Lecture Notes on Fourier Integral Operators: from local to global theory*

Lorenzo Zanelli
Centre de Mathématiques Laurent Schwartz
Ecole Polytechnique
Route de Saclay 91120 Palaiseau

lorenzo.zanelli@ens.fr

*First and Preliminary Version!

"*…The way out lay just in the possibility of attributing to the Hamilton principle, also, the operation of a wave mechanism on which the point-mechanical processes are essentially based, just as one had long become accustomed to doing in the case of phenomena relating to light and of the Fermat principle which governs them. Admittedly, the individual path of a mass point loses its proper physical significance and becomes as fictitious as the individual isolated ray of light. The essence of the theory, the minimum principle, however, remains not only intact, but reveals its true and simple meaning only under the wave-like aspect, as already explained. Strictly speaking, the new theory is in fact not new, it is a completely organic development, one might almost be tempted to say a more elaborate exposition, of the old theory.*"

Erwin Schrödinger
The fundamental idea of wave mechanics
Nobel Lecture, December 12, 1933.

Contents

1 Introduction

The aim of these Lecture Notes is to review the local and global theory of Fourier Integral Operators (FIO) as introduced by L. Hörmander [16], [17] and subsequently improved by J.J. Duistermaat [10] and F. Trèves [29]. This is a wide and general theory, and thus we provide here only a short and comprehensive (but rigorous) description.
From a general viewpoint, we can say that these operators naturally extend the set of Pseudodifferential Operators (PDO) and that this objective is realized by a link with the set of the canonical transformations and their graphs viewed as Lagrangian submanifolds of a symplectic manifold. In particular, the main idea is to require that FIO are integral operators exhibiting Lagrangian distribution kernels.
There exist meaningful applications of FIO in different frameworks, in particular to the study of hyperbolic type equations, and the related literature is quite large. In fact, as L. Hörmander underlined in [17], the original local notion of FIO is due to P.D. Lax in the paper [20] where the objective was the study of the singularities of hyperbolic differential equations.
In the last section of these Lecture Notes, we provide a resume the main results in the first papers as well as in the more recent ones involving the use of FIO to get local and global in time parametrices of the propagator of Schrödinger type equations.

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2 Pseudodifferential Operators

In this section we provide the standard setting about the theory of Pseudodifferential Operators on $\mathbb{R}^n$.

2.1 Settings

To begin, we recall the definition for the set of amplitudes functions.

Definition 2.1. ***Amplitudes*** *Let m, ρ be real numbers with $0 \leq \rho \leq 1$. We denote by $\Pi^m_\rho(\mathbb{R}^{3n})$ the set of all $a(x,y,\xi) \in C^\infty(\mathbb{R}^{3n};\mathbb{C})$ such that for all multiorders α,β,γ and some m' satisfy*

$$|\partial_x^\alpha \partial_y^\beta \partial_\xi^\gamma a(x,y,\xi)| \leq C_{\alpha,\beta,\gamma} \ \langle z \rangle^{m-\rho\,|\alpha+\beta+\gamma|} \ \langle x-y \rangle^{m'+\rho\,|\alpha+\beta+\gamma|},$$

where $z := (x,y,\xi)$, $\langle z \rangle := \sqrt{1+|z|^2}$ and $C_{\alpha,\beta,\gamma} > 0$.

The set of Pseudodifferential Operators associated with the above amplitudes can be introduced, as done by M.A. Shubin [32], in the following way

Definition 2.2. ***PDO***
Let $a \in \Pi^m_\rho(\mathbb{R}^{3n};\mathbb{C})$, the associated PDO is defined as

$$A(u) := (2\pi)^{-n} \int_{\mathbb{R}^{2n}} e^{i(x-y)\cdot\xi} \ a(x,y,\xi)u(y)dyd\xi, \qquad u \in \mathcal{S}(\mathbb{R}^n). \tag{2.1}$$

It can be easily proved that the map $A : \mathcal{S}(\mathbb{R}^n) \longrightarrow \mathcal{S}(\mathbb{R}^n)$, which is defined on the Schwartz space, is continuous by using an estimate of C^k-norms.

Looking at simbols $a \in \Pi^m_\rho(\mathbb{R}^n_x \times \mathbb{R}^n_\xi;\mathbb{R})$ in the form $a(x,\xi) := \sum_{1\leq j\leq n} \sum_{|\alpha|\leq m} a_\alpha(x)\xi_j^\alpha$ we localize a set of Differential Operators on $\mathbb{R}^n$, since it holds $A = a(x,\partial_x)$.

About the $L^2(\mathbb{R}^{2n})$-boundedness for a class of PDO, we recall the well known Calderon-Vaillancourt Theorem (see for example [23]).

Theorem 2.3. *Let $a \in \Pi^m_\rho(\mathbb{R}^{3n})$ be such that $\partial_z^\alpha a \in L^\infty(\mathbb{R}^{3n})$ for all $\alpha \in \mathbb{N}$. Then, the operator A defined in (2.1) is continuous with respect to the topology induced by the $L^2(\mathbb{R}^n)$-norm and extends to a bounded operator on $L^2(\mathbb{R}^n)$. Moreover,*

$$\|A\|_{L^2\to L^2} \leq C_n \sum_{|\alpha|\leq M_n} \|\partial_z^\alpha a\|_\infty \tag{2.2}$$

where $C_n, M_n > 0$ depend only on the dimension n.

2.2 Quantization

In the framework of semiclassical Analysis (see for example A. Martinez [23]) we have the following important notion

Definition 2.4. ***Quantization***
Let $0 < \hbar \leq 1$, $0 \leq t \leq 1$. A family of quantizations of the classical observables $b \in \Pi_\rho^m(\mathbb{R}^{2n};\mathbb{R})$ reads as

$$Op_\hbar^t(b)(u) := (2\pi\hbar)^{-n} \int_{\mathbb{R}^{2n}} e^{\frac{i}{\hbar}(x-y)\cdot\xi}\ b((1-t)x+ty,\xi)u(y)dyd\xi, \qquad u \in \mathcal{S}(\mathbb{R}^n).$$

The case $t=0$ is the so-called "standard" (or left) quantization, the case $t=1/2$ is known as the "Weyl" quantization and usually reads $Op_\hbar^W(b)$, whereas the case $t=1$ is the "right" quantization.

Remark 2.5. *The Weyl quantization naturally arises in Quantum Mechanics thanks to the link with the Heisemberg group (see for example G. Folland in [11]). Moreover, we remark that $Op_\hbar^W(b)$ is particularly useful in semiclassical Analysis since it is simmetric with respect to the $L^2(\mathbb{R}^n)$ scalar product.*

We now remind that the quantum evolution of an observable is closer and closer to its classical evolution as the Planck constant becomes negligible. This result is known in the literature as the Egorov's Theorem. In fact, it can be proved that the semiclassical asymptotic expansion for the propagation of quantum observables $Op_\hbar^W(b)$, for smooth Hamiltonians growing at most quadratically at infinity, is uniformly dominated at any order by an exponential term whose argument is linear in time. In particular, it arises necessarily a time obstruction $T(\hbar) \simeq \log(\hbar^{-1})$ which is called the "Ehrenfest time" for the validity of this semiclassical approximation.
To be more precise, let $H \in C^\infty(\mathbb{R}^{2n};\mathbb{R})$ be such that $\sup_{(x,\xi)\in\mathbb{R}^n} |\nabla^2 H(x,\xi)| < +\infty$, and take the quantum propagator $U_\hbar(t) := exp(-iOp_\hbar^W(H)t/\hbar)$. Then, for $t \in [0, T_N(\hbar))$ with $T_N(\hbar) := -2\log(\hbar)/(N-1)$, we have the semiclassical asymptotics

$$U_\hbar(-t) \circ Op_\hbar^W(b) \circ U_\hbar(t) = \sum_{j=0}^N \hbar^j Op_\hbar^W(b_j(t)) + \mathcal{R}_N(t) \tag{2.3}$$

for suitable simbols $b_j(t) \in \Pi_\rho^m(\mathbb{R}^{2n};\mathbb{R})$ and

$$\|\mathcal{R}_N(t)\|_{L^2\to L^2} \leq CN^{(6n+1)N}(-\hbar\log(\hbar))^N. \tag{2.4}$$

In particular, the lower order simbol reads $b_0(t,x,\xi) := b(\phi_H^t(x,\xi))$, namely the classical propagation of the simbol b. Whereas the higher order terms $b_j(t,x,\xi)$ are determined by the classical evolution but have a polynomial dependence on the derivatives of the flow ϕ_H^t with respect to the variables (x,ξ) up to the order $j-1$. About the precise setting of b_j we refer to D. Bambusi, S. Graffi and T. Paul [3].

3 Canonical transformations

3.1 Settings

Adopting standard notations, let M be a manifold and $T^\star M$ the corresponding cotangent bundle. We denote by $\omega = dp \wedge dx = \sum_{i=1}^n dp_i \wedge dx^i$ the 2–form on $T^\star M$ that defines its natural symplectic structure. As usual, a differomorphism $\mathcal{C} : T^\star M \longrightarrow T^\star M$ is a canonical transformation if the pull back of the symplectic form is preserved, namely the condition $\mathcal{C}^\star \omega = \omega$.
We say that $L \subset T^\star M$ is a Lagrangian submanifold if

$$\omega|_L = 0, \quad \dim(L) = n = \frac{1}{2}\dim(T^\star M). \tag{3.5}$$

A symplectic structure $\bar{\omega}$ on $T^\star M \times T^\star M \cong T^\star (M \times M)$ is the twofold pull–back of the standard symplectic 2–form on $T^\star M$ defined as $\bar{\omega} := pr_2^\star \omega - pr_1^\star \omega$ which in fact equals $\bar{\omega} = dp_2 \wedge dx_2 - dp_1 \wedge dx_1$. Similarly, $\Lambda \subset T^\star M \times T^\star M$ is called Lagrangian submanifold of $T^\star M \times T^\star M$ if

$$\bar{\omega}|_\Lambda = 0, \quad \dim(\Lambda) = 2n. \tag{3.6}$$

A diffeomorphism $\mathcal{C}$ on $T^* \mathbb{R}^n$ is canonical if and only if its graph

$$\Lambda = \{(y, \xi; x, p) \in T^* M \times T^* M \quad | \quad (x, p) = \mathcal{C}(y, \xi)\} \tag{3.7}$$

is a Lagrangian submanifold with respect to the induced symplectic structure $\bar{\omega}$.
We refer to an Hamiltonian as a C^2-function $H : T^\star M \longrightarrow \mathbb{R}$. It can be esily proved that the flow solving Hamilton's equations on $T^* M$

$$\dot{\gamma} = J\nabla H(\gamma), \quad \gamma(0) := \gamma_0 \in T^\star M,$$

is a one parameter group of canonical transformations

$$\phi_H^t : T^\star M \longrightarrow T^\star M, \qquad t \in \mathbb{R}.$$

The global well defined character of the flow is guaranteed by the global uniform Lipschitz behaviour of the Hamiltonian vector field $X_H := J\nabla H$ defined on $T^\star M$. In all the subsequent results involving Hamiltonian flows, if the particular form of H is not specified, we will assume this general assumption.
As we will see in the following sections, and in particualar in the last one, the local and global study of the time dependent family of Lagrangian submanifolds

$$\Lambda_t = \{(y, \xi; x, p) \in T^* M \times T^* M \quad | \quad (x, p) = \phi_H^t(y, \xi)\} \tag{3.8}$$

is very much important in classical mechanics and as a consequence in the semiclassical analysis of the related quantum flow.

3.2 Generating functions

We begin this subsection by recalling a well-known and important result of symplectic Geometry due to V.P. Maslov [25] and L. Hormander [16] which involves the local parametrization of arbitrary Lagrangian submanifolds.

Theorem 3.1. Local parametrization of Lagrangian submanifolds
Lel M be a manifold and $S \in C^2(M_x \times \mathbb{R}^k_\theta; \mathbb{R})$. Let $L \subset T^\star M$ be the locus defined as

$$L := \{(x,p) \in T^\star M \mid p = \nabla_x S(x,\theta),\ \ 0 = \nabla_\theta S(x,\theta)\ \}\,. \tag{3.9}$$

Suppose that

$$\mathrm{rank}\Big(\nabla^2_{x\theta} S(x,\theta)\ \nabla^2_{\theta\theta} S(x,\theta)\Big)\Big|_L = \max. \tag{3.10}$$

Then, L is a Lagrangian submanifold of $T^\star M$.
Conversely, let L be a Lagrangian submanifold of $T^\star M$ and take

$$L \hookrightarrow T^\star M \to M$$

$$\lambda \mapsto j(\lambda) = (x,p) \mapsto \pi(x,p) = x.$$

Then, for any $z_0 = (x_0, p_0) \in L$ there exists a local parametrization of type (3.9), namely

$$L \cap B_r(z_0) = \{(x,p) \in T^\star M \mid p = \nabla_x S(x,\theta),\ \ 0 = \nabla_\theta S(x,\theta)\,\}\,.$$

where the dimension k fulfills

$$k \geq \dim(M) - \mathrm{rank}[D(\pi \circ j)(z_0)].$$

Remark 3.2. *We undeline that the Lagrangian submanifold L is the geometrical object which is reasuming the local inverse functions of $x \mapsto \nabla_x S$. In view of the above result, we underline that the problem of the inversion of gradient maps is close to the problem of inversion of the Legendre transformation. We can say that the above family of functions S is a sort of weak analogue of the Hamiltonian function Legendre related to a Lagrangian function. In other words, the Lagrangian submanifold L can be interpreted as a sort of multi-valued Legendre relation.*

We introduce now the central objects of this section.

Definition 3.3. *A generating function for a Lagrangian submanifold $L \subset T^\star\mathbb{R}^n$ is a C^1 function $S : \mathbb{R}^n \times \mathbb{R}^k \longrightarrow \mathbb{R}$ such that*

(i) $L = \{(x,p) \in T^\star\mathbb{R}^n :\ \ p = \nabla_x S,\ \ 0 = \nabla_\theta S\ \}$,

(ii) $\mathrm{rank}\Big(\nabla^2_{x\theta} S(x,\theta)\ \nabla^2_{\theta\theta} S(x,\theta)\Big) = \max.$

The second condition is usually equivalently introduced by saying that zero (in $\mathbb{R}^k$) is a regular value of the map $(x,\theta) \longmapsto \nabla_\theta S(x,\theta)$.

In a similar way, we have the following

Definition 3.4. *A generating function for a Lagrangian submanifold* $\Lambda \subset T^\star\mathbb{R}^n \times T^\star\mathbb{R}^n$ *is a* C^1 *map* $\mathcal{S} : \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^k \longrightarrow \mathbb{R}$ *such that*

(i) $\Lambda = \{(x, p; y, \xi) \in T^\star\mathbb{R}^n \times T^\star\mathbb{R}^n : \ \ p = \nabla_x \mathcal{S}, \ \ \xi = -\nabla_y \mathcal{S}, \ \ 0 = \nabla_\theta \mathcal{S} \ \}$,

(ii) zero (in $\mathbb{R}^k$*) is a regular value of the map* $(x, y, \theta) \longmapsto \nabla_\theta \mathcal{S}(x, y, \theta)$.

As we will see in the next section, the original setting of local FIO involves phase functions as generating functions which are positive-omogeneous with repect to θ of degree one. On the other hand, a different class of generating functions naturally arises in symplectic geometry and as a consequence also in semiclassical Analysis (in particular, in the Schrödinger framework). Indeed, we recall that J. C. Sikorav ([34]) introduced the important notion of *generating function quadratic at infinity*. In these Lecture Notes we refer to the similar setting used in the work of Thèret [36], but involving not compact setting.

Definition 3.5 (GFQI). *A generating function* $S : \mathbb{R}^n \times \mathbb{R}^k \longrightarrow \mathbb{R}$ *is called*

- *weakly quadratic at infinity if there exists a* C^1*-function* $Q : \mathbb{R}^n \times \mathbb{R}^k \longrightarrow \mathbb{R}$ *in the form* $Q(x, \theta) := \langle Q(x)\theta, \theta\rangle$, *a* C^1 *function* $\langle b(x), \theta\rangle$ *and a* C^1*-bounded* $c(x, \theta)$ *such that* $P(x, \theta) := \langle Q(x)\theta, \theta\rangle + \langle b(x), \theta\rangle + c(x, \theta)$ *such that*

$$\|S(x, \cdot) - P(x, \cdot)\|_{C^1(\mathbb{R}^k)} \leq C, \quad \forall x \in \mathbb{R}^n. \tag{3.11}$$

- *quadratic at infinity if (3.11) is fulfilled with* $Q(x)$ *non degenerate.*

- *exactly quadratic at infinity if* $S(x, \theta) = \langle Q(x)\theta, \theta\rangle$, *with* $Q(x)$ *non degenerate outside of a compact set* $\mathcal{K} \subset \mathbb{R}^n$. *In particular, if* $Q(x) = Q$, S *is called special.*

We remember that there exists three operations preserving generating functions:

Definition 3.6. *Let* $S(x, \theta) : \mathbb{R}^n \times \mathbb{R}^k \longrightarrow \mathbb{R}$ *be a generating function for a Lagrangian submanifold* L. *We say that* $\widetilde{S}$ *is obtained from* S *by*

- *stabilization, if there exists a* C^1 *non degenerate quadratic function* $\langle a(x)v, v\rangle$, $v \in \mathbb{R}^h$ *and* $\widetilde{S}(x, \theta, v) = S(x, \theta) + \langle a(x)v, v\rangle$.

- *fibered diffeomorphism, if there exists a regular* $\theta : \mathbb{R}^n \times \mathbb{R}^k \longrightarrow \mathbb{R}^k$ *such that* $\theta(x, \cdot)$ *is a diffeomorphism and* $\widetilde{S}(x, v) = S(x, \theta(x, v))$.

- *addition of constant, if* $\widetilde{S}(x, \theta) = S(x, \theta) + C$.

Remark 3.7. *It is usual to consider the equivalence of two generating functions if they can be made equal after a succession of these three operations. Moreover, the uniqueness of quadratic generating functions for a Lagrangian submanifold is often considered with respect to this notion of equivalence.*

Now we recall the main results about existence and uniqueness of quadratic generating functions of the graphs of Hamiltonian isotopies on the cotangent bundle of closed manifolds M.

Theorem 3.8 (Sikorav [34])**.** *Let L be a closed Lagrangian submanifold in $T^\star M$ which admits a generating function exactly quadratic at infinity, $\phi_H^t : T^\star M \to T^\star M$ an hamiltonian isotopy. Then $\phi_H^t L$ also has a gfqi.*

Theorem 3.9 (Viterbo [37])**.** *Let us consider an hamiltonian isotopy $\phi_H^t : T^\star M \to T^\star M$, and $L_t = \phi_H^t \mathcal{O}_M$. Assuming S_1 and S_2 are two generating functions exactly quadratic at infinity for L_t, then they are equivalent.*

Remark 3.10. *We recall that Thèret ([36]) proved, in the same setting of Thereom 3.9, that any generating function quadratic at infinity is equivalent to a special one (see def. 3.5). This important fact implies that the previous two theorems still hold for generating functions quadratic at infinity.*

Moreover, if we set a smooth $H : T^\star M \to \mathbb{R}$, following Brunella [**?**] we underline that with an isometric embedding $i : M \hookrightarrow \mathbb{R}^N$ it can be defined a symplectic embedding $\mathcal{E} : T^\star M \hookrightarrow T^\star \mathbb{R}^d$ such that there exists an extension of the Hamiltonian dynamics. In what follows we state the precise result:

Theorem 3.11 (Brunella [4])**.** *Let $\phi_H^t : T^\star M \to T^\star M$ be an hamiltonian isotopy, then there exists a symplectic embedding $\mathcal{E} : T^\star M \hookrightarrow T^\star \mathbb{R}^d$ and an Hamiltonian isotopy $\psi_K^t : T^\star \mathbb{R}^d \to T^\star \mathbb{R}^d$, such that $\forall t \in [0, T]$:*

(i) $\mathcal{E} \circ \phi_H^t = \psi_K^t \circ \mathcal{E}$

(ii) ψ_K^t *leaves invariant* $T^\star \mathbb{R}^d|_{i(B)}$

moreover, if $V \subset \mathbb{R}^d$ is a neighborhood of $i(B)$, then we may choose every ψ^t with support contained in $T^\star \mathbb{R}^N|_V$.

In view of the above theorem, we underline the problem to determine smooth Hamiltonians $H : T^\star \mathbb{R}^n \to \mathbb{R}$ such that the graph of corresponding isotopies $\phi_H^t : T^\star \mathbb{R}^n \to T^\star \mathbb{R}^n$ exhibit existence and uniqueness of generating functions quadratic at infinity. An answer to this problem is given in the paper [13], where it is enquired the construction and equivalence of smooth global generating functions quadratic at infinity for the Lagrangian graph

$$\Lambda_t = \{(y, \xi; x, p) \in T^* \mathbb{R}^n \times T^* \mathbb{R}^n \quad | \quad (x, p) = \phi_H^t(y, \xi)\}, \tag{3.12}$$

namely $\mathcal{S} \in C^1([0, T] \times \mathbb{R}^{2n+k}; \mathbb{R})$ such that

$$\Lambda_t = \{(x, p; y, \xi) \in T^\star \mathbb{R}^n \times T^\star \mathbb{R}^n : \;\; p = \nabla_x \mathcal{S}, \;\; \xi = -\nabla_y \mathcal{S}, \;\; 0 = \nabla_\theta \mathcal{S} \;\} \tag{3.13}$$

as in the setting of Definition 3.5.

3.3 The graph of the Hamiltonian flow

In this subsection we deal with the following class of Hamiltonian functions

Definition 3.12. *$H \in C^\infty(\mathbb{R}^{2n};\mathbb{R})$ such that there exists an open set $\widetilde{\Omega} \subseteq \mathbb{R}^{2n}$ which is invariant under the Hamiltonian flow and such that for any $\alpha, \beta \in \mathbb{Z}^n_+$*

$$\sup_{|\alpha|+|\beta|\geq 2} \ \sup_{(x,p)\in\widetilde{\Omega}} |\partial_x^\alpha \partial_p^\beta H(x,p)| < +\infty.$$

For example, the class of confining mechanical Hamiltonian functions

$$H(x,p) := \frac{1}{2m}|p|^2 + V(x) \tag{3.14}$$

where $V \in C^\infty(\mathbb{R}^n;\mathbb{R})$ is such that:

$$\begin{array}{rcll} c_1\langle x\rangle^d \leq V(x) & \leq & c_2\langle x\rangle^d, & 0 < c_1 < c_2, \quad d \geq 2, \quad \forall\ |x| \geq R, \\ |\partial_x^\alpha V(x)| & \leq & c\langle x\rangle^{d-|\alpha|}, & \forall\ |\alpha| \geq 0. \end{array}$$

We can also take the class of mechanical Hamiltonians with potentials $V \in C^\infty(\mathbb{R}^n;\mathbb{R})$ such that

$$\sup_{x\in\mathbb{R}^n} |\nabla^2 V(x)| < +\infty. \tag{3.15}$$

We focus our attention on the graph of a Hamiltonian flow $\phi^t_H : T^\star\mathbb{R}^n \to T^\star\mathbb{R}^n$

$$\Lambda_t := \big\{(y,\eta;x,p) \in T^\star\mathbb{R}^n \times T^\star\mathbb{R}^n \mid (x,p) = \phi^t_H(y,\eta)\big\}$$

for H as in Definition 3.12 and time intervals $[0,T]$ arbitrary large. Our objective is to determine a class of global generating functions for Λ_t, namely

$$\Lambda_t = \{(y,\eta;x,p) \in T^\star\mathbb{R}^n \times T^\star\mathbb{R}^n \mid p = \nabla_x S, \quad y = \nabla_\eta S, \quad 0 = \nabla_\theta S(t,x,\eta,\theta)\} \tag{3.16}$$

In order to do so, we follow a variational approach and in particular the so-called Amann-Conley-Zhender reduction ([2], [9], [6], [38]).

First, we look at the boundary problem in the function space $H^1([0,T];\mathbb{R}^{2n})$ for the Hamilton's equation of motion:

$$\dot\gamma(s) = J\nabla H(\gamma(s)); \quad \gamma^x(t) = x, \quad \gamma^p(0) = \eta. \tag{3.17}$$

It can be easily seen that all the solutions of (3.17) belongs to the following set

$$\mathbf{\Gamma}_0(t,x,\eta) := \left\{ \left(x - \int_s^t \phi^x(\tau)\, d\tau; \eta + \int_0^s \phi^p(\tau)\, d\tau \right) \,\Big|\, \phi \in L^2([0,T];\mathbb{R}^{2n}) \right\} \tag{3.18}$$

Observe that the Hamilton-Helmholtz functional:

$$A[(\gamma^x,\gamma^p)] := \int_0^t \gamma^p(s)\dot\gamma^x(s) - H(\gamma^x(s),\gamma^p(s))\, ds \tag{3.19}$$

is well defined and continuous on the path space $W^{1,2}([0,T];\mathbb{R}^{2n})$. Now define the function $\mathcal{S} : [0,T] \times \mathbb{R}^{2n} \times L^2 \to \mathbb{R}$ by the above functional evaluated on $\gamma \in \mathbf{\Gamma}_0(t,x,\eta)$

$$\mathcal{S}(t,x,\eta,\phi) := \langle \gamma^x(0), \eta \rangle + \int_0^t \gamma^p(s)\dot{\gamma}^x(s) - H(\gamma^x(s), \gamma^p(s)) \; ds \tag{3.20}$$

It has been proved in [6], [14] that

$$\Lambda_t = \left\{ (y,\eta;x,p) \in T^\star\mathbb{R}^n \times T^\star\mathbb{R}^n \mid p = \nabla_x \mathcal{S}, \quad y = \nabla_\eta \mathcal{S}, \quad 0 = \frac{D\mathcal{S}}{D\phi}(t,x,\eta,\phi) \right\}$$

where $D\mathcal{S}/D\phi$ stands for the Gateaux derivative. In view of this fact, we read the problem (3.17) as a fixed point functional equation:

$$\phi = G(t,x,\eta,\phi) \tag{3.21}$$

where $G : [0,T] \times \mathbb{R}^{2n} \times L^2([0,T];\mathbb{R}^{2n}) \to L^2([0,T];\mathbb{R}^{2n})$ is given by

$$G(t,x,\eta,\phi^x,\phi^p) := \left(\frac{\eta}{m} + \frac{1}{m}\int_0^s \phi^p(\tau) \; d\tau, -\nabla V \left(x - \int_s^t \phi^x(\tau) d\tau \right) \right) \tag{3.22}$$

It can be easily seen that equation (3.21) is equivalent to the

$$0 = \frac{D\mathcal{S}}{D\phi}(t,x,\eta,\phi)$$

Remark 3.13. *This is an infinite dimensional setting, whereas we are looking for finite dimensional generating functions of Λ_t. However, this gives the motivation and the right variational approach that we are going to use in the subsequent part of this section.*

Before going further, we observe that for our class of Hamiltonians first of all we have to localize the study of the Hamiltonian flow, the related graph and the above fixed point equation within the invariant set $\widetilde{\Omega}$:

$$\mathbf{\Gamma}_1(t,x,\eta) := \left\{ \gamma \in \mathbf{\Gamma}_0(t,x,\eta) \mid \quad \gamma(s) \in \widetilde{\Omega}, \quad \forall s \in [0,t] \right\} \tag{3.23}$$

and look at the equation

$$\phi = G(t,x,\eta;\phi) \tag{3.24}$$

selecting only the solutions of the problem (3.17) in the set $\widetilde{\mathbf{\Gamma}}$.
Now, we set a finite dimensional ortogonal projector $\mathbb{P}_k : L^2 \to L^2$, define the dimension $K =: dim(\mathbb{P}_k L^2) = 2n(2k+1)$, indicate $v \in \mathbb{P}_k L^2$ and $\theta \in \mathbb{R}^K$ the coordinates of $v[\theta]$ with respect a fixed ortonormal basis of $\mathbb{P}_k L^2$. Define $\mathbb{Q}_k := id - \mathbb{P}_k$ and $f \in \mathbb{Q}_k L^2$. Decompose equation (3.21) into the following

$$\begin{aligned} v &= \mathbb{P}_k \circ G(t,x,\eta;\phi) & (3.25)\\ f &= \mathbb{Q}_k \circ G(t,x,\eta;\phi) & (3.26) \end{aligned}$$

where $v = \mathbb{P}_k\phi$ and $f = \mathbb{Q}_k\phi$.
We are now ready to recall the

Theorem 3.14. *If the lower bound*

$$k > C\ T^2 \sup_{|\alpha|+|\beta|\geq 2} \sup_{(y,\eta)\in\widetilde{\Omega}} |\partial_x^\alpha \partial_p^\beta \mathcal{H}(y,\eta)|^2 \tag{3.27}$$

is satysfied, then there exists a unique smooth $f = f(t,x,\eta,v)$

$$f : [0,T] \times \mathbb{R}^{2n} \times \mathbb{P}_k L^2([0,T];\mathbb{R}^{2n}) \to \mathbb{Q}_k L^2([0,T];\mathbb{R}^{2n})$$

solving the contraction equation on $\mathbb{Q}_k L^2([0,T];\mathbb{R}^{2n})$:

$$f = \mathbb{Q}_k \circ G(t,x,\eta; v+f)$$

Moreover, we have the L^2-norm estimates

$$\|\partial_x^\alpha \partial_\eta^\beta f(t,x,\eta,v)(\cdot)\|_{L^2} \leq C_{\alpha\beta}(T)$$

for some $C_{\alpha\beta}(T) > 0$ and $v \in \mathbb{P}_k L^2([0,T];\mathbb{R}^{2n})$.

□

The previous result allow to localize the finite space of curves

Definition 3.15. *By the identification*

$$\phi(t,x,\eta,\theta) \ := \ v[\theta] + f(t,x,\eta,v[\theta]) \tag{3.28}$$

where $\theta \in \mathbb{R}^K$ are the coordinates of $v \in \mathbb{P}_k L^2([0,T];\mathbb{R}^{2n})$ with dimension

$$K = 2n \cdot (2k+1) \tag{3.29}$$

we can reduce the space of curves (3.18) to the finite dimensional one

$$\mathbf{\Gamma}_2 := \left\{ \left(x - \int_s^t \phi^x(t,x,\eta,\theta)(\tau)\ d\tau; \eta + \int_0^s \phi^p(t,x,\eta,\theta)(\tau)\ d\tau \right) \Big| \, \theta \in \mathbb{R}^K \right\}$$

As a consequence, we realize the following inclusions:

$$\mathbf{\Gamma}_2(t,x,\eta) \subset \mathbf{\Gamma}_1(t,x,\eta) \subset \mathbf{\Gamma}_0(t,x,\eta) \tag{3.30}$$

This fact allow a finite dimensional variational reduction of the Action functional:

Definition 3.16.

$$S(t,x,\eta,\theta) := \langle \gamma^x(0), \eta \rangle + \int_0^t \gamma^p(s)\dot{\gamma}^x(s) - H(\gamma^x(s),\gamma^p(s))\ ds \tag{3.31}$$

with $\gamma = \gamma(t,x,\eta,\theta) \in \Gamma(t,x,\eta)$.

Theorem 3.17. *The function S as in (3.31) is a global generating function for*

$$\Lambda_t := \Big\{ (y,\eta;x,p) \in \widetilde{\Omega} \times \widetilde{\Omega} \mid (x,p) = \phi_H^t(y,\eta) \Big\}$$

namely we have the parametrization:

$$\Lambda_t = \Big\{ (y,\eta;x,p) \in \widetilde{\Omega} \times \widetilde{\Omega} \mid p = \nabla_x S, \quad y = \nabla_\eta S, \quad 0 = \nabla_\theta S(t,x,\eta,\theta) \Big\}$$

Moreover, it is fulfilled the rank condition:

$$\mathrm{rank}\Big(\nabla^2_{x\theta} S \ \nabla^2_{\eta\theta} S \ \nabla^2_{\theta\theta} S \Big)\Big|_{\Lambda_t} = \max.$$

Remark 3.18. *We remind that here we are dealing with phase functions obtained by a finite dimensional variational reduction of the Action functional. In this setting, we underline that for mechanical Hamiltonians as in Definition 3.12, the transition between the case of asymptotic quadratic potentials and the asymptotic polinomial potentials of higher order shows important features.*

- ◇ *First, we observe that if $d > 2$ we can take the sublevel set*

  $$\widetilde{\Omega}_E := \{(y,\eta) \in \mathbb{R}^{2n} \mid E_0 < H(y,\eta) < E\}, \quad E_0 := \inf_{(y,\eta)\in\mathbb{R}^{2n}} H(y,\eta).$$

  Moreover, if $E \to +\infty$ then $K(E)$ increases and the topology of the set Γ_E involves the whole infinite dimensional space $H^1([0,T];\mathbb{R}^{2n})$.

- ◇ *Second, we observe that in the case $d = 2$ we can construction the phase function not depending on bounded energy sublevels. Indeed, recalling condition (3.27), in this case the finite variational reduction (3.31) is globally well defined without requiring the constraint of the curves into bounded Ω_E.*

- ◇ *Third, as in the case $d = 2$, the same holds true for smooth potentials only satisfying $\|\nabla^2 V\|_\infty < +\infty$, which is an assumption used in many papers involving the semiclassical analysis of Schrödinger type problems; for example this is the case of smooth periodic potentials used for models in solid-state physics.*

4 Local theory of Fourier Integral Operators

Here we summarize the local theory of FIO, in particular we will refer to the books of Hörmander [16], [17], Duistermaat [10], Trèves [29], where the reader can find the exhaustive exposure.

4.1 Settings

Let $\Omega \subset \mathcal{R}^n$ be an open subset, it is customary to define $\mathcal{E}(\Omega) := C^\infty(\Omega)$ and $\mathcal{D}(\Omega) := C_0^\infty(\Omega)$ where $C_0^\infty(\Omega) := \bigcup_{K<<\Omega} \mathcal{D}_K$ and $\mathcal{D}_K := \{u \in C^\infty(\Omega) \mid supp\ u \in K\}$ with K is a compact subset in the topology of Ω; it can be easily shown that they have the structure of Fréchet spaces. In this settings, $\mathcal{E}'(\Omega)$ and $\mathcal{D}'(\Omega)$ are the sets of linear continuous functionals on these spaces.

Now we remember the definition of simbols space $S^m_{\rho,\delta}$ (as originally introduced by Hörmander and also mainly used by Duistermaat). This is in fact a generalization of the simbols used in the standard theory of Pseudodifferential Operators.

Definition 4.1. ***Symbols***
Let m, ρ, δ be real numbers with $0 \leq \rho \leq 1$, $0 \leq \delta \leq 1$. Then we denote by $S^m_{\rho,\delta}(\mathbb{R}^n \times \mathbb{R}^N)$ the set of all $a \in C^\infty(\mathbb{R}^n \times \mathbb{R}^N)$ such that for every compact set $K \subset X$ and all multiorders α β the estimate

$$|D_x^\beta D_\theta^\alpha a(x,\theta)| \leq C_{\alpha,\beta,K}\,(1+|\theta|)^{m-\rho|\alpha|+\delta|\beta|}, \quad x \in K, \theta \in \mathbb{R}^N,$$

is valid for some constant $C_{\alpha,\beta,K}$. The elements of $S^m_{\rho,\delta}$ are called symbols of order m and type (ρ,δ). If $\rho+\delta = 1$ we also use the notation S^m_ρ, and when $\rho = 1$ and $\delta = 0$ we sometimes write only S^m and talk about symbols of order m. Finally we set

$$S^\infty_{\rho,\delta} := \cup_m\ S^m_{\rho,\delta}, \quad S^{-\infty}_{\rho,\delta} := \cap_m\ S^m_{\rho,\delta}$$

We begin with the definition of Fourier Integrals, see [10], [16].

Definition 4.2. ***Fourier Integrals***
Let Ω be open subset of $\mathbb{R}^n$; the integral

$$I_\varphi(au) = \int_{\Omega\subset\mathbb{R}^n}\int_{\mathbb{R}^N} e^{i\varphi(x,\theta)} a(x,\theta) u(x)\ dx\ d\theta, \quad u \in C_0^\infty$$

is absolutely convergent if φ is real, $a \in S^\mu_\rho(\Omega \times \mathbb{R}^N)$ and $\mu + \rho < 0$. In this case $u \to I_\varphi(au)$ is continuous on $C_0^0(\Omega)$ and therefore define a distribution A on in Ω of order 0.

We continue with the following two definitions, due to Hörmander [16].

Definition 4.3. ***Phase functions***
Let Ω be an open subet of $\mathbb{R}^n$, $N \in \mathbb{N}$; we define the class Φ of phase functions

$$\phi : \Omega \times \Omega \times \mathbb{R}^N \to \mathbb{R}$$

such that:

(i) ϕ is real valued C^∞ function in $\Omega \times \Omega \times \mathbb{R}^N/\{0\}$;

(ii) ϕ is positive-omogeneous with repect to θ of degree one.

Definition 4.4. ***Fourier Integral Operators***
Let consider a simbol $a \in S^m_{\rho,\delta}$, and a phase function $\phi \in \Phi$. Now we can define the corresponding Fourier Integral Operator A on the functions $u \in C_0^\infty(\Omega)$:

$$Au(x) := \int_{\mathbb{R}^n} \int_{\mathbb{R}^N} e^{i\phi(x,y,\theta)} a(x,y,\theta) d\theta \ u(y) dy \tag{4.32}$$

We say that A has oder m and type (ρ, δ).

Now we remember a result about linear operators and distribution theory that will be used in the next theorem.

Theorem 4.5. *For every linear continuous operator A in the space $\mathcal{S}(\mathbb{R}^n)$ there exists a family of tempered distributions $\mathcal{A}(x, \cdot)$ depending on the parameter $x \in \mathbb{R}^n$ such that*

$$Av(x) = \langle \mathcal{A}(x,y), v(y) \rangle \quad \forall x \in \mathbb{R}^n, \tag{4.33}$$

Moreover for every $\mu \in \mathcal{S}'(\mathbb{R}^n)$, the map:

$$v \longmapsto \langle \mu, Av \rangle = \langle \mu(x), \langle \mathcal{A}(x,y), v(y) \rangle \rangle \quad v \in S(\mathbb{R}^n), \tag{4.34}$$

is a tempered distribution.

Definition 4.6. *The family of distributions $\mathcal{A}$ is said to be the Schwartz kernel of the linear operator A.*

We underline that Definition 4.4 is very much general, and for this reason it is useful to add non-degenerate conditions on the class of phase functions Φ in order to prove the well defined setting of the FIO.
Moreover we will see that many authors make different hypothesis on phase functions in order to prove more properties about the related FIO.

With respect to this problem, we recall an Hörmander's result (see [16]).

Theorem 4.7. ***Existence and continuity***

(i) *If $\phi \in \Phi$ has no critical points as function of (x, y, θ) with $\theta \neq 0$ then the oscillatory integral (4.32) exists in the sense of distributions. Moreover if*

$$m + N < k\rho, \quad m + N < k(1 - \delta), \tag{4.35}$$

then $\langle Au, v\rangle$ is a continuous bilinear form for the C_0^k topologies on u, v. Moreover A is a continuous linear map from $C_0^k(\Omega)$ to $\mathcal{D}'^k(\Omega)$ wich has a distribution kernel $K_A \in \mathcal{D}'^k(\Omega \times \Omega)$ given by the oscillatory integral

$$K_A(w) = \int_{\mathbb{R}^N} e^{i\phi(x,y,\theta)} a(y, \theta) w(x, y) dxdyd\theta, \tag{4.36}$$

(ii) *If $\phi \in \Phi$ has no critical points as function of (y, θ) with $\theta \neq 0$, then the oscillatory integral (4.32) exists in the sense of distributions. When (4.35) is valid then A is a continuous linear map from $C_0^k(\Omega)$ to $C(\Omega)$.*
If it is satisfied:

$$m + N + j < k\rho, \quad m + N + j < k(1 - \delta), \tag{4.37}$$

then A is a continuous linear map from $C_0^k(\Omega)$ to $C^j(\Omega)$.

(iii) *If $\phi \in \Phi$ has no critical points as function of (x, θ) with $\theta \neq 0$, then the adjoint of A has the properties listed in (ii) so A is continuous map from $\mathcal{E}'^j(\Omega)$ to $\mathcal{D}'^k(\Omega)$ when (4.37) is fulfilled. In particular A defines a continuous map from $\mathcal{E}'(\Omega)$ to $\mathcal{D}'(\Omega)$*

(iv) *Let R_ϕ be the set of all $(x, y) \in \Omega \times \Omega$ such that $\phi(x, y, \theta)$ has no critical point $\theta \neq 0$ as a function of θ. Then*

$$K_A(x, y) = \int_{\mathbb{R}^N} e^{i\phi(x,y,\theta)} a(y, \theta) d\theta \tag{4.38}$$

defines a function in $C^\infty(R_\phi)$ which is equal to the distribution (4.36) in R_ϕ. If $R_\phi = \Omega \times \Omega$, it follows that A is an integral operator with a C^∞ kernel, so A is a continuous map of $\mathcal{E}'(\Omega)$ into $C^\infty(\Omega)$.

This theorem is still quite general and it turns out that it is useful to encrease the hypothesis made on the phase functions, in order to obtain more features, for example continuity on $C_c^\infty(\Omega)$ and moreover L^2 local and global boundedness. With respect to this aim we begin by reporting the following theorem due to Trèves [29]:

Theorem 4.8. *Consider the Fourier Integral Operator A as in definition (4.32) related to a simbol $a \in S^m_{\rho,\delta}$ and a phase function $\phi \in \Phi$ as in (4.3). We assume the non-degeneracy condition: $d_{x,\theta}\phi$ and $d_{y,\theta}\phi$, the differentials of ϕ with respect to (x, θ) and (y, θ) respectively, do not vanish anywhere in $\Omega \times \Omega \times \mathbb{R}^N/\{0\}$.*
Then A is a continuous linear map from $C_c^\infty(\Omega) \to C^\infty(\Omega)$ which can be extended as a continuous map $\mathcal{E}'(\Omega) \to \mathcal{D}'(\Omega)$.

Some recent papers of Ruzhansky and Sugimoto (see [26] and [27]) deal with L^2-boundedness for FIO, within different hypothesis on phase functions and symbols. Here we provide a result on FIO directly generalizing a class of PDO.

Theorem 4.9. *Let T be defined by*

$$Tu(x) := \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} e^{i(x\cdot\xi + \phi(y,\xi))} a(x,y,\xi) u(y) dy d\xi \tag{4.39}$$

Let $\phi \in C^\infty(\mathbb{R}^{2n}; \mathbb{R})$ be such that $\forall |\gamma| \geq 1$

$$\left| \det\Big(\partial_x \partial_\xi \phi(x,\xi) \Big) \right| \geq C > 0, \tag{4.40}$$

$$|\partial_\xi^\gamma \phi(y,\xi)| \leq C_\gamma \langle y \rangle. \tag{4.41}$$

Let $a = a(x,y,\xi) \in C^\infty(\mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n)$ be such that

$$|\partial_x^\alpha \partial_y^\beta \partial_\xi^\gamma a(x,y,\xi)| \leq C_{\alpha\beta\gamma} \langle x \rangle^{-\alpha}, \tag{4.42}$$

for all α, β, γ. Then T is $L^2(\mathbb{R}^n)$ - bounded.

4.2 Properties

Here we summarize some important features of FIO, in particular we report the equivalence of phase functions theorem, and another result with the proof that the composition of two such operators is still of the same type. Moreover we will see how suitable equivalence classes of these operators are directly related to Lagrangian submanifolds, and it is exactly from this observation that arises the *Global* definition of Fourier Integral Operators that will be drawed in the section.

For the following theorem and lemma I refer to Düistermaat [10].

Theorem 4.10. ***Equivalence of phase functions***
Suppose $\varphi(x,\theta)$ and $\widetilde{\varphi}(x,\widetilde{\theta})$ are nondegenerate phase functions at $(x,\theta_0) \in \mathbb{R}^n \times \mathbb{R}^N \backslash \{0\}$ and at $(x,\widetilde{\theta}_0) \in \mathbb{R}^n \times \mathbb{R}^{\widetilde{N}} \backslash \{0\}$, with N and $\widetilde{N}$ suitable related. Let Γ and $\widetilde{\Gamma}$ be conic neighborhoods of (x_0,θ_0) and $(x_0,\widetilde{\theta}_0)$ such that $T_\varphi : C_\varphi \to \Lambda_\varphi$ and $T_{\widetilde{\varphi}} : C_{\widetilde{\varphi}} \to \Lambda_{\widetilde{\varphi}}$ are injective, respectively. If $\Lambda_\varphi = \Lambda_{\widetilde{\varphi}}$ then any Fourier Integral A, defined by the phase function φ and amplitude $a \in S^\mu_\rho(\mathbb{R}^n \times \mathbb{R}^N)$, $\rho > \frac{1}{2}$, with ess supp a contained in a sufficently conic neighborhood of (x_0,θ_0) is equal to a Fourier Integral defined by the phase function $\widetilde{\varphi}$ and an amplitude $\widetilde{a} \in S_\rho^{\mu + \frac{1}{2}(N-\widetilde{N})}(\mathbb{R}^n \times \mathbb{R}^N)$

The following lemma states exactly the suitable relation between N and $\widetilde{N}$ used in the previous theorem.

Lemma 4.11. *The number $\widetilde{N}$ of θ-variables verifies the inequality $N \geq k$, where k is the dimension of the intersection of the tangent spaces of Λ_φ, and the fiber of $T^\star(\mathbb{R}^n)$ at (x_0, ξ_0). Moreover, every Fourier integral defined by the phase function φ and amplitude $a \in S^\mu_\rho(\mathbb{R}^n \times \mathbb{R}^N)$, ess supp in a sufficently small conic neighborhood of (x_0, θ_0) can also be defined by a phase function in k variables and amplitude $b \in S_\rho^{\mu+\frac{1}{2}(N-k)}(\mathbb{R}^n \times \mathbb{R}^k)$.*

The above theorems show that one should rather speak of distributions A defined by a conic manifold Λ in $T^\star(\mathbb{R}^n)\backslash 0$, wich locally is equal to Λ_φ, φ a non degenerate phase function, instead of distributions defined by some phase functions φ.

Suppose $\Lambda \subset T^\star(\mathbb{R}^n \times \mathbb{R}^n)$ is a Lagrangian submanifold with two generating functions φ_1 and φ_2. We recall that the associated symplectic 2-form is $\bar{\omega} := dx \wedge d\xi - dy \wedge d\eta$. The corresponding submanifold

$$\Lambda' := \{(x, \xi; y, \eta) \in T^\star(\mathbb{R}^n) \times T^\star(\mathbb{R}^n) \mid (x, \xi; y, -\eta) \in \Lambda\},$$

is a Lagrangian submanifold with respect to the symplectic 2-form $\tilde{\omega} := dx \wedge d\xi + dy \wedge d\eta$. Next results states that the composition of two FIO is still a FIO, and the proof is contained in Duistermaat [10].

Theorem 4.12. ***Compostion of FIO***
Let X, Y, Z be open in $\mathbb{R}^{n_X}$, $\mathbb{R}^{n_Y}$, $\mathbb{R}^{n_Z}$ respectively. Let A_1 be a FIO:$C_0^\infty(Y) \to \mathcal{D}'(X)$ defined by a nondegenerate phase function φ_1 in an open cone $\Gamma_1 \in X \times Y \times \mathbb{R}^{N_1}\backslash\{0\}$ and an amplitude $a_1 \in S_\rho^{\mu_1}(X \times Y \times \mathbb{R}^{N_1})$, ess supp $a_1 \in \Gamma_1$. Similarly A_2 is a FIO $: C_0^\infty(Z) \to \mathcal{D}'(Y)$ defined by a nondegenerate phase function φ_2 in an open cone Γ_2 $\in Y \times Z \times \mathbb{R}^{N_2}\backslash\{0\}$ and an amplitude $a_1 \in S_\rho^{\mu_1}(Y \times Z \times \mathbb{R}^{N_2})$, ess supp $a_2 \in \Gamma_2$. Assume $\rho > \frac{1}{2}$ and:

$$\begin{aligned}
&The\ projection\ from\ \pi_{X\times Y}(suppa_1) \times \pi_{Y\times Z}(suppa_2) \cap X \times (diagY) \times Z \\
&into\ X \times Z\ is\ a\ proper\ mapping, && (4.43)\\
&\eta \neq 0\ if\ (x, \xi, y, \eta) \in \Lambda'_{\varphi_1}\ or\ (y, \eta, z, xi) \in \Lambda'_{\varphi_2}, && (4.44)\\
&\xi \neq 0\ or\ \zeta \neq 0\ if\ (x, \xi, y, \eta) \in \Lambda'_{\varphi_1}\ and\ (y, \eta, z, \zeta,) \in \Lambda'_{\varphi_2}, && (4.45)\\
&\Lambda'_{\varphi_1} \times \Lambda'_{\varphi_2}\ intersects\ T^\star X \times (diagT^\star Y) \times T^\star Z\ transversally. && (4.46)
\end{aligned}$$

Then $A_1 \circ A_2$ is well defined and, modulo an operator with C^∞ kernel, it is equal to a FIO: $C_0^\infty(Z) \to \mathcal{D}'(X)$ defined by a nondegenerate phase function φ in an open cone $\Gamma \in X \times Z \times \mathbb{R}^N\backslash\{0\}$, $N = N_1 + N_2 + n_Y$, and an amplitude $a \in S_\rho^\mu(Y \times Z \times \mathbb{R}^N)$, ess supp $a \in \Gamma$. Moreover,

$$\Lambda'_\varphi = \Lambda'_{\varphi_1} \circ \Lambda'_{\varphi_2}. \tag{4.47}$$

We now recall a general result about the asymptotic study of a class of oscillatory integrals depending on a parameter, more precisely written in the form

$$\mathrm{I}_\lambda \;:=\; \int_{\mathbb{R}^N} e^{i\lambda\phi(z)}\, a(z)\, dz \tag{4.48}$$

for smooth complex valued phase functions and smooth compactly supported amplitudes functions.

Theorem 4.13. ***Stationary Phase Theorem***
Let $K \subset \mathbb{R}^N$ a compact set, X an open neighborhood of K. Let $\lambda > 0$, phase function $\phi \in C^{k+1}(X;\mathbb{C})$ with $\mathrm{Im}(\phi) \geq 0$ and amplitude function $a \in C^k(K;\mathbb{C})$. Then,

$$\Big| \int_{\mathbb{R}^N} e^{i\lambda\phi(z)}\, (\mathrm{Im}\phi(z))^j a(z)\, dz \Big| \leq C\lambda^{-(j+k)} \sum_{|\alpha|\leq k} \sup_{z\in\mathbb{R}^N} |D^\alpha a(z)| \Big(|\nabla\phi(z)|^2 + \mathrm{Im}\phi(z) \Big)^{|\alpha|/2-k} \tag{4.49}$$

where $j, k \in \mathbb{N}$ and C is bounded when ϕ stays in a bounded subset of $C^{k+1}(X;\mathbb{C})$. When f is real valued the bove estimate reduces to

$$\Big| \int_{\mathbb{R}^N} e^{i\lambda\phi(z)}\, a(z)\, dz \Big| \;\leq\; C\lambda^{-k} \sum_{|\alpha|\leq k} \sup_{z\in\mathbb{R}^N} |D^\alpha a(z)| \cdot |\nabla\phi(z)|^{(|\alpha|-2k)}. \tag{4.50}$$

Remark 4.14.

1. *In the case of existence for some critical points of a real valued phase function, the estimate on the righthand side of (4.49) depends on the behaviour of the simbol $a(z)$ around the set of critical points of the phase functions*

$$\Sigma_\phi := \{ z \in X \subset \mathbb{R}^N \mid \nabla\phi(z) = 0 \}.$$

   In particular, $a(z)$ must have a suitable (depending on ϕ) vanishing behaviour around Σ_ϕ.

2. *The application of the Stationary Phase Theorem is very much useful in the study of FIO, and in particular for $L^2(\mathbb{R}^n)$-boundedness theorems.*

3. *The original idea of this result is due to Stokes and Kelvin, with subsequent refinement of Van der Corput. A well written proof, together with various corollaries, can be found in Hörmander [17].*

4. *In the case of $\hbar$-FIO, we have $\lambda = \hbar^{-1}$ and thus the estimate (4.49) becomes a semiclassical estimate of order $O(\hbar^{j+k})$. This is a central tool in the study quantum observables and related time evolution, as well as in the determination of the parametrices of the quantum dynamics.*

5 Global theory of Fourier Integral Operators

In this section we report two global, namely intrinsic, definitions of FIO, due to Hörmander [16] and Duistermaat [10].
Duistermaat provide a direct generalization of local definition, moreover he establish a caracterization of the set of FIO associated to a Lagrangian submanifold.
Hörmander provides a very general formulation for a Fourier Integral Operator that, on the other hand, it is not so easy to connect with local definition.

5.1 Preliminaries

Let V be an n-dimensional vector space over $\mathbb{R}$, $\Lambda^n V$ the space of the n-vectors in V, defined as the dual of the vector space of the n-linear alternanting forms: $V^n \to \mathbb{R}$. $\Lambda^n V$ is one dimensional over $\mathbb{R}$. For any $\alpha \in \mathbb{R}$ we call a *complex valued density of order* α each mapping $\rho : \Lambda^n V \backslash \{0\} \to \mathbb{C}$ such that $\rho(\lambda w) = |\lambda|^\alpha \rho(w)$ for each $w \in \Lambda^n V \backslash \{0\}$, $\lambda \in \mathbb{R}\backslash\{0\}$. The space of all densities of order α is 1-dimensional over $\mathbb{C}$ and will be denoted by $\Omega_\alpha(V)$.
Now let X a n-dimensional C^∞ manifold. The $\Omega_x(T^\star_x X)$, $x \in X$, are the fibers of a C^∞ complex line bundle $\Omega_\alpha(X)$ over X in a natural way. A C^∞ *density on* X *of order* α is now defined as a C^∞ section $\rho : X \to \Omega_\alpha(X)$. The space of C^∞ densities on X of order α will be denoted by $C^\infty(X, \Omega_\alpha)$. Note that after the choice of of a nowhere vanishing standard density of order α, the space $C^\infty(X, \Omega_\alpha)$ can be identified with $C^\infty(X)$. Using a partition of unity it is always contruct a strictly positive C^∞ density of order α on X if X is paracompact.
If $\rho \in C^\infty(X, \Omega_\alpha)$, $\sigma \in C^\infty(X, \Omega_\beta)$ then pointwise multiplication leads to a product $\rho \cdot \sigma \in C^\infty(X, \Omega_{\alpha+\beta})$. In particular

$$(\rho, \sigma) \to \int \rho \cdot \sigma \; dx$$

defines a continuous bilinear form on $C^\infty(X, \Omega_\alpha) \times C^\infty_0(X, \Omega_{1-\alpha})$ so $\sigma \to \int \rho \cdot \sigma \; dx$ is an element of $(C^\infty_0(X, \Omega_{1-\alpha}))'$ that will be also denoted by ρ. It follows that we have a continuous embedding: $C^\infty(X, \Omega_\alpha) \to (C^\infty_0(X, \Omega_{1-\alpha}))'$, and for this reason $(C^\infty_0(X, \Omega_{1-\alpha}))'$ is called the space $\mathcal{D}'(X, \Omega_\alpha)$ of *distribution densities of order* α.

5.2 Settings

By following Duistermaat [10], and Hörmander [16], here we report the Global definition of Fourier Integrals and Fourier Integral Operators.
To begin we remember that, in the local definition, a Fourier Integral is an integral that assume the form:

$$I_\varphi(au) = \int_{\Omega \subset \mathbb{R}^n} \int_{\mathbb{R}^N} e^{i\varphi(x,\theta)} a(x, \theta) u(x) \; dx \; d\theta, \quad u \in C^\infty_0$$

where Ω is an open subset of $\mathbb{R}^n$. It is absolutely convergent if φ is real, $a \in S^{\mu}_{\rho}(\Omega \times \mathbb{R}^N)$ and $\mu + \rho < 0$. In this case $u \to I_{\varphi}(au)$ is continuous on $C^0_0(\Omega)$ and therefore define a distribution A on in Ω of order 0.

While in the global setting we have the following (see [10]):

Definition 5.1. ***Global Fourier Integrals***
Let X be an n-dimensional smooth manifold, Λ an immersed conic Lagrangian submanifold in $T^\star(X)\backslash 0$. A Global Fourier Integral of order m and type ρ by Λ is a distribution $A \in \mathcal{D}'(X, \Omega^{\frac{1}{2}})$ such that

$$A = \sum_{j \in J} A_j, \tag{5.51}$$

where $A_j \in \mathcal{D}'(X, \Omega^{\frac{1}{2}})$ with locally finite supp A_j, and A_j is a FIO defined by a nondegenerate phase function φ_j on a open cone Γ_j in $X \times \mathbb{R}^{N_j}$ such that $(x, \theta) \mapsto (x, d_x\varphi_j(x, \theta))$ is a diffeomorphims from $C_{\phi_j} = \{(x, \theta) \in \Gamma_j \mid d_\theta \varphi_j(x, \theta) = 0\}$ onto an open cone in Λ. For the amplitude a_j it is required that $a_j \in S^{m - N_j/2 + n/4}(X \times \mathbb{R}^{N_j})$, cone supp $a_j \subset \Gamma_j$.
The space of all such Fourier integrals will be denoted by $I^m_\rho(X, \Lambda)$.

Now we refer to Hörmander [17], and I remember the following definition:

Definition 5.2. *Let X be a C^∞ manifold with $dimX = n$, and Y a submanifold of X. The space $I^m(X, Y; E)$ of conormal distribution sections of vector bundle E is the largest subspace of ${}^\infty H^{loc}_{-m-n/4}(X, E)$, which is left invarant by all first order differential operators tangent to the submanifold Y.*

It has proved (see [16], Th. 18.2.12) that it is even invariant under first order Pseudodifferential operators from E to E with principal symbol vanishing on the conormal bundle of Y. The definition is therefore applicable with no change to any Lagrangian manifold:

Definition 5.3. ***Lagrangian Distributions***
Let X be a C^∞ manifold with $dimX = n$, and $\Lambda \subset T^\star X/0$ a C^∞ closed conic Lagrangian submanifold, E a C^∞ vector bundle over X. Then the space $I^m(X, \Lambda; E)$ of Lagrangian distributions of E, of order m, is defined as the set of all $u \in \mathcal{D}'(X, E)$ such that:

$$L_1 ... L_N u \in {}^\infty H^{loc}_{-m-n/4}(X, E) \tag{5.52}$$

for all N and all properly supported $L_j \in \Psi^1(X; E, E)$ with principal symbols L^0_j vanishing on Λ.

Definition 5.4. *Let X, Y be two C^∞ manifolds and E, F two complex vector bundles on X, Y. Then every $\mathcal{D}'(X \times Y, C'; \Omega_{X \times Y} \otimes Hom(F, E))$ defines a continuous map*

$$\mathcal{A} : C^\infty_0(Y, \Omega^{\frac{1}{2}}_Y \otimes F) \to \mathcal{D}'(X, \Omega^{\frac{1}{2}}_Y \otimes E) \tag{5.53}$$

and conversely.

Here the fiber of the vector bundle $Hom(F,E)$ at (x,y) consists of the linear maps F_y to E_x. In particular if Λ is a conic Lagrangian submanifold of $T^\star(X\times Y)\backslash 0$ we can identify $I^m(X\times Y,\Lambda;\Omega_{X\times Y}\otimes Hom(F,E))$ with a space of such maps.
If we have $\Lambda\subset T^\star(X)\backslash 0\times T^\star(Y)\backslash 0$ then it follows (see [16], Th. 25.1.2) that $\mathcal{A}$ is even continuous map from $C_0^\infty(Y)$ to $C^\infty(X)$ which can be extended to a continuous map from $\mathcal{E}'$ to $\mathcal{D}'$ with

$$WF(\mathcal{A}u)=C(WF(u))\quad u\in\mathcal{E}'(Y,\Omega_Y^{\frac{1}{2}}\otimes F), \tag{5.54}$$

where

$$C=\Lambda'=\{(x,\xi,y,\eta)\ \in T^\star(X)\backslash 0\times T^\star(Y)\backslash 0\ ;(x,\xi,y,\eta)\in\Lambda\} \tag{5.55}$$

is a canonical relation from $T^\star(X)\backslash 0$ to $T^\star(X)\backslash 0$. We call $\Lambda=C'$ the twisted canonical relation.

Definition 5.5. ***Global Fourier Integral Operators***
Let C be a homogeneous canonical relation from $T^\star(Y)\backslash 0$ to $T^\star(Y)\backslash 0$ wich is closed in $T^\star(X\times Y)\backslash 0$ and let E,F be vectors bundles on X, Y. Then the operators with kernel belonging to $I^m(X\times Y,C';\Omega_{X\times Y}\otimes Hom(F,E))$ are called Fourier Integral Operators of order m from sections of F to sections of E, associated with the canonical relation C.

5.3 Characterization

Here I report a characterization, due to Duistermaat [10], about the set of Fuorier Integrals associated to a fixed Lagrangian submanifold of $T^\star(\mathbb{R}^n)$.

Definition 5.6. ***Principal symbol***
For a FIO A of order a defined by nondegenerate phase function φ and amplitude $a\in S_\rho^{m-N_j/2+n/4}(X\times\mathbb{R}^N)$, the principal symbol of order m is the element in

$$S_\rho^{m+n/4}(\Lambda,\Omega^{\frac{1}{2}}\otimes L)/S_\rho^{m+n/4+1-2\rho}(\Lambda,\Omega^{\frac{1}{2}}\otimes L), \tag{5.56}$$

given by

$$\Lambda\ni\alpha\mapsto e^{i\psi(\pi(\alpha),\alpha)}\langle ue^{-i\psi(x,\alpha)},A\rangle. \tag{5.57}$$

Here L is the complex Keller Maslov line bundle of Λ (see Keller...). While $\Lambda=\Lambda_\varphi$ is the conic Lagrangian manifold in $T^\star(X)\backslash 0$ defined by φ. $S_\rho^\mu(\Lambda,\Omega^{\frac{1}{2}}\otimes L)$ denotes the symbol space of sections of the complex line bundle $\Omega^{\frac{1}{2}}\otimes L$ over Λ, of growh order μ. Moreover, $u\in C_0^\infty(X\times\Lambda)$ and $\psi\in C^\infty(X\times\Lambda)$, $\psi(x,\alpha)$ is homogeneous of degree 1 in α and the graph $x\mapsto d_x\psi(x,\alpha)$ intersects Λ transversally at α. Regarding (5.56) as a function of u and ψ, it becomes an element of (5.56).

With respect to these definitions we have the following results:

Theorem 5.7. *If the immersion $\Lambda\mapsto T^\star(X)\backslash 0$ is proper and injective (that is an embedding), then the mapping $A\mapsto$ principal symbol of A, is an isomorphism:*

$$I_\rho^m(X,\Lambda)/I_\rho^m(X,\Lambda)\to S_\rho^{m+n/4}(\Lambda,\Omega^{\frac{1}{2}}\otimes L)/S_\rho^{m+n/4+1-2\rho}(\Lambda,\Omega^{\frac{1}{2}}\otimes L) \tag{5.58}$$

6 Parametrices for the quantum evolution

Here we summurize some recents papers, about the use of global FIO with non omogeneous phase functions to represent solutions of Schrödinger equations.
From a general viewpoint, we can mainly divide the large related literature into three parts. More precisely, the papers involving the use of oscillatory integrals with local in time and real phases, with the global in time and complex phases and finally with the global in time and real phases.
In particular, about the first approach we refer to the papers of J. Chazarain [8], D. Fujiwara [12], Kitada and Hitoshi Kumano-go [18]. About the second appoach we refer to A. Hassel and J. Wunsch [15], A. Laptev and I.M. Sigal [21], L. Kapitansky and Y. Safarov [19], T. Swart and V. Rousse [33], D. Robert [30]. About the last approach we refer to S. Graffi and L. Z. [14], L. Z. [40].

6.1 Real and local phases

In the local in time approach to the study of the Schrödinger equation (see for example J. Chazarain [8], D. Fujiwara [12], Kitada and Hitoshi Kumano-go [18]), the objective is the study of the problem

$$\begin{aligned} i\hbar\partial_t\psi(t,x) &= -\frac{\hbar^2}{2m}\Delta_x\psi(t,x)+V(x)\psi(t,x) \\ \psi(0,x) &= \varphi(x)\in\mathcal{S}(\mathbb{R}^n), \end{aligned} \tag{6.59}$$

with $|\partial_x^\alpha V(x)|\le C$, $|\alpha|=2$, for the time interval $t\in[0,T_0)$ where T_0 is the first time of appearence of the caustics phenomena in the phase space (see for example [1], [39]). Roughly speaking, the graph of the Hamiltonian flow can be globally parametrized with a generating function without θ-auxiliary parameters. Within this local in time setting, it can be constructed approximated parametrices for the Schrödinger propagator in the so-called WKB semiclassical asymptotics,

$$\psi(t,x)=(U_\hbar(t)\varphi)(x)=\int_{\mathbb{R}^n}U_\hbar(t,x,\eta)\hat{\varphi}_\hbar(\eta)d\eta$$

where $\hat{\varphi}_\hbar(\eta)$ is the $\hbar$-Fourier transform of the initial datum. Obviously, it can be equivalently studied the fundamental solution of the above problem, namely $\bar{U}_\hbar(t,x,y)$. The first equation naturally arising is the evolutive Hamilton-Jacobi equation

$$\frac{|\nabla_x S|^2}{2m}(t,x,\eta)+V(x)+\partial_t S(t,x,\eta)=0 \tag{6.60}$$

with the initial condition $S(0,x,\eta):=x\cdot\eta$.
The second equation is the coupled transport equation,

$$\partial_t\rho_0+\frac{1}{m}\nabla_x S\ \nabla_x\rho_0+\frac{1}{2m}\Delta_x S\ \rho_0=0, \tag{6.61}$$

with the initial condition $\rho_0(0,x,\eta) := 1$.
The third equation is a recoursive transport type equation

$$\partial_t \rho_j + \frac{1}{m}\nabla_x S \ \nabla_x \rho_j + \frac{1}{2m}\Delta_x S \ \rho_j = \frac{i}{2m}\Delta_x \rho_{j-1}, \tag{6.62}$$

with initial condition $\rho_j(0,x,\eta) := 0$, $j \in \mathbb{N}$, $j \geq 1$.
We are now ready to provide the family of operators

$$\hat{U}_{N,\hbar}(t)\varphi := (2\pi\hbar)^{-n} \sum_{j=0}^{N} \hbar^j \int_{\mathbb{R}^n} e^{\frac{i}{\hbar}S(t,x,\eta)} \ \rho_j(t,y,\eta)\hat{\varphi}_\hbar(\eta)\, d\eta \tag{6.63}$$

which satisfies, for $t \in [0,T_0)$, the estimate

$$\|U_\hbar(t) - \hat{U}_{N,\hbar}(t)\|_{L^2\to L^2} \leq R_N(T_0)\hbar^{N+1}.$$

In order to construct a parametrices for $U_\hbar(t)$ with $t \in [0,T]$ for $T \geq T_0$, we recall the semigroup property,

$$U_\hbar(t) = U_\hbar\left(\frac{t}{M}\right)^M, \quad M := [T/T_0].$$

Hence, for $t \in [0,T]$ it holds

$$\left\| U_\hbar(t) - \hat{U}_{N,\hbar}\left(\frac{t}{M}\right)^M \right\|_{L^2\to L^2} \leq \widetilde{R}_N(T)\, \hbar^{N+1}.$$

6.2 Complex and global phases

In the work of Laptev and Sigal it is considered an Hamiltonian function $H(t,x,\xi)$ real and smooth on $\mathbb{R} \times T^\star\mathbb{R}^n$, it is assumed that there are $m > 0$ and $C_{\mu\nu} > 0$ such that: for every $(t,x) \in \mathbb{R} \times \mathbb{R}^n$

$$|\partial_x^\mu \partial_\xi^\nu H(t,x,\xi)| < C_{\mu\nu}(1+|\xi|)^m.$$

Let $\hat{H}_\hbar(t)$ be the $\hbar$-PDO with symbol H, i.e.

$$\hat{H}_\hbar(t)f = (2\pi\hbar)^{-n} \int_{\mathbb{R}^{2n}} H(t,x,\xi) e^{\frac{i}{\hbar}(x-y)\cdot\xi} f(y)dyd\xi,$$

defined on the functions $f \in C_0^\infty(\mathbb{R}^n)$. Let $\widetilde{\Omega} \subseteq \mathbb{R}^{2n}$ be an open bounded set and $\mathcal{X}_{\widetilde{\Omega}}$ the characteristic function; the associated PDO reads

$$\hat{\mathcal{X}}_{\widetilde{\Omega}} f = (2\pi\hbar)^{-n} \int_{\mathbb{R}^{2n}} \mathcal{X}_{\widetilde{\Omega}}(y,\eta) e^{\frac{i}{\hbar}(x-y)\cdot\eta} f(y)dyd\eta,$$

Within this settings, it is considered the family of operators $U_\hbar(t)$ solving the Scrödinger equation in the functional form:

$$\begin{aligned} i\hbar\partial_t U_\hbar(t) &= \hat{H}_\hbar(t) \circ U_\hbar(t), & (6.64)\\ U_\hbar(0) &= \mathbb{I}. & (6.65) \end{aligned}$$

In order to localize a family of oscillatory integrals giving approximated parametrices, it is first defined the set of complex phase functions $\varphi(t,x,y,\xi) \in C^\infty([0,T]\times\mathbb{R}^n\times\widetilde{\Omega};\mathbb{C})$ related to the Hamiltonian flow $(x^t(y,\eta);\xi^t(y,\eta))$ and the action function

$$S(t,y,\eta) := \int_0^t \; h_\xi(s,x^s,\xi^s)\cdot\xi^s - h(s,x^s,\xi^s)\; ds, \tag{6.66}$$

by the following conditions

1. $\varphi(t,x^t(y,\eta),y,\eta) = S(t,y,\eta)$,

2. $\varphi_x(t,x^t(y,\eta),y,\eta) = \xi^t(y,\eta)$,

3. $-i\varphi_{xx}(t,x,y,\eta) \geq 0$ and is indipendent of x,

4. $\det\,\varphi_{xy}(t,x^t(y,\eta),y,\eta) \neq 0$ for $(t,y,\eta) \in [0,T)\times\widetilde{\Omega}$.

It is proved that these phase functions admit the representations:

$$\varphi(t,x,y,\xi) = S(t,y,\eta) + (x-x^t)\cdot\xi^t + i(x-x^t)\cdot B(x-x^t)/2 \tag{6.67}$$

where $B = B(y,\eta,t)$ are non-negative definite $n\times n$ matrix.

The main result of this paper consists on the construction of a family of L^2-bounded operator U_N with Schwartz kernel

$$U_{N,\hbar}(t,x,y) = (2\pi\hbar)^{-n}\int_{\mathbb{R}^n} e^{i\varphi/\hbar}\; u_{N,\hbar}(t,y,\eta)\; d\eta \tag{6.68}$$

and simbol $u_{N,\hbar} := \sum_{k=0}^N \hbar^k u^k_{N,\hbar}(t,y,\eta)$, φ in the class descripted above, and where it is fulfilled the approximation result:

$$\sup_{0\leq t\leq T} \|\,(U_\hbar(t) - U_{N,\hbar}(t))\circ\mathcal{X}_{\widetilde{\Omega}}\,\|_{L^2\to L^2} \leq C\,\hbar^N. \tag{6.69}$$

with $C = C(T,N,\widetilde{\Omega}) > 0$.

Remark 6.1. *Note that here it is not required the invariance of the set $\widetilde{\Omega} \subseteq \mathbb{R}^{2n}$ under the Hamiltonian flow. However, it is still an open problem to exhibit a more precise estimate of this constant with respect to $(T,N,\widetilde{\Omega})$.*

6.3 Real and global phases

Let us consider the Schrödinger equation:

$$\begin{aligned} i\hbar\partial_t\psi(t,x) &= -\frac{\hbar^2}{2m}\Delta\psi(t,x) + V(x)\psi(t,x), \\ \psi(0,x) &= \varphi(x), \end{aligned} \tag{6.70}$$

where $\varphi \in \mathcal{S}(\mathbb{R}^n)$ and the confining potential $V \in C^\infty(\mathbb{R}^n;\mathbb{R})$ is such that:

$$\begin{aligned} c_1\langle x\rangle^d \le V(x) &\le c_2\langle x\rangle^d, & 0 < c_1 < c_2, \quad d \ge 2, \quad \forall\ |x| \ge R, \\ |\partial_x^\alpha V(x)| &\le c\langle x\rangle^{d-|\alpha|}, & \forall\ |\alpha| \ge 0. \end{aligned}$$

Our objective is to determine, for arbitrary large time intervals $t \in [0,T]$, a family of semiclassical series of global Fourier Integral Operators for the one parameter unitary transformations $U(t) := e^{-i\hat{H}t/\hbar}$ where $\hat{H} = -\frac{\hbar^2}{2m}\Delta + V(x)$. More precisely, in this general setting we make a phase-space localization of the propagator within sublevels of the energy function $H(x,p) := \frac{1}{2m}|p|^2 + V(x)$. In fact, we follow a similar approach with respect to the paper of Laptev and Sigal in [21], by looking at bounded sets which are invariant under the Hamiltonian flow,

$$\widetilde{\Omega}_E := \{(y,\eta) \in \mathbb{R}^{2n} \mid E_0 < H(y,\eta) < E\}, \quad E_0 := \inf_{(y,\eta)\in\mathbb{R}^{2n}} H(y,\eta) < E,$$

and defining the related Pseudodifferential operators:

$$\mathcal{X}_E\varphi := (2\pi\hbar)^{-n}\int_{\mathbb{R}^{2n}} e^{\frac{i}{\hbar}\langle x-y,\eta\rangle}\chi_{\widetilde{\Omega}_E}(y,\eta)\varphi(y)dyd\eta.$$

The aim is to exhibit semiclassical approximations for $U(t)\circ\mathcal{X}_E$ through a family of global Fourier Integral Operators.
We are now ready to state the main result of the paper

Theorem 6.2. *For all $E > E_0$, $N \ge 0$ and $t \in [0,T]$ there exists a series of L^2-bounded global Fourier Integral Operators*

$$U_{j,E}(t)\varphi(x) := (2\pi\hbar)^{-n+j}\int_{\mathbb{R}^{2n}}\int_{\mathbb{R}^k} e^{\frac{i}{\hbar}S_E(t,x,\eta,\theta)-\langle y,\eta\rangle}\rho_{j,E}(t,x,y,\eta,\theta)d\theta\varphi(y)dyd\eta, \tag{6.71}$$

such that the following remainder estimate fulfills

$$\Big\| U(t)\circ\mathcal{X}_E - \sum_{j=0}^N U_{j,E}(t)\Big\|_{L^2\to L^2} \le C_N(T)\hbar^{N+1}. \tag{6.72}$$

with $C_N(T)$ not depending on the energy value. The phase S_E belongs to a class of global generating functions for the graph of the Hamiltonian flow $\phi_H^t : \mathbb{R}^{2n} \to \mathbb{R}^{2n}$ within the bounded energy set $\widetilde{\Omega}_E$, namely:

$$\begin{aligned} \Lambda_t^E &:= \Big\{(y,\eta;x,p) \in \widetilde{\Omega}_E \times \widetilde{\Omega}_E \mid (x,p) = \phi_H^t(y,\eta)\Big\} \\ &= \Big\{(y,\eta;x,p) \in \widetilde{\Omega}_E \times \widetilde{\Omega}_E \mid p = \nabla_x S_E,\ y = \nabla_\eta S_E,\ 0 = \nabla_\theta S_E\Big\} \end{aligned} \tag{6.73}$$

where $\theta \in \mathbb{R}^k$ and the dimension must fulfill the lower bound:

$$k > 2n + K_{d,E,N}\ T^4 \tag{6.74}$$

where $K_{E,N,d}$ is shown in Section 3.3. The zero order simbol $\rho_0 \in C^\infty([0,T] \times \mathbb{R}^{3n} \times \mathbb{R}^k; \mathbb{R})$ belongs to a class of solutions of the transport equation written in the following geometrical setting:

$$\partial_t \rho_{0,E} + \frac{1}{m}\nabla_x S_E\ \nabla_x \rho_{0,E} + \frac{1}{2m}\Delta_x S_E\ \rho_{0,E} = 0, \quad \nabla_\theta S_E = 0, \tag{6.75}$$

where the initial condition $\rho_{0,E}(0,x,y,\eta,\theta) := \sigma(\theta)\chi_{\widetilde{\Omega}_E}(y,\eta)$ is arbitrary fixed with a probability measure:

$$\sigma \in \mathcal{S}(\mathbb{R}^k; \mathbb{R}^+), \quad \int_{\mathbb{R}^k} \sigma(\theta) d\theta = 1, \quad \sigma(\theta) \leq c_d\, e^{-|\theta|^d}. \tag{6.76}$$

The higher order simbols $\rho_{j,E} \in C^\infty([0,T] \times \mathbb{R}^{3n} \times \mathbb{R}^k; \mathbb{R})$ belongs to a class of solutions of the following recoursive transport equations:

$$\partial_t \rho_{j,E} + \frac{1}{m}\nabla_x S_E\ \nabla_x \rho_{j,E} + \frac{1}{2m}\Delta_x S_E\ \rho_{j,E} = \frac{i}{2m}\Delta_x \rho_{j-1,E}, \quad \nabla_\theta S_E = 0, \tag{6.77}$$

with initial condition $\rho_{j,E}(0,x,y,\eta,\theta) = 0$, $\forall j \geq 1$.

In the asymptotic quadratic case $d = 2$, we can deal with a global FIO framework not depending on the energy. Indeed, in this case there exists a global generating function S not depending on the energy E and with finite auxiliary variables $\theta \in \mathbb{R}^k$ with global lower bound

$$k > 2n + \bar{K}_N T^4. \tag{6.78}$$

This fact allow a construction as in the previous theorem, but now without the Pseudodifferential operator $\mathcal{X}_E$ and with simbols not depending on y-variables. We can determine semiclassical series of global FIO directly representing the $\hbar$-Fourier transform $\mathcal{F}_\hbar$ of the foundamental solution of the propagator.

Theorem 6.3. *In the case $d = 2$, for all $N \geq 0$ and $t \in [0,T]$ we have a series of L^2-bounded global Fourier Integral Operators:*

$$U_j(t)\varphi(x) = (2\pi\hbar)^{-n+j} \int_{\mathbb{R}^n} \int_{\mathbb{R}^k} e^{\frac{i}{\hbar}S(t,x,\eta,\theta)} \rho_j(t,x,\eta,\theta) d\theta \hat{\varphi}_\hbar(\eta) d\eta, \tag{6.79}$$

where $\hat{\varphi}_\hbar := \mathcal{F}_\hbar \varphi$, and the remainder fulfills

$$\Big\| U(t) - \sum_{j=0}^{N} U_j(t) \Big\|_{L^2 \to L^2} \leq C_N(T)\hbar^{N+1}. \tag{6.80}$$

Remark 6.4.

1. *As a consequence of the above construction, the function S is a smooth solution for the Hamilton-Jacobi equation written in the geometrical setting:*

$$\frac{|\nabla_x S|^2}{2m} + V(x) + \partial_t S = 0, \quad \nabla_\theta S = 0, \tag{6.81}$$

with initial condition $S(0, x, \eta, \theta) = \langle x, \eta \rangle$. This enlarged geometrical framework allow to overcome the well known problem of the caustics. Indeed, here the global simbols simbols ρ_j solve transport type equations in the large time regime. This is an improvement with respect to the WKB method (see for example Chazarain [8], Fujiwara [12]) where the global in time construction is made by multiple composition of FIO's and so by the product of local in time simbolds, which not gives an intrinsic approach.

2. *The time interval $[0, T]$ is not depending on $\hbar$, so our construction works also beyond the so called Ehrenfest time $T(\hbar) \simeq -c \ \log \hbar$; this is a problem arising frequently in the literature, see for example the recent paper of Swart and Rousse [33] or Robert [30] for the parametrices of the propagator, or the paper of Bambusi, Graffi and Paul [3] for the evolution of the quantum observables.*

3. *The time interval $[0, T]$ is not depending on the energy values E. Indeed, our technique overcomes the obstruction shown by Yajima and Zang in [31] related to the limitation in time $T(E)$ for the approximated representation of the integral kernel of the propagator in the case of same class of superquadratic potentials.*

4. *This paper shows that the use of global FIO with complex valued phase functions are not necessary in order to determine global in time parametrices of the propagator, as instead it is suggested in the papers of Swart and Rousse [33], Robert [30], Laptev and Sigal [21], Kapitansky and Safarov [19].*

5. *The constant for the operator norm estimate in (6.72) does not depend on the energy, invariant set $\widetilde{\Omega}$ and $\hbar$; this is an improvement with respect to the same type of computation on the remainder operator of a semiclassical FIO series, shown by Laptev and Sigal in [21].*

6. *Here we improve our previous result [14] bsed on the same type of semiclassical series of global FIO for the propagator, but now it is exhibited for a larger class of confining potentials.*

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