}e^{-(\tau-s)}d\tau ds$$

The following estimate is obvious

$$\int_s^t e^{e^\tau(\log\beta+C)}e^{-(\tau-s)}d\tau\lesssim e^{-s}$$

as β is small. Assuming that

$$\upsilon\ll T^{-1}\tag{19}$$

and ϵ_2 is small, we have

$$\psi(t)\sim 1$$

and

$$x_\alpha(T,\alpha,\beta)>\left(\frac{1}{\beta}\right)^{e^{T/2}}\tag{20}$$

The estimate (20) is the key estimate that will guarantee the necessary growth.

Now, let us place a circle $S_\gamma(\tilde{x},\tilde{y})$ with radius γ and center at $(\tilde{x},\tilde{y})$ into the zone Ω_0. Consider also the line segment $l=[A_1,A_2]$, $A_1=(\tilde{x}-\gamma/2,\tilde{y})$, $A_2=(\tilde{x}+\gamma/2,\tilde{y})$ in the center, parallel to OX. We will track the evolution of this disc and this line segment under the flow. We have

$$x(T,A_2)-x(T,A_1)>\beta^{-e^{T/2}}|A_2-A_1|$$

From the positivity of $x_\alpha(T,\alpha,\beta)$ it follows that the image of l under the flow is a curve given by the graph of a smooth function $\Gamma(x)$. Thus, the image of l (call it l') has length at least $\beta^{-e^{T/2}}|A_2-A_1|$. Denote the distance from l' to $S'_\gamma(\tilde{x},\tilde{y})$, the image of the circle, by d. Then, the domain $\{\Gamma(x)-d<y(x)<\Gamma(x)+d, x\in(x(T,A_1),x(T,A_2))\}$ is inside $B'_\gamma(\tilde{x},\tilde{y})$. The area of this domain is at least

$$d\cdot\beta^{-e^{T/2}}|A_2-A_1|$$

Thus, assuming that the flow preserves the area, we have

$$d \lesssim \beta^{e^{T/2}}\gamma$$

Consequently, if we place a bump in Ω_0 such that the l and $S_\gamma(\tilde{x},\tilde{y})$ correspond to level sets, say, h_2 and h_1 (**and, what is crucial, $h_{1(2)}$ are essentially arbitrary** $0<h_1<h_2<0,0001$), then the original slope of at least $\sim |h_2-h_1|/\gamma$ will become not less than

$$\beta^{-e^{T/2}}\cdot(|h_2-h_1|/\gamma)$$

thus leading to double-exponential growth of arbitrarily large gradients.

Let us reiterate the order in which the parameters are chosen: we first fix any T, then small ϵ_2, then $\epsilon_1<\epsilon_2^{e^{2T}}$. This defines the set Ω_0. For the whole argument to work we need to collect all conditions on υ: (13), (16), (19) which leads to

$$\upsilon < \frac{\epsilon_1|\log\epsilon_2|}{\epsilon_2} \tag{21}$$

5. Small perturbations of a singular cross can also generate double exponential contraction in $\aleph$

Assume that the function θ_1 at any given time $t\in[0,T]$ is such that

$$\theta_1(t,x,y)=\theta_0^s(x,y)$$

outside the "cross"-domain $A=\{|x-\pi k|<\tau\}\cup\{|y-\pi l|<\tau\}$ where τ is small and k and l are arbitrary integers. Inside the domain A we only assume that θ_1 is bounded by one in absolute value, is even, and has zero average. Notice here that the Euler flow preserves the property of function to be even. Given fixed $\epsilon_{1(2)}$ and the domain $\aleph$ defined by these constants, we are going to show that the flow generated by θ_1 can be represented in the form (9) with $\upsilon(\tau)\to 0$ as $\tau\to 0$. We assume of course that $\tau\ll\epsilon_1$.

For that, we only need to study

$$F_1=\nabla\Delta^{-1}p,\quad p=\theta_1-\theta_0^s$$

Here are some obvious properties of F_1

1. $F_1(0)=0$ as θ_1 and θ_0^s are both even.
2. We have

$$F_1(z)\sim\int_A\left(\frac{\xi-z}{|\xi-z|^2}-\frac{\xi}{|\xi|^2}\right)p(\xi)d\xi$$

$$=\int_A \xi p(\xi)\frac{2\xi\cdot z-|z|^2}{|\xi-z|^2|\xi|^2}d\xi - z\int_A\frac{p(\xi)}{|z-\xi|^2}d\xi$$

and this expression is bounded by $|z|\tau\epsilon_1^{-2}+|z|^2\epsilon_1^{-2}\tau|\log\tau|$. Thus, by taking τ small, we can satisfy (10). How about (11)? For the Hessian, we have

$$|H\Delta^{-1}p|\lesssim\epsilon_1^{-2}\tau$$

and so

$$\upsilon\sim\epsilon_1^{-2}\tau|\log\tau|\ll 1 \tag{22}$$

defines the thickness. Thus, merely the condition on the size of the cross guarantees that the arguments in the previous section work.

6. The evolution of a small steep bump in $\aleph$

In this section, we assume that at a given moment $t \in [0,T]$, we have a smooth even function $\theta_2(t,x,y)$ with support in $\aleph \cup -\aleph$, with zero average, and

$$\|\theta_2\|_2 < \omega, \quad \|\nabla\theta_2\|_\infty < M$$

(here one should think about small ω and large M). We will study the flow generated by this function. Let

$$F_2 = \nabla\Delta^{-1}\theta_2$$

Here are some properties of F_2

1. $F_2(0) = 0$.

2. To estimate the Hessian of $\Delta^{-1}\theta_2$, consider the second order derivatives. For example,

$$(\Delta^{-1}\theta_2)_{\alpha\beta}(\alpha,\beta) \sim \int\limits_{(\alpha-\xi)^2+(\beta-\eta)^2<1} \frac{(\alpha-\xi)(\beta-\eta)}{((\alpha-\xi)^2+(\beta-\eta)^2)^2}\theta_2(\xi,\eta)d\xi d\eta =$$

$$= \int_{1>(\alpha-\xi)^2+(\beta-\eta)^2>\rho^2} \frac{(\alpha-\xi)(\beta-\eta)}{((\alpha-\xi)^2+(\beta-\eta)^2)^2}\theta_2(\xi,\eta)d\xi d\eta \quad +$$

$$\int\limits_{(\alpha-\xi)^2+(\beta-\eta)^2<\rho^2} \frac{(\alpha-\xi)(\beta-\eta)}{((\alpha-\xi)^2+(\beta-\eta)^2)^2}\left[\theta_2(\alpha,\beta) + \nabla\theta_2(\xi',\eta')\cdot(\xi-\alpha,\eta-\beta)\right]d\xi d\eta$$

The first term is controlled by $\omega\rho^{-1}$. By our assumption, the second term is dominated by $M\rho$. Optimizing in ρ we have

$$\|H\Delta^{-1}\theta_2\|_\infty \lesssim \sqrt{M\omega} \sim \upsilon$$

To guarantee the conditions that lead to double exponential growth with arbitrary a priory given M, we want to make ω so small that conditions (10) and (11) are satisfied with υ as small as we wish. The condition (11) is immediate and (10) follows from $F_2(0) = 0$, Lagrange formula and the estimate on the Hessian.

7. Stability results

It is well known that given $\theta_0 \in L^\infty$, the weak solution exists and the flow can be defined by the homeomorphic maps $\Phi_{\theta_0}(t,x,y)$ for all t so that $\theta(t,x,y) = \theta_0(\Phi^{-1}_{\theta_0}(t,x,y))$ where Φ_{θ_0} itself depends on θ_0. The continuity of this map though is rather poor ([3], theorem 2.3, p.99). In this section, we will first partially address the following question: which norm $\|\cdot\|_n$ should we take to guarantee that $\|\theta_0 - \theta_0^s\|_n \to 0$ implies $\max_{t\in[0,T]}\max_{z\in\mathbb{T}^2}|\Phi_{\theta_0}(t,z) - \Phi_{\theta_0^s}(t,z)| \to 0$?

We will only consider the special case when $\theta_0 = \theta_0^s$ outside the domain $\mathcal{D}$ of small area. Inside this domain we assume θ_0 to be bounded by some universal constant. The proof of Yudovich theorem (see, e.g., the argument on pp. 313–318, proof of Proposition 8.2, [2]) implies

$$\max_{t\in[0,T]}\max_{z\in\mathbb{T}^2}|\Phi_{\theta_0^s}(t,z) - \Phi_{\theta_0}(t,z)| \to 0$$

as $|\mathcal{D}| \to 0$.

This is the only stability result with respect to initial data that we are going to need in the argument below.

Theorem 7.1. *For any large T and λ, we can find smooth initial data θ_0 so that $\|\theta_0\|_\infty < 10$ and*

$$\max_{t\in[0,T]} \|\nabla\theta(t,\cdot)\|_\infty > \lambda^{e^{0,001T}} \|\nabla\theta_0\|_\infty$$

Proof. Fix any large $T > 0$ and find $\epsilon_{1(2)}$. For larger λ, we have to take smaller ϵ_2. Identify the domain Ω_0 and place a bump (call it $b(z)$) in $\Omega_0 \cup -\Omega_0$ so that the resulting function is even. Make sure that this bump has zero average, height h_2 and diameter of support h_1 so that the gradient initially is of the size h_2/h_1. Here $h_1 \ll h_2 \ll 1$ will be adjusted later.

Take a smooth even function $\omega(x,y)$ supported on $B_1(0)$ such that

$$\int_{\mathbb{T}^2} \omega(x,y)dxdy = 1$$

Take positive σ small and consider

$$\tilde\theta_\sigma(x,y) = \theta_0^s * \omega_\sigma \in C^\infty, \quad \omega_\sigma = \sigma^{-2}\omega(x/\sigma, y/\sigma)$$

We take $\sigma \ll \epsilon_1$ so $\tilde\theta_\epsilon(x,y)$ and $\theta_0^s(x,y)$ coincide in $\aleph$.

As the initial data for Euler dynamics we take a sum

$$\tilde\theta_\sigma(z) + b(z)$$

Then, since θ_0^s is stationary under the flow, the stability result guarantees that given any τ and keeping the same value of h_2/h_1, we can find σ and h_2 so small that over the time interval $[0,T]$ we satisfy

1. The "evolved bump" stays in the domain $\aleph$ (e.g., $\Phi_{\theta_0}(t)\Big(\operatorname{supp} b(z)\Big) \subset \aleph$).

2. Outside the cross of size τ and the support of the evolved bump b, the solution is identical to θ_0^s.

Fix the smoothing strength σ and $h_1 < h_1'$ so small to guarantee the small size of A, i.e. the smallness of τ. The value of τ must be small enough to ensure the double exponential scenario, the conditions (10) and (11). For that, we need (22).

Next, we proceed by contradiction. Assume that for all $t \in [0,T]$ we have $\|\nabla\theta\|_\infty < M = h_2/h_1\lambda^{0,001e^T}$. Then, as $\omega \lesssim h_1h_2$, we only need to take h_1 so small that $\sqrt{M\omega}$ is small enough to guarantee the double exponential scenario. This gives us a contradiction as the double exponential scenario makes the gradient's norm more than M. For the initial value,

$$\|\nabla\theta_0\|_\infty \sim \sigma^{-1} + h_2/h_1 \sim h_2/h_1$$

by arranging $h_{1(2)}$.

Here is a table of parameters for this construction:

$$\{T,\lambda\} \longrightarrow \epsilon_2 \longrightarrow \epsilon_1 \longrightarrow \{\sigma, h_{1(2)}\}$$

□

8. Discussion

In this paper we only proved that the double exponential growth is possible as long as the Sobolev norms are initially very large (in T). The interesting and important question is whether they can grow in the same double exponential rate starting with initial value ~ 1? We do not know the answer to this question yet but there are some potential scenarios in the contour dynamics [5] when the sharp corners are forming very fast. If one can prove that indeed the curvature of these contours grows very fast and the distance between them decays fast as well, then the methods of this paper might be applied to prove the double exponential growth of Sobolev norms etc. In case of the whole space $\mathbb{R}^2$, even infinite double exponential growth can probably be shown by considering domains very far apart in each of which the double exponential growth is observed with increasing $T_k \to \infty$.

9. Acknowledgment

This research was supported by NSF grants DMS-1067413 and DMS-0635607. The hospitality of the Institute for Advanced Study, Princeton, NJ is gratefully acknowledged.

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