On the Essential Spectrum of Phase-Space Anisotropic Pseudodifferential Operators

M. Măntoiu *

Address

Departamento de Matemáticas, Universidad de Chile,
Las Palmeras 3425, Casilla 653, Santiago, Chile

E-mail: Marius.Mantoiu@imar.ro

Abstract

A phase-space anisotropic operator in $\mathcal{H} = L^2(\mathbb{R}^n)$ is a self-adjoint operator whose resolvent family belongs to a natural C^*-completion of the space of Hörmander symbols of order zero. Equivalently, each member of the resolvent family is norm-continuous under conjugation with the Schrödinger unitary representation of the Heisenberg group. The essential spectrum of such a phase-space anisotropic operator is the closure of the union of usual spectra of all its "phase-space asymptotic localizations", obtained as limits over diverging ultrafilters of $\mathbb{R}^n \times \mathbb{R}^n$-translations of the operator. The result extends previous analysis of the purely configurational anisotropic operators, for which only the behavior at infinity in $\mathbb{R}^n$ was allowed to bo non-trivial.

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1 Introduction and main results

We are going to study self-adjoint operators acting in the Hilbert space $\mathcal{H} := L^2(\mathscr{X})$, where $\mathscr{X}$ is an n-dimensional real vector space. Let us also set $\Xi := \mathscr{X} \times \mathscr{X}^*$, where $\mathscr{X}^*$ denotes the dual of $\mathscr{X}$. For reasons coming from physics, we are going to call the spaces $\mathscr{X}$, $\mathscr{X}^*$ and Ξ the configuration, the momentum and the phase space, respectively. On Ξ there is a canonical symplectic form given by $[\![X, Y]\!] = [\![(x, \xi), (y, \eta)]\!] := y\cdot\xi - x\cdot\eta$, in terms of the duality $\mathscr{X} \times \mathscr{X}^* \ni (z, \zeta) \mapsto z \cdot \zeta := \zeta(z) \in \mathbb{R}$.

Our main result will be a formula giving the essential spectrum $\mathrm{sp}_{\mathrm{ess}}(H)$ of operators H affiliated to a remarkable C^*-algebra $\mathbf{B}^0(\mathcal{H})$ of bounded linear operators in $\mathcal{H}$ (this means that the resolvent family of H belongs to $\mathbf{B}^0(\mathcal{H})$). This formula will involve a certain type of limits of the operator H along suitable filters of the phase space Ξ.

To define $\mathbf{B}^0(\mathcal{H})$, we introduce first some notions and notations. We set $\mathbf{B}(\mathcal{H})$ for the C^*-algebra of linear bounded operators in $\mathcal{H}$ and $\mathbf{K}(\mathcal{H})$ for its ideal of compact operators. The group of unitary operators in $\mathcal{H}$ will be denoted by $\mathbf{U}(\mathcal{H})$. There is a projective representation $W : \Xi \to \mathbf{U}(\mathcal{H})$ with 2-cocycle $\kappa(X, Y) := \exp(i/2[\![X, Y]\!])$, given by

$$[W(x, \xi)u](y) := e^{i(y-x/2)\cdot\xi}u(y - x), \quad x, y \in \mathscr{X},\ \xi \in \mathscr{X}^*,\ u \in \mathcal{H} \tag{1.1}$$

and verifying $W(X)W(Y) = \kappa(X, Y)W(X + Y)$ for all $X, Y \in \Xi$. Associated to W, one has a (true) action of Ξ by automorphisms of the C^*-algebra $\mathbf{B}(\mathcal{H})$ given by

$$\mathscr{W}(X)T := W(X)TW(-X), \quad X \in \Xi,\ T \in \mathbf{B}(\mathcal{H}). \tag{1.2}$$

It is not norm continuous, so it defines a proper C^*-subalgebra $\mathbf{B}^0_{\mathscr{W}}(\mathcal{H}) \equiv \mathbf{B}^0(\mathcal{H})$ of continuous vectors

$$\mathbf{B}^0(\mathcal{H}) := \{T \in \mathbf{B}(\mathcal{H}) \mid \Xi \ni X \mapsto \mathscr{W}(X)T \in \mathbf{B}(\mathcal{H}) \text{ is } \|\cdot\| -\text{continuous}\}. \tag{1.3}$$

Our main result expresses the essential spectrum of any observable H affiliated to $\mathbf{B}^0(\mathcal{H})$ as the closure of the union of spectra of a family of "asymptotic obsevables" obtained as "limits at infinity" of H.

Theorem 1.1. *Let H be a self-adjoint operator in $\mathcal{H}$ affiliated to $\mathbf{B}^0(\mathcal{H})$. One has*

$$\mathrm{sp}_{\mathrm{ess}}(H) = \overline{\bigcup_{\mathcal{X}\in\delta(\Xi)} \mathrm{sp}(H_{\mathcal{X}})}, \tag{1.4}$$

where $\delta(\Xi)$ denotes the family of all ultrafilters on Ξ that are finer than the Fréchet filter (composed of complements of the relatively compact subsets of Ξ). For any $\mathcal{X} \in \delta(\Xi)$, one sets $H_{\mathcal{X}} := \lim_{X\to\mathcal{X}} \mathscr{W}(X)H$ in the strong resolvent sense.

The statement is modelled on previous results (see [5, 9, 10, 11, 12] and references therein) in which, as a rule, H has to be affiliated to the smaller algebra $\mathbf{E}(\mathcal{H})$ defined in (2.3). Under this assumption, its essential spectrum can be expressed using limits along diverging ultrafilters χ in the configuration space $\mathscr{X}$ applied to $\mathscr{W}(x, 0)H$. Some very partial information on full phase-space anisotropy is scattered through the existing publications and (1.4) is meant to answer a conjecture of Vladimir Georgescu. Connected results can be found in [15], in which however ultrafilters are not used and only bounded operators are treated.

An important ingredient for proving Theorem 1.1 is

Theorem 1.2. *Let $S \in \mathbf{B}^0(\mathcal{H})$ and $\mathcal{X} \in \delta(\Xi)$. Then the limit $\mathbf{T}_\mathcal{X} S := \lim_{X \to \mathcal{X}} \mathscr{W}(X) S$ exists in the $\mathbf{K}(\mathcal{H})$-strict topology and defines a C^*-morphism $\mathbf{T}_\mathcal{X} : \mathbf{B}^0(\mathcal{H}) \to \mathbf{B}^0(\mathcal{H})$. The product morphism $(\mathbf{T}_\mathcal{X})_{\mathcal{X}\in\delta(\Xi)} : \mathbf{B}^0(\mathcal{H}) \to \prod_{\mathcal{X}\in\delta(\Xi)} \mathbf{B}^0(\mathcal{H})$ has kernel $\mathbf{K}(\mathcal{H})$, so the quotient $\mathbf{B}^0(\mathcal{H})/\mathbf{K}(\mathcal{H})$ can be identified with a C^*-subalgebra of $\prod_{\mathcal{X}\in\delta(\Xi)} \mathbf{B}^0(\mathcal{H})$.*

The sign $\prod$ denotes a restricted product; its elements are families with a uniform bound on the norms. We recall that the strict topology on $\mathbf{B}^0(\mathcal{H})$ defined by the essential ideal $\mathbf{K}(\mathcal{H})$ is given by the family of seminorms

$$\left\{ \| S \|^K_{\mathbf{B}(\mathcal{H})} := \| KS \|_{\mathbf{B}(\mathcal{H})} + \| SK \|_{\mathbf{B}(\mathcal{H})} \mid K \in \mathbf{K}(\mathcal{H}) \right\}. \tag{1.5}$$

This strict topology is weaker than the norm topology, but stronger than the strong operator topology, so the limits $\mathbf{T}_\mathcal{X} S$ also exist strongly.

For insight we recall some basic facts about the Weyl calculus, which will also be an ingredient of the proof of Theorem 1.2. A correspondence between functions (and distributions) f the phase space Ξ and operators $\mathrm{Op}(f)$ acting on functions on the configuration space $\mathscr{X}$ is given formally by

$$[\mathrm{Op}(f)u](x) := \int_{\mathscr{X}} dy \int_{\mathscr{X}^*} d\eta \, e^{iy\cdot\eta} f\left(\frac{x+y}{2}, \eta\right) u(y). \tag{1.6}$$

Various interpretations [7] can be given to this formula, under various assumptions on f and u. We notice only that Op can be defined as an isomorphism between the space of tempered distributions $\mathcal{S}'(\Xi)$ and the space $\mathbf{L}[\mathcal{S}(\mathscr{X}); \mathcal{S}'(\mathscr{X})]$ of linear continuous operators from the Schwartz space $\mathcal{S}(\mathscr{X})$ to its dual $\mathcal{S}'(\mathscr{X})$. It also restricts to an isomorphism $\mathrm{Op} : \mathcal{S}(\Xi) \to \mathbf{L}[\mathcal{S}'(\mathscr{X}); \mathcal{S}(\mathscr{X})]$. On various subspaces of $\mathcal{S}'(\Xi)$ one introduces a multiplication $\sharp$ (the Weyl composition of symbols) satisfying $\mathrm{Op}(f)\mathrm{Op}(g) = \mathrm{Op}(f\sharp g)$. One of these spaces is certainly $\mathcal{S}(\Xi)$, a (Fréchet) *-algebra under $\sharp$ and complex conjugation.

Obviously $\mathbf{B}(\mathcal{H}) = \mathbf{B}\left[L^2(\mathscr{X})\right]$ can be identified to a subspace of $\mathbf{L}[\mathcal{S}(\mathscr{X}); \mathcal{S}'(\mathscr{X})]$. It follows that $\mathrm{Op}^{-1}[\mathbf{B}(\mathcal{H})] =: \mathfrak{B}(\Xi) \subset \mathcal{S}'(\Xi)$ is a C^*-algebra with a composition law extending $\sharp$ and with the norm $\| f \|_{\mathfrak{B}(\Xi)} := \| \mathrm{Op}(f) \|_{\mathbf{B}(\mathcal{H})}$. It is a rather obscure algebra, with a non-interesting ideal structure (the only proper ideal of $\mathbf{B}(\mathcal{H})$ is $\mathbf{K}(\mathcal{H})$). The slightly smaller one $\mathrm{Op}^{-1}[\mathbf{B}^0(\mathcal{H})] =: \mathfrak{B}^0(\Xi)$, however, is extremely interesting, its translational invariant ideal are well-understood and this is basic for getting the result above on the essential spectrum of self-adjoint operators affiliated to $\mathbf{B}^0(\mathcal{H}) = \mathrm{Op}\left[\mathfrak{B}^0(\Xi)\right]$. We speak of translational invariant ideals because the C^*-algebra $\mathfrak{B}^0(\Xi)$ itself is subject to the continuous action of Ξ by automorphisms $f(\cdot) \to [\mathcal{T}(X)f](\cdot) := f(\cdot - X)$, restricted from $\mathcal{S}'(\Xi)$ for instance. This follows from the definition of $\mathfrak{B}^0(\Xi)$ and from the formula $\mathscr{W}(X)[\mathrm{Op}(f)] = \mathrm{Op}[\mathcal{T}(X)f]$, since clearly $\mathscr{W}(X)\mathbf{B}^0(\mathcal{H}) = \mathbf{B}^0(\mathcal{H})$ for every $X \in \Xi$. The main feature that makes $\mathfrak{B}^0(\Xi)$ treatable is the fact that *it is obtained by Rieffel deformation from the Abelian C^*-algebra $BC^0(\Xi)$ of all bounded uniformly continuous functions on* Ξ; see Proposition 2.2.

Rieffel deformation [17] is an exact functor between categories of C^*-dynamical systems with group $\mathbb{R}^d$. Reducing the generality to fit to the present framework, assume that $(\mathcal{A}, \Theta, \Xi)$ is a C^*-dynamical system, i.e. the vector group Ξ acts strongly continuously by automorphisms on the C^*-algebra $\mathcal{A}$. On the C^∞ vectors $\mathcal{A}^\infty$ of the action, one uses the symplectic form on Ξ to deform the initial product to a new one (use oscillatory integrals)

$$f\# g := 2^{2n}\int_\Xi\int_\Xi dY dZ e^{2i[\![Y,Z]\!]}\Theta_Y(f)\Theta_Z(g). \tag{1.7}$$

Keeping the same involution, one gets a *-algebra structure on $\mathcal{A}^\infty$ which can be completed under a C^*-norm by techniques involving Hilbert modules. The action Θ of Ξ transfers to the resulting C^*-algebra $\mathfrak{A}$. The correspondence $\mathcal{A}\mapsto\mathfrak{A}$ can be raised also to a correspondence between Θ-equivariant morphisms. Equally important, all the ideals of $\mathcal{A}$ which are invariant under the action Θ are converted by deformation into invariant ideals of $\mathfrak{A}$.

The first interesting example is obtained by choosing $\mathcal{A} = BC^0(\Xi)$, acted continuously by Ξ by translations ($\Theta = \mathcal{T}$). In this case $\mathcal{A}^\infty = BC^\infty(\Xi)$, formed of all the C^∞ functions $f : \Xi \to \mathbb{C}$ with all the partial derivatives bounded. On $BC^\infty(\Xi)$ Rieffel's composition law $\#$ coincides with the Weyl multiplication $\sharp$. An expert in pseudodifferential operators would denote $BC^\infty(\Xi)$ by $S^0_{0,0}(\Xi)$, so $\mathfrak{A}$ forms in this case an operator algebra extension of the zero order pseudodifferential symbols. One might say that they have full phase-space anisotropy; elements of the Hörmander spaces $S^{-m}_{\rho,\delta}(\Xi)$, $m > 0$ of strictly negative order could be considered trivial at infinity with respect to $\xi \in \mathscr{X}^*$, having interesting (anisotropic) asymptotic behavior only in the variable $x \in \mathscr{X}$.

Since the Rieffel deformation of the C^*-algebra $BC^0(\Xi)$ coincides with $\mathfrak{B}^0(\Xi)$, it is isomorphic via the representation Op with the C^*-subalgebra $\mathbf{B}^0(\mathcal{H})$ of $\mathbf{B}(\mathcal{H})$. The image through Op of the Rieffel deformation $\mathfrak{K}(\mathcal{H})$ of the ideal $C_0(\Xi)$ of continuous functions decaying at infinity coincides with $\mathbf{K}(\mathcal{H})$.

Then, by general principles, the relevant quotient $\mathbf{B}^0(\mathcal{H})/\mathbf{K}(\mathcal{H})$ will be isomorphic to the Rieffel deformation of the quotient $BC^0(\Xi)/C_0(\Xi)$. The first quotient is relevant because the essential spectrum of an element of $\mathbf{B}^0(\mathcal{H})$ (or of an operator affiliated to it) coincides with the spectrum of its canonical image in $\mathbf{B}^0(\mathcal{H})/\mathbf{K}(\mathcal{H})$. This quotient is extremely complicated, but can be seen as a C^*-subalgebra of the product $\prod_{\mathcal{X}}\mathbf{B}^0(\mathcal{H})$. This follows after identifying the family of ideals $\mathbf{J}_{\mathcal{X}} = \ker\mathbf{T}_{\mathcal{X}}$ of $\mathbf{B}^0(\mathcal{H})$ indexed by ultrafilters $\mathcal{X}\in\delta(\Xi)$ and satisfying $\cap_{\mathcal{X}}\mathbf{J}_{\mathcal{X}} = \mathbf{K}(\mathcal{H})$. This is difficult to obtain for general C^*-subalgebras of $\mathbf{B}(\mathcal{H})$, but for the particular case $\mathbf{B}^0(\mathcal{H})$ the ground is prepared by Rieffel's deep analysis. It turns out that, by functoriality, it would be enough to find a corresponding family $\{\mathcal{J}_{\mathcal{X}} \mid \mathcal{X}\in\delta(\Xi)\}$ of Ξ-invariant ideals in $BC^0(\Xi)$ with the property that $\cap_{\mathcal{X}}\mathcal{J}_{\mathcal{X}} = C_0(\Xi)$. This and other topological issues are discussed in [9, 10], and implemented through the strategy we outlined above lead finally to Theorem 1.1.

We mention that many of the recent articles treating the essential spectrum of anisotropic operators have as a background an Abelian locally compact group $\mathscr{X}$ [9, 10, 14], or even rather general metric spaces $\mathscr{X}$ without a group structure [5, 8]. Rieffel's calculus is not yet developed in such a generality, so a lot of operators with a complicated phase-space behavior still beg for an adequate treatment.

This short paper is not the right opportunity to draw the history of studying the essential spectrum with (or without) algebraic techniques. Beside the articles already quoted, we send also to [1, 2, 13, 16, 18, 19] and to references therein for other results in this direction.

2 C^*-algebras of symbols and operators

Defining for $x \in \mathscr{X}$ and $\xi \in \mathscr{X}^*$ the unitary operators $U(x) \equiv e^{ix\cdot P} := W(-x,0)$ and $V(\xi) \equiv e^{iQ\cdot\xi} := W(0,\xi)$, one gets strongly continuous unitary representations $U : \mathscr{X} \to \mathbf{U}(\mathcal{H}),\ V : \mathscr{X}^* \to \mathbf{U}(\mathcal{H})$, satisfying for all $(x,\xi) \in \Xi$

$$U(x)V(\xi) = e^{ix\cdot\xi}V(\xi)U(x),$$

$$W(x,\xi) = e^{\frac{i}{2}x\cdot\xi}U(-x)V(\xi) = e^{-\frac{i}{2}x\cdot\xi}V(\xi)U(-x).$$

Their generators $P = (P_1 = -i\partial_1, \cdots P_n = -i\partial_n)$ and $Q = (Q_1, \ldots, Q_n)$ are the usual momentum and position operators in $\mathcal{H}$ satisfying the Canonical Commutation Relations

$$i[P_j, P_k] = 0, \quad i[Q_j, Q_k] = 0, \quad i[P_j, Q_k] = \delta_{jk}.$$

We associate to U and V the actions by automorphisms

$$\mathscr{U} : \mathscr{X} \to \mathrm{Aut}[\mathbf{B}(\mathcal{H})], \quad \mathscr{U}(x)T := U(x)TU(-x),$$

$$\mathscr{V} : \mathscr{X}^* \to \mathrm{Aut}[\mathbf{B}(\mathcal{H})], \quad \mathscr{V}(\xi)T := V(\xi)TV(-\xi).$$

The relation

$$\mathscr{W}(x,\xi) = \mathscr{U}(x)\mathscr{V}(\xi) = \mathscr{V}(\xi)\mathscr{U}(x), \quad (x,\xi) \in \Xi$$

follows immediately. The corresponding C^*-subalgebras of continuous vectors will be denoted, respectively, by $\mathbf{B}^0_{\mathscr{U}}(\mathcal{H})$ and $\mathbf{B}^0_{\mathscr{V}}(\mathcal{H})$. We recall that $\mathbf{B}^0(\mathcal{H}) \equiv \mathbf{B}^0_{\mathscr{W}}(\mathcal{H})$ has been defined at (1.3); obviously $\mathbf{B}^0(\mathcal{H}) = \mathbf{B}^0_{\mathscr{U}}(\mathcal{H}) \cap \mathbf{B}^0_{\mathscr{V}}(\mathcal{H})$. There are also smooth Fréchet *-algebras $\mathbf{B}^\infty_{\mathscr{U}}(\mathcal{H}), \mathbf{B}^\infty_{\mathscr{V}}(\mathcal{H})$ and $\mathbf{B}^\infty(\mathcal{H}) \equiv \mathbf{B}^\infty_{\mathscr{W}}(\mathcal{H})$. The last one, for instance, is composed of the bounded linear operators T for which $\Xi \ni X \mapsto \mathscr{W}(X)T \in \mathbf{B}(\mathcal{H})$ is C^∞ in norm.

Remark 2.1. It is easy to show that for all $x \in \mathscr{X}$, $\xi \in \Xi$, the transformations $V(\xi), U(x)$ and $W(x,\xi)$ defines (by restriction) automorphisms of $\mathcal{S}(\mathscr{X})$ and then (by transposition) automorphisms of $\mathcal{S}'(\mathscr{X})$. Hence $\mathscr{V}(\xi), \mathscr{U}(x)$ and $\mathscr{W}(x,\xi)$ will be automorphisms of $\mathbf{L}[\mathcal{S}(\mathscr{X}), \mathcal{S}'(\mathscr{X})]$ or of $\mathbf{L}[\mathcal{S}'(\mathscr{X}), \mathcal{S}(\mathscr{X})]$. The next relation, easy to check on $\mathcal{S}'(\Xi)$, is basic for our developments:

$$\mathscr{W}(X) \circ \mathrm{Op} = \mathrm{Op} \circ \mathcal{T}(X), \quad X \in \Xi. \tag{2.1}$$

Consider the C^*-subalgebra of $\mathfrak{B}(\Xi)$

$$\mathfrak{B}^0(\Xi) := \mathrm{Op}^{-1}\left[\mathbf{B}^0(\mathcal{H})\right] = \{f \in \mathfrak{B}(\Xi) \mid \lim_{X\to 0} \parallel \mathcal{T}(X)f - f \parallel_{\mathfrak{B}(\Xi)} = 0\}.$$

The next proposition identifies $\mathfrak{K}(\Xi) := \mathrm{Op}^{-1}[\mathbf{K}(\mathcal{H})]$ with $\mathfrak{C}_0(\Xi)$, the Rieffel deformation of $C_0(\Xi)$, as well as $\mathfrak{B}^0(\Xi) := \mathrm{Op}^{-1}[\mathbf{B}^0(\mathcal{H})]$ with $\mathfrak{BC}^0(\Xi)$, the Rieffel deformation of $BC^0(\Xi)$.

Proposition 2.2. *The Weyl calculus* Op *realizes a C^*-isomorphism between* $\mathfrak{BC}^0(\Xi)$ *and* $\mathbf{B}^0(\mathcal{H})$. *The image of* $\mathfrak{C}_0(\Xi)$ *through* Op *is precisely* $\mathbf{K}(\mathcal{H})$.

Proof. As said before, the C^*-algebra $\mathfrak{BC}^0(\Xi)$ contains the *-subalgebra $BC^\infty(\Xi)$ densely. By the Calderon-Vaillancourt Theorem [7], $\mathrm{Op} : BC^\infty(\Xi) \to \mathbf{B}(\mathcal{H})$ (defined as an oscillatory integral) is a well-defined representation. In [15, Prop. 2.6] it is shown that it extends to a faithful representation $\mathrm{Op} : \mathfrak{BC}^0(\Xi) \to \mathbf{B}(\mathcal{H})$. (The isometry of Op with respect to the Rieffel norm $\| \cdot \|_{\mathfrak{BC}^0(\Xi)}$ is also proven in a different way in [3].) The relation (2.1) shows that the space of smooth vectors of $\mathfrak{BC}^0(\Xi)$, i.e. $BC^\infty(\Xi)$, is sent by Op to $\mathbf{B}^\infty(\mathcal{H})$ which is dense into $\mathbf{B}^0(\mathcal{H})$. This already settles the first statement.

The second statement of the Proposition follows from the fact that $\mathrm{Op}[\mathcal{S}(\Xi)]$ is dense in $\mathbf{K}(\mathcal{H})$. □

On $\mathfrak{B}^0(\Xi)$ the seminorms

$$\Big\{\| f \|^k_{\mathfrak{B}^0(\Xi)} := \| f\sharp k \|_{\mathfrak{B}(\Xi)} + \| k\sharp f \|_{\mathfrak{B}(\Xi)} \mid k \in \mathfrak{K}(\Xi)\Big\} \tag{2.2}$$

define the strict topology associated to the essential ideal $\mathfrak{K}(\Xi)$. We are going to denote by $\mathfrak{B}^0(\Xi)_{\mathrm{str}}$ the space $\mathfrak{B}^0(\Xi)$ endowed with this topology. On the other hand, we set $\mathbf{B}^0(\mathcal{H})_{\mathrm{str}}$ for the space $\mathbf{B}^0(\mathcal{H})$ with the strict topology associated to the essential ideal $\mathbf{K}(\mathcal{H})$, given by the family (1.5) of seminorms. We will need below a useful result on the continuity of Op with respect to the strict topologies.

Corollary 2.3. *The mapping* $\mathrm{Op} : \mathfrak{B}^0(\Xi)_{\mathrm{str}} \to \mathbf{B}^0(\mathcal{H})_{\mathrm{str}}$ *is an isomorphism.*

Proof. This should be already clear. Working with the seminorms for instance, one shows easily that $\| \mathrm{Op}(f) \|^{\mathrm{Op}(k)}_{\mathbf{B}^0(\mathcal{H})} = \| f \|^k_{\mathfrak{B}^0(\Xi)}$ for every $f \in \mathfrak{B}^0(\Xi)$ and any $k \in \mathbf{B}^0(\mathcal{H})$. This follows from the definitions, from the relations $\mathrm{Op}(f)\mathrm{Op}(k) = \mathrm{Op}(f\sharp k)$ and $\mathrm{Op}(k)\mathrm{Op}(f) = \mathrm{Op}(h\sharp f)$ and from Proposition 2.2. □

Remark 2.4. The basic C^*-algebras considered until now in connection with the investigation of the essential spectrum of anisotropic operators are C^*-subalgebras of

$$\mathbf{E}(\mathcal{H}) := \{S \in \mathbf{B}^0(\mathcal{H}) \mid \| U(x)S^{(*)} - S^{(*)} \|_{\mathbf{B}(\mathcal{H})} \underset{x\to 0}{\longrightarrow} 0\}. \tag{2.3}$$

(The notation means that the condition is fulfilled both for T and T^*.) It is clear that $\mathbf{E}(\mathcal{H})\mathbf{B}^0_{\mathscr{U}}(\mathcal{H}) \subset \mathbf{E}(\mathcal{H}) \supset \mathbf{B}^0_{\mathscr{U}}(\mathcal{H})\mathbf{E}(\mathcal{H})$ so, *a fortiori*, $\mathbf{E}(\mathcal{H})$ is an ideal in $\mathbf{B}^0(\mathcal{H})$. It is known [9, 10] that $\mathfrak{E}(\Xi) := \mathrm{Op}^{-1}[\mathbf{E}(\mathcal{H})]$ coincides with the crossed product $BC^0(\mathscr{X}) \rtimes_\tau \mathscr{X}$ and it is also easy to see that it is the Rieffel deformation of $BC^0(\mathscr{X}) \otimes C_0(\mathscr{X}^*)$. They played a privileged role in [9, 10, 14] (even for Abelian locally compact groups $\mathscr{X}$) in the study of the essential spectrum of $\mathscr{X}$-anisotropic operators in $\mathcal{H} = L^2(\mathscr{X})$, but they are not enough to cover phase-space anisotropy.

Remark 2.5. Another natural ideal of $\mathbf{B}^0(\mathcal{H})$ is

$$\mathbf{F}(\mathcal{H}) := \{S \in \mathbf{B}^0(\mathcal{H}) \mid \| V(\xi)S^{(*)} - S^{(*)} \|_{\mathbf{B}(\mathcal{H})} \underset{\xi\to 0}{\longrightarrow} 0\},$$

for which obvious assertions can be made by analogy with $\mathbf{E}(\mathcal{H})$, both concerning the structure and the usefulness. The essential spectrum of self-adjoint operators H affiliated to $\mathbf{F}(\mathcal{H})$ would involve strong resolvent limits of $\mathscr{V}(\xi)H$ along ultrafilters finer than the Fréchet filter in the momentum space $\mathscr{X}^*$.

As a consequence of the Riesz-Kolmogorov criterion, one has $\mathbf{E}(\mathcal{H}) \cap \mathbf{F}(\mathcal{H}) = \mathbf{K}(\mathcal{H})$.

Remark 2.6. The large C^*-algebras $\mathbf{B}^0_{\mathscr{U}}(\mathcal{H})$ and $\mathbf{B}^0_{\mathscr{V}}(\mathcal{H})$ are also useful for studying anisotropic operators [5, 10], but since rather little is known about their ideal structure there are only partial results.

3 Proof of Theorems 1.2 and 1.1

We are going to divide the proof in several steps.

A. This first step is "purely topological"; we rely on results from [9, 10] in which we replace $\mathscr{X} = \mathbb{R}^n$ by $\Xi = \mathbb{R}^{2n}$. It is known [10, Sect. 5.1] that for any $\mathcal{X} \in \delta(\Xi)$ and any $f \in BC^0(\Xi)$ the limit

$$[\mathcal{T}_{\mathcal{X}}(f)]\,(Y) := \lim_{X\to\mathcal{X}} [\mathcal{T}(X)f]\,(Y) = \lim_{X\to\mathcal{X}} f(Y+X) \tag{3.1}$$

exists locally uniformly in $Y \in \Xi$. It defines a Ξ-morphism $\mathcal{T}_{\mathcal{X}} : BC^0(\Xi) \to BC^0(\Xi)$, meaning that it is a C^*-algebraic morphism and one has $\mathcal{T}_{\mathcal{X}}[\mathcal{T}(X)f] = \mathcal{T}(X)[\mathcal{T}_{\mathcal{X}}(f)]$ for all $X \in \Xi$ and all $f \in BC^0(\Xi)$. Denoting its kernel $\ker \mathcal{T}_{\mathcal{X}}$ by $\mathcal{L}_{\mathcal{X}}$, one has

$$\bigcap_{\mathcal{X}\in\delta(\Xi)} \mathcal{L}_{\mathcal{X}} = C_0(\Xi). \tag{3.2}$$

This implies immediately the embedding (a C^*-algebraical monomorphism)

$$\frac{BC^0(\Xi)}{C_0(\Xi)} \hookrightarrow \prod_{\mathcal{X}\in\delta(\Xi)} BC^0(\Xi), \qquad \widehat{f} \mapsto (\mathcal{T}_{\mathcal{X}}(f))_{\mathcal{X}\in\delta(\Xi)}\,. \tag{3.3}$$

B. We now use information from [17]. By Rieffel deformation one gets the Ξ-morphism $\mathfrak{T}_{\mathcal{X}} : \mathfrak{B}^0(\Xi) \to \mathfrak{B}^0(\Xi)$ with kernel $\ker \mathfrak{T}_{\mathcal{X}} =: \mathfrak{L}_{\mathcal{X}}$. To be precise, since $\mathcal{T}_{\mathcal{X}} : BC^0(\Xi) \to BC^0(\Xi)$ is a Ξ-morphism, it restricts to a morphism $\mathcal{T}_{\mathcal{X}} : BC^\infty(\Xi) \to BC^\infty(\Xi)$ of the Fréchet *-algebras of smooth vectors. We recall that the same $BC^\infty(\Xi)$ is the space of smooth vectors for the action of Ξ in the deformed C^*-algebra $\mathfrak{B}^0(\Xi)$ and that even the two natural Fréchet topologies coincide. Then $\mathcal{T}_{\mathcal{X}}$ extends to a C^*-morphism $\mathfrak{T}_{\mathcal{X}} : \mathfrak{B}^0(\Xi) \to \mathfrak{B}^0(\Xi)$; its kernel $\mathfrak{L}_{\mathcal{X}} := \ker \mathfrak{T}_{\mathcal{X}}$ is the closure in $\mathfrak{B}^0(\Xi)$ of $\mathcal{L}^\infty_{\mathcal{X}} = \mathcal{L}_{\mathcal{X}} \cap BC^\infty(\Xi)$. As a consequence of (3.2), (3.3), of the fact that $\mathfrak{K}(\Xi)$ is obtained from $C_0(\Xi)$ by deformation and of the properties of the Rieffel calculus, one gets

$$\bigcap_{\mathcal{X}\in\delta(\Xi)} \mathfrak{L}_{\mathcal{X}} = \mathfrak{K}(\Xi), \qquad \frac{\mathfrak{B}^0(\Xi)}{\mathfrak{K}(\Xi)} \hookrightarrow \prod_{\mathcal{X}\in\delta(\Xi)} \mathfrak{B}^0(\Xi). \tag{3.4}$$

C. We apply now Proposition 2.2. Taking into account the isomorphism $\mathrm{Op} : \mathfrak{B}^0(\Xi) \to \mathbf{B}^0(\mathcal{H})$, sending $\mathfrak{K}(\Xi)$ into $\mathbf{K}(\mathcal{H})$, one gets the morphism $\mathbf{T}_{\mathcal{X}} := \mathrm{Op} \circ \mathfrak{T}_{\mathcal{X}} \circ \mathrm{Op}^{-1} : \mathbf{B}^0(\mathcal{H}) \to \mathbf{B}^0(\mathcal{H})$ with kernel $\ker \mathbf{T}_{\mathcal{X}} =: \mathbf{L}_{\mathcal{X}} = \mathrm{Op}(\mathfrak{L}_{\mathcal{X}})$. Obviously

$$\bigcap_{\mathcal{X}\in\delta(\Xi)} \mathbf{L}_{\mathcal{X}} = \mathbf{K}(\mathcal{H}), \qquad \frac{\mathbf{B}^0(\mathcal{H})}{\mathbf{K}(\mathcal{H})} \hookrightarrow \prod_{\mathcal{X}\in\delta(\Xi)} \mathbf{B}^0(\mathcal{H}). \tag{3.5}$$

D. To finish the proof of Theorem 1.2, we only need to be more precise on the form of the morphism $\mathbf{T}_{\mathcal{X}}$. Relation (3.1) says that the limit $\mathcal{T}_{\mathcal{X}}(f) = C_0 - \lim_{\mathscr{X} \to \mathcal{X}} \mathcal{T}(X) f$ exists in the strict topology on $BC^0(\Xi)$ defined by the essential ideal $C_0(\Xi)$, given by the family of seminorms

$$\left\{ \| \, f \, \|^h_{BC^0(\Xi)} := \| \, fh \, \|_{BC(\Xi)} \mid h \in C_0(\Xi) \right\}. \tag{3.6}$$

It follows easily that the morphism $\mathfrak{T}_{\mathcal{X}}$ is given by $\mathfrak{T}_{\mathcal{X}}(f) = \mathfrak{K}\text{–}\lim_{X \to \mathcal{X}} \mathfrak{T}(X) f$, the limit being computed with respect to the $\mathfrak{K}(\Xi)$-strict topology on $\mathfrak{B}^0(\Xi)$. This is based on the (topological) coincidence [17] of the Fréchet dense subspaces of smooth vectors associated to the two actions: $\left[BC^0(\Xi)\right]^\infty_{\mathcal{T}} = \left[\mathfrak{B}^0(\Xi)\right]^\infty_{\mathfrak{T}}$ and $[C_0(\Xi)]^\infty_{\mathcal{T}} = [\mathfrak{K}(\Xi)]^\infty_{\mathfrak{T}}$.

Using this, the definition of $\mathbf{T}_{\mathcal{X}}$, Corollary 2.3 and formula (2.1), one can write

$$\begin{aligned} \mathbf{T}_{\mathcal{X}}[\mathrm{Op}(f)] :=& \mathrm{Op}\left[\mathfrak{T}_{\mathcal{X}}(f)\right] = \mathrm{Op}\left[\mathfrak{K}\text{–}\lim_{X \to \mathcal{X}} \mathfrak{T}(X) f\right] = \\ &= \mathbf{K}\text{–}\lim_{X \to \mathcal{X}} \mathrm{Op}\left[\mathfrak{T}(X) f\right] = \mathbf{K}\text{–}\lim_{X \to \mathcal{X}} \mathscr{W}(X)\left[\mathrm{Op}(f)\right]. \end{aligned}$$

We conclude that $\mathbf{T}_{\mathcal{X}}(S) := \mathbf{K}-\lim_{X \to \mathcal{X}} \mathscr{W}(X) S$ for any $S \in \mathbf{B}^0(\mathcal{H})$, in the $\mathbf{K}(\mathcal{H})$-strict topology.

E. Theorem 1.1 follows easily from Theorem 1.2. The essential spectrum of H coincides with the spectrum of its image (expressed at the level of resolvents) in the quotient $\mathbf{B}^0(\mathcal{H})/\mathbf{K}(\mathcal{H})$. This one can be computed in the product $\prod_{\mathcal{X} \in \delta(\Xi)} \mathbf{B}^0(\mathcal{H})$, so it is the closed union of spectra of all the components.

4 Affiliation

We give explicit affiliation criteria to the C^*-algebras $\mathfrak{B}^0(\Xi)$ and $\mathbf{B}^0(\mathcal{H})$. Some of them are (almost) obvious, others are rather simple adaptations of results from previous articles (mostly [10]), so we present them as a sequence of examples. It goes without saying that all the operators proven previously (as in [10, Sect.4]) to be affiliated to $\mathbf{E}(\mathcal{H})$ are also affiliated to $\mathbf{B}^0(\mathcal{H})$.

A. Clearly, every self-adjoint element of $\mathbf{B}^0(\mathcal{H})$ is affiliated to $\mathbf{B}^0(\mathcal{H})$. This includes, for instance, operators of the form $\mathrm{Op}(f)$, with $f \in BC^\infty(\Xi)_{\mathbb{R}}$. Other examples are $\varphi(Q)$ or $\psi(P)$ with $\varphi \in BC^0(\mathscr{X})_{\mathbb{R}}$ and $\psi \in BC^0(\mathscr{X}^*)_{\mathbb{R}}$ or self-adjoint linear combinations of products of such operators.

B. If H_0 is already shown to be affiliated, obviously $H = H_0 + H_1$ will be affiliated too for any $H_1 \in \mathbf{B}^0(\mathcal{H})$. Assume for instance that $\mathrm{Op}(f_0)$ is affiliated to $\mathbf{B}^0(\mathcal{H})$. The same will be true for $\mathrm{Op}(f_0 + f_1)$ for any real $f_1 \in BC^\infty(\Xi)$. In particular this happens for $H_1 = \lambda \in \mathbb{R}$, so the affiliation to $\mathbf{B}^0(\mathcal{H})$ of lower bounded operators H can be reduced to the case $H \geq 1$.

C. For real a, the convolution operator $a(P)$ is affiliated to $\mathbf{B}^0(\mathcal{H})$ iff the function $(a+i)^{-1}$ is uniformly continuous, since $\mathscr{W}(x,\xi)\left[(a(P)+i)^{-1}\right] = (a(P+\xi)+i)^{-1}$. Thus one needs to check that

$$\sup_{Y \in \Xi} \frac{|a(Y+X) - a(Y)|}{(1+|a(Y+X)|)(1+|a(Y)|)} \underset{X \to 0}{\longrightarrow} 0.$$

This happens, of course, when $a \in BC^0(\mathscr{X}^*)$, or when a is proper (diverges at infinity), since in this case $(a+i)^{-1} \in \mathbb{C} \otimes C_0(\mathscr{X}^*)$ and $a(P)$ will even be affiliated to $\mathbf{E}(\mathcal{H})$. There are, of course, many other opportunities for $(a+i)^{-1}$ to be uniformly continuous. Assume for instance, as in [10, 4.2], that a is C^1 and equivalent to a weight. If one has $|a'| \le C(1+|a|)$ for some constant C, then $(a+i)^{-1}$ is indeed uniformly continuous. For criteria involving higher order derivatives, see [10, Ex. 4.17]. Let us use a decomposition $\mathscr{X}^* = \mathscr{X}_1^* \times \cdots \times \mathscr{X}_m^*$ and pick real numbers $s_1, \dots, s_m$. The function $a(\xi) := \langle \xi_1 \rangle^{s_1} \dots \langle \xi_m \rangle^{s_m}$ leads to an operator $a(P)$ affiliated to $\mathbf{B}^0(\mathcal{H})$ independently of the signs of $s_1, \dots, s_m$. Another interesting example is $a(\xi) := \exp(s_1\xi_1 + \cdots + s_n\xi_n)$ in $\mathscr{X}^* = \mathbb{R}^n$. Many other very anisotropic combinations are possible.

D. Similar statements hold for the multiplication operator $b(Q)$. Of ourse this follows directly, since $\mathscr{W}(x,\xi)\left[(b(Q)+i)^{-1}\right] = (b(Q+x)+i)^{-1}$, but can also be deduced from a general symmetry principle: Assume that f is affiliated to $\mathfrak{B}^0(\Xi)$ and identify $\mathscr{X}^*$ with $\mathscr{X}$. Then the function $f^\circ(x,\xi) := f(\xi, x)$ is also affiliated to $\mathfrak{B}^0(\Xi)$.

E. Let H be a self-adjoint operator in $\mathcal{H}$ with domain $\mathcal{E}$ endowed with the graph norm. Denoting by $\mathcal{E}^*$ the (anti-)dual of $\mathcal{E}$, one gets canonical embeddings $\mathcal{E} \hookrightarrow \mathcal{H} \hookrightarrow \mathcal{E}^*$. Assume that $W(X)\mathcal{E} \subset \mathcal{E}$, $\forall X \in \Xi$. Then H is affiliated to $\mathbf{B}^0(\mathcal{H})$ if and only if $\| \, [W(X), H] \, \|_{\mathbf{B}(\mathcal{E},\mathcal{E}^*)} \underset{X \to 0}{\longrightarrow} 0$.

F. If only the form domain $\mathcal{G}$ of the self-adjoint operator H is invariant under W, then the relation $\| \, [W(X), H] \, \|_{\mathbf{B}(\mathcal{G},\mathcal{G}^*)} \equiv \| \, \mathscr{W}(X)H - H \, \|_{\mathbf{B}(\mathcal{G},\mathcal{G}^*)} \underset{X \to 0}{\longrightarrow} 0$ would imply that H is affiliated to the C^*-algebra $\mathbf{B}^0(\mathcal{H})$.

See [10, Def. 4.7, Cor. 4.8, Prop. 4.9] for the affiliation of abstract operators defined as form-sums $H = H_0 + H_1$.

5 Second order differential operators

We are interested in partial differential operators in $\mathcal{H} = L^2(\mathbb{R}^n)$ which are defined formally as $H_a := \sum_{j,k=1}^n P_j a_{jk}(Q) P_k$. Perturbations (especially by multiplication operators) can be added by the results reviewed in Section 4. It will always be assumed that the matrix $(a_{jk}(x))$ is positive definite and given by $L^1_{\rm loc}$-functions. Defining the quadratic form $q_a^{(0)}$ on $C_c^\infty(\mathscr{X})$ (the smooth compactly supported functions on $\mathscr{X} = \mathbb{R}^n$) by

$$q_a^{(0)}(u) := \int_{\mathbb{R}^n} dx \sum_{j,k=1}^n a_{jk}(x) \overline{(\partial_j u)(x)} (\partial_k u)(x),$$

we are also going to suppose that this quadratic form is closable. Generous explicit conditions on a insuring this can be found in [4, 20].

We define a norm on $C_c^\infty(\mathscr{X})$ by $\| \, u \, \|_a := \left(q_a^{(0)}(u) + \| \, u \, \|^2 \right)^{1/2}$ and denote by $\mathcal{G}_a$ the Hilbert space obtained by completing $C_c^\infty(\mathscr{X})$ with respect to $\| \, u \, \|_a$. One has canonically $\mathcal{G}_a \hookrightarrow \mathcal{H} \hookrightarrow \mathcal{G}_a^*$ and $q_a^{(0)}$ extends to a closed form $q_a : \mathcal{G}_a \to [0, \infty)$. A unique self-adjoint positive operator H_a is assigned to q_a, with $D(H_a^{1/2}) = \mathcal{G}_a$ and $\| \, H_a^{1/2} u \, \| = q_a(u)^{1/2}$, $\forall u \in \mathcal{G}_a$; it extends to a symmetric element of $\mathbf{B}(\mathcal{G}_a; \mathcal{G}_a^*)$. Just under the conditions above we say that H_a is *weakly elliptic*. If it is *uniformly elliptic* (i.e. $0 < c\,\mathrm{id} \leq a(\cdot) \leq c'\,\mathrm{id} < \infty$), it is known [6, 10] to be affiliated to $\mathbf{E}(\mathcal{H}) \subset \mathbf{B}^0(\mathcal{H})$. On the other hand, only the conditions $0 < a(\cdot) \leq c'\,\mathrm{id} < \infty$ are enough to conclude that H_a is affiliated to $\mathbf{B}^0_{\mathscr{U}}(\mathcal{H})$. This is not enough for a complete characterization of the essential spectrum of H_a, so we treat now affiliation to $\mathbf{B}^0(\mathcal{H})$.

Proposition 5.1. *Assume that $0 < a(\cdot) \leq c'\,\mathrm{id} < \infty$ and that there is a continuous function $C : \mathscr{X} \to (0, \infty)$ satisfying $C(0) = 1$ such that*

$$a(z + x) \leq C(x)a(z), \quad \forall x, z \in \mathscr{X}. \tag{5.1}$$

Then $W(X)\mathcal{G}_a \subset \mathcal{G}_a$ for all $X \in \Xi$ and H_a is affiliated to $\mathbf{B}^0(\mathcal{H})$.

Proof. The first assertion is very simple to check.
Then notice that, computing on $C_c^\infty(\mathscr{X})$, one has the identity

$$\begin{aligned}\mathscr{W}(X)H_a - H_a = &\sum_{j,k=1}^{n} P_j[a_{jk}(Q + x) - a_{jk}(Q)]P_k \\ &+ \sum_{j,k=1}^{n} \{\xi_j a_{jk}(Q + x)P_k + P_j a_{jk}(Q + x)\xi_k + a_{jk}(Q + x)\xi_j\xi_k\}\,.\end{aligned}$$

Using (5.1) it follows easily that

$$\langle u, [\mathscr{W}(X)H_a - H_a]u\rangle \leq D(X) \parallel u \parallel^2_{\mathcal{G}_a}, \quad \forall u \in C_c^\infty(\mathscr{X})$$

with $D(X) \to 0$ when $X \to 0$, implying that $\parallel \mathscr{W}(X)H_a - H_a \parallel_{\mathbf{B}(\mathcal{G}_a;\mathcal{G}_a^*)} \to 0$ when $X \to 0$.

Thus H_a is affiliated to $\mathbf{B}^0(\mathcal{H})$, by the criterion **F** of the preceding Section. □

Remark 5.2. This is far from optimal. If the coefficients $a(x)$ grow faster than $|x|^2$ at infinity, then H_a has a compact resolvent by [4, Cor. 1.6.7], so it is affiliated to $\mathbf{K}(\mathcal{H}) \subset \mathbf{E}(\mathscr{X}) \subset \mathbf{B}^0(\mathcal{H})$.

Remark 5.3. By [6, Th. 9], if there is a diverging sequence of points $(x_m)_{m\in\mathbb{N}}$ in the configuration space $\mathscr{X}$ and a diverging sequence $(r_m)_{m\in\mathbb{N}}$ of positive numbers such that

$$\lim_{m\to\infty} \left\{ \sup_{|x - x_m| \leq r_m} \parallel a(x) \parallel \right\} = 0,$$

then the operator H_a is not affiliated to the crossed product C^*-algebra $\mathbf{E}(\mathcal{H})$. This happens for instance if $\parallel a(x) \parallel \to 0$ when $x \to \infty$. In a huge number of such situations (5.1) is fulfilled and one really needs ultrafilters in phase space to describe the essential spectrum.

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