Covariant Fields of C^*-Algebras and Continuity of Spectra in Rieffel's Pseudodifferential Calculus

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Abstract

We show that Rieffel's functor sends covariant $\mathcal{C}(T)$-algebras in $\mathcal{C}(T)$-algebras and covariant continuous fields of C^*-algebras in continuous fields of C^*-algebras. We use this to prove spectral continuity results for families of Rieffel-type pseudodifferential operators.

Introduction

Let T be a locally compact topological space, always assumed to be Hausdorff. We denote by $\mathcal{C}(T)$ the Abelian C^*-algebra of all complex continuous functions on T that are arbitrarily small outside large compact subsets. A $\mathcal{C}(T)$-algebra [6, 18, 27] is a C^*-algebra $\mathcal{B}$ together with a non-degenerated injective morphism from $\mathcal{C}(T)$ to the center of $\mathcal{B}$ (multipliers are used if $\mathcal{B}$ is not unital). The main role of the concept of $\mathcal{C}(T)$-algebra consists in codifying in a simple and efficient way the idea that $\mathcal{B}$ is fibered in the sense of C^*-algebras over the base T [9, 26]. Actually $\mathcal{C}(T)$-algebras can be seen as upper semi-continuous fields of C^*-algebras over the base T; lower semi-continuity can also be put in this setting if one also uses the space of all primitive ideals [14, 18, 21, 25, 27]. We intend to put these concepts in the perspective of Rieffel quantization.

Rieffel's calculus [22, 23] is a machine that transforms functorially "simpler" C^*-algebras and morphisms into more complicated ones. The ingredients to do this are an action of the vector group $\Xi := \mathbb{R}^d$ by automorphisms of the "simple" algebra as well as a skew symmetric linear operator of Ξ. When morphisms are involved, they are always assumed to intertwine the existing actions.

Rieffel's machine is actually meant to be a quantization. The initial data are naturally defining a Poisson structure, regarded as a mathematical modelization of the observables of a classical physical system. After applying the machine to this classical data one gets a C^*-algebra seen as the family of observables of the same system, but written in the language of Quantum Mechanics. By varying a convenient parameter (Planck's constant $\hbar$) one can recover the Poisson structure (at $\hbar = 0$) from the C^*-algebras defined at $\hbar \neq 0$ in a way that satisfies certain natural axioms [13, 22, 23].

In a setting where all these concepts make sense, we prove in Theorem 3.3 and Proposition 3.4 their compatibility: By Rieffel quantization an upper semi-continuous fields of C^*-algebras is turned into an upper semi-continuous fields of C^*-algebras with fibers which are easy to identify; the proof uses $\mathcal{C}(T)$-algebras. Finally, using primitive ideals techniques, we show the analog of this result for lower semi-continuity. Putting everything together one gets

Theorem 0.1. *Rieffel quantization transforms covariant continuous fields of C^*-algebras into covariant continuous fields of C^*-algebras.*

We illustrate the result by some examples in Sections 5 and 6. Most of them involve an Abelian initial algebra $\mathcal{A}$. In this case the information is encoded in a topological dynamical system with locally compact

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space Σ and the upper semi-continuous field property can be read in the existence of a continuous covariant surjection $q : \Sigma \to T$; if this one is open, then lower semi-continuity also holds. If the orbit space of the dynamical system is Hausdorff, it serves as a good space T over which the Rieffel deformed algebra can be decomposed, with easily identified fibers. This can be used to show that the C^*-algebras of some compact quantum groups constructed in [24] can be written as continuous fields, some of the fiber being isomorphic to certain non-commutative tori.

The spirit of this quantization procedure is that of a pseudodifferential theory [10]. At least in simple situations the multiplication in the initial C^*-algebra is just point-wise multiplication of functions defined on some locally compact topological space, on which Ξ acts by homeomorphisms. The non-commutative product in the quantized algebra can be interpreted as a symbol composition of a pseudodifferential type. Actually the concrete formulae generalize and are motivated by the usual Weyl calculus.

This fact leads naturally to expect that topics or tools coming from the standard pseudodifferential theory could make sense and even work in this more general framework. In [16], some C^*-algebraic techniques of spectral analysis ([3, 4, 11, 15, 17] and references therein) were tuned with Rieffel quantization, getting results on spectra and essential spectra of certain self-adjoint operators that seemed to be out of reach by other methods. In the present article we continue the project by studying spectral continuity.

Roughly, the problem can be stated as follows: For each point t of the locally compact space T we are given a self-adjoint element (a classical observable) $f(t)$ of a C^*-algebra $\mathcal{A}(t)$, which is Abelian for most of the applications, and we assume some simple-minded continuity property in the variable t for this family. By quantization, $f(t)$ is turned into a quantum observable $\mathfrak{f}(t)$ belonging to a new, non-commutative C^*-algebra $\mathfrak{A}(t)$. We inquire if the family $S(t) := \mathrm{sp}\,[\mathfrak{f}(t)]$ of spectra computed in these new algebras vary continuously with t. Intuitively, outer continuity says that the family cannot suddenly expand: if for some t_0 there is a gap in the spectrum of $f(t_0)$ around a point $\lambda_0 \in \mathbb{R}$, then for t close to t_0 all the spectra $S(t)$ will have gaps around λ_0. On the other hand, inner continuity insures that if $f(t_0)$ has some spectrum in a non-trivial interval of $\mathbb{R}$, this interval will contain spectral points of all the elements $f(t)$ for t close to t_0. Although traditionally $\mathfrak{A}(t)$ is thought to be a C^*-algebra of bounded operators in some Hilbert space, the abstract situation is both natural and fruitful. One can work with abstract C^*-algebras $\mathfrak{A}(t)$ and then, if necessarily, they are represented faithfully in Hilbert spaces; the spectrum will be preserved under representation.

It comes out that such spectral continuity can be obtained from corresponding continuity properties of resolvent families of the elements $\mathfrak{f}(t)$ and this involves both inversion and norm in each complicated C^*-algebra $\mathfrak{A}(t)$. Things are smoothed out if the family $\{\mathfrak{A}(t) \mid t \in T\}$ has a priori continuity properties, that may be connected to concepts as $\mathcal{C}(T)$-algebra or (upper or lower semi)-continuous C^*-bundles. Usually such properties are more or less obvious for the initial family $\{\mathcal{A}(t) \mid t \in T\}$ and we hope to propagate them by the quantization mapping $\mathcal{A}(t) \mapsto \mathfrak{A}(t)$. We are going to investigate what happens when this mapping is Rieffel's quantization. Pioneering work on applying C^*-algebraic techniques to spectral continuity problems and applications to discrete physical systems may be found in [3, 5, 8]. Results on continuity of spectra for unbounded Schrödinger-like Hamiltonians (especially with magnetic fields) appear in [1, 2, 12, 19] and references therein. For our situation, which has a rather small overlap with these references, we also include an outer continuity result for *essential spectra* of Rieffel pseudodifferential operators. Continuity in Planck's constant $\hbar$, treated in [22] and in [16], is a very special case.

The full strength of these spectral techniques would require an extension of Rieffel's calculus to suitable families of unbounded elements. Hopefully this will be achieved in the future, and this would be the right opportunity to present detailed examples.

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1 Rieffel's pseudodifferential calculus; a short review

We start by describing briefly Rieffel quantization [22, 23]. The initial object, containing *the classical data*, is a quadruplet $(\mathcal{A}, \Theta, \Xi, [\![\cdot,\cdot]\!])$. The pair $(\Xi, [\![\cdot,\cdot]\!])$ will usually be taken to be a $2n$-dimensional symplectic vector space, but the skew-symmetric bilinear form $[\![\cdot,\cdot]\!]$ may be degenerate in most situations. On the other hand $(\mathcal{A}, \Theta, \Xi)$ is a C^*-*dynamical system*, meaning that the vector group acts strongly continuously by automorphisms of the (maybe non-commutative) C^*-algebra $\mathcal{A}$. Let us denote by $\mathcal{A}^\infty$ the family of elements f such that the mapping $\Xi \ni X \mapsto \Theta_X(f) \in \mathcal{A}$ is C^∞. It is a dense *-algebra of $\mathcal{A}$ and also a Fréchet algebra with the family of semi-norms

$$\| f \|_{\mathcal{A}}^{(k)} := \sum_{|\alpha|\le k} \frac{1}{|\alpha|!} \| \partial_X^\alpha [\Theta_X(f)]_{X=0} \|_{\mathcal{A}} \equiv \sum_{|\alpha|\le k} \frac{1}{|\alpha|!} \| \delta^\alpha(f) \|_{\mathcal{A}}, \quad k \in \mathbb{N}. \tag{1.1}$$

To quantize the above structure, one keeps the involution unchanged but introduce on $\mathcal{A}^\infty$ the product

$$f \# g := \pi^{-2n} \int_\Xi \int_\Xi dY dZ\, e^{2i[\![Y,Z]\!]}\, \Theta_Y(f)\, \Theta_Z(g), \tag{1.2}$$

suitably defined by oscillatory integral techniques. One gets a *-algebra $(\mathcal{A}^\infty, \#, ^*)$, which admits a C^*-completion $\mathfrak{A}$ in a C^*-norm $\|\cdot\|_{\mathfrak{A}}$ defined by Hilbert module techniques [22]. The action Θ leaves $\mathcal{A}^\infty$ invariant and extends to a strongly continuous action of the C^*-algebra $\mathfrak{A}$, that will also be denoted by Θ. The space $\mathfrak{A}^\infty$ of C^∞-vectors coincide with $\mathcal{A}^\infty$ and it is a Fréchet space with the family of semi-norms

$$\| f \|_{\mathfrak{A}}^{(k)} := \sum_{|\alpha|\le k} \frac{1}{|\alpha|!} \| \partial_X^\alpha [\Theta_X(f)]_{X=0} \|_{\mathfrak{A}} \equiv \sum_{|\alpha|\le k} \frac{1}{|\alpha|!} \| \delta^\alpha(f) \|_{\mathfrak{A}}, \quad k \in \mathbb{N}. \tag{1.3}$$

By Proposition 4.10 in [22], there exist $k \in \mathbb{N}$ and $C_k > 0$ such that

$$\| f \|_{\mathfrak{A}} \le C_k \| f \|_{\mathcal{A}}^{(k)}, \quad \forall f \in \mathcal{A}^\infty = \mathfrak{A}^\infty.$$

Replacing here f by $\delta^\alpha f$ for every multi-index α, it follows that on $\mathcal{A}^\infty$ the topology given by the semi-norms (1.1) is finer than the one given by the semi-norms (1.3). As a consequence of Theorem 7.5 in [22], the role of the C^*-algebras $\mathcal{A}$ and $\mathfrak{A}$ can be reversed: one obtains $\mathcal{A}$ as the quantization of $\mathfrak{A}$ by replacing the skew-symmetric form $[\![\cdot,\cdot]\!]$ by $-[\![\cdot,\cdot]\!]$. Thus $\mathcal{A}^\infty$ *and* $\mathfrak{A}^\infty$ *coincide as Fréchet spaces*.

The quantization transfers to Ξ-morphisms. Let $(\mathcal{A}_j, \Theta_j, \Xi, [\![\cdot,\cdot]\!])$, $j = 1, 2$, be two classical data and let $\mathcal{R} : \mathcal{A}_1 \to \mathcal{A}_2$ be a Ξ-morphism, *i.e.* a (C^*-)morphism intertwining the two actions Θ_1, Θ_2. Then $\mathcal{R}$ sends $\mathcal{A}_1^\infty$ into $\mathcal{A}_2^\infty$ and extends to a morphism $\mathfrak{R} : \mathfrak{A}_1 \to \mathfrak{A}_2$ that also intertwines the corresponding actions. In this way, one obtains a covariant functor. The functor is exact: it preserves short exact sequences of Ξ-morphisms. Namely, if $\mathcal{J}$ is a (closed, self-adjoint, two-sided) ideal in $\mathcal{A}$ that is invariant under Θ, then its quantization $\mathfrak{J}$ can be identified with an invariant ideal in $\mathfrak{A}$ and the quotient $\mathfrak{A}/\mathfrak{J}$ is canonically isomorphic to the quantization of the quotient $\mathcal{A}/\mathcal{J}$ under the natural quotient action.

We will refer to *the Abelian case* under the following circumstances: A continuous action Θ of Ξ by homeomorphisms of the locally compact Hausdorff space Σ is given. For $(\sigma, X) \in \Sigma \times \Xi$ we are going to use all the notations

$$\Theta(\sigma, X) = \Theta_X(\sigma) = \Theta_\sigma(X) \in \Sigma \tag{1.4}$$

for the X-transformed of the point σ. The function Θ is continuous and the homeomorphisms Θ_X, Θ_Y satisfy $\Theta_X \circ \Theta_Y = \Theta_{X+Y}$ for every $X, Y \in \Xi$.

We denote by $\mathcal{C}(\Sigma)$ the Abelian C^*-algebra of all complex continuous functions on Σ that are arbitrarily small outside large compact subsets of Σ. When Σ is compact, $\mathcal{C}(\Sigma)$ is unital. The action Θ of Ξ on Σ

induces an action of Ξ on $\mathcal{C}(\Sigma)$ (also denoted by Θ) given by $\Theta_X(f) := f \circ \Theta_X$. This action is strongly continuous, *i.e.* for any $f \in \mathcal{C}(\Sigma)$ the mapping

$$\Xi \ni X \mapsto \Theta_X(f) \in \mathcal{C}(\Sigma) \tag{1.5}$$

is continuous; thus we are placed in the setting presented above. We denote by $\mathcal{C}(\Sigma)^\infty \equiv \mathcal{C}^\infty(\Sigma)$ the set of elements $f \in \mathcal{C}(\Sigma)$ such that the mapping (1.5) is C^∞; it is a dense *-algebra of $\mathcal{C}(\Sigma)$. The general theory supplies a non-commutative C^*-algebra $\mathfrak{C}(\Sigma)$ acted continuously by the group Ξ, with smooth vectors $\mathfrak{C}^\infty(\Sigma) = \mathcal{C}^\infty(\Sigma)$.

2 Families of C^*-algebras

Now we give a short review of $\mathcal{C}(T)$-algebras and semi-continuous fields of C^*-algebras (see [6, 13, 14, 18, 21, 27] and references therein), outlining the connection between the two notions.

If $\mathcal{B}$ is a C^*-algebra, we denote by $\mathcal{M}(\mathcal{B})$ its multiplier algebra and by $\mathcal{ZM}(\mathcal{B})$ its center. If $\mathcal{B}_1, \mathcal{B}_2$ are two vector subspaces of $\mathcal{M}(\mathcal{B})$, we set $\mathcal{B}_1 \cdot \mathcal{B}_2$ for the vector subspace generated by $\{b_1 b_2 \mid b_1 \in \mathcal{B}_1, b_2 \in \mathcal{B}_2\}$. We are going to denote by $\mathcal{C}(T)$ the C^*-algebra of all complex continuous functions on the (Hausdorff) locally compact space T that decay at infinity.

Definition 2.1. *We say that $\mathcal{B}$ is a $\mathcal{C}(T)$-*algebra *if a non-degenerate monomorphism $\mathcal{Q} : \mathcal{C}(T) \to \mathcal{ZM}(\mathcal{B})$ is given.*

We recall that non-degeneracy means that the ideal $\mathcal{Q}[\mathcal{C}(T)] \cdot \mathcal{B}$ is dense in $\mathcal{B}$.

Definition 2.2. *By* upper semi-continuous field of C^*-algebras *we mean a family of epimorphisms of C^*-algebras $\left\{\mathcal{B} \xrightarrow{\mathcal{P}(t)} \mathcal{B}(t) \mid t \in T\right\}$ indexed by the locally compact topological space T and satisfying:*

1. *For every $b \in \mathcal{B}$ one has $\| b \|_{\mathcal{B}} = \sup_{t\in T} \| \mathcal{P}(t)b \|_{\mathcal{B}(t)}$.*
2. *For every $b \in \mathcal{B}$ the map $T \ni t \mapsto \| \mathcal{P}(t)b \|_{\mathcal{B}(t)}$ is upper semi-continuous and decays at infinity.*
3. *There is a multiplication $\mathcal{C}(T) \times \mathcal{B} \ni (\varphi, b) \to \varphi * b \in \mathcal{B}$ such that*

$$\mathcal{P}(t)[\varphi * b] = \varphi(t)\, \mathcal{P}(t)b , \quad \forall\, t \in T \,,\ \varphi \in \mathcal{C}(T) \,,\ b \in \mathcal{B} .$$

If, in addition, the map $t \mapsto \| \mathcal{P}(t)b \|$ is continuous for every $b \in \mathcal{B}$, we say that $\left\{\mathcal{B} \xrightarrow{\mathcal{P}(t)} \mathcal{B}(t) \mid t \in T\right\}$ is a continuous field of C^*-algebras.

The requirement 2 is clearly equivalent with the condition that for every $b \in \mathcal{B}$ and every $\epsilon > 0$ the subset $\{t \in T \mid \| \mathcal{P}(t)b \|_{\mathcal{B}(t)} \geq \epsilon\}$ is compact. One can rephrase 1 as $\cap_t \ker[\mathcal{P}(t)] = \{0\}$, so one can identify $\mathcal{B}$ with a C^*-algebra of sections of the field; this make the connection with other approaches, as that of [18] for example. It will always be assumed that $\mathcal{B}(t) \neq \{0\}$ for all $t \in T$.

We are going to describe briefly in which way the two definitions above are actually equivalent.

First let us assume that $\mathcal{B}$ is a $\mathcal{C}(T)$-algebra and denote by $\mathcal{C}_t(T)$ the ideal of all the functions in $\mathcal{C}(T)$ vanishing at the point $t \in T$. We get ideals $\mathcal{I}(t) := \overline{\mathcal{Q}\,[\mathcal{C}_t(T)] \cdot \mathcal{B}}$ in $\mathcal{B}$, quotients $\mathcal{B}(t) := \mathcal{B}/\mathcal{I}(t)$ as well as canonical epimorphisms $\mathcal{P}(t) : \mathcal{B} \to \mathcal{B}(t)$. One also sets

$$\varphi * b := \mathcal{Q}(\varphi)b , \quad \forall\, \varphi \in \mathcal{C}(T),\ b \in \mathcal{B} . \tag{2.1}$$

Then $\left\{\mathcal{B} \xrightarrow{\mathcal{P}(t)} \mathcal{B}(t) \mid t \in T\right\}$ is an upper semi-continuous field of C^*-algebras with multiplication $*$.

Conversely, if an upper semi-continuous field $\left\{\mathcal{B} \xrightarrow{\mathcal{P}(t)} \mathcal{B}(t) \mid t \in T\right\}$ is given, also involving the multiplication $*$, we set

$$\mathcal{Q} : \mathcal{C}(T) \to \mathcal{ZM}(\mathcal{B}), \quad \mathcal{Q}(\varphi)b := \varphi * b. \tag{2.2}$$

In this way one gets a $\mathcal{C}(T)$-algebra and each of the quotients $\mathcal{B}/\mathcal{I}(t)$ is isomorphic to the fiber $\mathcal{B}(t)$.

To discuss lower semi-continuity we need $\mathrm{Prim}(\mathcal{B})$, the space of all the primitive ideals (kernels of irreducible representations) of $\mathcal{B}$. The hull-kernel topology turns $\mathrm{Prim}(\mathcal{B})$ into a locally compact (non necessarily Hausdorff) topological space. We recall that *the hull application* $\mathcal{J} \mapsto h(\mathcal{J}) := \{\mathcal{K} \in \mathrm{Prim}(\mathcal{B}) \mid \mathcal{J} \subset \mathcal{K}\}$ realizes a decreasing bijection between the family of ideals of $\mathcal{B}$ and the family of closed subsets of $\mathrm{Prim}(\mathcal{B})$. Its inverse is *the kernel map* $\Omega \mapsto k(\Omega) := \cap_{\mathcal{K}\in\Omega}\mathcal{K}$, which is also decreasing.

The Dauns-Hofmann Theorem establishes the existence of a unique isomorphism $\Gamma : BC[\mathrm{Prim}(\mathcal{B})] \to \mathcal{ZM}(\mathcal{B})$, where $BC[\mathrm{Prim}(\mathcal{B})]$ is the C^*-algebra of bounded and continuous functions over $\mathrm{Prim}(\mathcal{B})$, such that for each $\mathcal{K} \in \mathrm{Prim}(\mathcal{B})$, $\Psi \in BC[\mathrm{Prim}(\mathcal{B})]$ and $b \in \mathcal{B}$ we have $\Gamma(\Psi)b + \mathcal{K} = \Psi(\mathcal{K})b + \mathcal{K}$. For a detailed study of the space $\mathrm{Prim}(\mathcal{B})$ and a self-contained proof of the Dauns-Hofmann Theorem, cf. sections A.2 and A.3 in [25]. Let us suppose that there is a continuous function $q : \mathrm{Prim}(\mathcal{B}) \to T$ with dense image. Then we can define $\mathcal{Q} : \mathcal{C}(T) \to \mathcal{ZM}(\mathcal{B})$ by $\mathcal{Q}(\varphi) = \Gamma(\varphi \circ q)$ and one can check that $\mathcal{Q}$ endows $\mathcal{B}$ with the structure of a $\mathcal{C}(T)$-algebra.

On the other hand, if we have a non-degenerate monomorphism $\mathcal{Q} : \mathcal{C}(T) \to \mathcal{ZM}(\mathcal{B})$, we can define canonically a continuous map $q : \mathrm{Prim}(\mathcal{B}) \to T$. One has $q(\mathcal{K}) = t$ if and only if $\mathcal{I}(t) \subset \mathcal{K}$, and we can recover $\mathcal{Q}$ from the above construction. Moreover *the map* $T \ni t \to \| b(t) \|_{\mathcal{B}(t)} \in \mathbb{R}_+$ *is continuous for every* $b \in \mathcal{B}$ *(so we have a continuous field of* C^**-algebras) if and only if* q *is open*. For the proof of this facts see propositions C.5 and C.10 in [27].

3 Covariant $\mathcal{C}(T)$-algebras and upper semi-continuity under Rieffel quantization

Let T be a locally compact Hausdorff space and $(\mathcal{A}, \Theta, \Xi, [\![\cdot,\cdot]\!])$ a classical data. The canonical C^*-dynamical system defined by Rieffel quantization is $(\mathfrak{A}, \Theta, \Xi)$.

Definition 3.1. *We say that* $\mathcal{A}$ *is a* covariant $\mathcal{C}(T)$-algebra *with respect to the action* Θ *if a non-degenerate monomorphism* $\mathcal{Q} : \mathcal{C}(T) \to \mathcal{ZM}(\mathcal{A})$ *is given (so it is a* $\mathcal{C}(T)$*-algebra) and in addition one has*

$$\Theta_X[\mathcal{Q}(\varphi)f] = \mathcal{Q}(\varphi)\,[\Theta_X(f)], \quad \forall\, f \in \mathcal{A},\ X \in \Xi,\ \varphi \in \mathcal{C}(T). \tag{3.1}$$

We intend to prove that the Rieffel quantization transforms covariant $\mathcal{C}(T)$-algebras into covariant $\mathcal{C}(T)$-algebras. For this and for a further result identifying the emerging quotient algebras, we are going to need

Lemma 3.2. *Let* I *be an ideal of* $\mathcal{C}(T)$ *and denote by* $\overline{\mathcal{Q}(I) \cdot \mathcal{A}}$ *the closure of* $\mathcal{Q}(I) \cdot \mathcal{A}$ *in the* C^**-algebra* $\mathcal{A}$. *Then* $\mathcal{Q}(I) \cdot \mathcal{A}^\infty$ *is dense in* $\left(\overline{\mathcal{Q}(I) \cdot \mathcal{A}}\right)^\infty \equiv \left(\overline{\mathcal{Q}(I) \cdot \mathcal{A}}\right) \cap \mathcal{A}^\infty$ *for the Fréchet topology inherited from* $\mathcal{A}^\infty$.

Proof. By the covariance condition $\overline{\mathcal{Q}(I) \cdot \mathcal{A}}$ is an invariant ideal of $\mathcal{A}$.

The proof uses regularization. Consider *the integrated form of* Θ, i.e. for each $\Phi \in C_c^\infty(\Xi)$ (compactly supported smooth function) and $g \in \mathcal{A}$ define

$$\Theta_\Phi(g) = \int_\Xi dY\, \Phi(Y)\Theta_Y(g).$$

Note that for every $X \in \Xi$ one has

$$\Theta_X\left[\Theta_\Phi(g)\right] = \int_\Xi dY\, \Phi(Y - X)\Theta_Y(g).$$

Then $\Theta_\Phi(g) \in \mathcal{A}^\infty$ and for each multi-index μ we have

$$\delta^\mu \left[\Theta_\Phi(g)\right] = (-1)^{|\mu|}\Theta_{\partial^\mu\Phi}(g) \quad \text{and} \quad \| \delta^\mu \left[\Theta_\Phi(g)\right] \|_{\mathcal{A}} \leq \| \partial^\mu \Phi \|_{L^1(\Xi)} \| g \|_{\mathcal{A}} \ .$$

One of the deepest theorems about smooth algebras, the Dixmier-Malliavin Theorem [7], say that $\mathcal{A}^\infty$ is generated (algebraically) by the set of all the elements of the form $\Theta_\Phi(g)$ with $\Phi \in C_c^\infty(\Xi)$ and $g \in \mathcal{A}$. Replacing $\mathcal{A}$ with $\overline{\mathcal{Q}(I)\cdot\mathcal{A}}$, for $f \in \left(\overline{\mathcal{Q}(I)\cdot\mathcal{A}}\right)^\infty$ there exist $\Phi_1,\dots,\Phi_m \in C_c^\infty(\Xi)$ and $f_1,\dots,f_m \in \overline{\mathcal{Q}(I)\cdot\mathcal{A}}$ such that $f = \sum_{i=1}^m \Theta_{\Phi_i}(f_i)$. Let $\epsilon > 0$ and fix a multi-index α. Choose $g_1,\dots,g_m \in \mathcal{Q}(I)\cdot\mathcal{A}$ such that for each i

$$\| f_i - g_i \|_{\mathcal{A}} \leq \frac{\epsilon}{m \| \partial^\alpha \Phi_i \|_{L^1(\Xi)}} \,.$$

Then

$$\left\| \delta^\alpha \left(f - \sum_{i=1}^m \Theta_{\Phi_i}(g_i) \right) \right\|_{\mathcal{A}} = \left\| \sum_{i=1}^m \Theta_{\partial^\alpha\Phi_i}(f_i - g_i) \right\|_{\mathcal{A}} \leq \sum_{i=1}^m \| \partial^\alpha \Phi_i \|_{L^1(\Xi)} \| f_i - g_i \|_{\mathcal{A}} \leq \epsilon \,.$$

Thus we only need to prove that for each $\Phi \in C_c^\infty(\Xi)$ and $g \in \mathcal{Q}(I)\cdot\mathcal{A}$ the element $\Theta_\Phi(g)$ belongs to $\mathcal{Q}(I)\cdot\mathcal{A}^\infty$. Let $\varphi_1,\dots,\varphi_j \in I$ and $h_1,\dots,h_j \in \mathcal{A}$ such that $g = \sum_{i=1}^j \mathcal{Q}(\varphi_i)h_i$. Then

$$\Theta_\Phi(g) = \sum_{i=1}^j \Theta_\Phi\left[\mathcal{Q}(\varphi_i)h_i\right] \,,$$

and by covariance, for each index i one has

$$\Theta_\Phi\left[\mathcal{Q}(\varphi_i)h_i\right] = \int_\Xi dY \Phi(Y) \mathcal{Q}(\varphi_i)\Theta_X(h_i) = \mathcal{Q}(\varphi_i)\left[\Theta_\Phi(h_i)\right] \in \mathcal{Q}(I)\cdot\mathcal{A}^\infty \,.$$

□

Theorem 3.3. *Rieffel quantization transforms covariant $\mathcal{C}(T)$-algebras into covariant $\mathcal{C}(T)$-algebras: there exists a non-degenerate monomorphism $\mathfrak{Q} : \mathcal{C}(T) \to \mathcal{Z}\mathcal{M}(\mathfrak{A})$ satisfying for all $\varphi \in \mathcal{C}(T)$, $f \in \mathcal{A}$ and $X \in \Xi$ the covariance relation $\Theta_X[\mathfrak{Q}(\varphi)f] = \mathfrak{Q}(\varphi)\left[\Theta_X(f)\right]$.*

Proof. The action Θ of Ξ on $\mathcal{A}$ extends canonically to an action by automorphisms of the multiplier algebra $\mathcal{M}(\mathcal{A})$, also denoted by Θ, which is not strongly continuous in general. But, tautologically, it restricts to a strongly continuous action $\Theta : \Xi \to \mathrm{Aut}[\mathcal{M}_0(\mathcal{A})]$ on the C^*-subalgebra

$$\mathcal{M}_0(\mathcal{A}) := \{ m \in \mathcal{M}(\mathcal{A}) \mid \Xi \ni X \mapsto \Theta_X(m) \in \mathcal{M}(\mathcal{A}) \text{ is norm continuous} \} \,. \tag{3.2}$$

In these terms, the covariance condition on $\mathcal{Q}$ says simply that for any $\varphi \in \mathcal{C}(T)$ the multiplier $\mathcal{Q}(\varphi)$ is a fixed point for all the automorphisms Θ_X. As a very weak consequence one has $\mathcal{Q}[\mathcal{C}(T)] \subset \mathcal{M}_0(\mathcal{A})^\infty$, with an obvious notation for the smooth vectors.

Proposition 5.10 from [22] applied to the unital C^*-algebra $\mathcal{M}_0(\mathcal{A})$ says that the Rieffel quantization of $\mathcal{M}_0(\mathcal{A})$ is a C^*-subalgebra of $\mathcal{M}(\mathfrak{A})$. Consequently one has $\mathcal{Q}[\mathcal{C}(T)] \subset \mathcal{M}_0(\mathcal{A})^\infty \subset \mathcal{M}(\mathfrak{A})$ and this supplies a candidate $\mathfrak{Q} : \mathcal{C}(T) \to \mathcal{M}(\mathfrak{A})$. This is obviously an injective map and the range is only composed of fixed points, which insures covariance.

Let us set for a moment $\mathcal{M} := \mathcal{M}_0(\mathcal{A})$, with multiplication $\cdot$, and denote by $\mathfrak{M} \subset \mathcal{M}(\mathfrak{A})$ its Rieffel quantization, with multiplication legitimately denoted by $\#$. For smooth elements $m, n \in \mathcal{M}^\infty = \mathfrak{M}^\infty$, one of them being a fixed point central in $\mathcal{M}$, one has $m\#n = m\cdot n = n\cdot m = n\#m$ (Corollary 2.13 in [22]). Thus the mapping $\mathfrak{Q}$ is once again a monomorphism and its range is contained in $\mathcal{Z}\mathfrak{M}$. A density argument with respect to the strict topology implies that $\mathfrak{Q}[\mathcal{C}(T)] \subset \mathcal{Z}\mathcal{M}(\mathfrak{A})$.

Now we only need to show non-degeneracy, i.e. the fact that $\mathfrak{Q}[\mathcal{C}(T)]\cdot\mathfrak{A}$ is dense in $\mathfrak{A}$. We show the even stronger assertion that $\mathcal{Q}[\mathcal{C}(T)]\cdot\mathcal{A}^\infty=\mathfrak{Q}[\mathcal{C}(T)]\cdot\mathfrak{A}^\infty$ is dense in $\mathfrak{A}$. This would follow if we knew that $\mathcal{Q}[\mathcal{C}(T)]\cdot\mathcal{A}^\infty$ is dense in $\mathfrak{A}^\infty$ with respect to its Fréchet topology given by the semi-norms (1.3); then we use denseness of $\mathfrak{A}^\infty$ in the weaker C^*-norm topology of $\mathfrak{A}$.

We recall from section 1 that $\mathcal{A}^\infty$ and $\mathfrak{A}^\infty$ coincide even as Fréchet spaces. Therefore one is reduced to showing that $\mathcal{Q}[\mathcal{C}(T)]\cdot\mathcal{A}^\infty$ is dense in $\mathcal{A}^\infty$ for its Fréchet topology. Taking $\mathcal{I}=\mathcal{C}(T)$ in Lemma 3.2, we find out that $\mathcal{Q}[\mathcal{C}(T)]\cdot\mathcal{A}^\infty$ is dense in $\left(\overline{\mathcal{Q}[\mathcal{C}(T)]\cdot\mathcal{A}}\right)\cap\mathcal{A}^\infty$, which equals $\mathcal{A}^\infty$ since $\mathcal{Q}$ has been assumed non-degenerate. This finishes the proof. □

If $\mathcal{A}$ is a covariant $\mathcal{C}(T)$-algebra, then $\mathcal{I}(t):=\overline{\mathcal{Q}\left[\mathcal{C}_t(T)\right]\cdot\mathcal{A}}$ is an invariant ideal of $\mathcal{A}$. We can apply Rieffel quantization to $\mathcal{I}(t)$, to $\mathcal{A}(t):=\mathcal{A}/\mathcal{I}(t)$ (with the obvious actions of Ξ) and to the projection $\mathcal{P}(t):\mathcal{A}\rightarrow\mathcal{A}(t)$. One gets C^*-algebras $\mathfrak{I}_t$, $\mathfrak{A}_t$ as well as the morphism $\mathfrak{P}_t:\mathfrak{A}\rightarrow\mathfrak{A}_t$. By Theorem 7.7 at [22] the kernel of $\mathfrak{P}_t$ is $\mathfrak{I}_t$, so $\mathfrak{A}_t$ can be identified to the quotient $\mathfrak{A}/\mathfrak{I}_t$.

On the other hand, by Theorem 3.3, we have ideals $\mathfrak{I}(t):=\overline{\mathfrak{Q}\left[\mathcal{C}_t(T)\right]\cdot\mathfrak{A}}$ in $\mathfrak{A}$ as well as quotients $\mathfrak{A}(t):=\mathfrak{A}/\mathfrak{I}(t)$ to which we associates projections $\mathfrak{A}\xrightarrow{\mathfrak{P}(t)}\mathfrak{A}(t)$. However, one gets

Proposition 3.4. *With notation as above, for each $t\in T$ we have $\mathfrak{I}(t)=\mathfrak{I}_t$.*

In particular, the fibers $\mathfrak{A}(t)=\mathfrak{A}/\mathfrak{I}(t)$ of the $\mathcal{C}(T)$-algebra $\mathfrak{A}$ are isomorphic to the Rieffel quantization $\mathfrak{A}_t$ of the fibers $\mathcal{A}(t)=\mathcal{A}/\mathcal{I}(t)$ of $\mathcal{A}$ and for each $f\in\mathfrak{A}$ the mapping $t\mapsto\parallel\mathfrak{P}(t)f\parallel_{\mathfrak{A}(t)}=\parallel\mathfrak{P}_tf\parallel_{\mathfrak{A}_t}$ is upper semi-continuous.

Proof. We recall that $\mathcal{I}(t)^\infty$ and $\mathfrak{I}(t)^\infty$ coincide as Fréchet spaces. By Lemma 3.2, $\mathfrak{Q}\left[\mathcal{C}_t(T)\right]\cdot\mathfrak{A}^\infty$ is dense in $\mathfrak{I}(t)^\infty$, thus in $\mathfrak{I}(t)$, and $\mathcal{Q}\left[\mathcal{C}_t(T)\right]\cdot\mathcal{A}^\infty$ is dense in $\mathcal{I}(t)^\infty=\mathfrak{I}(t)^\infty$, thus also dense in $\mathfrak{I}_t$.

By construction one has $\mathfrak{Q}\left[\mathcal{C}_t(T)\right]\cdot\mathfrak{A}^\infty=\mathcal{Q}\left[\mathcal{C}_t(T)\right]\cdot\mathcal{A}^\infty$; consequently $\mathfrak{I}(t)=\mathfrak{I}_t$ for every $t\in T$ and the proof is finished. □

Remark 3.5. For obvious reasons, we are going to say that $\left\{\mathcal{A}\xrightarrow{\mathcal{P}(t)}\mathcal{A}(t)\mid t\in T\right\}$ and $\left\{\mathfrak{A}\xrightarrow{\mathfrak{P}(t)}\mathfrak{A}(t)\mid t\in T\right\}$ are *covariant upper semi-continuous fields of C^*-algebras*. The intrinsic definition, in the first case for instance, would be the following: $\left\{\mathcal{A}\xrightarrow{\mathcal{P}(t)}\mathcal{A}(t)\mid t\in T\right\}$ is required to be an upper semi-continuous field of C^*-algebras and we also ask the action Θ to leave invariant all the ideals $\mathcal{I}(t)=\ker[\mathcal{P}(t)]$. It is easily seen that this is equivalent to require the covariance of the associated $\mathcal{C}(T)$-structure. This makes the connection with Definition 3.1 in [21].

For section C^*-algebras of an upper semi-continuous field it is known [27] that each irreducible representation factorizes through one of the fibers. Therefore we get

Corollary 3.6. *Let $(\mathcal{A},\Theta,\Xi,[\![\cdot,\cdot]\!])$ be a classical data and assume that $\mathcal{A}$ is a $\mathcal{C}(T)$-algebra with respect to a Hausdorff locally compact space T, with fibers $\{\mathcal{A}(t)\mid t\in T\}$. Denote, respectively, by $\mathfrak{A}$ and $\mathfrak{A}(t)$ the corresponding quantized C^*-algebras. Then any irreducible representation of $\mathfrak{A}$ factorizes through one of the algebras $\mathfrak{A}(t)$.*

The $\mathcal{C}(T)$-structure $\mathfrak{Q}$ of $\mathfrak{A}$, given by Theorem 3.3, defines canonically the map $\mathfrak{q}:\mathrm{Prim}(\mathfrak{A})\rightarrow T$, as explained at the end of section 2. If $\pi:\mathfrak{A}\rightarrow\mathbb{B}(\mathcal{H})$ is the irreducible Hilbert space representation of $\mathfrak{A}$, then the point t in Corollary 3.6 is $\mathfrak{q}[\ker(\pi)]$.

4 Lower semi-continuity under Rieffel quantization

We keep the previous setting and inquire now if lower semi-continuity of the mappings $t\mapsto\parallel\mathcal{P}(t)f\parallel_{\mathcal{A}(t)}$ for all $f\in\mathcal{A}$ implies lower semi-continuity of the mappings $t\mapsto\parallel\mathfrak{P}(t)f\parallel_{\mathfrak{A}(t)}$ for all $f\in\mathfrak{A}$. We start by noticing that $\mathrm{Prim}(\mathcal{A})$ and $\mathrm{Prim}(\mathfrak{A})$ are canonically endowed with continuous actions of the group Ξ ; once again these actions will be denoted by Θ . By the discussion at the end of section 2 we are left with proving

Proposition 4.1. *Suppose that $\mathcal{Q} : \mathcal{C}(T) \to \mathcal{Z}(\mathcal{A})$ is a covariant $\mathcal{C}(T)$-algebra structure on $\mathcal{A}$ and that the associated function $q : \mathrm{Prim}(\mathcal{A}) \to T$ is open. Then the function $\mathfrak{q} : \mathrm{Prim}(\mathfrak{A}) \to T$ associated to $\mathfrak{Q} : \mathcal{C}(T) \to \mathcal{Z}(\mathfrak{A})$ is also open.*

Proof. We remark first that q is Θ-covariant (Lemma 8.1 in [27]), i.e. one has $q \circ \Theta_X = q$ for every $X \in \Xi$. Consequently, if $\mathcal{O} \subset \mathrm{Prim}(\mathcal{A})$ is an open set, then $\Theta_\Xi(\mathcal{O}) := \{\Theta_X(\mathcal{K}) \mid X \in \Xi, \mathcal{K} \in \mathcal{O}\}$ will also be an open set and $q(\mathcal{O}) = q\left[\Theta_\Xi(\mathcal{O})\right)]$. So q will be open iff it sends open *invariant* subsets of $\mathrm{Prim}(\mathcal{A})$ into open subsets of T . The same is true for $\mathfrak{q} : \mathrm{Prim}(\mathfrak{A}) \to T$. But the most general open subset of $\mathrm{Prim}(\mathcal{A})$ has the form

$$\mathcal{O}_\mathcal{J} := \{\mathcal{K} \in \mathrm{Prim}(\mathcal{A}) \mid \mathcal{J} \not\subset \mathcal{K}\} = h(\mathcal{J})^c$$

for some ideal $\mathcal{J}$ of $\mathcal{A}$, being the complement of the hull $h(\mathcal{J})$ of this ideal. In addition, $\mathcal{O}_\mathcal{J}$ is Θ-invariant iff $\mathcal{J}$ is an invariant ideal. We also recall that Rieffel quantization establishes a one-to-one correspondence between invariant ideals of $\mathcal{A}$ and invariant ideals of $\mathfrak{A}$.

So let $\mathcal{J}$ be an invariant ideal in $\mathcal{A}$ and $\mathfrak{J}$ its quantization (an invariant ideal in $\mathfrak{A}$). We would like to show that $q\left(\mathcal{O}_\mathcal{J}\right) = \mathfrak{q}\left(\mathcal{O}_\mathfrak{J}\right)$; by the discussion above this would imply that q and $\mathfrak{q}$ are simultaneously open. Using the fact that $q(\mathcal{K}) = t$ if and only if $\mathcal{I}(t) \subseteq \mathcal{K}$ and similarly for $\mathfrak{q}$, one gets

$$q\left(\mathcal{O}_\mathcal{J}\right) = \{t \in T \mid \exists \mathcal{K} \in \mathrm{Prim}(\mathcal{A}),\ \mathcal{J} \not\subset \mathcal{K},\ \mathcal{I}(t) \subset \mathcal{K}\}$$

and

$$\mathfrak{q}\left(\mathcal{O}_\mathfrak{J}\right) = \{t \in T \mid \exists \mathfrak{K} \in \mathrm{Prim}(\mathfrak{A}),\ \mathfrak{J} \not\subset \mathfrak{K},\ \mathfrak{I}(t) \subset \mathfrak{K}\}\,.$$

Using the hull application and the fact that both the hull and the kernel are decreasing, one can write

$$t \notin q\left(\mathcal{O}_\mathcal{J}\right) \iff h[\mathcal{I}(t)] \cap h[\mathcal{J}]^c = \emptyset \iff h[\mathcal{I}(t)] \subset h[\mathcal{J}] \iff \mathcal{I}(t) \supset \mathcal{J}$$

and

$$t \notin q\left(\mathcal{O}_\mathfrak{J}\right) \iff h[\mathfrak{I}(t)] \cap h[\mathfrak{J}]^c = \emptyset \iff h[\mathfrak{I}(t)] \subset h[\mathfrak{J}] \iff \mathfrak{I}(t) \supset \mathfrak{J}\,.$$

To finish the proof one only needs to notice that the Rieffel quantization of invariant ideals preserves inclusions. □

Remark 4.2. The definition of *a covariant continuous field of C^*-algebras* is naturally obtained by adding the lower semi-continuity condition to the definition of an upper semi-continuous field of C^*-algebras contained in Remark 3.5. Using this notion, *Theorem 0.1 is now fully justified.*

Remark 4.3. Crossed products associated to actions of $\mathscr{X} := \mathbb{R}^n$ on C^*-algebras are obtained from Rieffel's quantization procedure, as it is explained in [22], Example 10.5. From the results of the present Chapter one could infer rather easily, as a particular case, that (informally) *the crossed product by a continuous field of C^*-algebras is a continuous field of crossed products.* Such results exist in a much greater generality, including (twisted) actions of amenable locally compact groups [18, 20, 21, 27], so we are not going to give details.

The C^*-dynamical system $(\mathcal{A}, \Theta, \Xi)$ being given, one could try one of the choices $T = \mathrm{Orb}[\mathrm{Prim}(\mathcal{A})]$ (*the orbit space*) or $T = \mathrm{Quorb}[\mathrm{Prim}(\mathcal{A})]$ (*the quasi-orbit space*), both associated to the natural action of Ξ on the space $\mathrm{Prim}(\mathcal{A})$. We recall that, by definition, *a quasi-orbit* is the closure of an orbit and we refer to [27] for all the fairly standard assertions we are going to make about these spaces. The two spaces are quotients of $\mathrm{Prim}(\mathcal{A})$ with respect to obvious equivalence relations. Endowed with the quotient topology they are locally compact, but they may fail to possess the Hausdorff property. O the positive side, both *the orbit map* $p : \mathrm{Prim}(\mathcal{A}) \to \mathrm{Orb}[\mathrm{Prim}(\mathcal{A})]$ and *the quasi-orbit map* $q : \mathrm{Prim}(\mathcal{A}) \to \mathrm{Quorb}[\mathrm{Prim}(\mathcal{A})]$ are continuous open surjections. So one can state:

Corollary 4.4. *If the quasi-orbit space of the dynamical system* $(\mathrm{Prim}(\mathcal{A}), \Theta, \Xi)$ *is Hausdorff, then the deformed* C^**-algebra* $\mathfrak{A}$ *can be expressed as a continuous field of* C^**-algebras over the base* $\mathrm{Quorb}[\mathrm{Prim}(\mathcal{A})]$.

A similar statement holds with "quasi-orbit" replaced by "orbit" and with $\mathrm{Quorb}[\mathrm{Prim}(\mathcal{A})]$ *replaced by* $\mathrm{Orb}[\mathrm{Prim}(\mathcal{A})]$.

The space $\mathrm{Quorb}[\mathrm{Prim}(\mathcal{A})]$ is surely Hausdorff if the action of Ξ on $\mathrm{Prim}(\mathcal{A})$ is proper. Notice that, when $\mathrm{Orb}[\mathrm{Prim}(\mathcal{A})]$ is assumed to be Hausdorff, the orbits will be automatically closed, as inverse images by p of points; so one would actually have $\mathrm{Orb}[\mathrm{Prim}(\mathcal{A})] = \mathrm{Quorb}[\mathrm{Prim}(\mathcal{A})]$.

5 The Abelian case

The most important is the Abelian case, that has been described at the end of section 1.

We assume given a continuous surjection $q : \Sigma \to T$. Then we have the disjoint decomposition of Σ in closed subsets

$$\Sigma = \sqcup_{t\in T}\Sigma_t\,, \quad \Sigma_t := q^{-1}(\{t\})\,. \tag{5.1}$$

Associated to the canonical injections $j_t : \Sigma_t \to \Sigma$, we have associated restriction epimorphisms

$$\mathcal{R}(t) : \mathcal{C}(\Sigma) \to \mathcal{C}(\Sigma_t)\,, \quad \mathcal{R}(t)f := f|_{\Sigma_t} = f \circ j_t\,, \ \forall\, t \in T\,. \tag{5.2}$$

We give conditions on the topological data (Σ, q, T) in order to get a continuous field of Abelian C^*-algebras.

Proposition 5.1. *If* q *is continuous,* $\left\{\mathcal{C}(\Sigma) \overset{\mathcal{R}(t)}{\longrightarrow} \mathcal{C}(\Sigma_t) \mid t \in T\right\}$ *is an upper semi-continuous field of commutative* C^**-algebras. If* q *is also open, the field is continuous.*

Proof. Obviously $\cap_{t\in T}\ker[\mathcal{R}(t)] = \{0\}$, since $f|_{\Sigma_t} = 0\,,\ \forall\, t \in T$ implies $f = 0$. On the other hand, setting

$$\varphi * f := (\varphi \circ q) f\,, \quad \forall\, \varphi \in \mathcal{C}(T)\,,\ f \in \mathcal{C}(\Sigma)\,, \tag{5.3}$$

we get immediately $\mathcal{R}(t)(\varphi * f) = \varphi(t)\, \mathcal{R}(t) f$, $\forall\, t \in T$.

We need to study continuity properties of the mapping

$$T \ni t \mapsto n_f(t) := \|\mathcal{R}(t)f\|_{\mathcal{C}(\Sigma_t)} = \sup_{\sigma\in\Sigma_t}|f(\sigma)| = \inf\left\{\|f + h\|_{\mathcal{C}(\Sigma)} \ \mid\ h \in \mathcal{C}(\Sigma),\ h|_{\Sigma_t} = 0\right\} \in \mathbb{R}_+\,.$$

The last expression for the norm can be justified directly easily, but it also follows from the canonical isomorphism $\mathcal{C}(\Sigma_t) \cong \mathcal{C}(\Sigma)/\mathcal{C}_{\Sigma_t}(\Sigma)$, where $\mathcal{C}_{\Sigma_t}(\Sigma)$ is the ideal of functions $h \in \mathcal{C}(\Sigma)$ such that $h|_{\Sigma_t} = 0$.

We first assume that q is only continuous. For every $S \subset T$ we set $\Sigma_S := q^{-1}(S)$. It is easy to see by Stone-Weierstrass Theorem that

$$\mathcal{C}_{(t)}(\Sigma) := \{h \in \mathcal{C}(\Sigma) \mid \exists \text{ an open neighborhood } U \text{ of } t \text{ such that } h|_{\Sigma_{\overline{U}}} = 0\}$$

is a self-adjoint 2-sided ideal dense in $\mathcal{C}_{\Sigma_t}(\Sigma)$. Let $t_0 \in T$ and $\varepsilon > 0$; by density and the definition of $\inf$

$$\exists\, h \in \mathcal{C}_{(t_0)}(\Sigma) \ \text{ such that } \ n_f(t_0) + \varepsilon \ \geq\ \|f + h\|_{\mathcal{C}(\Sigma)}\,.$$

Let U be the open neighborhood of t_0 for which $h|_{\Sigma_{\overline{U}}} = 0$. For any $t \in U$ one also has $h \in \mathcal{C}_{(t)}(\Sigma)$, so

$$n_f(t) = \inf\left\{\|f + g\|_{\mathcal{C}(\Sigma)} \ \mid\ g \in \mathcal{C}_{(t)}(\Sigma)\right\} \ \leq\ \|f + h\|_{\mathcal{C}(\Sigma)} \ \leq\ n_f(t_0) + \varepsilon$$

and this is upper semi-continuity.

Let us also suppose q open, let $t_0 \in T$ and $\varepsilon > 0$. By the definition of sup, there exists $\sigma_0 \in \Sigma_{t_0}$ such that $|f(\sigma_0)| \geq n_f(t_0) - \varepsilon/2$. Since f is continuous, there is a neighborhood V of σ_0 in Σ such that

$$|f(\sigma)| \geq |f(\sigma_0)| - \varepsilon/2 \geq n_f(t_0) - \varepsilon, \quad \forall \sigma \in V.$$

Since q is open, $U := q(V)$ is a neighborhood of t_0. For every $t \in U$ we have $\Sigma_t \cap V \neq \emptyset$, so for such t

$$n_f(t) \geq \sup\{|f(\sigma)| \mid \sigma \in \Sigma_t \cap V\} \geq n_f(t_0) - \varepsilon$$

and this is lower semi-continuity. □

Remark 5.2. The result also follows from the fact that $\mathcal{C}(\Sigma)$ is a $\mathcal{C}(T)$-algebra for the injective morphism

$$\mathcal{Q} : \mathcal{C}(T) \to BC(\Sigma) \cong \mathcal{M}[\mathcal{C}(\Sigma)], \quad \mathcal{Q}(\varphi) := \varphi \circ q.$$

We have identified the multiplier algebra of $\mathcal{C}(T)$ with the unital C^*-algebra of all bounded continuous complex functions defined on Σ. The direct topological proof of Proposition 5.1 seemed to us more suitable.

We recall now that an action Θ of Ξ on Σ by homeomorphisms is given.

Definition 5.3. *We say that the continuous surjection q* is Θ-covariant *if it satisfies the equivalent conditions:*

1. *Each Σ_t is Θ-invariant.*
2. *For each $X \in \Xi$ one has $q \circ \Theta_X = q$.*
3. *For all $X \in \Xi$ and $\varphi \in \mathcal{C}(T)$ one has $\Theta_X[\mathcal{Q}(\varphi)] = \mathcal{Q}(\varphi)$.*

The equivalence of the three conditions is straightforward. We conclude that $\mathcal{C}(\Sigma)$ is a covariant $C(T)$-algebra (cf. Definition 3.1). The Rieffel-quantized C^*-algebras $\mathfrak{C}(\Sigma)$ and $\mathfrak{C}(\Sigma_t)$ as well as the epimorphisms $\mathfrak{R}(t) : \mathfrak{C}(\Sigma) \to \mathfrak{C}(\Sigma_t)$ were introduced in Section 1. If one wants to avoid the language of $\mathcal{C}(T)$-algebras, by Remark 3.5, it should be noticed that all the ideals $\mathcal{I}(T) := \ker[\mathcal{R}(t)] = \{f \in \mathcal{C}(T) \mid f|_{\Sigma_t} = 0\}$ are left invariant by the action Θ.

Applying now Proposition 5.1, the results obtained in Section 3 and Theorem 7.3, one gets

Corollary 5.4. *Assume that the mapping $q : \Sigma \to T$ is a Θ-covariant continuous surjection. Then $\left\{\mathfrak{C}(\Sigma) \xrightarrow{\mathfrak{R}(t)} \mathfrak{C}(\Sigma_t) \mid t \in T\right\}$ is a covariant upper semi-continuous field of non-commutative C^*-algebras.*

If q is also open, then the field is continuous.

Let us assume now that the orbit space $\mathrm{Orb}(\Sigma)$ is Hausdorff. Any orbit, being the inverse image of a point in $\mathrm{Orb}(\Sigma)$, will be closed in Σ and invariant; it will also be homeomorphic to the quotient of Ξ by the corresponding stability group. As a precise particular case of Corollary 4.4 one can state:

Corollary 5.5. *If the orbit space of the dynamical system (Σ, Θ, Ξ) is Hausdorff, then the deformed C^*-algebra $\mathfrak{C}(\Sigma)$ can be expressed as a continuous field of C^*-algebras over the base space $\mathrm{Orb}(\Sigma)$. The fiber over $\mathcal{O} \in \mathrm{Orb}(\Sigma)$ is the deformation of the Abelian algebra $\mathcal{C}(\mathcal{O}) \cong \mathcal{C}(\Xi/\Xi_{\mathcal{O}})$.*

Remark 5.6. It is known that the orbit space is Hausdorff if the action Θ is proper, including the case in which Σ is a Hausdorff locally compact group on which the closed subgroup Ξ acts by left translations. More generally, assume that the action Θ factorizes through a compact group $\widehat{\Xi}$, i.e. the kernel of Θ contains a closed co-compact subgroup Z of Ξ (with $\widehat{\Xi} = \Xi/Z$). Then the orbit space under the initial action is the same as the orbit space of the action of the compact quotient, so it is proper and Corollary 5.5 applies.

6 Some examples

Let $\mathscr{A}$ be a C^*-algebra and T a locally compact space. On

$$\mathcal{A} \equiv \mathcal{C}(T;\mathscr{A}) := \{f : T \to \mathscr{A} \mid f \text{ is continuous and small at infinity}\} \tag{6.1}$$

we consider the natural structure of C^*-algebra. It clearly defines a continuous field of C^*-algebras

$$\left\{\mathcal{C}(T;\mathscr{A}) \xrightarrow{\delta(t)} \mathscr{A} \mid t \in T\right\}, \qquad \delta(t)f := f(t)\,.$$

The associated $\mathcal{C}(T)$-structure is given by $[\mathcal{Q}(\varphi)f]\,(t) := \varphi(t)f(t)$ for $\varphi \in \mathcal{C}(T)$, $f \in \mathcal{A}$, $t \in T$.

For each $t \in T$ an action θ^t of Ξ on $\mathscr{A}$ is given; we require for each $f \in \mathcal{A}$ the condition

$$\sup_{t\in T} \parallel \theta^t_X[f(t)] - f(t) \parallel_{\mathscr{A}} \underset{X\to 0}{\longrightarrow} 0\,. \tag{6.2}$$

Then obviously

$$\Theta : \Xi \to \mathrm{Aut}(\mathcal{A})\,, \qquad [\Theta_X(f)](t) := \theta^t_X[f(t)] \tag{6.3}$$

defines a continuous action of the vector group Ξ on $\mathcal{A}$. Each of the kernels

$$\mathcal{I}(t) := \ker[\delta(t)] = \{f \in \mathcal{C}(T;\mathscr{A}) \mid f(t) = 0\}$$

is Θ-invariant, so one actually has a covariant continuous field of C^*-algebras (see Remarks 3.5 and 4.2).

It makes sense to apply Rieffel quantization, getting C^*-algebras (respectively) $\mathfrak{A} \equiv \mathfrak{C}(T;\mathscr{A})$ from $(\mathcal{A} \equiv \mathcal{C}(T;\mathscr{A}),\Theta)$ and $\mathfrak{A}(t)$ from $(\mathscr{A},\theta^t)$ for all $t \in T$. From the results above one concludes that $\left\{\mathfrak{A} \xrightarrow{\Delta(t)} \mathfrak{A}(t) \mid t \in T\right\}$ *is also a covariant field of C^*-algebras*. For each t we denoted by $\Delta(t)$ the Rieffel quantization of the morphism $\delta(t)$.

A particular case, considered in [22], Ch. 8, consists in taking $T := \mathrm{End}(\Xi)$ the space of all linear maps $t : \Xi \to \Xi$; it is a locally compact (finite-dimensional vector) space with the obvious operator norm. If an initial action θ of Ξ on $\mathscr{A}$ is fixed, the choice $\theta^t_X := \theta_{tX}$ verify all the requirements above. Therefore one gets a covariant continuous field of C^*-algebras indexed by $\mathrm{End}(\Xi)$. This is basically [22] Theorem 8.3; we think that our treatment gives a simpler and more unified proof of this result, especially concerning the lower semi-continuous part. In particular, for any $f \in \mathcal{C}[\mathrm{End}(\Xi);\mathscr{A}]$, one has $\lim_{t\to 0} \parallel f(t) \parallel_{\mathfrak{A}(t)} = \parallel f(0) \parallel_{\mathscr{A}}$. An interesting particular case is obtained restricting the arguments to the compact subspace $T_0 := \{t = \sqrt{\hbar}\,\mathrm{id}_\Xi \mid \hbar \in [0,1]\} \subset T$. The number $\hbar$ corresponds to the Plank constant and, even for constant $f : [0,1] \to \mathscr{A}$, the relation $\lim_{\hbar\to 0} \parallel f \parallel_{\mathfrak{A}(\hbar)} = \parallel f \parallel_{\mathscr{A}}$ has an important physical interpretation concerning the semiclassical behavior of the Quantum Mechanical formalism. We refer to [13, 22, 23] for much more on this topic.

Remark 6.1. A way to convert this Example in a less trivial one is as follows:

For every $t \in T$ pick $\mathcal{B}(t)$ to be a C^*-subalgebra of $\mathscr{A}$ which is invariant under the action θ^t. Construct the C^*-subalgebra $\mathcal{B}$ of $\mathcal{A}$ defined as $\mathcal{B} := \{f \in \mathcal{C}(T;\mathscr{A}) \mid f(t) \in \mathcal{B}(t),\ \forall\, t \in T\}$, which is obviously invariant under the action Θ. One gets a covariant continuous field of C^*-algebras $\left\{\mathcal{B} \xrightarrow{\mathcal{P}(t)} \mathcal{B}(t) \mid t \in T\right\}$, where $\mathcal{P}(t)$ is a restriction of the epimorphism $\delta(t)$. The general theory developed in this chapter supplies another covariant continuous field of C^*-algebras $\left\{\mathfrak{B} \xrightarrow{\mathfrak{P}(t)} \mathfrak{B}(t) \mid t \in T\right\}$, where $\mathfrak{B}(t)$ is the quantization of $\mathcal{B}(t)$ and it can be identified with an invariant C^*-subalgebra of $\mathfrak{A}(t)$.

In [24] one constructs C^*-algebras which can be considered quantum versions of a certain class of compact connected Lie groups. We will have nothing to say about the extra structure making them quantum groups; we are only going to apply the results above to present these C^*-algebras as continuous fields.

Let Σ be a compact connected Lie group, containing *a toral subgroup*, i.e. a connected closed Abelian subgroup H. Such a toral group is isomorphic to an n-dimensional torus $\mathbb{T}^n$. Assume given a continuous group epimorphism $\eta : \mathbb{R}^n \to H$ (for example the exponential map defined on the Lie algebra $\mathfrak{H} \equiv \mathbb{R}^n$). We use η to define an action of $\Xi := \mathbb{R}^n \times \mathbb{R}^n$ on Σ by $\Theta_{(x,y)}(\sigma) := \eta(-x)\sigma\eta(y)$. Then, by applying Rieffel deformation to $\mathcal{A} := \mathcal{C}(\Sigma)$ using the action Θ (and a certain type of skew-symmetric operator on Σ), one gets the C^*-algebra $\mathfrak{A} := \mathfrak{C}(\Sigma)$ which, endowed with suitable extra structure, is regarded as a quantum group corresponding to Σ.

It is obvious that the action factorizes through the compact group $H \times H$. Thus the orbit space $\mathrm{Orb}(\Sigma)$ is Hausdorff and Remark 5.6 and Corollary 5.5 serve to express $\mathfrak{C}(\Sigma)$ as a continuous field of C^*-algebras. For the stability group of any orbit $\mathcal{O}$ one can write $\Xi_{\mathcal{O}} \supset \ker(\Theta) \supset \ker(\eta) \times \ker(\eta)$, thus $\mathcal{O} \cong \Xi/\Xi_{\mathcal{O}}$ is the continuous image of $H \times H$.

An interesting particular case, taken from [24], involves the construction of a quantum version of the compact Lie group $\Sigma := \mathbb{T} \times SU(2)$. Here $\mathbb{T}$ is the 1-torus, the group $SU(2)$ contains diagonally a second copy of $\mathbb{T}$ and can be parametrised by the 3-sphere $S^3 := \{(z, w) \in \mathbb{C}^2 \mid z^2 + w^2 = 1\}$, and so Σ contains a 2-torus. Initially $\Xi = \mathbb{R}^4$ acts on Σ in the given way, but it is shown in [24] (using results from [22]) that the same deformed algebra is obtained by the action

$$\Theta' : \Xi' := \mathbb{R}^2 \to \mathrm{Homeo}\left(\mathbb{T} \times S^3\right), \qquad \Theta'_{(x,y)}(\eta; z, w) := \left(e^{-2\pi i x}\zeta; z, e^{4\pi i y}w\right).$$

The orbit space is homeomorphic with the closed unit ball $T := \{z \in \mathbb{C} \mid |z| \le 1\}$. The orbits corresponding to $|z| < 1$ are 2-tori, while the orbits corresponding to $|z| = 1$ (implying $w = 0$) are 1-tori. If we set $\mathcal{A} := \mathcal{C}(\mathbb{T} \times SU(2))$, then the quantized C^*-algebra $\mathfrak{A} \cong \mathfrak{C}(\mathbb{T} \times SU(2))$ deserves to be called *a quantum* $\mathbb{T} \times SU(2)$. The deformation of the continuous functions on any of the 2-tori leads to a quantum tori. By multiplying the symplectic form $[\![\cdot,\cdot]\!]$ with an irrational number β one can make this non-commutative torus $\mathfrak{C}_\beta(\mathbb{T}^2)$ irrational, which serves to show that the corresponding quantum $\mathbb{T} \times SU(2)$ (obtained for such a β) is not of type I. But applying the results obtained here one also gets the detailed information: *The algebra* $\mathfrak{C}(\mathbb{T} \times SU(2))$ *can be written over the base T as a continuous field of non-commutative 2-tori and Abelian C^*-algebras (corresponding to the one-dimensional orbits).*

Many other particular cases can be worked out in detail. We propose to the reader the example $\Sigma := SU(2) \times SU(2)$.

7 Spectral continuity

Let us introduced the concept of continuity for families of sets that will be useful below.

Definition 7.1. *Let T be a Hausdorff locally compact topological space and $\{S(t) \mid t \in T\}$ a family of compact subsets of $\mathbb{R}$.*

1. *The family is called* outer continuous *if for any $t_0 \in T$ and any compact subset K of $\mathbb{R}$ such that $K \cap S(t_0) = \emptyset$, there exists a neighborhood V of t_0 with $K \cap S(t) = \emptyset$, $\forall t \in V$.*

2. *The family $\{S(t) \mid t \in T\}$ is called* inner continuous *if for any $t_0 \in T$ and any open subset A of $\mathbb{R}$ such that $A \cap S(t_0) \neq \emptyset$, there exists a neighborhood W of t_0 with $A \cap S(t) \neq \emptyset$, $\forall t \in W$.*

3. *If the family is both inner and outer continuous, we say simply that it is* continuous.

In applications the sets $S(t)$ are spectra of some self-adjoint elements $f(t)$ of (non-commutative) C^*-algebras $\mathfrak{A}(t)$. The next result states technical conditions under which one gets continuity of such families of spectra. It is taken from [1] and it has been inspired by the treatment in [3]. We include the proof for the convenience of the reader.

Proposition 7.2. *For any $t \in T$ let $f(t)$ be a self-adjoint element in a C^*-algebra $\mathfrak{A}(t)$ with norm $\| \cdot \|_{\mathfrak{A}(t)}$ and inversion $g \mapsto g^{(-1)_{\mathfrak{A}(t)}}$. We denote by $S(t) \subset \mathbb{R}$ the spectrum of $f(t)$ in $\mathfrak{A}(t)$.*

1. *Assume that for any $z \in \mathbb{C} \setminus \mathbb{R}$ the mapping*
$$T \ni t \mapsto \left\| (f(t) - z)^{(-1)_{\mathfrak{A}(t)}} \right\|_{\mathfrak{A}(t)} \in \mathbb{R}_+ \tag{7.1}$$
is upper semi-continuous. Then the family $\{S(t) \mid t \in T\}$ is outer continuous.

2. *Assume that for any $z \in \mathbb{C} \setminus \mathbb{R}$ the mapping (7.1) is lower semi-continuous. Then the family $\{S(t) \mid t \in T\}$ is inner continuous.*

Proof. We use the functional calculus for self-adjoint elements in the C^*-algebra $\mathfrak{A}(t)$ to define $\chi[f(t)]$ for every continuous function $\chi : \mathbb{R} \to \mathbb{C}$ decaying at infinity. Notice that
$$(f(t) - z)^{(-1)_{\mathfrak{A}(t)}} = \chi_z[f(t)], \quad \text{with} \quad \chi_z(\lambda) := (\lambda - z)^{-1}.$$
By a standard argument relying on Stone-Weierstrass Theorem, one deduces that the map $t \mapsto \|\chi[f(t)]\|_{\mathfrak{A}(t)}$ has the same continuity properties (upper or lower semi-continuity, respectively) as (7.1).

Let us suppose now upper semi-continuity in t_0 and assume that $S(t_0) \cap K = \emptyset$ for some compact set K. By Urysohn's Lemma, there exists $\chi \in C_0(\mathbb{R})_+$ with $\chi|_K = 1$ and $\chi|_{S(t_0)} = 0$, so $\chi[f(t_0)] = 0$. Choose a neighborhood V of t_0 such that for $t \in V$
$$\| \chi[f(t)] \|_{\mathfrak{A}(t)} \leq \| \chi[f(t_0)] \|_{\mathfrak{A}(t)} + \frac{1}{2} = \frac{1}{2}.$$
If for some $t \in V$ there exists $\lambda \in K \cap S(t)$, then
$$1 = \chi(\lambda) \leq \sup_{\mu \in S(t)} \chi(\mu) = \| \chi[f(t)] \|_{\mathfrak{A}(t)} \leq \frac{1}{2},$$
which is absurd.

Let us assume now lower semi-continuity in t_0. Pick an open set $A \subset \mathbb{R}$ such that $S(t_0) \cap A \neq \emptyset$ and let $\lambda \in S(t) \cap A$. By Urysohn's Lemma there exist a positive function $\chi \in C_0(\mathbb{R})$ with $\chi(\lambda) = 1$ and $\text{supp}(\chi) \subset A$; thus $\| \chi[f(t_0)] \| \geq 1$. Suppose moreover that for any neighborhood $W \subset I$ of t_0 there exists $t \in W$ such that $S(t) \cap A = \emptyset$ and thus $\chi[f(t)] = 0$. This clearly contradicts the lower semi-continuity of $t \mapsto \| \chi[f(t)] \|_{\mathfrak{A}(t)}$. We conclude thus the inner continuity condition for the family $S(t)$. □

Proving these properties of the resolvents is a priory a difficult task, since this involves working both with norms and composition laws that depend on t. But putting together the information obtained until now, we get our abstract result concerning spectral continuity:

Theorem 7.3. *Let $\left\{ \mathcal{A} \xrightarrow{\mathcal{P}(t)} \mathcal{A}(t) \mid t \in T \right\}$ be a covariant upper semi-continuous field of C^*-algebras indexed by a Hausdorff locally compact space T and let f be a smooth self-adjoint element of $\mathcal{A}$. For any $t \in T$ we denote by $\mathfrak{A}(t)$ the Rieffel quantization of $\mathcal{A}(t)$ and consider $f(t) := \mathcal{P}(t)f$ as an element of $\mathcal{A}(t)^\infty = \mathfrak{A}(t)^\infty \subset \mathfrak{A}(t)$, with spectrum $S(t)$ computed in $\mathfrak{A}(t)$. Then the family $\{S(t) \mid t \in T\}$ is outer continuous.*

If the field is continuous, the family of subsets will also be continuous.

Proof. The results of the first chapter allow us to conclude that the quantized field $\left\{\mathfrak{A}\xrightarrow{\mathfrak{P}(t)}\mathfrak{A}(t)\mid t\in T\right\}$ has the same continuity properties as the original one.

For any $z\in\mathbb{C}\setminus\mathbb{R}$ one has $(f-z)^{(-1)_{\mathfrak{A}}}\in\mathfrak{A}$ and $(f(t)-z)^{(-1)_{\mathfrak{A}(t)}}=\mathfrak{P}(t)\left[(f-z)^{(-1)_{\mathfrak{A}}}\right]$. Therefore the assumptions of Proposition 7.2 are fulfilled both in the upper semi-continuous and in the lower semi-continuous case, so we obtain the desired continuity properties for the family $\{S(t)\mid t\in T\}$. □

Of course, the conclusion also holds for non-smooth self-adjoint elements $f\in\mathfrak{A}$. Very often they are much less "accessible" than the smooth elements, being obtained by an abstract completion procedure, so we only make the statements for C^∞ vectors.

Specializing to the Abelian case and using the notations of section 5, one gets

Corollary 7.4. *Let $f\in\mathcal{C}^\infty(\Sigma)$ a real function and for each $t\in T$ denote by $S(t)$ the spectrum of $f(t):=f|_{\Sigma_t}\in\mathcal{C}^\infty(\Sigma_t)=\mathfrak{C}^\infty(\Sigma_t)$ seen as an element of the non-commutative C^*-algebra $\mathfrak{C}(\Sigma_t)$. Then the family $\{S(t)\mid t\in T\}$ of compact subsets of $\mathbb{R}$ is outer continuous.*

If q is also open, the family of subsets is continuous.

Remark 7.5. One can use Exemple 10.2 in [22] to identify quantum tori as Rieffel-type quantizations of usual tori. One is naturally placed in the setting above and can reproduce some known spectral continuity results [8, 3] on generalized Harper operators.

The standard approach of Quantum Mechanics asks for Hilbert space operators. This can be achieved by representing faithfully the C^*-algebras $\mathfrak{A}(t)$ in a Hilbert space of L^2-functions in a way that generalizes the Schrödinger representation. We are going to get continuity results for both spectra and essential spectra of the emerging self-adjoint operators. We work in the following

Framework.

1. $(\mathcal{C}(\Sigma),\Theta,\Xi,[\![\cdot,\cdot]\!])$ is an Abelian classical data.
2. Ξ is symplectic, given in a Lagrangean decomposition $\Xi=\mathscr{X}\times\mathscr{X}^*\ni X=(x,\xi)$, $Y=(y,\eta)$, where $\mathscr{X}$ is a n-dimensional real vector space, $\mathscr{X}^*$ is its dual and the symplectic form on Ξ is given in terms of the duality between $\mathscr{X}$ and $\mathscr{X}^*$ by $[\![(x,\xi),(y,\eta)]\!]:=y\cdot\xi-x\cdot\eta$.
3. $q:\Sigma\to T$ is a Θ-covariant continuous surjection. We also assume that each $\Sigma_t:=q^{-1}(\{t\})$ is *a quasi-orbit*, i.e. there is a point $\sigma\in\Sigma_t$ such that the orbit $\mathcal{O}_\sigma:=\Theta_\Xi(\sigma)$ is dense in Σ_t (we say that σ *generates the quasi-orbit* Σ_t).
4. We fix a real element $f\in\mathcal{C}^\infty(\Sigma)$. For each $t\in T$ and for any point σ generating the quasi-orbit Σ_t we define $f(t):=f|_{\Sigma_t}$ and $f_\sigma(t):=f(t)\circ\Theta_\sigma:\Xi\to\mathbb{R}$.
5. We set $H_\sigma(t):=\mathfrak{Op}\left[f_\sigma(t)\right]$ (self-adjoint operator in the Hilbert space $\mathcal{H}:=L^2(\mathscr{X})$), by applying to $f_\sigma(t)$ the usual Weyl pseudodifferential calculus. We denote by $S(t)$ the spectrum of $H_\sigma(t)$.

Some explanations are needed. It is easy to see that each $f_\sigma(t)$ belongs to $BC^\infty(\Xi)$, i.e. it is a smooth function with bounded derivatives of any order. Therefore, using oscillatory integrals, one can define the self-adjoint operator in $L^2(\mathscr{X})$

$$[H_\sigma(t)u](x)\equiv\left[\mathfrak{Op}(f_\sigma(t))u\right](x):=(2\pi)^{-n}\int_{\mathscr{X}}\mathrm{d}y\int_{\mathscr{X}^*}\mathrm{d}\xi\,e^{i(x-y)\cdot\xi}\,[f_\sigma(t)]\left(\frac{x+y}{2},\xi\right)u(y). \tag{7.2}$$

This operator is bounded by the Calderón-Vaillancourt Theorem [10]. Using the notation (1.4), we see that for every $X\in\Xi$ one has $[f_\sigma(t)](X):=f\left[\Theta_X(\sigma)\right]$; this depends on $t\in T$ through σ and only involves the values of f on the dense subset $\mathcal{O}_\sigma$ of Σ_t. The same is true about $H_\sigma(t)$, which can be written

$$[H_\sigma(t)u](x) = (2\pi)^{-n} \int_{\mathscr{X}} \mathrm{d}y \int_{\mathscr{X}^*} \mathrm{d}\xi\, e^{i(x-y)\cdot\xi} f\left[\Theta_{\left(\frac{x+y}{2},\xi\right)}(\sigma)\right] u(y)\,. \tag{7.3}$$

It is shown in [16] that if σ and σ' are both generating the same quasi-orbit Σ_t, then the operators $H_\sigma(t)$ and $H_{\sigma'}(t)$ are isospectral (but not unitarily equivalent in general). Thus the compact set $S(t)$ only depend on t and not on the choice of the generating element σ.

Theorem 7.6. *Assume the Framework above, with σ compact. Then the family $\{S(t) \mid t \in T\}$ is outer continuous.*

If q is also open, than the family is continuous.

Proof. By Corollary 7.4, it would be enough to show for every t that $S(t)$ coincides with the spectrum of $f(t) \in \mathfrak{C}(\Sigma_t)$. For this we define

$$\mathcal{N}_\sigma : \mathcal{C}^\infty(\Sigma_t) \to BC^\infty(\Xi), \quad \mathcal{N}_\sigma(g) := g \circ \Theta_\sigma$$

and then set

$$\mathfrak{Op}_\sigma := \mathfrak{Op} \circ \mathcal{N}_\sigma : \mathcal{C}^\infty(\Sigma_t) \to \mathbb{B}(\mathcal{H})\,.$$

Then one has $H_\sigma(t) := \mathfrak{Op}\,[f_\sigma(t)] = \mathfrak{Op}_\sigma\,[f(t)]$. It is not quite trivial, but it has been shown in [16], that $\mathfrak{Op}_\sigma$ extends to a faithful representation of the Rieffel quantized C^*-algebra $\mathfrak{C}(\Sigma_t)$ in $\mathcal{H}$. Faithfulness is implied by the fact that σ generates the quasi-orbit Σ_t, which results in the injectivity of $\mathcal{N}_\sigma$, conveniently extended to $\mathfrak{C}(\Sigma_t)$. It follows then that $\mathrm{sp}\,[H_\sigma(t)] = \mathrm{sp}\,[f(t)]$, as required, so the family $\{S(t) \mid t \in T\}$ has the desired continuity properties. □

We recall that *the essential spectrum* of an operator is the part of the spectrum composed of accumulation points or infinitely-degenerated eigenvalues. Let us denote by $S^{\mathrm{ess}}(t)$ the essential spectrum of $H_\sigma(t)$; once again this only depends on t. To discuss the continuity properties of this family of sets we are going to need some preparations relying mainly on results from [16].

First we write each Σ_t as a disjoint Θ-invariant union $\Sigma_t = \Sigma_t^{\mathrm{g}} \sqcup \Sigma_t^{\mathrm{n}}$. The elements σ_1 of Σ_t^{g} are *generic points for* Σ_t, meaning that each of them is generating Σ_t. The points $\sigma_2 \in \Sigma_t^{\mathrm{n}}$ are *non-generic*, i.e. the closure of the orbit $\mathcal{O}_{\sigma_2}$ is strictly contained in Σ_t.

Let us now fix a point $t \in T$ and a generating element $\sigma \in \Sigma_t$. The monomorphism $\mathcal{N}_\sigma$ extends to an isomorphism between $\mathcal{C}(\Sigma_t)$ and a C^*-subalgebra $\mathcal{B}_\sigma(t)$ of the C^*-algebra $BC_{\mathrm{u}}(\Xi)$ of all the bounded uniformly continuous complex functions on Ξ. It is shown in Lemma 2.2 from [16] that only two possibilities can occur, and this is independent of σ: either $\mathcal{C}(\Xi) \subset \mathcal{B}_\sigma(t)$ (and then t is called *of the first type*), or $\mathcal{C}(\Xi) \cap \mathcal{B}_\sigma(t) = \{0\}$ (and then we say that t is *of the second type*). Correspondingly, one has the disjoint decomposition $T = T_I \sqcup T_{II}$.

Theorem 7.7. *Assume the Framework above. Then the family $\{S^{\mathrm{ess}}(t) \mid t \in T\}$ is outer continuous.*

Proof. One must rephrase the essential spectrum $S^{\mathrm{ess}}(t) := \mathrm{sp}_{\mathrm{ess}}[H_\sigma(t)]$ in convenient C^*-algebraic terms. Assume first that t is of the second type. By [16], Proposition 3.4, the discrete spectrum of $H_\sigma(t)$ is void, thus one has $S^{\mathrm{ess}}(t) = S(t)$. If t is of the first type, the subset Σ_t^{n} is invariant under the action Θ and it is also closed by Proposition 2.5 in [16]. Denoting by $f^{\mathrm{n}}(t)$ the restriction of $f(t)$ to Σ_t^{n}, one gets an element of $\mathcal{C}^\infty(\Sigma_t^{\mathrm{n}}) \subset \mathfrak{C}(\Sigma_t^{\mathrm{n}})$ with spectrum $S^{\mathrm{n}}(t)$. But Theorem 3.7 in [16] states among others that $S^{\mathrm{n}}(t)$ coincides with $S^{\mathrm{ess}}(t)$.

We need to construct now a suitable restricted dynamical system. Let us consider the decomposition

$$\Sigma = \Big(\bigsqcup_{t\in T_I} \Sigma_t\Big) \sqcup \Big(\bigsqcup_{t\in T_{II}} \Sigma_t\Big) = \Big(\bigsqcup_{t\in T_I} \Sigma_t^{\mathrm{g}}\Big) \sqcup \Big\{\Big(\bigsqcup_{t\in T_I} \Sigma_t^{\mathrm{n}}\Big) \sqcup \Big(\bigsqcup_{t\in T_{II}} \Sigma_t\Big)\Big\} =: \Sigma^{\mathrm{d}} \sqcup \Sigma^{\mathrm{ess}}\,.$$

One might set $\Sigma_t^{\rm ess} := \Sigma_t^{\rm n}$ if $t \in T_I$ and $\Sigma_t^{\rm ess} := \Sigma_t$ if $t \in T_{II}$. Notice that each $\Sigma_t^{\rm ess}$ is not void. This is clear for $t \in T_{II}$, since q has been supposed surjective. If $t \in T_I$ and $\Sigma_t^{\rm n} = \emptyset$, then $\Sigma_t = \Sigma_t^{\rm g}$ is minimal and compact, so $t \in T_{II}$ by Lemma 2.3 in [16], which is absurd. The disjoint union $\Sigma^{\rm ess} := \sqcup_{t\in T}\Sigma_t^{\rm ess}$ is a compact dynamical system under the restriction of the action Θ of Ξ and $q^{\rm ess} := q|_{\Sigma^{\rm ess}} : \Sigma^{\rm ess} \to T$ is a covariant continuous surjection. Thus we can apply the previous results and conclude that $\{\mathfrak{C}(\Sigma^{\rm ess}) \to \mathfrak{C}(\Sigma_t^{\rm ess}) \mid t \in T\}$ is an upper semi-continuous field of C^*-algebras; the arrows are Rieffel quantizations of obvious restriction maps.

From all these applied to $f|_{\Sigma^{\rm ess}} \in \mathfrak{C}(\Sigma^{\rm ess})$ it follows that the family $\left\{S^{\rm ess}(t) = {\rm sp}\left[f(t)|_{\Sigma^{\rm ess}(t)}\right] \mid t \in T\right\}$ is outer continuous. □

Remark 7.8. Even in simple situations, the surjective restriction of a continuous open surjection may not be open. So $q^{\rm ess}$ may fail to be open and in general we don't obtain inner continuity for the family of essential spectra. On the other hand, if openness of the restriction $q^{\rm ess}$ is required, one clearly gets the inner continuity. Since only the dynamical system $(\Sigma^{\rm ess}, \Theta, \Xi)$ is involved in controlling the family of essential spectra, some assumptions weaker than those above wold suffice; an example is outlined below.

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