A local limit theorem for a transient chaotic walk in a frozen environment☆

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Abstract

This paper studies particle propagation in a one-dimensional inhomogeneous medium where the laws of motion are generated by chaotic and deterministic local maps. Assuming that the particle's initial location is random and uniformly distributed, this dynamical system can be reduced to a random walk in a one-dimensional inhomogeneous environment with a forbidden direction. Our main result is local a limit theorem which explains in detail why, in the long run, the random walk's probability mass function does not converge to a Gaussian density, although the corresponding limiting distribution over a coarser diffusive space scale is Gaussian.

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1. Introduction

1.1. A chaotic dynamical system

This paper studies a particle moving in a continuous inhomogeneous medium which is composed of a linear chain of cells modeled by the unit intervals $[k, k+1)$ of the positive real line. Each interval $[k, k+1)$ is assigned a label ω_k and a map U_{ω_k} which determines the dynamics of the particle as long as the particle remains in the interval. The sequence of labels $\omega = (\omega_k)_{k \in \mathbb{Z}_+}$, called an *environment*, is assumed to be either nonrandom, or a realization of a random sequence that is frozen during the particle's lifetime.

We are interested in the case in which the local dynamical rules U_{ω_k} are chaotic in the sense that the distance between two initially nearby particles grows at an exponential rate. More concretely, we shall focus on a model where a particle located at $x_n \in [k, k+1)$ at time n jumps to $x_{n+1} = k + U_{\omega_k}(x_n - k)$. Here $\omega_k \in (0, 1)$ and U_{ω_k} is the piecewise affine map from $[0, 1)$ onto $[0, 2)$ such that $U_{\omega_k}[0, 1 - \omega_k) = [0, 1)$ and $U_{\omega_k}[1 - \omega_k, 1) = [1, 2)$. The dynamical system generated by the local rules is compactly expressed by $x_{n+1} = \mathcal{U}_\omega(x_n)$, where the global map $\mathcal{U}_\omega$ on the positive real line is defined by

$$\mathcal{U}_\omega(x) = [x] + U_{\omega_{[x]}}(x - [x]), \tag{1.1}$$

and $[x]$ denotes the integral part of x, see Figure 1.

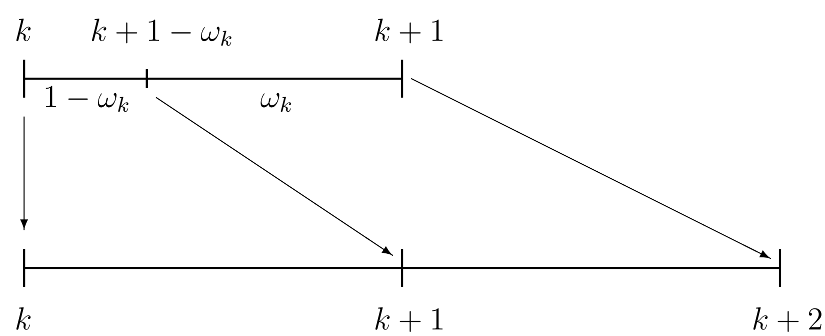

Figure 1: The map $\mathcal{U}_\omega$ acts piecewise affinely on the interval $[k, k+1)$.

The above model belongs to the realm of extended dynamical systems, a somewhat vaguely defined yet highly active field of research (e.g. Chazottes and Fernandez [1]). Telltale characteristics of such systems are a noncompact or high-dimensional phase space and the lack of relevant finite invariant measures. Our principal motivation is to study the impact of environment inhomogeneities on the long-term behavior of extended dynamical systems. Concrete models include neural oscillator networks (Lin, Shea-Brown, and Young [2]) and the Lorentz gas with randomly placed scatterers (Chernov and Dolgopyat [3]; Cristadoro, Lenci, and Seri [4]), to name a few. In this paper, we shall restrict the analysis to the affine dynamical model in (1.1), to keep the presentation simple and clear.

1.2. Random initial data

Because the local maps U_{ω_k} are chaotic, predicting the particle's future location with any useful accuracy over any reasonably long time horizon would require a precise knowledge of its initial position — a sheer impossibility in practice. Therefore, it is natural to take the statistical point of view and study the stochastic process defined by

$$\begin{aligned} x_0 &\overset{d}{=} \text{Uniform}[0,1), \\ x_{n+1} &= \mathcal{U}_\omega(x_n). \end{aligned} \tag{1.2}$$

To analyze the time evolution of the above process, we must impose some regularity conditions on the environment. In particular, those conditions guarantee ballistic motion, and one might guess that the distribution approaches Gaussian in the long run. To test this hypothesis, we have plotted in Figure 2 numerically computed histograms of x_n at time $n = 2^{13}$ in two frozen environments, using the intervals $[k, k+1)$ as bins. Rather surprisingly, the histograms do not appear Gaussian. A similar phenomenon was recently observed by Simula and Stenlund [5, 6].

1.3. Summary of main results

The main contribution of the paper is to explain the emergence of the histograms in Figure 2. This is accomplished by first reducing the continuum dynamical system to a unidirectional random walk on the integers (Theorem 2.1), and then deriving a local limit law (Theorem 2.4) that completely explains the behavior observed in Figure 2. As a byproduct, we also obtain a law of large numbers (Theorem 2.3) and a central limit law (Theorem 2.5) for the walk. These limit laws are valid for all frozen environments — random or nonrandom — which satisfy certain statistical regularity properties. We

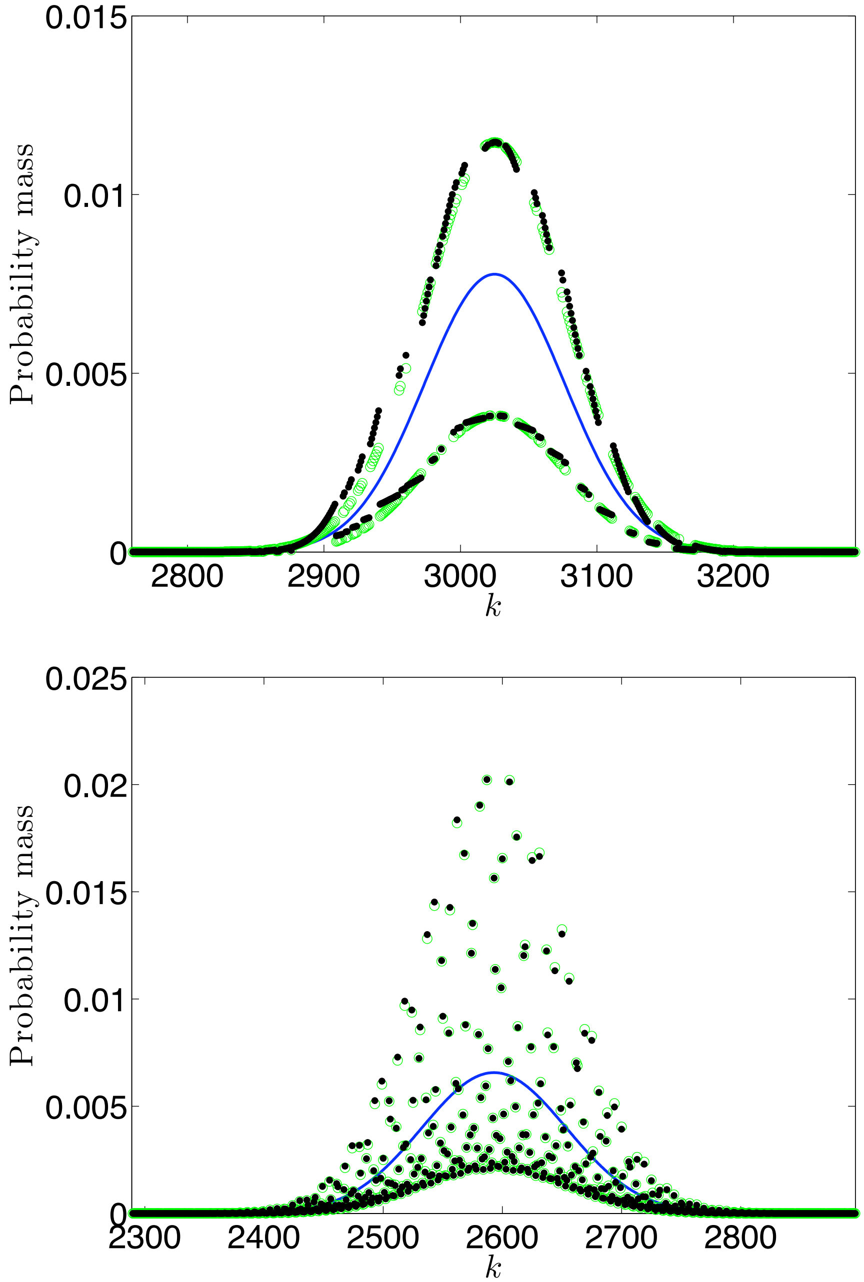

Figure 2: Histograms of x_n (black dots) at time $n = 2^{13}$ in two frozen environments ω. Top: ω is a realization of a Markov chain with values in $\left\{\frac{1}{4}, \frac{3}{4}\right\}$ and $\mathbb{P}(\omega_{k+1} = \omega_k \,|\, \omega_k) = \frac{4}{5}$. Bottom: ω is nonrandom, with $\omega_k = \frac{11}{20} + \frac{9}{20}\sin k$. The green circles are obtained by modulating the blue Gaussian density by the factor ω_k^{-1}/μ appearing in Theorem 2.4.

also devote a separate section to the analysis of random environments, where we show (Theorem 2.7) that the three aforementioned limit laws are valid for almost all realizations of a stationary random environment under suitable moment and mixing conditions. Because the random walk is unidirectional (it never steps backwards), limit theorems for its hitting times are immediate consequences of classical limit laws for independent random variables. Translating the limit theorems of the hitting times into limit theorems of the walk location form the main task in proving the results; see Section 3.3 for a general outline of the proofs.

1.4. Related work

We shall discuss here only literature most closely related to transient one-dimensional random walks in quenched random environments; for a broad picture of the theory of random walks in random environments, see e.g. Bolthausen and Sznitman [7], Sznitman [8], and Zeitouni [9]. Laws of large numbers and averaged central limit theorems for random walks in random environments have been known already for a long time (e.g. Solomon [10]; Kesten, Kozlov, and Spitzer [11]), whereas the literature on quenched central limit theorems is more recent. Buffet and Hannigan [12] proved a quenched central limit theorem for a pure birth process in an independently scattered random environment under moment conditions later relaxed by Horváth and Shao [13, 14]; this model is a direct continuous-time analogue of the random walk studied here. Quenched central limit theorems for more general one-dimensional transient random walks were proved only very recently, independently by Goldsheid [15] and Peterson [16] (see also Alili [17] for a result concerning a special quasiperiodic environment). Rassoul-Agha and Seppäläinen [18] obtained a similar result for multidimensional random walks with a forbidden direction, which in the one-dimensional case corresponds to the unidirectional walk analyzed in this paper. Dolgopyat, Keller, and Liverani [19] have obtained a quenched central limit theorem for environments changing in time and space.

Our approach differs from most earlier works in that we separately analyze the two degrees of randomness involved in random walks in quenched random environments. In the first part, we extract a set of statistical regularity properties for a given environment that are sufficient for proving the limit laws, while treating the environment as nonrandom. In the second part, we derive conditions for the probability distribution of the random environment that yield almost surely regular realizations. A key result for the second part is a law of large numbers for the moving averages of a stationary sequence (Lemma 4.2), which is proved with the help of Peligrad's extension [20] of the Baum–Katz theorem [21] (see Bingham [22] for a nice survey). The major advantage of this approach is a clarified picture on how different sources of randomness affect the random walk's behavior in quenched random environments.

Local limit theorems for random walks in homogeneous environments (e.g. Spitzer [23]) can be proved as simple consequences of Gnedenko's classical theorem (e.g. [24, Chapter 9] or [25, Section 3.5]), because such walks are just sums of independent random variables. A generalization for a periodic environment was derived by Takenami [26], and for a periodic graph recently by Kazami and Uchiyama [27]. In contrast, local limit theorems for random walks in aperiodic inhomogeneous environments appear nonexistent. To the best of our knowledge, Theorem 2.4 and Theorem 2.7 are the first local limit laws concerning transient random walks in aperiodic nonrandom or quenched random environments. Although our analysis is restricted to a very special instance of a random walk, the model is still rich enough to capture several interesting phenomena, such as the need for nonlinear centering and a non-Gaussian modulating factor, and we believe that the results could serve as useful benchmarks when testing hypotheses concerning more general random walks.

Regarding extended dynamical systems, we have found two earlier local limit theorems, both corresponding to homogeneous environments. Szász and Varjú [28] have considered Lorentz processes with periodic configurations of scatterers, while Bardet, Gouëzel, and Keller [29] study rather different type of systems: small (possibly inhomogeneous) perturbations of weakly coupled, translation invariant, coupled map lattices; see Nagaev [30] and Guivarc'h [31] for some of the original techniques. Let us finally stress that for more classical, probability measure preserving, dynamical systems, various types of limit theorems have been proved for many decades. Yet such systems, too, continue to be studied vigorously, with important recent developments (e.g. Chazottes and Gouëzel [32]; Gouëzel [33, 34]).

1.5. Organization of the paper

The rest of the paper is organized as follows. Section 2 presents the main results. The proofs for nonrandom environments are given in Section 3, and the proofs for quenched random environments in Section 4. Section 5 concludes the paper, and Appendix A contains basic facts on generalized inverses of increasing sequences.

2. Main results

2.1. Representation as a random walk

The distribution of the dynamical system (1.2) at any time instant can be completely characterized in terms of the following simple unidirectional random walk (discrete-time pure birth process) on the integers. Given an environment $\omega \in (0,1)^{\mathbb{Z}_+}$, let $(X_n)_{n\in\mathbb{Z}_+}$ be a random walk in $\mathbb{Z}_+$ having the initial state $X_0 = 0$ and transitions

$$k \mapsto \begin{cases} k, & \text{with probability } 1-\omega_k, \\ k+1, & \text{with probability } \omega_k. \end{cases} \tag{2.1}$$

We denote by P_ω the distribution of the walk in the path space $\mathbb{Z}_+^{\mathbb{Z}_+}$. Note that if the environment ω is a realization of a random sequence, the process (X_n) can be identified as a random walk in a random environment, and the distribution P_ω is usually called the *quenched law* of the random walk. The expectation and variance with respect to P_ω are denoted by E_ω and Var_ω, respectively. When presenting general facts in probability theory, we write P and E for the measure and expectation.

Theorem 2.1. *For any environment ω, the value of the dynamical system (1.2) at any time instant n has the same distribution as $X_n + R$, where (X_n) is the random walk in $\mathbb{Z}_+$ defined by (2.1), and R is a uniformly distributed random variable in $[0,1)$ independent of X_n.*

Proof. The proof follows by induction and Lemma 2.2 below. $\square$

Lemma 2.2. *Let X be a positive random variable such that (i) $[X]$ and $\{X\} = X - [X]$ are independent, and (ii) $\{X\}$ is uniformly distributed in $[0,1)$. Then the same is true for $\mathcal{U}_\omega X$, and moreover,*

$$P([\mathcal{U}_\omega X] = l \,|\, [X] = k) = \begin{cases} 1-\omega_k, & \text{if } l = k, \\ \omega_k, & \text{if } l = k+1, \end{cases} \tag{2.2}$$

whenever $P([X] = k) > 0$.

Proof. Denote $X = K+R$, where K is a positive random integer independent of R, and R is uniformly distributed in $[0,1)$. Assume first that $K = k$ for some nonrandom integer k. Then $\mathcal{U}_\omega X = k + U_{\omega_k} R$. Moreover, a simple calculation based on the definition of U_{ω_k} shows that for all $r \in [0,1)$,

$$P([\mathcal{U}_\omega X] = l,\, \{\mathcal{U}_\omega X\} \le r) = \begin{cases} (1-\omega_k)r, & \text{if } l = k, \\ \omega_k r, & \text{if } l = k+1, \\ 0, & \text{else.} \end{cases} \tag{2.3}$$

Hence $\mathcal{U}_\omega X$ satisfies (i), (ii), and (2.2) in the case where $[X]$ is nonrandom. The general case follows by conditioning on X. □

2.2. Limit theorems for regular frozen environments

In this section we shall analyze the random walk (X_n) in a fixed, sufficiently regular environment, which may either be nonrandom, or a realization of a random sequence. More precisely, we shall in general assume that the environment $\omega \in (0,1)^{\mathbb{Z}_+}$ is such that

$$\omega_k^{-1} = O(k^\lambda) \tag{2.4}$$

for some $0 \le \lambda < 1/2$; and

$$k^{-1} \sum_{j=0}^{k-1} \omega_j^{-1} = \mu + o(k^{-\lambda}(\log k)^{-1/2}), \tag{2.5}$$

$$k^{-1} \sum_{j=0}^{k-1} (1-\omega_j)\omega_j^{-2} = \sigma^2 + o(k^{-\lambda}(\log k)^{-1/2}), \tag{2.6}$$

for some constants $\mu > 1$ and $\sigma^2 > 0$. Moreover, we assume that

$$k^{-1} \sum_{j=0}^{k-1} \omega_j^{-3} = O(1), \tag{2.7}$$

and that the environmental moving averages satisfy

$$\max_{j:|j|\le ub(k)} \left| \sum_{\ell=k}^{k+j-1} (\omega_\ell^{-1} - \mu) \right| = o(k^{1/2-\lambda}) \tag{2.8}$$

for all $u > 0$, where $b(k) = (k \log k)^{1/2}$. (In the special case with $\lambda = 0$ it suffices to use $b(k) = k^{1/2}$). Concrete examples of environments that satisfy the above regularity conditions shall be given in Section 2.3.

The quantity ω_k^{-1} represents the mean sojourn time of the particle in the interval $[k, k+1)$ (see Section 3.1 for more details). Therefore, the constant μ appearing in (2.5) may be interpreted as the inverse of the particle's traveling speed. The following result confirms this intuition.

Theorem 2.3 (Law of large numbers). *For any environment ω satisfying (2.4) – (2.5),*

$$P_\omega(n^{-1}X_n \to \mu^{-1}) = 1. \tag{2.9}$$

The main result of the paper is the following limit theorem for the random walk (X_n) defined by (2.1), or alternatively (by virtue of Theorem 2.1), for the dynamical system defined by (1.2).

Theorem 2.4 (Local limit theorem). *For any environment ω satisfying (2.4) – (2.8),*

$$P_\omega(X_n = k) \;=\; \frac{\omega_k^{-1}}{\mu} \cdot \frac{1}{\sqrt{2\pi\tilde{\sigma}^2 n}}\, e^{-\frac{(k-k_n^\omega)^2}{2\tilde{\sigma}^2 n}} \;+\; o(n^{-1/2}), \tag{2.10}$$

uniformly with respect to $k \geq 0$, where $\tilde{\sigma}^2 = \sigma^2/\mu^3$, and the centering factors k_n^ω are given by

$$k_n^\omega = \min\left\{ k \geq 0 : \sum_{j=0}^{k-1} \omega_j^{-1} \geq n \right\}. \tag{2.11}$$

Two features in Theorem 2.4, which distinguish it from classical limit theorems, call for special attention. First, the centering factors k_n^ω depend on the environment, and are in general nonlinear functions of n. Second, the modulating factor ω_k^{-1}/μ in (2.10) causes the asymptotic shape of the probability mass function of X_n to be non-Gaussian. This modulating factor explains the behavior observed in Figure 2.

In contrast, when looking at the probability distribution of the walk over a coarser diffusive space scale, the non-Gaussian modulating factor in Theorem 2.4 averages out asymptotically, and we end up with a standard Gaussian limiting distribution.

Theorem 2.5 (Central limit theorem)**.** *For any environment ω satisfying (2.4) – (2.8) for some constants $\mu > 1$ and $\sigma^2 > 0$,*

$$P_\omega\left(\frac{X_n - k_n^\omega}{\tilde{\sigma}\sqrt{n}} \le x\right) \to \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-t^2/2}\, dt \tag{2.12}$$

for all $x \in \mathbb{R}$, where $\tilde{\sigma}^2 = \sigma^2/\mu^3$, and k_n^ω are given by (2.11).

Remark 2.6. The centering factor k_n^ω may be identified as a generalized inverse (Appendix A) of the function $k \mapsto E_\omega T_k$, where T_k denotes the hitting time of the walk into site k (Section 3.1). A quick inspection of the proofs in Section 3 shows that Theorem 2.4 remains true if k_n^ω is replaced by $\ell_n^\omega = k_n^\omega + o(n^{1/2-\lambda})$, where λ is as in (2.4); use the inequality $|e^{-x^2} - e^{-y^2}| \le |x - y|$ for $k \le n$ and the proof of Lemma 3.6 for $k > n$. Moreover, Theorem 2.5 remains true if k_n^ω is replaced by $\ell_n^\omega = k_n^\omega + o(n^{1/2})$.

2.3. Limit theorems for quenched random environments

In this section we assume that the environment ω is a realization of a stationary random sequence in $(0,1)^{\mathbb{Z}_+}$, and denote by $\mathbb{P}$ its distribution on $(0,1)^{\mathbb{Z}_+}$. The expectation with respect to $\mathbb{P}$ is denoted by $\mathbb{E}$. We shall assume that

$$\mathbb{E}(\omega_0^{-1})^q < \infty \quad \text{for some } q > 5. \tag{2.13}$$

To guarantee that the environmental averages converge to their mean values rapidly enough, we assume that

$$\sum_{k \ge 1} \phi^{1/2}(k) < \infty, \tag{2.14}$$

where the mixing coefficients $\phi(k)$ are defined by

$$\phi(k) = \sup_m \sup_{A \in \mathcal{F}_0^m, B \in \mathcal{F}_{m+k}^\infty, \mathbb{P}(A) > 0} |\mathbb{P}(B|A) - \mathbb{P}(B)|, \tag{2.15}$$

and where $\mathcal{F}_0^m = \sigma(\omega_j, j \le m)$ and $\mathcal{F}_m^\infty = \sigma(\omega_j, j \ge m)$ (e.g. Bradley [35]).

The following result summarizes three limit theorems for the quenched random walk in a stationary strongly mixing random environment.

Theorem 2.7. *The law of large numbers (2.9), the local limit law (2.10), and the central limit law (2.12) are valid with $\mu = \mathbb{E}\omega_0^{-1}$ and $\sigma^2 = \mathbb{E}(1-\omega_0)\omega_0^{-2}$ for almost every realization of a stationary random environment satisfying (2.13) and (2.14).*

Especially, the limit laws summarized by Theorem 2.7 hold in the following cases:

- Independently scattered stationary environments (environments where the site labels ω_k are independent and identically distributed).
- Uniformly ergodic environments as discussed in Goldsheid [15].
- Environments which are realizations of finite-state irreducible aperiodic stationary Markov chains, or more general Markov chains satisfying Doeblin's condition (e.g. [35]).

Although in many applications it is natural to assume that the environment is stationary, the limits of Theorem 2.7 remain valid under looser conditions, as is clear from the results of Section 2.2.

Alternative versions of the law of large numbers and the central limit law in Theorem 2.7, where the centering factors k_n^ω are replaced by $E_\omega X_n$, can be proved as consequences of Theorems 3.1 and 5.4 in Rassoul-Agha and Seppäläinen [18], if we additionally assume that

$$\mathbb{P}\Big(\inf_k \omega_k \geq \delta\Big) = 1 \quad \text{for some } \delta > 0.$$

This so-called nonnestling assumption is close in spirit to the uniform ellipticity of nearest-neighbor random walks in random environments; in the context of our model it corresponds to the special case $\lambda = 0$ in (2.4). Although we believe that the local and central limit laws in Theorems 2.4, 2.5, and 2.7 remain generally valid also for the alternative centering $n \mapsto E_\omega X_n$, we prefer to use the centering $n \mapsto k_n^\omega$ defined in (2.11), because these factors are easily computed from the environment. Analogous central limit laws for nearest-neighbor walks were recently independently found by Goldsheid [15] and Peterson [16].

If we were only interested in the law of large numbers (2.9), we could do with less assumptions in Theorem 2.7. For example, as our proof in Section 4 shows, the moment condition (2.13) would only be needed for $q > 2$. The mixing assumption (2.14) could be relaxed as well, see for example Bingham [22] for more details.

3. Proofs for regular nonrandom environments

This section is devoted to proving Theorems 2.3–2.5 for the random walk (X_n) in an environment that satisfies the regularity assumptions (2.4) – (2.8).

The environment ω shall be fixed once and for all during the whole section — here we do not care whether it is a realization of a random sequence or not.

The section is organized as follows. Section 3.1 describes some preliminaries on the hitting times of the walk, and Section 3.2 gives the proof of the law of large numbers. The proof of the local limit theorem is split into Sections 3.3–3.7, and the proof of the central limit theorem is in Section 3.8.

3.1. Hitting times of the walk

We denote the hitting time of (X_n) into site k by $T_k = \min\{n \geq 0 : X_n = k\}$, and the sojourn time at site k by $\tau_k = T_{k+1} - T_k$. Because the walk never moves backwards, the equivalence

$$X_n = k \quad \text{if and only if} \quad T_k \leq n < T_{k+1} \tag{3.1}$$

is valid for all k and n. Moreover, the sojourn times are independent, and τ_k has a geometric distribution on $\{1, 2, \dots\}$ with success probability ω_k. Hence the mean and the variance of τ_k are given by $E_\omega \tau_k = \omega_k^{-1}$ and $\mathrm{Var}_\omega(\tau_k) = (1-\omega_k)\omega_k^{-2}$, respectively. The mean and the variance of T_k are denoted by $\mu_k = E_\omega T_k$ and $\sigma_k^2 = \mathrm{Var}_\omega(T_k)$, so that

$$\mu_k = \sum_{j=0}^{k-1} \omega_j^{-1} \quad \text{and} \quad \sigma_k^2 = \sum_{j=0}^{k-1} (1-\omega_j)\omega_j^{-2}. \tag{3.2}$$

The following result transforms the realization-by-realization relationship (3.1) into one concerning the probability mass functions.

Lemma 3.1. *For any environment ω and any $k, n \geq 0$,*

$$P_\omega(X_n = k) = \omega_k^{-1} P_\omega(T_{k+1} = n+1).$$

Proof. Note that $P_\omega(\tau_k > m) = \omega_k^{-1} P_\omega(\tau_k = m+1)$ for all $m \geq 0$. Because T_k and τ_k are independent, we find by applying (3.1) and conditioning on T_k that

$$\begin{aligned} P_\omega(X_n = k) &= P_\omega(T_k \leq n,\, T_k + \tau_k > n) \\ &= E_\omega 1_{\{T_k \leq n\}} P_\omega(T_k + \tau_k > n \,|\, T_k) \\ &= \omega_k^{-1} E_\omega 1_{\{T_k \leq n\}} P_\omega(T_k + \tau_k = n+1 \,|\, T_k) \\ &= \omega_k^{-1} P_\omega(T_k \leq n, T_k + \tau_k = n+1). \end{aligned}$$

This implies that claim, because $T_{k+1} = T_k + \tau_k$ and $\tau_k \geq 1$ almost surely. $\square$

3.2. Proof of Theorem 2.3

By (2.4) we see that $\mathrm{Var}_\omega(\tau_k) \le ck^{2\lambda}$ for some constants $c > 0$ and $\lambda < 1/2$, so that $\sum_{k=0}^\infty k^{-2}\,\mathrm{Var}_\omega(\tau_k) < \infty$, which implies as a consequence of Kolmogorov's variance criterion (e.g. [36, Cor. 4.22]) that $k^{-1}\sum_{j=0}^{k-1}(\tau_j - \omega_j^{-1}) \to 0$ almost surely. By (2.5), we conclude that the hitting times of the walk satisfy the following law of large numbers:

$$k^{-1}T_k \to \mu \quad \text{almost surely.} \tag{3.3}$$

Further, by (3.1), we see that $X_n = T^{\leftarrow}(n+1) - 1$, where $T^{\leftarrow}$ denotes the generalized inverse of the sequence (T_k) defined in Appendix A. Lemma A.1 together with (3.3) now shows that $n^{-1}T^{\leftarrow}(n) \to \mu^{-1}$ almost surely, and the proof of Theorem 2.3 is complete. □

3.3. Outline of proof for the local limit theorem

The starting point of the proof is Lemma 3.1 in Section 3.1, which reduces the problem into analyzing the probability mass function of the hitting times T_k. Because T_k is a sum of independent random variables, we may apply a classical local limit theorem (e.g. Petrov [37]) in Section 3.4 to conclude that $P_\omega(T_k = n) \approx f_k(n)$ for large values of k, where

$$f_k(n) = (2\pi\sigma_k^2)^{-1/2}\, e^{-\frac{(n-\mu_k)^2}{2\sigma_k^2}}, \tag{3.4}$$

and μ_k and σ_k^2 denote the mean and the variance of T_k given by (3.2).

In the rest of the proof we need to transform the Gaussian density f_k of the time variable into a Gaussian density of the space variable. This will be accomplished in two steps. We show in Section 3.5 that $f_k(n) \approx g_k(n)$, where

$$g_k(n) = (2\pi n\sigma^2/\mu)^{-1/2}\, e^{-\frac{(\mu_k-n)^2}{2n\sigma^2/\mu}}, \tag{3.5}$$

and further in Section 3.6 that $g_k(n) \approx \mu^{-1}h_n(k)$, where

$$h_n(k) = (2\pi n\sigma^2/\mu^3)^{-1/2}\, e^{-\frac{(k-k_n)^2}{2n\sigma^2/\mu^3}} \tag{3.6}$$

is the Gaussian density appearing in the statement of Theorem 2.4 (we write k_n in place of k_n^ω for convenience).

The approximation $g_k(n) \approx \mu^{-1}h_n(k)$ is a subtle part in the argument, where we have been guided by the following intuition. By approximating

$\mu_{k_n} \approx n$, we obtain $\mu_k - n \approx \mu_k - \mu_{k_n} = \sum_{\ell=k_n}^{k_n+j-1} \omega_\ell^{-1}$, where $j = k - k_n$. Therefore, the difference of the square roots of the exponents in (3.5) and (3.6) can be approximated by

$$\frac{(\mu_k - n)}{\sqrt{2n\sigma^2/\mu}} - \frac{\mu(k-k_n)}{\sqrt{2n\sigma^2/\mu}} \approx (2n\sigma^2/\mu)^{-1/2} \sum_{\ell=k_n}^{k_n+j-1} (\omega_\ell^{-1} - \mu).$$

Condition (2.8) has been tailored to guarantee that the above difference is small for values of k such that $\mu_k \approx n$.

An essential technical complication is the fact that the modulating factor ω_k^{-1} in (2.10) (see also Lemma 3.1) is unbounded. To overcome this difficulty, we have strived to obtain sharp estimates. Identifying the set of reasonable sufficient assumptions (2.4) – (2.8) has played a crucial role in the proof.

3.4. Local limit theorem for the hitting times

By applying a classical local limit theorem for sums of independent random variables (Petrov [37, Theorem VII.5]; see also Davis and McDonald [38]), we obtain the following local limit theorem for the hitting times.

Lemma 3.2. *For any environment ω satisfying (2.6) and (2.7),*

$$\sup_{n\geq 0} |P_\omega(T_k = n) - f_k(n)| = O(k^{-1}), \tag{3.7}$$

where the functions f_k are defined by (3.4).

Proof. Observe first that by (2.6),

$$\liminf_{k\to\infty} k^{-1} \sum_{j<k} \mathrm{Var}_\omega(\tau_j) = \sigma^2 > 0,$$

and that

$$\limsup_{k\to\infty} k^{-1} \sum_{j<k} E_\omega|\tau_j - E_\omega\tau_j|^3 < \infty$$

because of $E_\omega|\tau_j - E_\omega\tau_j|^3 \leq 3\omega_j^{-3}$ and (2.7). Therefore, to apply [37, Theorem VII.5], it suffices to verify (note that $P_\omega(\tau_j = 1) \geq P_\omega(\tau_j = m)$ for all j and m) that

$$\frac{1}{\log k} \sum_{j<k} P_\omega(\tau_j = 1) P_\omega(\tau_j = 2) \to \infty. \tag{3.8}$$

By writing $\omega_j^{-2}(1-\omega_j) = x_j y_j$, where $x_j = \omega_j^{2/5}(1-\omega_j)^{1/5}$ and $y_j = \omega_j^{-12/5}(1-\omega_j)^{4/5}$, and applying Hölder's inequality with conjugate exponents 5 and 5/4, we see that

$$\sum_{j<k} \omega_j^{-2}(1-\omega_j) \leq \left(\sum_{j<k} \omega_j^2(1-\omega_j)\right)^{1/5} \left(\sum_{j<k} \omega_j^{-3}(1-\omega_j)\right)^{4/5}.$$

After dividing both sides above by k, and applying (2.6) and (2.7), we see that

$$\liminf_{k\to\infty} k^{-1} \sum_{j<k} \omega_j^2(1-\omega_j) > 0,$$

which implies (3.8), because $P_\omega(\tau_j = 1)P_\omega(\tau_j = 2) = \omega_j^2(1-\omega_j)$. The claim now follows by applying [37, Theorem VII.5] and recalling that $\sigma_k^{-1} = O(k^{-1/2})$ by (2.6). □

The following result is an analogue of Lemma 3.2, where the time variable n instead of the space variable k tends to infinity.

Lemma 3.3. *For any $\lambda \in [0, 1/2)$ and any environment ω satisfying (2.5), (2.6), and (2.7), the Gaussian densities f_k defined by (3.4) satisfy*

$$\sup_{k\geq 1} k^\lambda |P_\omega(T_k = n) - f_k(n)| = O(n^{\lambda-1}).$$

Proof. Fix an $\epsilon \in (0, \mu^{-1})$, and note that

$$\sup_{k\geq \epsilon n} k^\lambda |P_\omega(T_k = n) - f_k(n)| = O(n^{\lambda-1}) \tag{3.9}$$

by Lemma 3.2. Therefore, we only need to analyze $P_\omega(T_k = n)$ and $f_k(n)$ for $1 \leq k \leq \epsilon n$. As a preliminary, note that, because $\epsilon < \mu^{-1}$ and $\mu_n/n \to \mu$ by (2.5), we may fix a constant $c_1 < 1$ and an integer n_0 such that $\mu_{[\epsilon n]} \leq c_1 n$ for all $n \geq n_0$.

Assume now that $k \leq \epsilon n$ and $n \geq n_0$. Then $n - \mu_k \geq n - \mu_{[\epsilon n]} \geq (1-c_1)n$, and moreover, $\sigma_k^2 \leq \sigma_+^2 \epsilon n$, where $\sigma_+^2 = \sup_{\ell\geq 1}(\sigma_\ell^2/\ell)$ is finite by (2.6). Therefore,

$$(n-\mu_k)^2/\sigma_k^2 \geq c_2 n, \tag{3.10}$$

where $c_2 = (1-c_1)^2/(\sigma_+^2\epsilon) > 0$. A rough estimate together with Chebyshev's inequality shows that

$$P_\omega(T_k = n) \leq P_\omega(|T_k - \mu_k| \geq |n-\mu_k|) \leq (n-\mu_k)^{-2}\sigma_k^2,$$

so by (3.10), we conclude that

$$k^\lambda P_\omega(T_k = n) \le c_2^{-1}\epsilon^\lambda n^{\lambda-1}. \tag{3.11}$$

By applying (3.10) once more, we see that

$$k^\lambda f_k(n) \le (2\pi\sigma_-^2)^{-1/2}k^{\lambda-1/2}e^{-c_2 n/2} \le (2\pi\sigma_-^2)^{-1/2}e^{-c_2 n/2}, \tag{3.12}$$

where $\sigma_-^2 = \inf_{\ell\ge 1}(\sigma_\ell^2/\ell)$ is strictly positive by (2.6). The proof is now completed by combining the estimates (3.11) and (3.12) with (3.9). $\square$

3.5. Variance of the hitting times

Lemma 3.4. *For any $\lambda \in [0, 1/2)$ and any environment ω satisfying (2.5) and (2.6),*

$$\max_{1\le k\le n} k^\lambda |f_k(n) - g_k(n)| = o(n^{-1/2}), \tag{3.13}$$

where the functions f_k and g_k are defined by (3.4) and (3.5), respectively.

Proof. The proof is split into two parts according to whether or not $|n-\mu_k| \le u(n\log n)^{1/2}$, where u is a large constant to be determined later.

(i) Assume that $k \le n$ is such that $|n-\mu_k| \le u(n\log n)^{1/2}$. Using the triangle inequality and the inequality $|e^{-x^2} - e^{-y^2}| \le |x-y|$, we see that

$$\begin{aligned}
&(2\pi n)^{1/2}\,|f_k(n) - g_k(n)| \\
&\quad\le \left|(n/\sigma_k^2)^{1/2} - (\mu/\sigma^2)^{1/2}\right| + (\mu/\sigma^2)^{1/2}\left|e^{-\frac{(n-\mu_k)^2}{2\sigma_k^2}} - e^{-\frac{(n-\mu_k)^2}{2n\sigma^2/\mu}}\right| \\
&\quad\le \left(1 + (2\sigma^2/\mu)^{-1/2}\frac{|n-\mu_k|}{\sqrt{n}}\right)\left|(n/\sigma_k^2)^{1/2} - (\mu/\sigma^2)^{1/2}\right|.
\end{aligned}$$

Consequently, assuming that n is large enough so that $u(\log n)^{1/2} \ge 1$,

$$(2\pi n)^{1/2}\,|f_k(n) - g_k(n)| \le c_1 u(\log n)^{1/2}\left|(n/\sigma_k^2)^{1/2} - (\mu/\sigma^2)^{1/2}\right|$$

where $c_1 = 1 + (2\sigma^2/\mu)^{-1/2}$.

Observe next that, assuming n is large enough so that $n - u(n\log n)^{1/2} \ge n/2$,

$$k = (\mu_k/k)^{-1}(n - (n-\mu_k)) \ge (2\mu_+)^{-1}n, \tag{3.14}$$

where $\mu_+ = \sup_{\ell \geq 1}(\mu_\ell/\ell)$. Further,

$$\left|(n/\sigma_k^2)^{1/2} - (\mu/\sigma^2)^{1/2}\right| = \frac{|n/\sigma_k^2 - \mu/\sigma^2|}{(n/\sigma_k^2)^{1/2} + (\mu/\sigma^2)^{1/2}} \leq (\sigma^2/\mu)^{1/2}|n/\sigma_k^2 - \mu/\sigma^2|.$$

Now using (3.14) we find that $\sigma_k^2 \geq c_2^{-1}n$ for $c_2 = 2\mu_+/\sigma_-^2$. Therefore,

$$\left|n/\sigma_k^2 - \mu/\sigma^2\right| = |(n-\mu_k)/\sigma_k^2 + d(k)| \leq c_2 u n^{-1/2}(\log n)^{1/2} + |d(k)|,$$

where

$$d(k) = \frac{\mu_k/k}{\sigma_k^2/k} - \frac{\mu}{\sigma^2}.$$

As a consequence,

$$n^{1/2}k^\lambda \left|f_k(n) - g_k(n)\right| \leq c_3 u n^\lambda (\log n)^{1/2}\left(un^{-1/2}(\log n)^{1/2} + |d(k)|\right) \quad (3.15)$$

for all large enough n, where $c_3 = c_1 \max(c_2, 1)$. By combining (2.5) and (2.6), we find that $|d(k)| = o(k^{-\lambda}(\log k)^{-1/2})$. Because $(2\mu_+)^{-1}n \leq k \leq n$ by (3.14), we conclude that the right side above tends to zero as $n \to \infty$.

(ii) Assume now that $k \leq n$ is such that $|n - \mu_k| > u(n \log n)^{1/2}$. Note that $\sigma_-^2 k \leq \sigma_k^2 \leq \sigma_+^2 k$, where $\sigma_-^2 = \inf_{\ell \geq 1}(\sigma_\ell^2/\ell)$ and $\sigma_+^2 = \sup_{\ell \geq 1}(\sigma_\ell^2/\ell)$ are finite and strictly positive by (2.6). Therefore, the exponent in the definition of f_k is bounded by

$$\frac{(\mu_k - n)^2}{2\sigma_k^2} \geq (2\sigma_+^2)^{-1}u^2 \log n.$$

Consequently,

$$n^{1/2}k^\lambda f_k(n) \leq c_4 k^{\lambda - 1/2} n^{1/2 - c_5 u^2} \leq c_4 n^{1/2 - c_5 u^2}, \quad (3.16)$$

where $c_4 = (2\pi\sigma_-^2)^{-1/2}$ and $c_5 = (2\sigma_+^2)^{-1}$. For the function g_k we immediately see that

$$n^{1/2}k^\lambda g_k(n) \leq c_6 n^{1/2 - c_7 u^2}, \quad (3.17)$$

where $c_6 = (2\pi\sigma^2/\mu)^{-1/2}$ and $c_7 = (2\sigma^2/\mu)^{-1}$. The proof is now finished by choosing $u > 0$ large enough so that $c_5 u^2 > 1/2$ and $c_7 u^2 > 1/2$, and combining the estimates (3.16) and (3.17) with (3.15). $\square$

3.6. From hitting times to walk locations

Lemma 3.5. *For any environment ω satisfying (2.5) and (2.8) for some $\lambda \in [0, 1/2)$,*

$$\max_{1 \le k \le n} k^{\lambda} |g_k(n) - \mu^{-1} h_n(k)| = o(n^{-1/2}),$$

where the functions g_k and h_n are defined by (3.4) and (3.6), respectively.

Proof. We will show the claim by treating separately the cases $|k - k_n| \le ub(k_n)$ and $|k - k_n| > ub(k_n)$, where $b(k) = (k \log k)^{1/2}$ is as in (2.8), and $u > 0$ is a large constant to be determined later.

(i) Assume that $k \le n$ is such that $|k - k_n| \le ub(k_n)$. Using the inequality $|e^{-x^2} - e^{-y^2}| \le |x - y|$, we see that

$$|g_k(n) - \mu^{-1} h_n(k)| \le c_1 n^{-1} |d(k, n)|,$$

where $c_1 = \pi^{-1/2}(2\sigma^2/\mu)^{-1}$, and $d(k, n) = (n - \mu_k) - \mu(k_n - k)$. Note that

$$|d(k,n)| = \left| \sum_{\ell=k}^{k_n - 1} (\omega_\ell^{-1} - \mu) + n - \mu_{k_n} \right| \le \left| \sum_{\ell=k}^{k_n-1} (\omega_\ell^{-1} - \mu) \right| + \omega_{k_n - 1}^{-1}, \qquad (3.18)$$

where the latter inequality is due to $\mu_{k_n} - \omega_{k_n-1}^{-1} < n \le \mu_{k_n}$. To analyze the right side of (3.18), observe that, because $k_n \to \infty$, we see using (2.8) that

$$\max_{k : |k - k_n| \le ub(k_n)} \left| \sum_{\ell=k}^{k_n - 1} (\omega_\ell^{-1} - \mu) \right| = \max_{j : |j| \le ub(k_n)} \left| \sum_{\ell=k_n}^{k_n + j - 1} (\omega_\ell^{-1} - \mu) \right| = o(k_n^{1/2 - \lambda}).$$

The above limiting relation also shows (substitute $j = 1$) that $\omega_{k_n-1}^{-1} = o(k_n^{1/2-\lambda})$. Because $k_n/n \to \mu^{-1}$ (by (2.5) and Lemma A.1), it follows that

$$\max_{k : |k - k_n| \le ub(k_n)} |d(k, n)| = o(n^{1/2 - \lambda}),$$

and therefore,

$$n^{1/2} \max_{1 \le k \le n : |k - k_n| \le ub(k_n)} k^{\lambda} |g_k(n) - \mu^{-1} h_n(k)| \to 0. \qquad (3.19)$$

(ii) Assume that $k \le n$ is such that $|k - k_n| > ub(k_n)$ for some $u \ge 2$ and $n \ge n_0$, where n_0 has been chosen large enough so that $k_n \ge \frac{1}{2}\mu^{-1} n$ for all $n \ge n_0$ (this is possible by virtue of (2.5) and Lemma A.1). Observe first that, because $\omega_k^{-1} \geq 1$ and $|k - k_n| \geq 2$, we see by Lemma A.2 that $|\mu_k - n| \geq |k - k_n| - 1 \geq \frac{1}{2}|k - k_n|$. Hence, the exponent in the definition of $g_k(n)$ is bounded by

$$\frac{(n-\mu_k)^2}{2n\sigma^2/\mu} \geq c_1 \frac{(k-k_n)^2}{k_n} \geq c_1 u^2 \log k_n \geq -c_1 u^2 \log(2\mu) + c_1 u^2 \log n,$$

where $c_1 = (16\sigma^2)^{-1}$. The same bound is also valid for the exponent in the definition of $h_n(k)$, because $\mu^3 > \mu$. Therefore, we obtain

$$n^{1/2} k^{\lambda} |g_k(n) - \mu^{-1} h_n(k)| \leq c_2 (2\mu)^{c_1 u^2} n^{\lambda - c_1 u^2}, \tag{3.20}$$

where $c_2 = 2(2\pi\sigma^2/\mu)^{-1/2}$. The right side above tends to zero as $n \to \infty$, if in addition to $u \geq 2$, we also require that $u > (\lambda/c_1)^{1/2}$. The proof is now completed by combining (3.19) and (3.20). □

Lemma 3.6. *For any $\lambda \geq 0$ and any environment ω satisfying (2.5), there exists a constant $c > 0$ such that the functions $h_n(k)$ defined by (3.6) satisfy*

$$\sup_{k>n} k^{\lambda} h_n(k) = o(e^{-cn}).$$

Proof. Recalling (by (2.5) and Lemma A.1) that $k_n/n \to \mu^{-1} < 1$, we may fix a positive constant $c_1 < 1$ and an integer n_0 such that $k_n \leq c_1 n$ for all $n \geq n_0$. Assume now that $k > n$ and $n \geq n_0$. Then, $k - k_n \geq (1 - c_1)k$, so that the exponent in the expression of $h_n(k)$ is bounded by

$$\frac{(k-k_n)^2}{2n\sigma^2/\mu^3} \geq \frac{(1-c_1)^2}{2\sigma^2/\mu^3} k.$$

As a consequence,

$$k^{\lambda} h_n(k) \leq c_2 n^{-1/2} k^{\lambda} e^{-c_3 k},$$

where $c_2 = (2\pi\sigma^2/\mu^3)^{-1/2}$ and $c_3 = (1-c_1)^2(2\sigma^2/\mu^3)^{-1}$. Because the function $t \mapsto t^{\lambda} e^{-c_3 t}$ is decreasing on the interval $[\lambda/c_3, \infty)$, we conclude that

$$\sup_{k>n} k^{\lambda} h_n(k) \leq c_2 n^{\lambda - 1/2} e^{-c_3 n},$$

for all $n \geq \max(n_0, \lambda/c_3)$, so the claim follows. □

3.7. Proof of Theorem 2.4

By combining Lemmas 3.3, 3.4, and 3.5 we see that

$$\max_{1\le k\le n} k^\lambda \left| P_\omega(T_k = n) - \mu^{-1} h_n(k)\right| = o(n^{-1/2}),$$

where the functions h_n are defined by (3.6). Hence by Lemma 3.6 and the fact that $P_\omega(T_k = n) = 0$ for all $k > n$, we conclude that

$$\sup_{k\ge 1} k^\lambda \left| P_\omega(T_k = n) - \mu^{-1} h_n(k)\right| = o(n^{-1/2}).$$

Further, because $\omega_k^{-1} \le c_1 k^\lambda \le c_1(k+1)^\lambda$ by (2.4), we see by Lemma 3.1 that

$$\sup_{k\ge 0} \left| P_\omega(X_n = k) - (\omega_k^{-1}/\mu) h_{n+1}(k+1)\right| = o(n^{-1/2}).$$

We complete the proof of Theorem 2.4 by showing below that

$$\sup_{k\ge 0} k^\lambda |h_{n+1}(k+1) - h_n(k)| = o(n^{-1/2}). \tag{3.21}$$

Let us write $h_n(k) = c_1 n^{-1/2} e^{-c_2 \alpha_{k,n}^2}$, where $c_1 = (2\pi\sigma^2/\mu^3)^{-1/2}$, $c_2 = (2\sigma^2/\mu^3)^{-1}$, and $\alpha_{k,n} = n^{-1/2}(k - k_n)$. Note that

$$\alpha_{k+1,n+1} - \alpha_{k,n} = (n+1)^{-1/2}(k_n - k_{n+1} + 1) + (k - k_n)((n+1)^{-1/2} - n^{-1/2}).$$

Note that $0 \le k_{n+1} - k_n \le 1$ (Lemma A.2) and $|k - k_n| \le (1 + c_3)n$ for all $k \le n$, where $c_3 = \sup_{\ell\ge 0}(k_\ell/\ell)$ is finite by Lemma A.1. Therefore by applying the inequality $n^{-1/2} - (n+1)^{-1/2} \le \frac{1}{2} n^{-3/2}$, we see that

$$|\alpha_{k+1,n+1} - \alpha_{k,n}| \le (2 + (1 + c_3)/2) n^{-1/2}$$

for all $k \le n$. This estimate combined with the inequality $|e^{-x^2} - e^{-y^2}| \le |x - y|$ now shows that

$$\max_{k\le n} k^\lambda |h_{n+1}(k+1) - h_n(k)| = O(n^{\lambda-1}).$$

Together with Lemma 3.6, we now conclude the validity of (3.21), and the proof of Theorem 2.4 is complete. □

3.8. Proof of Theorem 2.5

It suffices to show that

$$P_\omega\left(x < \frac{X_n - k_n}{\tilde{\sigma}\sqrt{n}} \le y\right) \to \frac{1}{\sqrt{2\pi}} \int_x^y e^{-t^2/2}\, dt$$

for all $x < y$, where $\tilde{\sigma}^2 = \sigma^2/\mu^3$. Note that the left side above can be written as $\sum_{k\in I_n} P_\omega(X_n = k)$, where the set $I_n = I_n(x, y)$ is defined by

$$I_n = \left\{k \in \mathbb{Z} : [x\tilde{\sigma}\sqrt{n}] + 1 \le k - k_n \le [y\tilde{\sigma}\sqrt{n}]\right\},$$

and by Theorem 2.4,

$$\sum_{k\in I_n} P_\omega(X_n = k) - \mu^{-1} \sum_{k\in I_n} \omega_k^{-1} h_n(k) \to 0.$$

Note that for any real numbers $x < y$ and any sequence $(a_k)_{k\in\mathbb{Z}}$,

$$\left| \int_x^y a_{[t]}\, dt - \sum_{k=[x]+1}^{[y]} a_k \right| \le |a_{[x]}| + |a_{[y]}|. \tag{3.22}$$

By applying (3.22), using (2.4), and performing a change of variables, we see that

$$\begin{aligned}\sum_{k\in I_n} \omega_k^{-1} h_n(k) &= \int_{x\tilde{\sigma}\sqrt{n}}^{y\tilde{\sigma}\sqrt{n}} \omega_{k_n+[u]}^{-1} h_n(k_n + [u])\, du + O(n^{\lambda-1/2}) \\ &= \tilde{\sigma}\sqrt{n} \int_x^y \omega_{k_n+[\tilde{\sigma}t\sqrt{n}]}^{-1} h_n(k_n + [\tilde{\sigma}t\sqrt{n}])\, dt + o(1),\end{aligned}$$

where $O(n^{\lambda-1/2}) = o(1)$ due to $\lambda < 1/2$. Further, $|t^2 - [t]^2| \le 2|t|$ shows that

$$\begin{aligned}h_n(k_n + [\tilde{\sigma}t\sqrt{n}]) &= (2\pi\tilde{\sigma}^2 n)^{-1/2} \exp\left(-\frac{[t\tilde{\sigma}\sqrt{n}]^2}{2n\tilde{\sigma}^2}\right) \\ &= (2\pi\tilde{\sigma}^2 n)^{-1/2} e^{-t^2/2} O\big(e^{|t|/(\tilde{\sigma}\sqrt{n})}\big) \\ &= (2\pi\tilde{\sigma}^2 n)^{-1/2} e^{-t^2/2}(1 + O(n^{-1/2})),\end{aligned}$$

because we assume that $t \in (x, y)$.

The next lemma finishes the proof by showing that

$$\frac{1}{\sqrt{2\pi}} \int_x^y (\omega_{k_n+[\tilde{\sigma} t\sqrt{n}]}^{-1}/\mu)\, e^{-t^2/2}\, dt \to \frac{1}{\sqrt{2\pi}} \int_x^y e^{-t^2/2}\, dt.$$

The intuition is that, restricting integration to sufficiently small subintervals of (x, y) and making n large, $e^{-t^2/2}$ is virtually constant, while the fluctuations of $\omega_{k_n+[\tilde{\sigma} t\sqrt{n}]}^{-1}$ average out in the integral.

Lemma 3.7. *On the interval* (x, y)*, the probability measures defined by* $\mathrm{m}_n(dt) = Z_n^{-1}\mu^{-1}\omega_{k_n+[\tilde{\sigma} t\sqrt{n}]}^{-1}\, dt$ *converge weakly to the uniform probability measure* $\mathrm{m}(dt) = (y - x)^{-1}\, dt$ *and the normalizing factor* $Z_n \to y - x$*.*

Proof. We must show that $\mathrm{m}_n((x, s]) \to \mathrm{m}((x, s])$ for all $s \in (x, y)$. By a change of variables and (3.22) we see that

$$\int_x^s \omega_{k_n+[\tilde{\sigma} t\sqrt{n}]}^{-1}\, dt = \tilde{\sigma}^{-1} n^{-1/2} \sum_{k\in I_n(x,s)} \omega_k^{-1} + o(1),$$

because $\omega_k^{-1} = O(k^\lambda)$ and $\lambda < \frac{1}{2}$. The center of $I_n(x, s)$ is $c_n = \mu^{-1} n(1+o(1))$. For $u > 0$ large enough and $b(n)$ as in (2.8), $|I_n(x, s)| = [s\tilde{\sigma}\sqrt{n}] - [x\tilde{\sigma}\sqrt{n}] < ub(n)$. Therefore, (2.8) implies that

$$n^{-1/2} \sum_{k\in I_n(x,s)} \omega_k^{-1} = n^{-1/2} |I_n(x, s)| \mu + o(1).$$

Because $n^{-1/2}|I_n(x, s)| \to (s - x)\tilde{\sigma}$, we get $\mu^{-1} \int_x^s \omega_{k_n+[\tilde{\sigma} t\sqrt{n}]}^{-1}\, dt \to s - x$, and $Z_n \to y - x$ follows by taking $s = y$. This finishes the proof. $\square$

4. Proofs for quenched random environments

In this section we prove Theorem 2.7 for the random walk (X_n) in a quenched random environment satisfying the assumptions (2.13)–(2.14). Section 4.1 contains some preliminary facts on the growth rate and moving averages of stationary sequences, and the proof of Theorem 2.7 is given in Section 4.2.

4.1. Growth rate and moving averages of stationary sequences

The following result establishes a bound on the growth rate of a stationary sequence in terms of its moments.

Lemma 4.1. *Let* $(\xi_0, \xi_1, \dots)$ *be a stationary random sequence such that* $\mathbb{E}|\xi_0|^q < \infty$ *for some* $q > 0$*. Then* $\xi_k = o(k^{1/q})$ *almost surely.*

Proof. Fix $\epsilon > 0$. By stationarity and Fubini's theorem we see that

$$\sum_{k=0}^{\infty} \mathbb{P}(k^{-1/q}|\xi_k| > \epsilon) = \sum_{k=0}^{\infty} \mathbb{P}((\epsilon^{-1}|\xi_0|)^q > k) = \mathbb{E}[(\epsilon^{-1}|\xi_0|)^q] < \infty,$$

where $[x]$ denotes the integer part of x. Because ϵ was arbitrarily chosen, the claim follows by the Borel–Cantelli lemma. □

The next result analyzes the moving averages of a stationary random sequence in terms of its moments and mixing rate. The result, proven with the help Peligrad's law of large numbers [20] (see also Bingham [22] for a nice survey), is similar in spirit to Kiesel [39, Thm 1], but tailored to fit our needs.

Lemma 4.2. *Let* $(\xi_0, \xi_1, \dots)$ *be a stationary random sequence such that* $\mathbb{E}|\xi_0|^q < \infty$ *for some* $q \geq 1$*, and for which the mixing coefficients as defined in (2.15) satisfy*

$$\sum_{n=1}^{\infty} \phi^{1/\kappa}(2^n) < \infty \quad \textit{for some } \kappa \geq 2. \tag{4.1}$$

Then

$$\max_{1 \leq j \leq uk^s} \left| \sum_{\ell=k+1}^{k+j} (\xi_\ell - \mathbb{E}\xi_0) \right| = o(k^r) \tag{4.2}$$

almost surely for any $s > 0$*,* $u > 0$*, and*

$$r \geq \max\left(\frac{1+s}{q},\ \frac{s}{2} + \frac{1}{\kappa - 1}\right). \tag{4.3}$$

Proof. Denote the left side of (4.2) by M_k, and fix an arbitrary $\epsilon > 0$. Note that by stationarity,

$$\sum_{k=1}^{\infty} \mathbb{P}(k^{-r} M_k > \epsilon) = \sum_{k=1}^{\infty} \mathbb{P}\Big(\max_{1 \leq j \leq uk^s} |S_j| > \epsilon k^r\Big), \tag{4.4}$$

where $S_j = \sum_{\ell=1}^{j}(\xi_\ell - \mathbb{E}\xi_0)$. The right side in (4.4) is finite if and only if

$$\int_1^\infty \mathbb{P}\Big(\max_{1\le j\le ut^s} |S_j| > \epsilon t^r\Big)\, dt < \infty,$$

and by a change of variables, the above is further equivalent to

$$\int_1^\infty t^{1/s-1}\mathbb{P}\Big(\max_{1\le j\le t} |S_j| > \epsilon_1 t^{r/s}\Big)\, dt < \infty,$$

where $\epsilon_1 = u^{-r/s}\epsilon$. We conclude that $\sum_{k=1}^\infty \mathbb{P}(k^{-r}M_k > \epsilon)$ is finite if and only if

$$\sum_{n=1}^\infty n^{\alpha p-2}\mathbb{P}\Big(\max_{1\le j\le n} |S_j| > \epsilon_1 n^\alpha\Big) < \infty, \tag{4.5}$$

where $\alpha = r/s$ and $p = (1+s)/r$.

The parameters r and s have been transformed into α and p to conform with the notations used in [20]. Observe that $\alpha p > 1$, and moreover, (4.3) implies that $p \le q$, $\alpha > 1/2$, and $[(\alpha p - 1)/(\alpha - 1/2)] + 1 \le \kappa$. Hence, Peligrad's law of large numbers [20, Thm 2] yields the validity of (4.5), and therefore, $\sum_{k=1}^\infty \mathbb{P}(k^{-r}M_k > \epsilon)$ is finite. Because ϵ was arbitrarily chosen, the claim follows by the Borel–Cantelli lemma. □

4.2. Proof of Theorem 2.7

By virtue of Theorems 2.3–2.5, we only need to verify that the regularity conditions (2.4) – (2.8) are valid for $\mathbb{P}$-almost every realization of the random environment.

Recall that $\mathbb{E}\omega_0^{-q} < \infty$ for some $q > 5$ by (2.13). Consequently, Lemma 4.1 implies that $\omega_k^{-1} = o(k^{1/q})$, so that especially, (2.4) holds with $\lambda = 1/q$. Because $q > 5$, the random variables $m_k = \omega_k^{-1}$ and $s_k^2 = (1-\omega_k)\omega_k^{-2}$ have finite second moments, so the assumption (2.14) on the mixing coefficients implies that the stationary random sequences (m_k) and (s_k^2) satisfy the law of the iterated logarithm [40, Section 12]. As a consequence, (2.5) and (2.6) are valid $\mathbb{P}$-almost surely with $\mu = \mathbb{E}m_0 > 1$ and $\sigma^2 = \mathbb{E}s_0^2 > 0$.

Because $\mathbb{E}\omega_0^{-3}$ is finite, the pointwise ergodic theorem implies that $k^{-1}\sum_{j=0}^{k-1}\omega_k^{-3}$ converges, so that (2.7) holds.

To verify (2.8), note that because ϕ is decreasing, the sum in (4.1) equals $\int_1^\infty \phi^{1/\kappa}(2^{[x]})\, dx \le \int_1^\infty \phi^{1/\kappa}([2^{x-1}])\, dx = (\log 2)^{-1}\int_1^\infty \phi^{1/\kappa}([y])y^{-1}\, dy$. Using Hölder's inequality, the last integral can be bounded from above by

$\left(\int_1^\infty \phi^{1/2}([y])\,dy\right)^{2/\kappa}\left(\int_1^\infty y^{-p}\,dy\right)^{1/p}$ with $p = (1-2/\kappa)^{-1}$, which is finite due to (2.14). Therefore, condition (4.1) of Lemma 4.2 holds for any $\kappa > 2$. Next, because $q > 5$, we may choose an exponent $s > 1/2$ such that $s < 1 - 2q^{-1}$ and $s \le q/2 - 2$. Our choice of s implies that the exponent $r = 1/2 - q^{-1}$ satisfies (4.3) for some large enough κ. Hence by Lemma 4.2, it follows that

$$\max_{j:|j|\le uk^s}\left|\sum_{\ell=k}^{k+j-1}(\omega_\ell^{-1}-\mu)\right| = o(k^r)$$

almost surely for all $u > 0$. Because $(k\log k)^{1/2} = o(k^s)$, it follows that (2.8) holds with $\lambda = 1/q$. □

5. Conclusions

We studied the propagation of a particle in a one-dimensional inhomogeneous medium, where the motion is induced by chaotic and fully deterministic local rules, and the initial condition is the sole source of randomness. The spatially varying local rules constitute an environment, which is frozen during the particle's lifetime. This model falls into the framework of extended dynamical systems which lack physically observable invariant measures. Defining the local rules via piecewise affine maps allows to reduce the model to a simple unidirectional random walk on the integers.

The main result of the paper shows that the probability mass function of the random walk approaches a modulated Gaussian density, where the modulating factor is explicitly given in terms of the local properties of the environment. In contrast, when looking at the walk over a coarser diffusive space scale, the non-Gaussian modulating factor averages out asymptotically, and the distribution of the walk approaches a standard Gaussian distribution.

Although our analysis is restricted to a special instance of a random walk, we believe that the obtained results could serve as useful benchmarks when testing hypotheses concerning more general extended dynamical systems and random walks in random environments.

A. A generalized inverse

Let $(a(k))_{k=0}^\infty$ be an increasing sequence such that $a(0) = 0$ and $\lim_{k\to\infty} a(k) = \infty$, and define its generalized inverse by

$$a^{\leftarrow}(n) = \min\{k \in \mathbb{Z}_+ : a(k) \ge n\}. \tag{A.1}$$

Then also $a^{\leftarrow}(0) = 0$ and $\lim_{n\to\infty} a^{\leftarrow}(n) = \infty$. The following result summarizes some basic properties of the inverse.

Lemma A.1. *If* $\lim_{k\to\infty} a(k)/k = \mu$*, then* $\lim_{n\to\infty} a^{\leftarrow}(n)/n = 1/\mu$*.*

Proof. By definition, $a(a^{\leftarrow}(n) - 1) < n \le a(a^{\leftarrow}(n))$ for all n such that $a^{\leftarrow}(n) > 0$, so that

$$\frac{a(a^{\leftarrow}(n) - 1)}{a^{\leftarrow}(n)} < \frac{n}{a^{\leftarrow}(n)} \le \frac{a(a^{\leftarrow}(n))}{a^{\leftarrow}(n)}.$$

Because $a^{\leftarrow}(n) \to \infty$, it follows that $a(a^{\leftarrow}(n))/a^{\leftarrow}(n) \to \mu$. Therefore, the above bounds imply $n/a^{\leftarrow}(n) \to \mu$, and the proof is complete. □

Lemma A.2. *Assume that* $c = \inf_k(a(k+1) - a(k)) > 0$*. Then for all positive integers* k *and* n*,*

$$|a(k) - n| \ge c(|k - a^{\leftarrow}(n)| - 1), \tag{A.2}$$

and

$$a^{\leftarrow}(n+1) - a^{\leftarrow}(n) \le \lceil c^{-1} \rceil. \tag{A.3}$$

Proof. If $k \ge a^{\leftarrow}(n)$, then $a(k) - n \ge a(k) - a(a^{\leftarrow}(n)) \ge c(k - a^{\leftarrow}(n))$. If $k < a^{\leftarrow}(n)$, then $n - a(k) > a(a^{\leftarrow}(n) - 1) - a(k) \ge c(a^{\leftarrow}(n) - k - 1)$. Hence (A.2) follows.

To prove (A.3), fix a positive integer n, and denote $\ell = a^{\leftarrow}(n)$. Then $a(\ell) \ge n$ and $a(\ell + \lceil c^{-1} \rceil) - a(\ell) \ge c\lceil c^{-1} \rceil \ge 1$, so that $a(\ell + \lceil c^{-1} \rceil) \ge n + 1$. Hence, $a^{\leftarrow}(n+1) \le \ell + \lceil c^{-1} \rceil$. □

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