REGULARITY PROPERTIES OF NONLOCAL MINIMAL SURFACES VIA LIMITING ARGUMENTS

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Abstract. We prove an improvement of flatness result for nonlocal minimal surfaces which is independent of the fractional parameter s when $s \to 1^-$.

As a consequence, we obtain that all the nonlocal minimal cones are flat and that all the nonlocal minimal surfaces are smooth when the dimension of the ambient space is less or equal than 7 and s is close to 1.

The purpose of this paper is to study some regularity properties of nonlocal minimal surfaces as they approach the classical minimal surfaces.

Let $n \geqslant 2$ and $s \in (0,1)$. Given two non-overlapping (measurable) subsets A and B of $\mathbb{R}^n$, we define

$$\mathcal{L}(A,B) := \int_A \int_B \frac{1}{|x-y|^{n+s}}\, dy\, dx.$$

Given a bounded open set $\Omega \subset \mathbb{R}^n$ and a set $E \subseteq \mathbb{R}^n$, we let

$$\mathcal{J}_s(E,\Omega) := \mathcal{L}(E\cap\Omega, (\mathcal{C}E)\cap\Omega) + \mathcal{L}(E\cap\Omega, (\mathcal{C}E)\cap(\mathcal{C}\Omega)) + \mathcal{L}(E\cap(\mathcal{C}\Omega), (\mathcal{C}E)\cap\Omega).$$

We say that E is s-minimal in Ω if for any $\widetilde{E} \subseteq \mathbb{R}^n$ for which $\widetilde{E}\cap(\mathcal{C}\Omega) = E\cap(\mathcal{C}\Omega)$ one has that

$$\mathcal{J}_s(E,\Omega) \leqslant \mathcal{J}_s(\widetilde{E},\Omega).$$

That is, E is s-minimal if it minimizes the functional among competitors which agree outside Ω.

The functional $\mathcal{J}_s$ has been recently introduced in [4] as a model for nonlocal minimal surfaces, and its relation with the classical minimal surfaces has been established in [6, 1], both in the geometric sense and in the Gamma–convergence framework.

Besides their neat geometric motivation, such nonlocal minimal surfaces also arise as limit interfaces of nonlocal phase segregation problems, see [9, 10].

The main difficulty in the framework we consider is, of course, the nonlocal aspect of the contributions in the functional. The counterpart of this difficulty, however, is given by the fact that the functional is well defined for every (measurable) set – in particular, there is no need to introduce Caccioppoli sets in this case. Nevertheless, in spite of the results of [4, 6, 1], several regularity issues for s-minimizers are still open.

The purpose of this paper is to develop some regularity theory when s is close to 1 by a compactness argument, taking advantage of the regularity theory of the classical minimal surfaces. Our main result is the following improvement of flatness:

Theorem 1. *Let $s_o \in (0,1)$, $\alpha \in (0,1)$ and $s \in [s_o, 1)$. Let E be s-minimal in B_1. There exists $\varepsilon_\star > 0$, possibly depending on n, s_o and α, but independent of s, such that if*

$$\partial E \cap B_1 \subseteq \{|x\cdot e_n| \leqslant \varepsilon_\star\} \tag{0.1}$$

then ∂E is a $C^{1,\alpha}$-graph in the e_n-direction.

The crucial part of Theorem 1 is that its threshold $\varepsilon_\star$ is independent of s as $s \to 1^-$: in fact, for a fixed s, an improvement of flatness whose threshold depends on s has been obtained in [4] (see Theorem 6.1 there). The techniques used to prove Theorem 1 (hence to obtain a threshold independently of s as $s \to 1^-$) are a uniform measure estimate for the oscillation, and a Calderón–Zygmund iteration. Both these tools have somewhat a classical flavor, but they need to be appropriately, and deeply, modified here: in particular, some fine estimates performed in [5] turn out to be very useful here in order to obtain bounds that are independent of s, and the iteration is not straightforward, but it has to distinguish two cases according to the size of the cubes involved, and the technical difficulties arising in the course of the proof turn out to be quite challenging.
As a consequence of Theorem 1, we obtain several regularity and rigidity results for s-minimal surfaces, such as:

Theorem 2. *Let $n \leqslant 7$.*
There exists $\epsilon_o > 0$ such that if $s \in (1-\epsilon_o, 1)$ then any s-minimal cone is a hyperplane.

Theorem 3. *Let $n \leqslant 7$.*
There exists $\epsilon_o > 0$ such that if $s \in (1-\epsilon_o, 1)$ then any s-minimal set is locally a $C^{1,\alpha}$-hypersurface.

Theorem 4. *Let $n = 8$.*
There exists $\epsilon_o > 0$ such that if $s \in (1-\epsilon_o, 1)$ then any s-minimal set is locally $C^{1,\alpha}$, everywhere except, at most, at countably many isolated points.

Theorem 5. *There exists $\epsilon_o > 0$ such that if $s \in (1-\epsilon_o, 1)$ then any s-minimal set is locally $C^{1,\alpha}$ outside a closed set Σ, with $\mathcal{H}^d(\Sigma) = 0$ for any $d > n-8$.*

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1. Notation

A point $x \in \mathbb{R}^n$ will be often written in coordinates as $x = (x', x_n) \in \mathbb{R}^{n-1} \times \mathbb{R}$.

The complement of a set $\Omega \subseteq \mathbb{R}^n$ will be denoted by $\mathcal{C}\Omega := \mathbb{R}^n \setminus \Omega$. For any $P \in \mathbb{R}^n$ and $\rho > 0$, we define the cylinder

$$K_\rho(P) := \{|x' - P'| < \rho\} \times \{|x_n - P_n| < \rho\}.$$

We also set $K_\rho := K_\rho(0)$.
The $(n-1)$-dimensional cube of side R centered at $x'_o \in \mathbb{R}^{n-1}$ will be denoted by $Q_R(x_o)$.
If $\nu \in \mathrm{S}^{n-1}$, given $x \in \mathbb{R}^n$, we define its projection along ν, that is $\pi_\nu x := x - (x \cdot \nu)\nu$.
Given a set $E \subset \mathbb{R}^n$, we denote by $d_E(x)$ the signed distance of a point $x \in \mathbb{R}^n$; we will take the sign convention that $d_E(x) \geqslant 0$ if $x \in \mathcal{C}E$.
If $\Sigma \subset \mathbb{R}^n$ is a C^2-portion of hypersurface, we define $\mathcal{H}(P)$ to be the mean curvature of Σ at P (with the convention that $\mathcal{H}$ equals the sum of all the principal curvatures).
The k-dimensional Lebesgue measure of a (measurable) set $A \subseteq \mathbb{R}^k$ will be denoted by $|A|$.
We let ϖ be the $(n-2)$-dimensional Hausdorff measure of the boundary of the $(n-1)$-dimensional unit ball.
Often, we will denote by c, C a suitable positive constant, that we allow ourselves the latitude of renaming at each step of the computation.

2. Proof of Theorem 1

Now we start the proof of Theorem 1, which is based on several steps.
First, we need to approximate our s-minimal surface with a graph. As soon as s approaches 1, a flat s-minimal surface approach a classical, smooth, minimal surface, and this will allow us to keep the Lipschitz norm of this approximating graph under control.
Then, we perform an estimate on the detachment of this graph from its tangent hyperplane: this bound (together with a suitable auxiliary function and an estimate relating the integral equation with the classical mean curvature equation in the limit) provides an Alexandrov-Bakelman-Pucci type theory that bounds the oscillation of the graph in measure.
This may be repeated at finer and finer scales via dyadic decomposition, by possibly taking advantage of the closeness to the smooth minimal surface when the size of the cubes become too small. In this way, one obtains a pointwise control on the oscillation of the approximating graph (and so of the original s-minimal surface), leading to the proof of Theorem 1.
Below are the full details or the proof.

2.1. **Building a graph via the distance function.** One of the difficulties of our framework is that the s-minimal surfaces we are dealing with are not necessarily graphs. To get around this problem, we follow an idea of [3] and we consider level sets of the distance function in an appropriate scaling (this may be seen as a sup-convolution technique).
For this, we recall the following classical geometric observation on the regularity of the level sets of the distance function:

Lemma 6. *Let $E \subset \mathbb{R}^n$. Assume that*

$$\{x_n \leqslant -\gamma\} \cap K_r \subseteq E \cap K_r \subseteq \{x_n \leqslant \gamma\} \cap K_r, \tag{2.1}$$

for some $r > \gamma > 0$.
Let $\delta \in (0, r/4)$ and $\mathcal{S}^\pm := \{x \in \mathbb{R}^n \text{ s.t. } d_E(x) = \pm\delta\}$.
Then, there exist $c \in (0,1)$ and $C \in (1,+\infty)$ such that if $\gamma/\delta < c$ then $\mathcal{S}^\pm \cap K_{r-2\delta}$ is a Lipschitz graph in the nth direction with Lipschitz constant bounded by $C\sqrt{\gamma/\delta}$.

Furthermore, $\mathcal{S}^-$ (resp., $\mathcal{S}^+$) may be touched at any point of $K_{r-2\delta}$ by a tangent paraboloid from above (resp., below).

Proof. We focus on $\mathcal{S}^-$, the case of $\mathcal{S}^+$ being analogous. We would like to show that for any x, $z \in \mathcal{S}^- \cap K_{r-2\delta}$

$$x_n - z_n \leqslant C\sqrt{\frac{\gamma}{\delta}}\,|x' - z'|, \tag{2.2}$$

from which the desired result follows by possibly exchanging the roles of x and z.
For this, we argue like this. For any $x \in \mathcal{S}^- \cap K_{r-2\delta}$, the ball of radius δ centered at x is tangent to ∂E at some point $y(x) \in \partial E \cap K_r$, and, conversely,

$$\text{the ball of radius } \delta \text{ centered at } y(x) \text{ is tangent to } \mathcal{S}^- \text{ at } x. \tag{2.3}$$

Let $e_n := (0, \dots, 1)$. Since $x + \delta e_n \in B_\delta(x)$, we have that $x + \delta e_n$ must lie below ∂E. Hence, by (2.1),

$$x_n + \delta \leqslant \gamma. \tag{2.4}$$

Similarly, since $y(x) \in \partial E$, we obtain from (2.1) that

$$y_n(x) \geqslant -\gamma. \tag{2.5}$$

By (2.4) and (2.5),

$$y_n(x) - x_n \geqslant \delta - 2\gamma. \tag{2.6}$$

In the same way, we see that

$$y_n(z) - z_n \geqslant \delta - 2\gamma. \tag{2.7}$$

Now, if $|x' - z'| \geqslant \sqrt{\gamma\delta}$, we use (2.1) and (2.6) to deduce that

$$\begin{aligned} x_n - z_n &\leqslant (x_n - y_n(x)) + |y_n(x)| + |y_n(z)| + |y_n(z) - z_n| \\ &\leqslant (2\gamma - \delta) + \gamma + \gamma + |y(z) - z| \\ &\leqslant (2\gamma - \delta) + \gamma + \gamma + \delta \\ &= 4\gamma \leqslant 4\sqrt{\frac{\gamma}{\delta}}\,|x' - z'|, \end{aligned}$$

which proves (2.2) in this case.
So, we may focus on the case in which

$$|x' - z'| \leqslant \sqrt{\gamma\delta}. \tag{2.8}$$

Then, from (2.7),

$$\delta^2 = |y(z) - z|^2 = |y'(z) - z'|^2 + |y_n(z) - z_n|^2 \geqslant |y'(z) - z'|^2 + (\delta - 2\gamma)^2,$$

which gives

$$|y'(z) - z'| \leqslant 2\sqrt{\gamma\delta}. \tag{2.9}$$

Hence

$$|x' - y'(z)| \leqslant |x' - z'| + |z' - y'(z)| \leqslant 3\sqrt{\gamma\delta},$$

due to (2.8) and (2.9), and so, in particular,

$$|x' - y'(z)| \leqslant \frac{\delta}{100}. \tag{2.10}$$

So, we can define

$$p := \big(x', y_n(z) - \sqrt{\delta^2 - |y'(z) - x'|^2}\big). \tag{2.11}$$

We observe that

$$p \in \partial B_\delta(y(z)). \tag{2.12}$$

By (2.10), we have that x must be below $B_\delta(y(z))$, hence (2.12) implies that

$$x_n \leqslant p_n. \tag{2.13}$$

Now, we define $P := (p - y(z))/\delta$ and $Z := (z - y(z))/\delta$. We observe that P, $Z \in \partial B_1$, due to (2.12). Also, P_n, $Z_n \leqslant 0$, due to (2.7) and (2.11). Moreover, $|P'| + |Z'| \leqslant 1/50$ thanks to (2.9), (2.10) and (2.11). As a consequence

$$|P_n - Z_n| \leqslant 100\,|P' - Z'|^2.$$

By scaling back, this gives that

$$|p_n - z_n| \leqslant \frac{100}{\delta}\,|p' - z'|^2 = \frac{100}{\delta}\,|x' - z'|^2 \leqslant 100\sqrt{\frac{\gamma}{\delta}}\,|x' - z'|,$$

where (2.8) was used once again. From this and (2.13), we infer that

$$x_n - z_n \leqslant p_n - z_n \leqslant 100\sqrt{\frac{\gamma}{\delta}}\,|x' - z'|,$$

which gives (2.2) in this case too.
Then, the desired Lipschitz property is a consequence of (2.2), and the existence of a tangent paraboloid follows from (2.3). □

A global version of Lemma 6 is given by the following result:

Corollary 7. *Let $E_\star \subseteq \mathbb{R}^n$. Suppose that $\partial E_\star \cap K_2$ is a $C^{1,\alpha}$-graph in the nth direction, for some $\alpha > 0$, and let $M_\star$ be its $C^{1,\alpha}$-norm.*
Then, there exists $c_\star \in (0,1)$, possibly depending on $M_\star$, such that the following holds.
Let γ, $\delta \in (0, 1/4)$, $E \subseteq \mathbb{R}^n$ and suppose that

$$E \cap K_2 \text{ lies in a } \gamma\text{-neighborhood of } E_\star. \tag{2.14}$$

Let $\mathcal{S}^\pm := \{x \in \mathbb{R}^n \text{ s.t. } d_E(x) = \pm\delta\}$.
Then, $\mathcal{S}^\pm \cap K_1$ is a Lipschitz graph in the nth direction, provided that $\gamma/\delta < c_\star$, $\delta < c_\star \gamma^{1/(1+\alpha)}$ and $\gamma < c_\star$.
More precisely, there exists a constant $C > 1$ for which $\mathcal{S}^\pm \cap K_1$ is a Lipschitz graph in the nth direction and the Lipschitz norm of $\mathcal{S}^\pm \cap K_1$ is controlled by $C\sqrt{\gamma/\delta} + M_o$, where M_o is the Lipschitz norm of $\partial E_\star \cap K_2$.
Furthermore, $\mathcal{S}^-$ (resp., $\mathcal{S}^+$) may be touched at any point of $K_{r-2\delta}$ by a tangent paraboloid from above (resp., below). Finally, for any $|x'| \leqslant 1/2$,

$$u^+(x') - u^-(x') \leqslant 2(2 + M_o)(\gamma + \delta). \tag{2.15}$$

Proof. Since $\partial E_\star \cap K_2$ is $C^{1,\alpha}$, it separates with power $(1+\alpha)$ from its tangent hyperplane, with multiplicative constant $M_\star$. Then, we take $r := (\gamma/M_\star)^{1/(1+\alpha)}$ and we cover $\partial E_\star \cap K_2$ with cylinders K_r, centered at points of $\partial E_\star$ and rotated parallel to the tangent plane of $\partial E_\star$.
By construction, in each of these cylinders, $\partial E_\star$ separates no more than $M_\star r^{1+\alpha} = \gamma$ from its tangent hyperplane, and so E is 2γ-close to such hyperplane. Therefore, Lemma 6 applies (with γ there replaced by 2γ). Consequently, in each of these cylinders, $\mathcal{S}^\pm$ is a Lipschitz graph with respect to the normal direction ν of $\partial E_\star$ (and its Lipschitz norm is bounded by $C\sqrt{\gamma/\delta}$ with respect to ν).
This proves the first part of Corollary 7. It remains to prove (2.15). For this, we fix $|\bar{x}'| \leqslant 1/2$ and we set $P^\pm := (\bar{x}', u^\pm(\bar{x}')) \in \mathcal{S}^\pm$. Then, we take $Q^\pm \in \partial E$ that realizes the distance, i.e. $|P^\pm - Q^\pm| = \delta$. By (2.14), we find points $R^\pm \in \partial E_\star$ such that $|R^\pm - Q^\pm| \leqslant \gamma$. Notice that

$$|(R^\pm)' - (P^\pm)'| \leqslant |(R^\pm)' - (Q^\pm)'| + |(Q^\pm)' - (P^\pm)| \leqslant \gamma + \delta.$$

Therefore, since $(P^+)' = (P^-)' = u(\bar{x})$,

$$|(R^+)' - (R^-)'| \leqslant |(R^+)' - (P^+)'| + |(P^-)' - (R^-)'| \leqslant 2(\gamma + \delta).$$

So, since $\partial E_\star$ is a Lipschitz graph,

$$|R^+_n - R^-_n| \leqslant M_o|(R^+)' - (R^-)'| \leqslant 2M_o(\gamma + \delta).$$

In particular,

$$|R^+ - R^-| \leqslant 2(1 + M_o)(\gamma + \delta)$$

and so

$$\begin{aligned} & |P^+ - P^-| \\ \leqslant\ & |P^+ - Q^+| + |Q^+ - R^+| + |R^+ - R^-| + |R^- - Q^-| + |Q^- - P^-| \\ \leqslant\ & 2(1 + M_o)(\gamma + \delta) + 2\gamma + 2\delta, \end{aligned}$$

which gives (2.15). □

2.2. **Detachment from the tangent hyperplane.** Next result is one of the cornerstones of our procedure since it manages to reconstruct a geometry similar to the one obtained in Lemma 8.1 of [5]. In spite of his technical flavor, it basically states under which conditions we can say that a functions separates from a tangent hyperplane quadratically in a ring, independently of s as $s \to 1^-$.

Lemma 8. *Fix $\overline{C} \geqslant 1$. Let ε, $R > 0$ and $\bar{x}' \in \mathbb{R}^{n-1}$.*
Let $u : \mathbb{R}^{n-1} \to \mathbb{R}$ be a Lipschitz function, with

$$|\nabla u(x')| \leqslant \overline{C} \tag{2.16}$$

a.e. $|x' - \bar{x}'| \leqslant R$ and let $\bar{x}_n := u(\bar{x}')$, $\bar{x} := (\bar{x}', \bar{x}_n)$ and $E := \{x_n < u(x')\}$.
Assume that

$$(1 - s) \int_{B_R(\bar{x})} \frac{\chi_E(y) - \chi_{\mathcal{C}E}(y)}{|\bar{x} - y|^{n+s}}\, dy \leqslant \frac{\varepsilon}{R^s}. \tag{2.17}$$

Suppose that there exists $\mathcal{P} \in C^{1,1}(\mathbb{R}^{n-1})$ such that

$$|\nabla \mathcal{P}(x')| + R\,|D^2\mathcal{P}(x')| \leqslant \varepsilon \tag{2.18}$$

a.e. $|x' - \bar{x}'| \leqslant R$,

$$\mathcal{P}(\bar{x}') = u(\bar{x}') \text{ and } \mathcal{P}(x') \leqslant u(x') \text{ in } |x' - \bar{x}'| \leqslant R. \tag{2.19}$$

Then, there exists a constant $C \geqslant 1$, only depending on n and $\overline{C}$, such that[1] *the following result holds, as long as $\varepsilon \in (0, 1/C)$. There exists a $(n-1)$-dimensional ring $S_r := \{|x' - \bar{x}'| \in (r/C, r)\}$, with $r \in (0, R]$, such that, for any $M > 0$ we have*

$$\frac{\left|S_r \cap \left\{u(x') - \bar{x}_n - \nabla\mathcal{P}(\bar{x}') \cdot (x' - \bar{x}') > \dfrac{M\varepsilon r^2}{R}\right\}\right|}{\big|S_r\big|} \leqslant \frac{C}{M}. \tag{2.20}$$

Proof. We consider the normal vector of the graph of $\mathcal{P}$ at $\bar{x}'$, to wit

$$\nu := \frac{(-\nabla\mathcal{P}(\bar{x}'), 1)}{\sqrt{|\nabla\mathcal{P}(\bar{x}')|^2 + 1}}.$$

Let also

$$\begin{aligned} P &:= \{x_n < \mathcal{P}(x')\}, \\ L &:= \{x_n < \nabla\mathcal{P}(\bar{x}') \cdot (x' - \bar{x}') + \bar{x}_n\} \\ \text{and } A &:= \bar{x} + \left\{|x \cdot \nu| \leqslant \frac{4\varepsilon}{R}|\pi_\nu x|^2\right\}. \end{aligned}$$

We notice that A is just the translation and the rotation of the set

$$\left\{|x_n| \leqslant \frac{4\varepsilon}{R}|x'|^2\right\}.$$

and so, for any $\rho > r > 0$,

$$\int_{B_\rho(\bar{x})\setminus B_r(\bar{x})} \chi_A(y)\,dy \leqslant \int_{|y'|\leqslant\rho} \left[\int_{|y_n|\leqslant(4\varepsilon/R)|y'|^2} dy_n\right] dy' \leqslant \frac{C\varepsilon\rho^{n+1}}{R}. \tag{2.21}$$

On the other hand, since L is a halfspace passing through $\bar{x}$, the following cancellations hold:

$$\int_{B_\rho(\bar{x})\setminus B_r(\bar{x})} \chi_L(y) - \chi_{\mathcal{C}L}(y)\,dy = 0 \text{ and } \int_{B_\rho(\bar{x})\setminus B_r(\bar{x})} \frac{\chi_L(y) - \chi_{\mathcal{C}L}(y)}{|\bar{x} - y|^{n+s}}\,dy = 0. \tag{2.22}$$

Moreover, by (2.19), we have that $P \subseteq E$, thus

$$\chi_E \geqslant \chi_P \text{ and so } \chi_{\mathcal{C}E} \leqslant \chi_{\mathcal{C}P}. \tag{2.23}$$

[1]The reader may compare (2.20) here and (8.1) in [5]. Notice that such an estimate, roughly speaking, says that u separates quadratically from its tangent hyperplane in a ring, up to a set with small density – and the the constants are independent of s.

From this, a general geometric argument implies a uniform quadratic detachment in a whole ball with smaller radius (see (8.2) and (8.3) in [5]) and consequently a linear bound on the image of the subdifferential of the convex envelope (see (8.4) in [5]), and this is the necessary ingredient for the Alexandrov-Bakelman-Pucci theory to work (see Sections 8, 9 and 10 in [5]). In our framework, u will be the level set of the distance from an s-minimal surface: we will add to it the auxiliary function of Section 2.4 and consider the touching point of the convex envelope. These points, by construction are touched from below by a hyperplane, so u is touched from below by a smooth function, which motivates the setting of Lemma 8.

Also, the quadratic detachment of $\mathcal{P}$ from its tangent plane given by (2.18) implies that $(L \setminus A) \cap B_R \subseteq P \cap B_R$ and $(\mathcal{C}P) \cap B_R \subseteq ((\mathcal{C}L) \cup A) \cap B_R$. Therefore, in B_R,

$$\chi_L - \chi_A \leqslant \chi_{L\setminus A} \leqslant \chi_P \text{ and } \chi_{\mathcal{C}P} \leqslant \chi_{(\mathcal{C}L)\cup A} \leqslant \chi_{\mathcal{C}L} + \chi_A. \tag{2.24}$$

So, from (2.23) and (2.24), we obtain that, in B_R,

$$\chi_E - \chi_{\mathcal{C}E} \geqslant \chi_P - \chi_{\mathcal{C}P} \geqslant \chi_L - \chi_{\mathcal{C}L} - 2\chi_A. \tag{2.25}$$

Now, for any $m \in \mathbb{N}$, let

$$\begin{aligned} r_m &:= \frac{R}{\big((2+\overline{C})n\big)^m}, \\ R_m &:= B_{r_m}(\bar{x}) \setminus B_{r_{m+1}}(\bar{x}) \\ \text{and } b_m &:= \int_{R_m} \frac{\chi_E(y) - \chi_{\mathcal{C}E}(y)}{|\bar{x}-y|^{n+s}}\, dy. \end{aligned}$$

We claim that there exists $m \in \mathbb{N}$ such that

$$b_m \leqslant \frac{C_o \varepsilon r_m^{1-s}}{R}, \tag{2.26}$$

for a suitable constant $C_o \geqslant 1$. The proof is by contradiction: if not, we have

$$\begin{aligned} &\int_{B_R(\bar{x})} \frac{\chi_E(y) - \chi_{\mathcal{C}E}(y)}{|\bar{x}-y|^{n+s}}\, dy \\ &\quad = \sum_{m=0}^{+\infty} b_m \geqslant \frac{C_o\varepsilon}{R} \sum_{m=0}^{+\infty} r_m^{1-s} = \frac{C_o\varepsilon}{R^s} \sum_{m=0}^{+\infty} \big((2+\overline{C})n\big)^{-(1-s)m} \\ &\quad = \frac{C_o\varepsilon}{R^s} \cdot \frac{1}{1 - \big((2+\overline{C})n\big)^{-(1-s)}} > \frac{C_o\varepsilon}{R^s} \cdot \frac{1}{C(1-s)}. \end{aligned}$$

This is in contradiction with (2.17) if C_o is large, and so (2.26) is established. From now on, m will be the one given by (2.26), and C_o will be simply C.

Now, we make use of (2.25), (2.22) and (2.21) to obtain that

$$\begin{aligned} &\int_{R_m} \Big(\chi_E(y) - \chi_{\mathcal{C}E}(y)\Big) \left(\frac{1}{|\bar{x}-y|^{n+s}} - \frac{1}{r_m^{n+s}}\right) dy \\ &\quad \geqslant \int_{R_m} \Big(\chi_L(y) - \chi_{\mathcal{C}L}(y) - 2\chi_A(y)\Big) \left(\frac{1}{|\bar{x}-y|^{n+s}} - \frac{1}{r_m^{n+s}}\right) dy \\ &\quad = \int_{R_m} \Big(-2\chi_A(y)\Big) \left(\frac{1}{|\bar{x}-y|^{n+s}} - \frac{1}{r_m^{n+s}}\right) dy \\ &\quad \geqslant -2 \int_{R_m} \frac{\chi_A(y)}{|\bar{x}-y|^{n+s}}\, dy \\ &\quad \geqslant -\frac{C}{r_m^{n+s}} \int_{R_m} \chi_A(y)\, dy \\ &\quad \geqslant -\frac{C\varepsilon r_m^{1-s}}{R}. \end{aligned}$$

Combining this with (2.26), we conclude that

$$\begin{aligned}\frac{|E\cap R_m|-|(\mathcal{C}E)\cap R_m|}{r_m^{n+s}} &= \int_{R_m}\frac{\chi_E(y)-\chi_{\mathcal{C}E}(y)}{r_m^{n+s}}\,dy\\ &= b_m-\int_{R_m}\Big(\chi_E(y)-\chi_{\mathcal{C}E}(y)\Big)\left(\frac{1}{|\bar x-y|^{n+s}}-\frac{1}{r_m^{n+s}}\right)\\ &\leqslant \frac{C\varepsilon r_m^{1-s}}{R}\end{aligned}$$

that is

$$|E\cap R_m|-|(\mathcal{C}E)\cap R_m|\leqslant\frac{C\varepsilon r_m^{n+1}}{R}. \tag{2.27}$$

Now we prove that

$$\int_{\{r_{m+1}\leqslant|x'-\bar x'|\leqslant r_m/(\overline{C}\sqrt{n})\}} u(x')-\bar x_n-\nabla\mathcal{P}(\bar x')\cdot(x'-\bar x')\,dx'\leqslant\frac{C\varepsilon r_m^{n+1}}{R}. \tag{2.28}$$

To this scope, we observe that

$$K_{r_m/\sqrt{n}}\subseteq B_{r_m}\subseteq K_{r_m}$$

and $r_{m+1}<r_m/(\overline{C}\sqrt{n})$. Hence

$$S_m:=\left\{r_{m+1}<|x'-\bar x'|<r_m/\sqrt{n}\right\}\times\left\{|x_n-\bar x_n|<r_m/\sqrt{n}\right\}\subseteq R_m. \tag{2.29}$$

Let $\alpha:=\chi_E-\chi_L=\chi_{\mathcal{C}L}-\chi_{\mathcal{C}E}$. We recall that

$$\alpha+\chi_A\geqslant 0 \text{ in } R_m, \tag{2.30}$$

due to (2.23) and (2.24).
Accordingly, by (2.21), (2.22), (2.29) and (2.30),

$$\begin{aligned}&|E\cap R_m|-|(\mathcal{C}E)\cap R_m|\\ &=\int_{R_m}\chi_E(y)-\chi_{\mathcal{C}E}(y)\,dy-0\\ &=\int_{R_m}\chi_E(y)-\chi_{\mathcal{C}E}(y)\,dy-\int_{R_m}\chi_L(y)-\chi_{\mathcal{C}L}(y)\,dy\\ &=2\int_{R_m}\alpha(y)\,dy\\ &=2\int_{R_m}\alpha(y)+\chi_A(y)\,dy-2\int_{R_m}\chi_A(y)\,dy\\ &\geqslant 2\int_{S_m}\alpha(y)+\chi_A(y)\,dy-\frac{C\varepsilon r_m^{n+1}}{R}.\end{aligned} \tag{2.31}$$

Now, we use (2.16) and (2.18) to see that, if $|y'-\bar x'|<r_m/(\overline{C}\sqrt{n})$, we have

$$\begin{aligned}&\nabla\mathcal{P}(\bar x')\cdot(y'-\bar x')\geqslant-|y'-\bar x'|>-r_m/\sqrt{n}\\ \text{and }\ &u(y')-\bar x_n=u(y')-u(\bar x')\leqslant\overline{C}|y'-\bar x'|<r_m/\sqrt{n}.\end{aligned} \tag{2.32}$$

So, fixed y', with $|y'-\bar{x}'|\in\big(r_{m+1},r_m/(\overline{C}\sqrt{n})\big)$ we see that $\alpha(y',y_n)=1$ when (y',y_n) is trapped between E and $\mathcal{C}L$ (notice that it cannot exit S_m from either the top or the bottom, by (2.32)), i.e., when

$$\bar{x}_n+\nabla\mathcal{P}(\bar{x}')\cdot(\bar{x}')(y'-\bar{x}')\leqslant y_n<u(y').$$

So, recalling (2.30) and integrating first in dy_n, we have that

$$\begin{aligned}\int_{S_m}\alpha(y)+\chi_A(y)\,dy &\geqslant \int_{\big\{|y'-\bar{x}'|\in(r_{m+1},r_m/(\overline{C}\sqrt{n}))\big\}}\Big(u(y')-\bar{x}_n-\nabla\mathcal{P}(\bar{x}')\cdot(x'-\bar{x}')\Big)^+\,dy'\\ &\geqslant \int_{\big\{|y'-\bar{x}'|\in(r_{m+1},r_m/(\overline{C}\sqrt{n}))\big\}}u(y')-\bar{x}_n-\nabla\mathcal{P}(\bar{x}')\cdot(x'-\bar{x}')\,dy'.\end{aligned}$$

This, (2.31) and (2.27) imply (2.28).
Then, (2.20) follows from (2.28) and the Chebyshev Inequality, taking $r:=r_m/(\overline{C}\sqrt{n})$, $S_r:=\{|x'-\bar{x}'|\in(r_{m+1},r_m/(\overline{C}\sqrt{n}))\}$ and noticing that $|S_r|\sim r_m^{n-1}$. □

2.3. **The mean curvature as a limit equation.** In this section, we show that the integral equation of s-minimal surfaces converges, in a somewhat uniform way, to the classical mean curvature equation as $s\to 1^-$, and we remark that the estimates improve as the surfaces gets flatter and flatter. An estimate of this kind will be useful in the computation of the forthcoming Lemma 10.

Lemma 9. *Let $s\in[1/10,1)$. Let $\alpha\in(0,1)$. Let $F\subset\mathbb{R}^n$, $x_o\in\partial F$, and suppose that $\partial F\cap B_1(x_o)$ is a $C^{2,\alpha}$-graph in some direction, with $C^{2,\alpha}$-norm bounded by some $M>0$.*
Then, there exists $C\geqslant 1$, only depending on α and n, such that

$$\left|\mathcal{H}(x_o)-\frac{(n-1)(1-s)}{\varpi}\int_\Omega\frac{\chi_F(y)-\chi_{\mathcal{C}F}(y)}{|x_o-y|^{n+s}}\,dy\right|\leqslant\frac{CM(1-s)}{r},\tag{2.33}$$

where

$$r:=\min\left\{\frac{1}{n},\frac{1}{2M}\right\}.\tag{2.34}$$

In particular, if $M\in(0,1]$,

$$\left|\mathcal{H}(x_o)-\frac{(n-1)(1-s)}{\varpi}\int_\Omega\frac{\chi_F(y)-\chi_{\mathcal{C}F}(y)}{|x_o-y|^{n+s}}\,dy\right|\leqslant CM(1-s).\tag{2.35}$$

Proof. Without loss of generality, up to a translation and a rotation, which leave our problem invariant, we may take $x_o=0$ and the tangent hyperplane of ∂F at 0 to be $\{x_n=0\}$. In this way, we write ∂F as the graph $x_n=g(x')$, for $|x'|\leqslant 1/\sqrt{n}$, with $\nabla g(0)=0$ and $\mathcal{H}(0)=\Delta g(0)$. Up to a rotation of the horizontal coordinates, we also suppose that $D^2g(0)$ is diagonal, with eigenvalues $\lambda_1,\dots,\lambda_{n-1}$. In this way

$$g(y')=\frac12\sum_{i=1}^{n-1}\lambda_i y_i^2+h(y'),$$

and $|h(y')|\leqslant M|y'|^{2+\alpha}$. So, for any $|y'|\leqslant r$,

$$|g(y')|\leqslant Mr^2\leqslant\frac{r}{2}.\tag{2.36}$$

We observe that, by rotational symmetry,

$$\int_{\{|y'|\leqslant r\}} y_j^2\, |y'|^{-(n+s)}\, dy' = \int_{\{|y'|\leqslant r\}} y_1^2\, |y'|^{-(n+s)}\, dy'$$

for any $j = 1, \dots, n-1$ and therefore, by summing up in j,

$$\begin{aligned}\frac{\varpi r^{1-s}}{1-s} &= \int_{\{|y'|\leqslant r\}} |y'|^{2-(n+s)}\, dy' \\ &= (n-1)\int_{\{|y'|\leqslant r\}} y_1^2\, |y'|^{-(n+s)}\, dy' = (n-1)\int_{\{|y'|\leqslant r\}} y_i^2\, |y'|^{-(n+s)}\, dy'\end{aligned}$$

for any $i = 1, \dots, n-1$. Therefore

$$\int_{\{|y'|\leqslant r\}} \sum_{i=1}^{n-1} \lambda_i y_i^2\, |y'|^{-(n+s)}\, dy' = \frac{\varpi r^{1-s}\mathcal{H}(0)}{(n-1)(1-s)}. \tag{2.37}$$

Let now

$$G_s(\tau) := \int_0^\tau \frac{dt}{(1+t^2)^{(n+s)/2}}.$$

We observe that $G_s(0) = 0$, $G_s'(0) = 1$ and $|G_s''(\tau)| = (n+s)(1+t^2)^{-(n+s+2)/2}|t| \leqslant (n+1)|t|$. Therefore, a Taylor expansion gives

$$G_s(\tau) = \tau + \widetilde{G}_s(\tau),$$

with $|\widetilde{G}_s(\tau)| \leqslant C|\tau|^3$. Therefore, if we write

$$\widetilde{g}(y') := \frac{g(y')}{|y'|} = \frac{1}{2|y'|}\sum_{i=1}^{n-1} \lambda_i y_i^2 + \widetilde{h}(y')$$

with $|\widetilde{h}(y')| = |h(y')|/|y'| \leqslant M|y'|^{1+\alpha}$, we have that

$$\begin{aligned} G_s(\widetilde{g}(y')) &= \widetilde{g}(y') + \widetilde{G}_s(\widetilde{g}(y')) \\ &= \frac{1}{2|y'|}\sum_{i=1}^{n-1} \lambda_i y_i^2 + \widetilde{h}(y') + \widetilde{G}_s(\widetilde{g}(y')) \\ &= \frac{1}{2|y'|}\sum_{i=1}^{n-1} \lambda_i y_i^2 + \ell(y'),\end{aligned}$$

with

$$|\ell(y')| \leqslant |\widetilde{h}(y')| + C|\widetilde{g}(y')|^3 \leqslant CM(|y'|^{1+\alpha} + |y'|^3) \leqslant CM|y'|^{1+\alpha}$$

for any $|y'| \leqslant r$. As a consequence of this and (2.37),

$$\int_{\{|y'|\leqslant r\}} \frac{G_s(\widetilde{g}(y'))}{|y'|^{n+s-1}}\, dy' = \frac{\varpi r^{1-s}\mathcal{H}(0)}{2(n-1)(1-s)} + \varepsilon_1 \tag{2.38}$$

with $|\varepsilon_1| \leqslant CMr^{1+\alpha-s}/(1+\alpha-s) \leqslant CM$. Now, since the map $(0,+\infty) \ni t \mapsto 1-e^{-t}$ is concave, we have that $1-e^{-t} \in [0,t]$, hence

$$1 - r^{1-s} \in \big[0, (1-s)\log r^{-1}\big].$$

Accordingly, we may write (2.38) as

$$\int_{\{|y'|\leqslant r\}} \frac{G_s(\widetilde{g}(y'))}{|y'|^{n+s-1}}\,dy' = \frac{\varpi \mathcal{H}(0)}{2(n-1)(1-s)} + \varepsilon_2 \tag{2.39}$$

with $|\varepsilon_2| \leqslant CM(1+\log r^{-1})$.
Now, we recall (2.36), we integrate in the vertical coordinate and we substitute $t := y_n/|y'|$ to obtain that

$$\begin{aligned}
&\int_{K_r} \frac{\chi_F(y)-\chi_{\mathcal{C}F}(y)}{|y|^{n+s}}\,dy \\
=\;& \int_{|y'|\leqslant r}\left[\int_{-r}^{g(y')} \frac{dy_n}{(|y'|^2+|y_n|^2)^{(n+s)/2}} - \int_{g(y')}^{r} \frac{dy_n}{(|y'|^2+|y_n|^2)^{(n+s)/2}}\right] dy' \\
=\;& \int_{|y'|\leqslant r} \frac{1}{|y'|^{n+s}}\left[\int_{-r}^{g(y')} \frac{dy_n}{(1+(|y_n|/|y'|)^2)^{(n+s)/2}} - \int_{g(y')}^{r} \frac{dy_n}{(1+(|y'|/|y_n|)^2)^{(n+s)/2}}\right] dy' \\
=\;& \int_{|y'|\leqslant r} \frac{1}{|y'|^{n+s-1}}\left[\int_{-r/|y'|}^{\widetilde{g}(y')} \frac{dt}{(1+t^2)^{(n+s)/2}} - \int_{\widetilde{g}(y')}^{r/|y'|} \frac{dt}{(1+t^2)^{(n+s)/2}}\right] dy' \\
=\;& \int_{|y'|\leqslant r} \frac{1}{|y'|^{n+s-1}}\left[G_s(\widetilde{g}(y')) - G_s(-r/|y'|) - G_s(r/|y'|) + G_s(\widetilde{g}(y'))\right] dy'.
\end{aligned}$$

Therefore, since G_s is odd,

$$\int_{K_r} \frac{\chi_F(y)-\chi_{\mathcal{C}F}(y)}{|y|^{n+s}}\,dy = 2\int_{|y'|\leqslant r} \frac{G_s(\widetilde{g}(y'))}{|y'|^{n+s-1}}\,dy' = \frac{\varpi \mathcal{H}(0)}{(n-1)(1-s)} + \varepsilon_3 \tag{2.40}$$

with $|\varepsilon_3| \leqslant CM(1+\log r^{-1})$, due to (2.39).
Now, we point out the following cancellation:

$$\begin{aligned}
\left|\int_{K_r\setminus B_r} \frac{\chi_F(y)-\chi_{\mathcal{C}F}(y)}{|y|^{n+s}}\,dy\right| &\leqslant \int_{(K_r\setminus K_{r/\sqrt{n}})\cap\{|y_n|\leqslant M|y'|\}} \frac{1}{|y|^{n+s}}\,dy \\
&\leqslant CM\int_{r/\sqrt{n}}^{r} \rho^{-1-s}\,ds = \frac{CM(n^{s/2}-1)}{sr^s} \leqslant \frac{CM}{r}.
\end{aligned}$$

Accordingly, we can write (2.40) as

$$\int_{B_r} \frac{\chi_F(y)-\chi_{\mathcal{C}F}(y)}{|y|^{n+s}}\,dy = \frac{\varpi \mathcal{H}(0)}{(n-1)(1-s)} + \varepsilon_4$$

with $|\varepsilon_4| \leqslant CM(1+\log r^{-1}+r^{-1}) \leqslant CMr^{-1}$. This proves (2.33).
Then, (2.35) follows from (2.33) and (2.34), by observing that, if $M \in (0,1]$, we have that $r = 1/n$ so it does not depend on M. □

2.4. **Construction of an auxiliary function.** The purpose of this section is to obtain a special function, which is positive in a large ball, and that satisfies the correct inequality with respect to the integral operator of (2.17) in a smaller ball. This is needed to apply an appropriate variation of the local Alexandrov-Bakelman-Pucci theory of [2, 5], in order to localize the set in which the solution we are considering becomes positive. Indeed, the following function is the one that replaces the auxiliary functions in Lemma 4.1 of [2] and Corollary 9.3 of [5] for our framework (here, some technical complications also arise since the operator in (2.44) is both nonlocal and nonlinear in its dependence on the sets):

Lemma 10. *Fix $R > 0$ and constants $c_1, \dots, c_5 > 0$. Fix also $c_0 \in (0, c_1)$. There exists $C \geqslant 1$ (possibly depending on $c_0, \dots, c_5 > 0$ but independent of R) such that, if $1 - s$, $\varepsilon \in (0, 1/C]$, the following results hold.*
There exists $\Phi \in C^\infty(\mathbb{R}^{n-1}, [-C\varepsilon R, C\varepsilon R])$ satisfying the following conditions:

$$\begin{gathered}\Phi(x') > \varepsilon R \text{ if } |x'| \geqslant (c_1 + c_2)R,\ \Phi(x') \leqslant -4\varepsilon R \text{ if } |x'| \leqslant c_1 R, \text{ and} \\ \sup_{\mathbb{R}^{n-1}} |\nabla \Phi| + R\,|D^2\Phi| \leqslant C\varepsilon.\end{gathered} \tag{2.41}$$

Also, let L be an affine function with

$$|\nabla L| \leqslant \frac{1}{C}, \tag{2.42}$$

set

$$\widetilde{\Phi} := L - \Phi \text{ and } F := \{x_n < \widetilde{\Phi}(x')\}. \tag{2.43}$$

Then

$$(1 - s) \int_{B_{c_3 R}(x)} \frac{\chi_F(y) - \chi_{\complement F}(y)}{|x - y|^{n+s}}\, dy \geqslant \frac{c_4 \varepsilon}{R^s} \tag{2.44}$$

for any $x \in \partial F \cap \{c_0 R < |x'| \leqslant (c_1 + c_2 + c_5)R\}$.

Proof. Up to replacing $\Phi(x')$ with $R\Phi(x'/R)$, we may and do consider just the case $R = 1$. Then, the function we will construct is depicted in Figure 1.

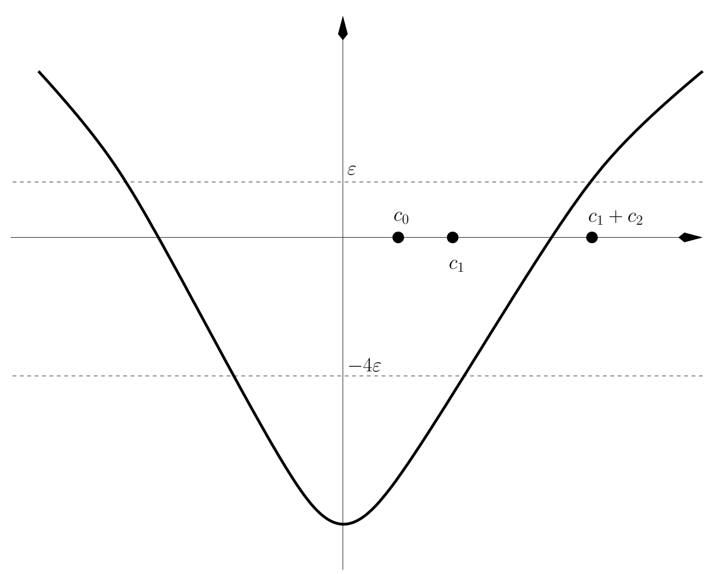

Figure 1: *The auxiliary function Φ (with $R = 1$).*

More explicitly, we take Φ to be smooth, radial, radially decreasing, satisfying (2.41) with $R = 1$, and in fact

$$\|\Phi\|_{C^{2,\alpha}(\mathbb{R}^{n-1})} \leqslant C(1+\mu_q)\varepsilon,$$

and such that

$$\Phi(x') = \varepsilon\left(\mu_q + 2 - \frac{c_0^q \mu_q}{|x'|^q}\right)$$

if $|x'| > c_0$. Here, $q > n-3$ is a free parameter and $\mu_q > 0$ will be chosen appropriately at the end of the proof. We observe that, if $|x'| > c_0$,

$$\begin{aligned} |\partial_i \Phi| &\leqslant \varepsilon q \mu_q c_0^q \, |x'|^{-q-1}, \\ |\partial^2_{ij} \Phi| &\leqslant \varepsilon q(q+3)\mu_q c_0^q \, |x'|^{-q-2} \\ \text{and} \quad -\Delta\widetilde{\Phi} = \Delta\Phi &= -\varepsilon q(q-n+3)\mu_q \, c_0^q |x'|^{-q-2}. \end{aligned}$$

Accordingly, if $|x'| > c_0$,

$$\sqrt{1+|\nabla\widetilde{\Phi}|^2} \in [1,2]$$

$$\text{and} \qquad \Delta\widetilde{\Phi} + \left|\frac{(D^2\widetilde{\Phi}\nabla\widetilde{\Phi})\cdot\nabla\widetilde{\Phi}}{1+|\nabla\widetilde{\Phi}|^2}\right| \geqslant \frac{\varepsilon q(q+3-n)\mu_q}{4} c_0^q \, |x'|^{-q-2}$$

as long as ε is small enough, thanks to (2.42). Hence, we estimate the mean curvature of ∂F at some point x with $|x'| \in (c_0, c_1 + c_2 + c_5]$ as

$$\begin{aligned} \mathcal{H}(x) &= \frac{1}{\sqrt{1+|\nabla\widetilde{\Phi}|^2}}\left(\Delta\widetilde{\Phi} - \frac{(D^2\widetilde{\Phi}\nabla\widetilde{\Phi})\cdot\nabla\widetilde{\Phi}}{1+|\nabla\widetilde{\Phi}|^2}\right) \\ &\geqslant \frac{\varepsilon\mu_q}{C} \end{aligned}$$

Therefore, if $x \in \partial F$, $|x'| \in (c_0, c_1 + c_2 + c_5]$, we have that

$$(1-s)\int_{B_1(x)} \frac{\chi_F(y) - \chi_{\mathcal{C}F}(y)}{|x-y|^{n+s}}\, dy \geqslant \frac{\varepsilon\mu_q}{C^2} - C(1+\mu_q)\varepsilon(1-s) \geqslant \frac{\varepsilon\mu_q}{C^3}$$

thanks to (2.35) in Lemma 9, as long as $1-s$ and ε are small enough. This and a suitable choice of μ_q give (2.44). □

2.5. **Measure estimates for the oscillation.** We obtain the following measure estimate. Such result may be seen as the counterpart, in our framework, of the measure estimate in Lemma 4.5 of [2] and Lemmata 8.6 and 10.1 of [5].

Lemma 11. *Fix $\overline{C} \geqslant 1$. Let $\kappa \in \mathbb{R}$ and $R > 0$. Let $u : \mathbb{R}^{n-1} \to \mathbb{R}$ be a Lipschitz function, with*

$$|\nabla u(x')| \leqslant \overline{C} \tag{2.45}$$

a.e. $|x'| \leqslant 3R$, and

$$u(x') \geqslant \kappa \text{ for any } |x'| \leqslant R. \tag{2.46}$$

Let $E := \{x_n < u(x')\}$.

Assume that, for any $x \in \partial E \cap B_{4n}$,

$$(1-s)\int_{B_R(x)} \frac{\chi_E(y)-\chi_{\mathcal{C}E}(y)}{|x-y|^{n+s}}\,dy \leqslant \frac{\varepsilon}{R^s}. \tag{2.47}$$

Then, if

$$\inf_{Q_{3R}} u \leqslant \kappa + \varepsilon R \tag{2.48}$$

we have that

$$\Big|\big\{u-\kappa \leqslant M\varepsilon R\big\}\cap Q_R\big\}\Big| \geqslant \mu R^{n-1}, \tag{2.49}$$

for appropriate universal constants $M > 1$ *and* $\mu \in (0,1)$, *as long as* $1-s$ *and* $\varepsilon \in (0, 1/C]$, *with* $C \geqslant 1$ *suitably large.*
Here, M, μ *and* C *only depend on* n *and* $\overline{C}$.

Proof. Up to translation, we may suppose that $\kappa = 0$. Let Φ be as in Lemma 10 (with $c_0, \dots, c_5$ to be conveniently chosen in what follows). Let $v := u + \Phi$ and $\Gamma : \mathbb{R}^{n-1} \to \mathbb{R}$ be the convex envelope of $v^- := \min\{v, 0\}$ in $B_{6\sqrt{n}R}$, that is

$$\Gamma(x) := \begin{cases} \sup\limits_{\Xi} \ell(x) & \text{if } |x'| < 6\sqrt{n}R,\\ 0 & \text{if } |x'| \geqslant 6\sqrt{n}R, \end{cases}$$

where Ξ above is a short-hand notation for all the affine functions ℓ such that $\ell(y') \leqslant v^-(y')$ for any $|y'| < 6\sqrt{n}R$ (see pages 23–27 of [2] for the basic properties of the convex envelope). Let $\mathcal{T}$ be the touching set between v and Γ, i.e.

$$\mathcal{T} := \{x' \in \mathbb{R}^{n-1} \text{ s.t. } \Gamma(x') = v(x')\}.$$

Let

$$m_o := -\inf_{Q_{3R}} v.$$

Notice that $v \leqslant u - 4\varepsilon R$ in Q_{3R}, due to (2.41) (for this we choose $c_1 := 3\sqrt{n}/2$ in Lemma 10, so that $Q_{3R} \subseteq \{|x'| \leqslant c_1 R\}$).
Therefore, by (2.48),

$$\inf_{Q_{3R}} v \leqslant -2\varepsilon R,$$

so $m_o \geqslant 2\varepsilon R$.
We recall that all the hyperplanes with slope bounded by $m_o/(CR)$ belong to $\nabla\Gamma(B_{6\sqrt{n}R})$ (see page 24 of [2] and also (3.9) there), hence

$$\varepsilon^{n-1} \leqslant C\left(\frac{m_o}{R}\right)^{n-1} \leqslant C|\nabla\Gamma(\mathcal{T})|. \tag{2.50}$$

Now, for any $\bar{x}' \in \mathcal{T}$, we let $L(x') := v(\bar{x}') + \nabla\Gamma(\bar{x}')\cdot(x' - \bar{x}')$ and $\mathcal{P} := L - \Phi$. We point out that $v > 0$ in $\{|x'| \geqslant 3\sqrt{n}R\}$, thanks to (2.41) and (2.46) (for this, we choose $c_2 := 3\sqrt{n}/2$ in Lemma 10, so that $c_1 + c_2 =:= 3\sqrt{n}$).
In particular, since $\Gamma \leqslant 0$, we see that $\bar{x}' \in \mathcal{T} \subseteq \{|x'| \leqslant 3\sqrt{n}R\}$.
Also, from (2.41), we have

$$|D^2\mathcal{P}| = |D^2\Phi| \leqslant \frac{C\varepsilon}{R}. \tag{2.51}$$

Moreover, v is above Γ which is above L in $B_{6\sqrt{n}R}$, by convexity, therefore, for any $e \in S^{n-1}$

$$0 \geqslant \Gamma(\bar{x}'+Re) \geqslant L(\bar{x}'+Re) = v(\bar{x}') + R\nabla\Gamma(\bar{x}')\cdot e \geqslant -C\varepsilon R + R\nabla\Gamma(\bar{x}')\cdot e$$

that is $\nabla\Gamma(\bar{x}')\cdot e \leqslant C\varepsilon$. So, since e is an arbitrary unit vector, we get that

$$|\nabla L| = |\nabla\Gamma(\bar{x}')| \leqslant C\varepsilon, \tag{2.52}$$

and so, by (2.41),

$$|D\mathcal{P}| \leqslant C\varepsilon. \tag{2.53}$$

Now we observe that

$$\mathcal{T} \subseteq Q_R. \tag{2.54}$$

The proof is by contradiction: if not, $u+\Phi \geqslant L$ in $\{|x'| \leqslant 6\sqrt{n}R\}$, with equality at some $\bar{x}'$ with $\bar{x}' \notin Q_R$. In particular, $|\bar{x}'| \geqslant R/2$. Then, we can use Lemma 10, with F as in (2.43) (notice that (2.42) is satisfied here due to (2.52)). For this, we set $\bar{x} := (\bar{x}', u(\bar{x}')) \in \partial F$, and we choose $c_0 := 1/4$, $c_4 := 2$ and $c_5 := 100\sqrt{n}$ in Lemma 10. In this way since $E \cap B_{6\sqrt{n}R} \supseteq F \cap B_{6\sqrt{n}R}$, we deduce from (2.44) that

$$(1-s)\int_{B_R(\bar{x})} \frac{\chi_E(y) - \chi_{\mathcal{C}E}(y)}{|x-y|^{n+s}}\,dy \geqslant (1-s)\int_{B_R(\bar{x})} \frac{\chi_F(y) - \chi_{\mathcal{C}F}(y)}{|x-y|^{n+s}}\,dy \geqslant \frac{2\varepsilon}{R^s}.$$

This is in contradiction with (2.47) and so it establishes (2.54).
Also, given $\bar{x}' \in \mathcal{T}$, we have that $\mathcal{P}(\bar{x}') = v(\bar{x}') - \Phi(\bar{x}') = u(\bar{x}')$ and

$$\mathcal{P} \leqslant \Gamma - \Phi \leqslant v - \Phi = u.$$

This, (2.45), (2.47), (2.51) and (2.53) say that the hypotheses of Lemma 8 are fulfilled (up to scaling ε to $C\varepsilon$). As a consequence, by (2.20), for any M large enough,

$$\frac{\left|S^{(\bar{x}')} \cap \left\{u(x') - u(\bar{x}') - \nabla\mathcal{P}(\bar{x}')\cdot(x'-\bar{x}') > \dfrac{M\varepsilon r_{\bar{x}'}^2}{R}\right\}\right|}{\left|S^{(\bar{x}')}\right|} \leqslant \frac{C}{M} \tag{2.55}$$

for a suitable ring $S^{(\bar{x}')} := \left\{|x'-\bar{x}'| \in \left(r_{\bar{x}'}/C, r_{\bar{x}'}\right)\right\}$ and a suitable $r_{\bar{x}'} \in (0,R]$.
On the other hand, by (2.41),

$$-\Phi(x') + \Phi(\bar{x}') + \nabla\Phi(\bar{x}')\cdot(x'-\bar{x}') \geqslant -\frac{\varepsilon r_{\bar{x}'}^2}{R} \geqslant -\frac{M\varepsilon r_{\bar{x}'}^2}{2R}$$

if $x' \in S^{(\bar{x}')}$, as long as M is big enough. Consequently, using that v lies above Γ and that $\bar{x}' \in \mathcal{T}$, we have that

$$\begin{aligned}
& \Gamma(x') - \Gamma(\bar{x}') - \nabla\Gamma(\bar{x}')\cdot(x'-\bar{x}') - \frac{M\varepsilon r_{\bar{x}'}^2}{2R} \\
\leqslant\; & \Gamma(x') - \Phi(x') - \Gamma(\bar{x}') + \Phi(\bar{x}') - \Big(\nabla\Gamma(\bar{x}') - \nabla\Phi(\bar{x}')\Big)\cdot(x'-\bar{x}') \\
\leqslant\; & v(x') - \Phi(x') - v(\bar{x}') + \Phi(\bar{x}') - \Big(\nabla\Gamma(\bar{x}') - \nabla\Phi(\bar{x}')\Big)\cdot(x'-\bar{x}') \\
=\; & u(x') - u(\bar{x}') - \nabla\mathcal{P}(\bar{x}')\cdot(x'-\bar{x}').
\end{aligned}$$

The latter estimate and (2.55) imply that

$$\frac{\left|S^{(\bar{x}')}\cap\left\{\Gamma(x')-\Gamma(\bar{x}')-\nabla\Gamma(\bar{x}')\cdot(x'-\bar{x}')>\frac{M\varepsilon r_{\bar{x}'}^2}{2R}\right\}\right|}{\left|S^{(\bar{x}')}\right|}\leqslant\frac{C}{M}.$$

So, by taking M appropriately large and using Lemma 8.4 of [5] we deduce that

$$\Gamma(x')-\Gamma(\bar{x}')-\nabla\Gamma(\bar{x}')\cdot(x'-\bar{x}')\leqslant\frac{C\varepsilon r_{\bar{x}'}^2}{R} \tag{2.56}$$

for any $|x'-\bar{x}'|<r_{\bar{x}'}/2$.

In particular, for any $|x'-\bar{x}'|<r_{\bar{x}'}/4$, we set $\rho:=r_{\bar{x}'}/4$, we plug the point $x'+\rho e$ inside (2.56), we use the convexity of Γ twice and we obtain

$$\begin{aligned}\frac{C\varepsilon\rho^2}{R} &\geqslant \Gamma(x'+\rho e)-\Gamma(\bar{x}')-\nabla\Gamma(\bar{x}')\cdot(x'+\rho e-\bar{x}')\\ &\geqslant \Gamma(x')+\rho\nabla\Gamma(x')\cdot e\\ &\qquad -\Gamma(\bar{x}')-\nabla\Gamma(\bar{x}')\cdot(x'+\rho e-\bar{x}')\\ &\geqslant \Gamma(\bar{x}')+\nabla\Gamma(\bar{x}')\cdot(x'-\bar{x}')\\ &\qquad +\rho\nabla\Gamma(x')\cdot e\\ &\qquad -\Gamma(\bar{x}')-\nabla\Gamma(\bar{x}')\cdot(x'+\rho e-\bar{x}')\\ &= \rho\big(\nabla\Gamma(x')-\nabla\Gamma(\bar{x}')\big)\cdot e.\end{aligned}$$

So, since e is an arbitrary unit vector, it follows that

$$|\nabla\Gamma(x')-\nabla\Gamma(\bar{x}')|\leqslant\frac{C\varepsilon r_{\bar{x}'}}{R}$$

for any $|x'-\bar{x}'|<r_{\bar{x}'}/4$, that is: the $(n-1)$-dimensional ball of radius $r_{\bar{x}'}/4$ centered at $\bar{x}'$ (which we now call $B^{(\bar{x}')}$) which is sent, via the map $\nabla\Gamma$, inside the $(n-1)$-dimensional ball of radius $C\varepsilon r_{\bar{x}'}/R$ centered at $\nabla\Gamma(\bar{x}')$ (we observe that the latter is a ball smaller by a scale factor $C\varepsilon/R$, and let us call $\widetilde{B}^{(\bar{x}')}$ such a ball).

Now we cover $\nabla\Gamma(\mathcal{T})$ with a countable, finite overlapping system of these balls, say $\{\widetilde{B}^{(j)}\}_{j\in\mathbb{N}}$. By the previous observations, these covering induces a countable, finite overlapping covering of $\mathcal{T}$, say $\{B^{(j)}\}_{j\in\mathbb{N}}$, with $|\widetilde{B}^{(j)}|\leqslant C(\varepsilon/R)^{n-1}|B^{(j)}|$. So, we obtain the measure estimate

$$|\nabla\Gamma(\mathcal{T})|\leqslant\sum_{j\in\mathbb{N}}|\widetilde{B}^{(j)}|\leqslant C\left(\frac{\varepsilon}{R}\right)^{n-1}\sum_{j\in\mathbb{N}}|B^{(j)}|. \tag{2.57}$$

On the other hand, we observe that, if $|x'-\bar{x}'|\leqslant r_{\bar{x}'}$, then

$$\begin{aligned}u(x') &\leqslant u(x')-\Gamma(x')\\ &\leqslant u(x')-\Gamma(\bar{x}')-\nabla\Gamma(\bar{x}')\cdot(x'-\bar{x}')\\ &= u(x')-u(\bar{x}')-\Phi(\bar{x}')-\big(\nabla\mathcal{P}(\bar{x}')+\nabla\Phi(\bar{x}')\big)\cdot(x'-\bar{x}')\\ &\leqslant u(x')-u(\bar{x}')-\Phi(x')-\nabla\mathcal{P}(\bar{x}')\cdot(x'-\bar{x}')+\frac{C\varepsilon}{R}|x'-\bar{x}'|^2\\ &\leqslant u(x')-u(\bar{x}')-\nabla\mathcal{P}(\bar{x}')\cdot(x'-\bar{x}')+C\varepsilon R\end{aligned}$$

thanks to the convexity of Γ and (2.51). Therefore

$$\begin{aligned} & S^{(\bar{x}')} \cap \left\{ u(x') - u(\bar{x}') - \nabla \mathcal{P}(\bar{x}') \cdot (x' - \bar{x}') \leqslant \frac{M\varepsilon r_{\bar{x}'}^2}{R} \right\} \\ \subseteq\ & S^{(\bar{x}')} \cap \{ u(x') \leqslant C\varepsilon R \} \\ \subseteq\ & B^{(\bar{x}')} \cap \{ u(x') \leqslant C\varepsilon R \}. \end{aligned} \tag{2.58}$$

Also, by (2.55)

$$\begin{aligned} & \left| S^{(\bar{x}')} \cap \left\{ u(x') - u(\bar{x}') - \nabla \mathcal{P}(\bar{x}') \cdot (x' - \bar{x}') \leqslant \frac{M\varepsilon r_{\bar{x}'}^2}{R} \right\} \right| \\ & \qquad \geqslant \left(1 - \frac{C}{M} \right) |S^{(\bar{x}')}| \geqslant \frac{|S^{(\bar{x}')}|}{2} \geqslant \frac{|B^{(\bar{x}')}|}{C}. \end{aligned}$$

This and (2.58) give that

$$|B^{(\bar{x}')}| \leqslant C |B^{(\bar{x}')} \cap \{ u(x') \leqslant C\varepsilon R \}|.$$

Gathering this estimate, (2.50) and (2.57), and using the finite overlapping property of $\{B^{(j)}\}_{j\in\mathbb{N}}$, we conclude that

$$\begin{aligned} \varepsilon^{n-1} & \leqslant C |\nabla \Gamma(\mathcal{T})| \leqslant C \left(\frac{\varepsilon}{R} \right)^{n-1} \sum_{j\in\mathbb{N}} |B^{(j)}| \\ & \leqslant C \left(\frac{\varepsilon}{R} \right)^{n-1} \sum_{j\in\mathbb{N}} \left| B^{(j)} \cap \{ u \leqslant C\varepsilon R \} \right| \leqslant C \left(\frac{\varepsilon}{R} \right)^{n-1} \left| \bigcup_{j\in\mathbb{N}} B^{(j)} \cap \{ u \leqslant C\varepsilon R \} \right|. \end{aligned} \tag{2.59}$$

Accordingly, (2.49) is a consequence of (2.59) and (2.54). □

2.6. **Uniform improvement of flatness.** The cornerstone of the regularity theory of [4] is Lemma 6.9 there, to wit a Harnack Inequality, according to which s-minimal surfaces become more and more flat when we get closer and closer to any of their points. However, the estimates in Lemma 6.9 of [4] are all uniform when s is bounded away from both 0 and 1, but they do degenerate as $s \to 1^-$ (see, in particular, the estimate on I_1 on page 1129 of [4]), therefore such result cannot be applied directly in our framework.

For this scope, we provide the following result, which is a version of Lemma 6.9 of [4] with uniform estimates as $s \to 1^-$. In fact, the reader may compare Lemma 12 here below with Lemma 6.9 in [4]: the only difference is that the estimates here are uniform as $s \to 1^-$.

Our proof is completely different from the one in [4] and it is based on the uniformity of the results obtained in the preceding sections, together with a Calderón–Zygmund iteration, which needs to distinguish between two scales of the dyadic cubes.

Lemma 12. *Fix $s_o \in (0,1)$ and $\alpha \in (0,1)$. Then, there exist $K \in \mathbb{N}$ and $d \in (0,1)$ which only depend on n, α and s_o, for which the following result holds.*

Let $a := 2^{-K\alpha}$. Let E be a set with s-minimal perimeter in $B_{2^{K+1}}$, with $s \in [1/10, 1)$. Assume that

$$\partial E \cap B_1 \subseteq \{ |x_n| \leqslant a \} \tag{2.60}$$

and, for any $i \in \{0, \dots, K\}$,

$$\partial E \cap B_{2^i} \subseteq \{|x \cdot \nu_i| \leqslant a 2^{i(1+\alpha)}\} \tag{2.61}$$

for some $\nu_i \in \mathrm{S}^{n-1}$. *Then*

$$\begin{aligned} &\textit{either } \partial E \cap B_d \subseteq \{x_n \leqslant a(1-d^2)\} \\ &\textit{or } \partial E \cap B_d \subseteq \{x_n \geqslant a(-1+d^2)\}. \end{aligned} \tag{2.62}$$

Proof. The proof is not simple, but the naive idea is to argue by contradiction, supposing that there is a sequence of E_j's that oscillate too much. Then one performs the following steps:

- By [6], one gets a sequence $s_j \to 1^-$ for which E_j approaches a classical minimal surface $E_\star$;
- By (7), one shadows E_j with level sets of distance functions $u_j^\pm$ from above and below, and the graphs of $u_j^\pm$ are close to $\partial E_\star$ as $s_j \to 1^-$;
- Since (by contradiction) we assumed E_j to oscillate too much, there are points of E_j (and so of the graphs of $u_j^\pm$) that stay very close to the bottom and the top of the cylinder of height a;
- Accordingly, from the fact that there is a point for which u_j^- is close to the bottom, we deduce that u_j^- is close to the bottom in a rather large set: for this, one needs to use a dyadic cube argument – when the cubes are reasonably big, one can repeat Lemma 11, and when the cubes get too small one takes advantage of the regularity theory for the classical minimal surface $E_\star$;
- Analogously, from the fact that there is a point for which u_j^+ is close to the top, we deduce that u_j^+ is close to the top in a rather large set;
- In particular, we find a point for which u_j^+ is close to the top and u_j^- close to the bottom, that is $u_j^+ - u_j^-$ is of the order of a;
- This is in contradiction with (2.15) and so it completes the proof.

We remark that, in these arguments, there are two uncorrelated scales involved. One is the flatness of order one (which, in the course of the proof, will be dominated by a configuration of cylinders whose ratio between the height and the base is some $\varepsilon^\star$); the other is the one induced by the criticality ratio for the minimal surfaces flatness condition (which is some universal ε_o). Of course, both these configurations are somewhat induced by the trapping of the surface in a strip of small size a. The interplay between these two scales is what allows us to choose the critical s in an independent way, and so to decouple the ratio of the scales involved. Finally , this implies also that as ε_o improves, we can apply the decrease of oscillation more and more times, so that in the vertical blow up limit we get a Hölder graph, that is harmonic in viscosity sense.
Below is the full detail discussion. The proof is by contradiction. If the claim were false, since the estimates of Lemma 6.9 of [4] are uniform when $s \geqslant 1/10$ is bounded away from 1, it follows that there exist

$$s_j \to 1^-, \tag{2.63}$$

and a sequence E_j of s_j-minimal surfaces in $B_{2^{K+1}}$ such that

$$\partial E_j \cap B_1 \subseteq \{|x_n| \leqslant a\} \tag{2.64}$$

and, for any $i \in \{0, \dots, K\}$,

$$\partial E_j \cap B_{2^i} \subseteq \{|x \cdot \nu_i| \leqslant a 2^{i(1+\alpha)}\}. \tag{2.65}$$

for suitable $\nu_i \in \mathrm{S}^{n-1}$, but

$$\partial E_j \cap B_d \cap \{x_n \geqslant a(1-d^2)\} \neq \varnothing \text{ and } \partial E_j \cap B_d \subseteq \{x_n \geqslant a(-1+d^2)\} \neq \varnothing. \tag{2.66}$$

Up to replacing E_j with its complement, we suppose that $E_j \cap B_1$ lies below its boundary in the nth direction. By (2.63) and Theorem 7 in [6], we have that χ_{E_j} converges in $L^1(B_{(9/7)2^K})$ to some $E_\star$ (possibly up to subsequence). Therefore (see the Remark after Corollary 17 in [6]) E_j approaches $E_\star$ uniformly in $B_{(8/7)2^K}$ and then, by Theorem 6 in [6], we have that $E_\star$ is a classical minimal surface in B_{2^K}.

We will define γ_j to be the distance between E_j and $E_\star$ in B_{2^K}: by construction

$$\lim_{j \to +\infty} \gamma_j = 0. \tag{2.67}$$

Let also

$$\delta_j := a \gamma_j^{1/(1+\alpha)},$$

and notice that

$$\lim_{j \to +\infty} \delta_j = 0. \tag{2.68}$$

Now, we observe that $K\alpha > 4(1+\alpha)$ if K is large enough, and so we can take $K' \in \mathbb{N}$ such that

$$\frac{K\alpha}{2(1+\alpha)} - 1 < K' \leqslant \frac{K\alpha}{2(1+\alpha)}. \tag{2.69}$$

Now, we denote by ε_o the flattening constants of the classical minimal surfaces (see, e.g., [3] and references therein) according to which if a minimal surface is trapped in a cylinder whose ratio between the height and the base is below ε_o, then the minimal surface is a $C^{1,\alpha}$-graph in half the cylinder. By (2.65), (2.69) and the uniform convergence of E_j, we see that, for large K (possibly in dependence of ε_o),

$$\begin{aligned} \partial E_\star \cap B_{2^{K'}} &\subseteq \{|x \cdot \nu_{K'}| \leqslant 2^{-K\alpha} 2^{K'(1+\alpha)}\} \\ &\subseteq \{|x \cdot \nu_{K'}| \leqslant 2^{-K\alpha/2}\} \subseteq \{|x \cdot \nu_{K'}| \leqslant \varepsilon_o\}, \end{aligned}$$

and so

$$\partial E_\star \cap B_{2^{K'-1}} \text{ is a } C^{1,\alpha}\text{-graph}. \tag{2.70}$$

Now, we use Corollary 7 with $\gamma := \gamma_j$ and $\delta := \delta_j$: for this, we define

$$\mathcal{S}_j^{\pm} := \{x \in \mathbb{R}^n \text{ s.t. } d_{E_j}(x) = \pm \delta_j\} \tag{2.71}$$

and we deduce from (2.70) and Corollary 7 that $\mathcal{S}_j^{\pm} \cap B_{2^{K'-2}}$ is

$$\text{the graph of a uniformly Lipschitz function, say } u_j^{\pm}. \tag{2.72}$$

Also, from (2.15), (2.67) and (2.68), we have that

$$u_j^+(x') - u_j^-(x') \leqslant C\delta_j \tag{2.73}$$

for any $|x'| \leqslant 1$, as long as j is large enough.

Now we will concentrate on u_j^- (the case of u_j^+ being specular): we set $E_j^- := \{x_n < u^-(x')\}$, so that $\partial E_j^- = \mathcal{S}_j^-$. From (2.66) and the fact that $\mathcal{S}_j^-$ lies below E_j, we obtain that there exists $\zeta' \in \mathbb{R}^{n-1}$ with

$$|\zeta'| \leqslant d \tag{2.74}$$

and

$$u_j^-(\zeta') \leqslant a(-1+d^2). \tag{2.75}$$

Now, we use the following notation: given any $x \in \mathcal{S}_j^-$, let $y(x) \in \partial E_j$ such that $|y(x) - x| = \delta_j$, and let $\nu(x) := y(x) - x$. Then

$$E_j^- + \nu(x) \subseteq \overline{E}. \tag{2.76}$$

Indeed, if $p \in E_j^- + \nu(x)$, we have that $p - \nu(x) \in E_j^-$ and so $\overline{B_{\delta_j}(p-\nu(x))} \subseteq \overline{E}_j$. Then, since $|\nu(x)| = \delta_j$, we have $p \in \overline{B_{\delta_j}(p-\nu(x))} \subseteq \overline{E}_j$, proving (2.76).
Moreover ∂E_j has zero Lebesgue measure (see, e.g., Corollary 4.4(i) of [4]), thus we infer from (2.76) that

$$\chi_{E_j^-+\nu(x_o)} \leqslant \chi_{E_j} \qquad \text{and} \qquad \chi_{\mathcal{C}(E_j^-+\nu(x_o))} \geqslant \chi_{\mathcal{C}E_j}. \tag{2.77}$$

Therefore, using (2.77), the Euler-Lagrange equation satisfied by E (see Theorem 5.1 of [4]) and the change of variable $z := x + \nu(x_o)$, we obtain

$$\begin{aligned} \int_{\mathbb{R}^n} \frac{\chi_{E_j^-}(x) - \chi_{\mathcal{C}(E_j^-)}(x)}{|x-x_o|^{n+s_j}}\,dx &= \int_{\mathbb{R}^n} \frac{\chi_{E_j^-+\nu(x_o)}(z) - \chi_{\mathcal{C}(E_j^-+\nu(x_o))}(z)}{|z-y(x_o)|^{n+s_j}}\,dz \\ &\leqslant \int_{\mathbb{R}^n} \frac{\chi_E(z) - \chi_{\mathcal{C}E}(z)}{|z-y(x_o)|^{n+s_j}}\,dz \leqslant 0 \end{aligned} \tag{2.78}$$

for any $x_o \in \partial E_j^- \cap B_C$. On the other hand, by (2.65), we have that $|x_o \cdot \nu_i| \leqslant Ca2^{i(1+\alpha)}$, and so

$$\begin{aligned} \partial E_j \cap B_{2^i}(x_o) &\subseteq \partial E_j \cap B_{2^{i+C}} \\ &\subseteq \{|x\cdot\nu_i| \leqslant Ca2^{i(1+\alpha)}\} \subseteq \{|(x-x_o)\cdot\nu_i| \leqslant Ca2^{i(1+\alpha)}\} \end{aligned}$$

for any $1 \leqslant i \leqslant K - C$. Therefore, for j large,

$$\partial E_j^- \cap B_{2^i}(x_o) \subseteq \{|(x-x_o)\cdot\nu_i| \leqslant Ca2^{i(1+\alpha)}\}$$

for any $1 \leqslant i \leqslant K - C$. As a consequence, we obtain the following cancellation:

(2.79)

$$\begin{aligned}
&\left| \int_{\mathcal{C}B_1(x_o)} \frac{\chi_{E_j^-}(x) - \chi_{\mathcal{C}(E_j^-)}(x)}{|x - x_o|^{n+s_j}} \, dx \right| \\
&\leqslant \sum_{i=1}^{K-C} \left| \int_{B_{2^i}(x_o) \setminus B_{2^{i-1}}(x_o)} \frac{\chi_{E_j^-}(x) - \chi_{\mathcal{C}(E_j^-)}(x)}{|x - x_o|^{n+s_j}} \, dx \right| + \left| \int_{\mathcal{C}B_{2^{K-C}}(x_o)} \frac{\chi_{E_j^-}(x) - \chi_{\mathcal{C}(E_j^-)}(x)}{|x - x_o|^{n+s_j}} \, dx \right| \\
&\leqslant C \left[\sum_{i=1}^{K-C} \int_{\substack{B_{2^i}(x_o) \setminus B_{2^{i-1}}(x_o) \\ \{|(x - x_o)\cdot \nu_i| \leqslant C a 2^{i(1+\alpha)}\}}} \frac{1}{|x - x_o|^{n+s_j}} \, dx + \int_{\mathcal{C}B_{2^{K-C}}(x_o)} \frac{1}{|x - x_o|^{n+s_j}} \, dx \right] \\
&\leqslant C \left[\sum_{i=1}^{K-C} \int_{2^{i-1}}^{2^i} \frac{a 2^{i(1+\alpha)} \rho^{n-2}}{\rho^{n+s_j}} \, d\rho + \int_{2^{K-C}}^{+\infty} \frac{\rho^{n-1}}{\rho^{n+s_j}} \, d\rho \right] \\
&\leqslant Ca
\end{aligned}$$

provided that j is big enough (in particular, s_j is larger than α).

Therefore, by (2.78) and (2.79), for any $x_o \in \partial E_j^- \cap B_C$,

$$\int_{B_1(x_o)} \frac{\chi_{E_j^-}(x) - \chi_{\mathcal{C}(E_j^-)}(x)}{|x - x_o|^{n+s_j}} \, dx \leqslant Ca. \tag{2.80}$$

With this, we are in position to obtain a finer bound in measure, often referred to with the name of "L^β-estimate" (see, e.g., Lemma 4.6 of [2] and Lemma 9.2 of [5] for the corresponding results for fully nonlinear or fractional operators, the proof of which is based on related, but quite different, techniques). Such estimate will be based on a Calderón–Zygmund type dyadic cube decomposition. According to the different scales involved, we use either a repeated version of Lemma 11 or the vicinity of the classical minimal surface $E_\star$ to deduce the necessary rigidity features.

Here are the details of such L^β-estimate. We take μ and M as in Lemma 11, and we fix a large integer k_o such that

$$(1 - \mu)^{k_o} \leqslant \frac{1}{4}. \tag{2.81}$$

Then, we choose

$$d := \frac{1}{2M^{k_o}}, \tag{2.82}$$

we set $a_j := a + \delta_j + \gamma_j$, and we claim that, for any $k \in \mathbb{N}$, with $1 \leqslant k \leqslant k_o$, we have that

$$\left| \left\{ u_j^- + a_j \geqslant \frac{a_j M^{k-k_o}}{2} \right\} \cap Q_1 \right| \leqslant (1 - \mu)^k \tag{2.83}$$

as long as j is large enough.

Indeed, when $k = 1$, (2.83) is a consequence of (2.49), by applying Lemma 11 here with $\varepsilon := da_j$, $\kappa := -a_j$ and $R := 1$ – for this recall (2.75), (2.80) and (2.82) in order to check (2.47) and (2.48), and consider the complement set in (2.49): such configuration is sketched in Figure 2.

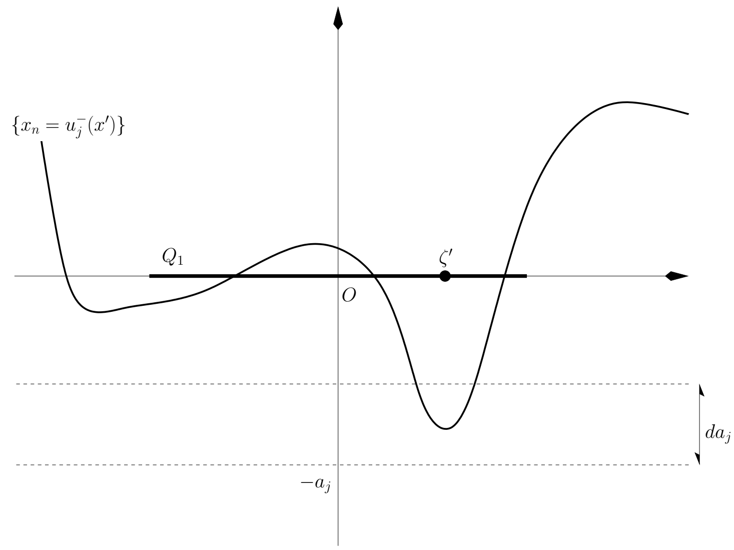

Figure 2: *Proving* (2.83) *when* $k = 1$.

Then, we proceed by induction, by supposing that (2.83) holds for $k-1$, and we prove it for $k \leqslant k_o$. For simplicity, we just perform the step from $k = 1$ to $k = 2$ (the others are analogous). For this, we define

$$A := \left\{ u_j^- + a_j > \frac{a_j M^{2-k_o}}{2} \right\} \cap Q_1 \quad \text{and} \quad B := \left\{ u_j^- + a_j > \frac{a_j M^{1-k_o}}{2} \right\} \cap Q_1.$$

Notice that

$$A \subseteq B \subseteq Q_1 \tag{2.84}$$

and

$$|A| \leqslant \left| \left\{ u_j^- + a_j > \frac{a_j M^{1-k_o}}{2} \right\} \cap Q_1 \right| \leqslant 1 - \mu, \tag{2.85}$$

since we know that (2.83) holds when $k = 1$.

Now we take a dyadic cube decomposition of Q_1, with the notation that if Q is one of the cubes of the family, its predecessor is denoted by $\widetilde{Q}$. We claim that

$$\text{if } |A \cap Q| > (1-\mu)|Q| \text{ then } \widetilde{Q} \subseteq B. \tag{2.86}$$

Notice that if (2.86) holds, then, by Lemma 4.2 of [2] (applied here with $\delta := 1 - \mu$) and the inductive assumption (that is, in this case, (2.83) with $k = 1$), we have that

$$\begin{aligned} &\left| \left\{ u_j^- + a_j > \frac{a_j M^{2-k_o}}{2} \right\} \cap Q_1 \right| = |A| \\ &\qquad \leqslant (1-\mu)|B| = (1-\mu) \left| \left\{ u_j^- + a_j > \frac{a_j M^{1-k_o}}{2} \right\} \cap Q_1 \right| \leqslant (1-\mu)^2. \end{aligned}$$

This would complete the induction necessary for the proof of (2.83), hence we focus on the proof of (2.86).
For the proof of (2.86), we argue by contradiction, by supposing that

$$|A \cap Q| > (1-\mu)|Q| \tag{2.87}$$

but there exists $\xi' \in \widetilde{Q} \setminus B$, i.e.

$$u_j^-(\xi') + a_j \leqslant \frac{a_j M^{1-k_o}}{2}. \tag{2.88}$$

We denote by ℓ the width of Q (which is, say, centered at some $x'_\star \in \mathbb{R}^{n-1}$). We need to distinguish two cases, according to the scale of the cube Q, namely, we distinguish whether or not $a_j/\ell \leqslant \varepsilon^\star$, using either Lemma 11 or the minimal surface rigidity (here $\varepsilon^\star$ is a small quantity, say the minimum between the threshold for the classical minimal surface regularity ε_o, as introduced after (2.69), and the small constants given by Lemma 11: a precise requirement about this will be taken after (2.90)).
If

$$a_j/\ell \leqslant \varepsilon^\star, \tag{2.89}$$

we use Lemma 11. For this scope, given $x_o \in \partial E_j^- \cap B_C$, we notice that

$$\left|\int_{B_1\setminus B_\ell(x_o)} \frac{\chi_{E_j^-}(x) - \chi_{\complement(E_j^-)}(x)}{|x-x_o|^{n+s_j}}\,dx\right| = \left|\int_{(B_1\setminus B_\ell(x_o))\cap\{|x_n|\leqslant Ca_j\}} \frac{\chi_{E_j^-}(x) - \chi_{\complement(E_j^-)}(x)}{|x-x_o|^{n+s_j}}\,dx\right|$$
$$\leqslant C\int_{(\complement B_\ell(x_o))\cap\{|x_n|\leqslant Ca_j\}} \frac{1}{|x'-x'_o|^{n+s_j}}\,dx \leqslant Ca_j \int_\ell^{+\infty} \frac{\rho^{n-2}}{\rho^{n+s}}\,d\rho \leqslant \frac{Ca_j}{\ell^{1+s}}.$$

As a consequence, recalling (2.80),

$$(1-s_j)\int_{B_\ell(x_o)} \frac{\chi_{E_j^-}(x) - \chi_{\complement(E_j^-)}(x)}{|x-x_o|^{n+s_j}}\,dx \leqslant \frac{C(1-s_j)a_j}{\ell^{1+s}}. \tag{2.90}$$

With this, we are in position to apply Lemma 11 with $\kappa := -a_j$, $R := \ell$ and $\varepsilon := a_j M^{1-k_o}/(2\ell)$ – notice indeed that (2.47) follows from (2.90), (2.48) follows from (2.88) and, recalling (2.89), we see that $\varepsilon \leqslant \varepsilon^\star M^{1-k_o}/2$ which is small if so is $\varepsilon^\star$: this configuration is represented in Figure 3.
So, we obtain from (2.49) that

$$\begin{aligned}|A\cap Q| &= \left|\Big\{u_j^- + a_j > \frac{a_j M^{2-k_o}}{2}\Big\}\cap Q\right|\\ &= \left|\Big\{u_j^- - \kappa > M\varepsilon R\Big\}\cap Q\right| \leqslant (1-\mu)|Q|,\end{aligned}$$

which is in contradiction with (2.87). This proves (2.86) if (2.89) holds true.
Now we deal with the case in which $a_j/\ell \geqslant \varepsilon^\star$, and we fix $\theta \in (0,1)$ to be chosen suitably small in the sequel. We set $p := a_j/(\theta^2\varepsilon^\star)$. Notice that, for small θ, we have that $p > 10a_j/\varepsilon^\star \geqslant 10\ell$. Also, the ratio between a_j and p is below $\theta^2\varepsilon^\star$, hence a minimal surface that is trapped inside $\{|x'| \leqslant p\} \times \{|x_n| \leqslant 8a_j\}$ is the graph of a function ω, with $|\nabla\omega| \leqslant \theta^{3/2}\varepsilon^\star$. Accordingly,

$$\begin{gathered}\text{the oscillation of } \omega \text{ in } \{|x'_i| \leqslant 6\ell\}\\ \text{is bounded by } \theta\varepsilon^\star\ell \leqslant \theta a_j.\end{gathered} \tag{2.91}$$

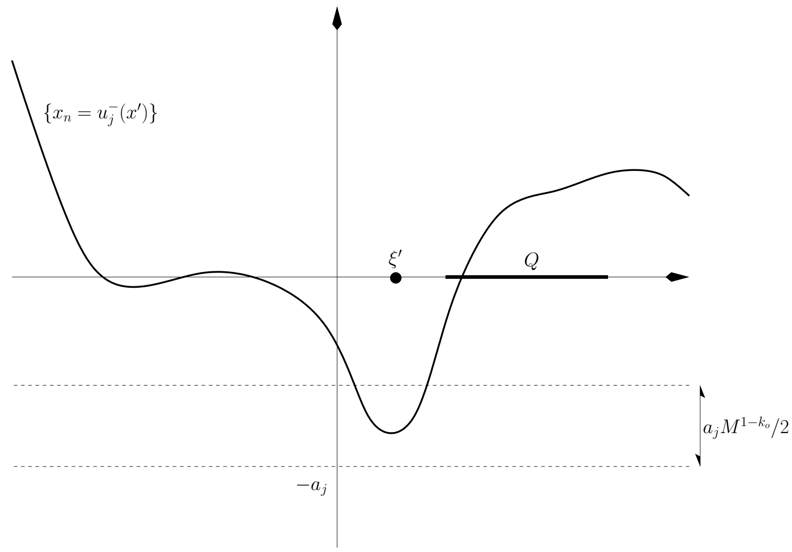

Figure 3: *Proving the inductive step of* (2.83) *when* $a_j/\ell \leqslant \varepsilon^\star$.

Keeping this in mind, we take j so large that γ_j, i.e. the distance between E_j and $E_\star$ is less than $\theta^2\varepsilon^\star p/2$ (recall (2.67)). Also, for large j, we have that the graph of u_j^- is at distance δ_j less than $\theta^3\varepsilon^\star p/2$ from E_j, and so less than $\theta^3\varepsilon^\star p$ from $E_\star$ (recall (2.68) and (2.71)).
Accordingly, $\partial E^\star \cap \{|x_i'| \leqslant 6\ell\}$ is trapped in a slab of width $4a_j + 2\theta^3\varepsilon^\star p < 8a_j$, and, by (2.88), its boundary contains a point with vertical entry below $(a_j M^{1-k_o}/2) + \theta^3\varepsilon^\star p$. Then, by (2.91), the whole of $\partial E^\star \cap \{|x_i'| \leqslant 4\ell\}$ has vertical entry below

$$-a_j + (a_j M^{1-k_o}/2) + \theta^3\varepsilon^\star p + \theta a_j.$$

Consequently, the graph of u^- on Q would stay below

$$\begin{aligned} & -a_j + (a_j M^{1-k_o}/2) + \theta^3\varepsilon^\star p + \theta a_j + \theta^3\varepsilon^\star p \\ & \quad = -a_j + (a_j M^{1-k_o}/2) + 3\theta a_j < -a_j + (a_j M^{2-k_o}/2), \end{aligned}$$

as long as we choose $\theta < M^{1-k_o}(M-1)/6$. Hence, $A \cap Q = \varnothing$, which is in contradiction with (2.87).
This ends the proof of (2.86), and therefore the one of (2.83).
As a consequence, by taking $k := k_o$ in (2.83) and recalling (2.81), we obtain that

$$\left| \left\{ u_j^- < -\frac{a_j}{2} \right\} \cap Q_1 \right| \geqslant \frac{3}{4} \tag{2.92}$$

for large j. A mirror argument on u_j^+ gives that

$$\left| \left\{ u_j^+ > \frac{a_j}{2} \right\} \cap Q_1 \right| \geqslant \frac{3}{4} \tag{2.93}$$

for large j. So, by (2.92) and (2.93), there must exist y_j' such that $u_j^-(y_j') \leqslant -a_j/2$ and $u_j^+(y_j') \geqslant a_j/2$, hence

$$u_j^+(y_j') - u_j^-(y_j') \geqslant a_j \geqslant a/2.$$

This is in contradiction with (2.73), and so the proof of Lemma 12 is completed. □

2.7. **Completion of the proof of Theorem 1.** Thanks to Lemma 12, we have obtained a statement analogous to the one of Lemma 6.9 of [4], but with uniform estimates. Then, the argument from Lemma 6.10 to the end of Section 6 in [4] also yield the proof of Theorem 1 here.

3. Proof of Theorem 2

The proof is by contradiction. We suppose that there are s_k-minimal cones E_k that are not hyperplanes, with $s_k \to 1^-$. By dimensional reduction (see Theorem 10.3 of [4]), we may focus on the case in which E_k is singular at the origin.

From [6], up to subsequence, we have that E_k approaches locally uniformly a classical cone of minimal perimeter. Since $n \leqslant 7$, we have that such a cone is a halfspace, say $\{x_n < 0\}$ (see, e.g., Section 1.5.2 of [8]). So, for large k, we have that (0.1) holds true for E_k, namely

$$\partial E_k \cap B_1 \subseteq \{|x \cdot e_n| \leqslant \varepsilon_\star\}.$$

Therefore, by Theorem 1, we obtain that ∂E_k is smooth, i.e. E_k is a hyperplane, for infinitely many k's. This is a contradiction with our assumptions and it proves Theorem 2.

4. Proof of Theorem 3

Let E be s-minimal. We take the blow up of E and we obtain a minimal cone E' (see Theorem 9.2 of [4]).

By Theorem 2, we know that E' is a hyperplane. Then, ∂E is $C^{1,\alpha}$, thanks to Theorem 9.4 in [4]. This ends the proof of Theorem 3.

5. Proof of Theorems 4 and 5

The proofs of Theorems 4 and 5 follow now verbatim the ones of Theorems 11.7 and 11.8 in [7] (the only difference is that the dimensional reduction is performed via Theorem 10.3 of [4], and the regularity needed in low dimension is assured here by Theorem 2).

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