DIFFUSION ALONG TRANSITION CHAINS OF INVARIANT TORI AND AUBRY-MATHER SETS

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Abstract. We describe a topological mechanism for the existence of diffusing orbits in a dynamical system satisfying the following assumptions: (i) the phase space contains a normally hyperbolic invariant manifold diffeomorphic to a two-dimensional annulus, (ii) the restriction of the dynamics to the annulus is an area preserving monotone twist map, (iii) the annulus contains sequences of invariant tori that form transition chains, i.e., the unstable manifold of each torus has a topologically transverse intersection with the stable manifold of the next torus in the sequence, (iv) the transition chains of tori are interspersed with gaps created by resonances, (v) within each gap there is a designated, finite collection of Aubry-Mather sets. Under these assumptions, there exist trajectories that follow the transition chains, cross over the gaps, and follow the Aubry-Mather sets within each gap, in any prescribed order. This mechanism is related to the Arnold diffusion problem in Hamiltonian system. In particular, we prove the existence of diffusing trajectories in the large gap problem of Hamiltonian systems. The argument is topological and constructive.

1. Introduction

In this paper we study a topological mechanism of diffusion and chaotic orbits related to the Arnold diffusion problem. We consider a normally hyperbolic invariant manifold diffeomorphic to a two-dimensional annulus. We assume that the dynamics restricted to the annulus is given by an area preserving monotone twist map. We assume that inside the annulus there exist invariant primary tori (homotopically nontrivial invariant closed curves) with the property that the unstable manifold of each torus has topologically transverse intersections with the stable manifolds of all sufficiently close tori. Sequences of such tori and their heteroclinic connections form transition chains of tori. The successive transition chains of tori are interspersed with gaps. We assume that each gap is a region in the annulus bounded by two invariant primary tori which contains no invariant primary torus in its interior, or that each gap is separated by finitely many invariant primary tori, where each torus is either isolated, or it consists of a hyperbolic periodic orbit together with its stable and unstable manifolds that are assumed to coincide. Within each gap we prescribe a finite collection of Aubry-Mather sets. We prove the existence of orbits that shadow the primary tori in each transition chain, cross over the gaps that separate the successive transition chains, and also shadow the specified Aubry-Mather sets within each gap.

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The motivation for this result is the Arnold diffusion problem of Hamiltonian systems. This problem asserts that all sufficiently small perturbations of generic, integrable Hamiltonian systems exhibit orbits along which the action variable changes substantially; also, there exist chaotic orbits that can be coded through symbolic dynamics. A classification of nearly integrable systems proposed in [12] distinguishes between a priori stable systems, in which the unperturbed system can be expressed in terms of action-angle variables only, and a priori unstable systems, in which the unperturbed system contains both action-angle and hyperbolic variables. In the a priori stable case analytical results have been announced in [43]. In the a priori unstable case there have been several analytical results in the last several years (see [12, 48, 45, 46, 16, 3]). Some of the approaches involve variational methods, geometric methods, or topological methods. See [18] for an overview on the Arnold diffusion problem, applications, and additional references.

It is relevant in applications to detect, combine, and compare different mechanisms of diffusion displayed by concrete systems. In many models, as well as in numerical experiments, diffusion can only be observed for perturbations of sizes much larger than those considered by the analytical approaches [13, 38, 37, 35, 27]. For these types of problems, the geometric and topological approaches are particularly advantageous as they yield constructive methods to detect diffusion, quantitative estimates on diffusing orbits, and explicit conditions that can be verified in concrete examples.

In this paper we describe a general method to establish the existence of diffusing orbits for a large class of dynamical systems. The dynamical systems under consideration are not assumed to be small perturbations of integrable Hamiltonians. Moreover, some systems that are not Hamiltonian can be considered. Our method requires the existence of certain geometric objects that organize the dynamics, and employs topological tools to establish the existence of diffusing orbits. The existence of the geometric objects can be verified in concrete systems through analytical methods, or through numerical methods, or through a combination of thereof.

We illustrate our method in the case of a Hamiltonian system consisting of a pendulum and a rotator with a small periodic coupling. We give an analytic argument for the existence of diffusing orbits. In [14] the topological method is applied to show, with the aid of a computer, the existence of diffusing orbits in the spatial restricted three-body problem, where the two primaries are the Sun and the Earth; this model is not nearly integrable. Similar ideas appear in [9, 19, 21, 20].

The diffusing orbits detected by this approach follow transition chains of invariant tori up to the gaps between these transition chains, and then cross the gaps following the inner dynamics restricted to the annulus. The orbits that cross the gaps follow Birkhoff connecting orbits that go from one boundary of the gap to the other, or Mather connecting orbits, that shadow a prescribed sequence of Aubry-Mather set inside the gap, or homoclinic orbits.

The diffusing orbits that we obtain are similar to those found through variational methods, as in [10, 11]; they are however not action minimizing. Also, we do not need the unperturbed Hamiltonian to have positive-definite normal torsion.

This paper completes the investigations undertaken in [24, 25] where some non-generic conditions, or conditions difficult to verify in concrete systems, were assumed. The present paper assumes very general conditions and provides a new topological mechanism of diffusion based on Aubry-Mather sets.

2. Main result

In this section we state the assumptions and the main result of this paper. After the statement, the assumptions and the main result are explained and exemplified.

(A1) M is an n-dimensional C^1-manifold, and $f : M \to M$ is a C^1-smooth map.

(A2) There exists a 2-dimensional submanifold Λ in M, homeomorphic to an annulus. On Λ there is a system of angle-action coordinate (ϕ, I), with $\phi \in \mathbb{T}^1$ and $I \in [0, 1]$. We assume that Λ is normally hyperbolic in M relative to f, with $\dim(W^s(x)) = n_s$ and $\dim(W^u(x)) = n_u$ for all $x \in \Lambda$, where $W^s(x)$, $W^u(x)$ denote the stable and unstable manifolds of a point $x \in \Lambda$, respectively, and $n = n_u + n_s + 2$.

(A3) The restriction $f_{|\Lambda}$ of f to Λ is an area preserving, monotone twist map, with respect to the angle-action coordinates (ϕ, I).

(A4) The stable and unstable manifolds of Λ, $W^s(\Lambda)$ and $W^u(\Lambda)$, have a differentiably transverse intersection along a 2-dimensional homoclinic channel Γ that is C^1-smooth. We denote by S the scattering map associated to Γ (see Section 6).

(A5) There exists a bi-infinite sequence of Lipschitz primary invariant tori $\{T_i\}_{i\in\mathbb{Z}}$ in Λ, and a bi-infinite, increasing sequence of integers $\{i_k\}_{k\in\mathbb{Z}}$ with the following properties:
   - (i) Each torus T_i intersects the domain U^- and the range U^+ of the scattering map S associated to Γ.
   - (ii) For each $i \in \{i_k + 1, \dots, i_{k+1} - 1\}$, the image of T_i under the scattering map S topologically transverse to T_{i+1}.
   - (iii) For each torus T_i with $i \in \{i_k + 2, \dots, i_{k+1} - 1\}$, the restriction of f to T_i is topologically transitive.
   - (iv) Each torus T_i with $i \in \{i_k + 2, \dots, i_{k+1} - 1\}$, can be C^0-approximated from both sides by other primary invariant tori from Λ.

   We will refer to a finite sequence $\{T_i\}_{i=i_k+1,\dots,i_{k+1}}$ as above as a transition chain of tori.

(A6) The region in Λ between T_{i_k} and T_{i_k+1} contains no invariant primary torus in its interior.

(A7) Inside each region between T_{i_k} and T_{i_k+1} there exists a finite collection of Aubry-Mather sets $\{\Sigma_{\omega_1^k}, \Sigma_{\omega_2^k}, \dots, \Sigma_{\omega_{s_k}^k}\}$, where $s_k \geq 1$, and ω_s^k denotes the rotation number of $\Sigma_{\omega_s^k}$. Each Aubry-Mather set $\Sigma_{\omega_s^k}$ is assumed to lie on some essential circle $C_{\omega_s^k}$, with the circles $C_{\omega_s^k}$ mutually disjoint for all $s \in \{1, \dots, s_k\}$, and $C_{\omega_s^k}$ below $C_{\omega_{s'}^k}$ for all $\omega_s^k < \omega_{s'}^k$.

Instead of (A6) we can consider the following condition:

(A6′) The region Λ_k in Λ between T_{i_k} and T_{i_k+1} contains finitely many invariant primary tori $\{\Upsilon_{h_1^k}, \dots, \Upsilon_{h_{l_k}^k}\}$, where $l_k \geq 1$, satisfying the following properties:
   - (i) Each $\Upsilon_{h_j^k}$ is either one of the following:
     - (a) an isolated invariant primary torus,
     - (b) a hyperbolic periodic orbit together with its stable and unstable manifolds that are assumed to coincide,
   - (ii) Each $\Upsilon_{h_j^k}$ satisfies the following condition:

(a) If $\Upsilon_{h^k_j}$ is a primary invariant torus, then for each $h \in \{h^k_1, \dots, h^k_{l_k}\}$, the inverse image $S^{-1}(\Upsilon_h)$ forms with Υ_h a topological disk below Υ_h, which is mapped by S onto another topological disk above Υ_h, bounded by Υ_h and $S(\Upsilon_h)$.

(b) If $\Upsilon_{h^k_j}$ is a hyperbolic periodic orbit together with its stable and unstable manifolds, then it consists of two curves $\Upsilon^{\text{lower}}_{h^k_j}, \Upsilon^{\text{upper}}_{h^k_j}$ that only have in common the points of the periodic orbit. Then we assume that for each $h \in \{h^k_0, \dots, h^k_{l_{k+1}}\}$ and each $\alpha \in \{\text{lower}, \text{upper}\}$, the inverse image $S^{-1}(\Upsilon^\alpha_h)$ forms with Υ^α_h a topological disk below Υ^α_h, which is mapped by S onto another topological disk above Υ^α_h, bounded by Υ^α_h and $S(\Upsilon^\alpha_h)$.

Now we state the main result of the paper.

Theorem 2.1. *We assume a discrete dynamical system $f : M \to M$, and a sequence of invariant primary tori $(T_i)_{i\in\mathbb{Z}}$ in Λ, satisfying the properties (A1) – (A6), or (A1)-(A5) and (A6′), from above. Then for each sequence $(\epsilon_i)_{i\in\mathbb{Z}}$ of positive real numbers, there exist a point $z \in M$ and a bi-infinite increasing sequence of integers $(N_i)_{i\in\mathbb{Z}}$ such that*

$$d(f^{N_i}(z), T_i) < \epsilon_i, \text{ for all } i \in \mathbb{Z}. \tag{2.1}$$

In addition, if assumption (A7) is satisfied, given a finite sequence of positive integers $\{n^k_s\}_{s=1,\dots,s_k}$ for every $k \in \mathbb{Z}$, there exist $z \in M$ and $(N_i)_{i\in\mathbb{Z}}$ as in (2.1)*, and a finite sequence of positive integers $\{m^k_s\}_{s=1,\dots,s_k}$ for every $k \in \mathbb{Z}$, such that, for each k and each $s \in \{1, \dots, s_k\}$, we have*

$$\pi_\phi(f^j(w^k_s)) < \pi_\phi(f^j(z)) < \pi_\phi(f^j(\bar{w}^k_s)), \tag{2.2}$$

for all j with $N_{i_k} + \sum_{t=0}^{s-1} n^k_t + \sum_{t=0}^{s-1} m^k_t \le j \le N_{i_k} + \sum_{t=0}^{s} n^k_t + \sum_{t=0}^{s-1} m^k_t$, where $w^k_s, \bar{w}^k_s \in \Sigma_{\omega^k_s}$.

Theorem 2.1 asserts that if the conditions (A1)-(A6), or (A1)-(A5) and (A6') are satisfied, then there exists orbits that shadows all tori in the transition chains in the prescribed order, and also cross over the large gaps that separate the successive transition chains. In particular, there exist orbits that travel arbitrarily far with respect to the action variable, and there also exist chaotic orbits. Additionally, if some Aubry-Mather sets are prescribed inside each gap that separates successive transition chains, as in condition (A7), then there exist orbits that, besides shadowing the transition chains, they also shadow the Aubry-Mather sets in the prescribed order. Note that the tori in the transition chains are shadowed in the sense that the diffusing orbit gets arbitrarily close to these tori. However, the Aubry-Mather sets are shadowed in the sense of the cyclical ordering: for each prescribed Aubry-Mather set, the diffusing orbit stays between the orbits of two points in the Aubry-Mather set, relative to the ϕ-coordinate, for any prescribed time interval.

Now we explain each assumption.

Assumption (A1) describes a smooth, discrete dynamical system. In applications, the map f represents the first return map to a Poincaré section associated to a flow. In many examples of interest the flow is a Hamiltonian flow.

Assumption (A2) prescribes the existence of a normally hyperbolic invariant manifold Λ for f, diffeomorphic to an annulus, and parametrized by some angle-action coordinates (ϕ, I). Since Λ is normally hyperbolic, the map f is topologically conjugate to its linearization near Λ (see [44]). This topological conjugacy induces a (non-smooth) system of linearized coordinates (z_c, z_s, z_u) in a neighborhood $\mathcal{N}$ of Λ in M, with $(z_c, 0, 0) \in \Lambda$, $z_s \in E^s_{(z_c,0,0)}$ and $z_u \in E^u_{(z_c,0,0)}$, where E^s, E^u denote the stable and unstable bundles associated to the normally hyperbolic invariant manifold Λ. For a point $z = (z_c, z_s, z_u) \in \mathcal{N}$, we will denote by $\pi_\phi(z)$ the ϕ-coordinate of the point $(z_c, 0, 0) \in \Lambda$; the notation $\pi_I(z)$ is defined similarly. In many examples of nearly integrable Hamiltonian systems, e.g. [2], one can identify a normally hyperbolic invariant manifold Λ_0 in the unperturbed system, and use the standard theory of normal hyperbolicity to establish the persistence of a normally hyperbolic invariant manifold Λ_ε diffeomorphic to Λ_0 for the perturbed system, for all sufficiently small perturbation parameters $\varepsilon \neq 0$. There also exist examples, e.g. from celestial mechanics, were the existence of a normally hyperbolic manifold can be established through a computer assisted proof (see [8]). We note that the assumption (A2) does not require that the stable and unstable manifolds of Λ have equal dimensions, thus our setting includes dynamical systems that are not Hamiltonian.

Assumption (A3) is satisfied automatically in examples like the weakly-coupled pendulum-rotator system considered in [16], or the periodically perturbed geodesic flow on a torus considered in [15, 32]. Some properties of area preserving, monotone twist maps of the annulus are reviewed in Section 4.

Assumption (A4) asserts that the stable and unstable manifolds of Λ have a transverse intersection Γ along a homoclinic manifold Γ. In perturbed systems one often uses a Melnikov method to establish the existence, and the persistence for all sufficiently small values of the perturbation, of a transverse intersection of the invariant manifolds. Assumption (A4) also provides technical conditions for the existence of the scattering map (see [17]). These conditions are generic.

Assumption (A5) prescribes the existence of a bi-infinite collection $\{T_i\}_{i\in\mathbb{Z}}$ of invariant primary Lipschitz tori that can be grouped into transition chains of the type $\{T_i\}_{i=i_k+1,\ldots,i_{k+1}}$.

Assumption (A5)-(i) requires that each torus T_i intersects the domain and the range of the scattering map.

Assumption (A5)-(ii) requires that each torus in a transition chain $\{T_i\}_{i=i_k+1,\ldots,i_{k+1}}$ is mapped by the scattering map topologically transversally across the next torus in the chain. Since the tori are only Lipschitz, topological transversality (topological crossing) is used in place of differentiable transversality. The definition of topological crossing can be found in [7]. Roughly speaking, two manifolds are topologically crossing if they can be made differentiably transverse with non-zero oriented intersection number by the means of a sufficiently small homotopy. In our case the two manifolds are two 1-dimensional arcs of $S(T_i)$ and of T_{i+1} in Λ. From (A5)-(ii), it follows as in [17] that $W^u(T_i)$ has a topologically transverse intersection point with $W^s(T_{i+1})$. Condition (A5)-(iii) requires that each torus in $\{T_i\}_{i=i_k+1,\ldots,i_{k+1}}$, except for the end tori, are topologically transitive. Assumption (A5)-(iv) says that all tori in the transition chain except for the end ones can be C^0-approximated from both ways by some other invariant primary tori in Λ, not necessarily from the transition chain. This means that for each $i \in \{i_k + 2, \ldots, i_{k+1} - 1\}$

there exist two sequences of invariant primary tori $(T_{j_l^-(i)})_{l\geq 1}, (T_{j_l^+(i)})_{l\geq 1}$ in Λ that approach T_i in the C^0-topology, such that the annulus bounded by $T_{j_l^-(i)}$ and $T_{j_l^+(i)}$ contains T_i in its interior for all l.

Assumption (A6) says that every pair of successive transition chains $\{T_i\}_{i=i_{k-1}+1,\dots,i_k}$ and $\{T_i\}_{i=i_k+1,\dots,i_{k+1}}$ is separated by the region between T_{i_k} and T_{i_k+1} which contains no invariant primary torus in its interior. A region in an annulus that is bounded by two invariant primary tori and contains no invariant primary torus in its interior is referred as a Birkhoff Zone of Instability (BZI). The boundary tori have in general only Lipschitz regularity.

Assumptions (A5) and (A6) describe a geometric structure that is typical for the large gap problem for a priori unstable Hamiltonian systems. In such systems, Melnikov theory implies that $W^u(\Lambda)$ intersects transversally $W^s(\Lambda)$ at an angle of order ε, where ε is the size of perturbation. The KAM theorem yields a Cantor family of smooth, invariant primary tori that survives the perturbation. The family of tori is interrupted by 'large gaps' of order $\varepsilon^{1/2}$ located at the resonant regions. Using the transverse intersection between $W^u(\Lambda)$ and $W^s(\Lambda)$, one can find heteroclinic connections between KAM tori that are sufficiently close, within order ε, from one another, and thus form transition chains of tori. Since the large gaps are of order $\varepsilon^{1/2}$ and the splitting size of $W^u(\Lambda)$, $W^s(\Lambda)$ is only order ε, the transition chain mechanism cannot be extended across the large gaps. In our model, the large gaps are modeled by BZI's, as in assumption (A6). This can be achieved by extending the transition chains to maximal transition chains, that go from the boundary of one large gap to the boundary of the next large gap. The intermediate tori in the chain can be chosen as KAM tori: therefore the assumption that these tori are topologically transitive and are C^0-approximable from both sides by other tori is satisfied in such cases. This may not be the case for the tori at the ends of the transition chains.

Assumption (A7) says that inside each BZI between T_{i_k} and T_{i_k+1} there is a prescribed collection of Aubry-Mather sets $\{\Sigma_{\omega_1^k}, \Sigma_{\omega_2^k}, \dots, \Sigma_{\omega_{s_k}^k}\}$ that is vertically ordered. The vertical ordering means that the Aubry-Mather sets lie on essential (non-invariant) circles $C_{\omega_s^k}$ that are graphs over the ϕ-coordinate of the annulus, and with $C_{\omega_s^k}$ below $C_{\omega_{s'}^k}$ provided $\omega_s^k < \omega_{s'}^k$; we write $C_{\omega_s^k} \prec C_{\omega_{s'}^k}$. The vertical ordering of the Aubry-Mather sets is shown for example in [26].

Assumption (A6′) is a relaxation of (A6). Instead of requiring that the region in Λ between T_{i_k} and T_{i_k+1} is a BZI, it allows the existence of finitely many invariant primary tori $\{\Upsilon_{h_j^k}\}_{j=1,\dots,l_k}$ that separate the region into disjoint components. These invariant primary tori are either isolated or else they consist of hyperbolic periodic points together with their stable and unstable manifolds which are assumed to coincide. We require that the image of each $\Upsilon_{h_j^k}$ under S satisfies a certain transversality condition with $\Upsilon_{h_{j+1}^k}$ that allows one to use the scattering map in order to move points from one side of the set to the other side of the set. We note that isolated invariant tori and hyperbolic periodic points whose stable and unstable manifolds coincide do not occur in generic systems.

3. Application

We apply Theorem 2.1 to show the existence of diffusing orbits in an example of a nearly integrable Hamiltonian system. Let

$$H_\varepsilon(p,q,I,\phi,t) \quad = \quad h_0(I) \pm (\frac{1}{2}p^2 + V(q)) + \varepsilon h(p,q,I,\phi,t;\varepsilon), \tag{3.1}$$

where $(p,q,I,\phi,t) \in \mathbb{R}\times\mathbb{T}^1\times\mathbb{R}\times\mathbb{T}^1\times\mathbb{T}^1$ and h is a trigonometric polynomial in (ϕ,t). Here $h_0(I)$ represents a rotator, $P_\pm(p,q) = \pm(\frac{1}{2}p^2+V(q))$ represents a pendulum, and εh a small, periodic coupling. We assume that V, h_0 and h are uniformly C^r for some r sufficiently large. We assume that V is periodic in q of period 1 and has a unique non-degenerate global maximum; this implies that the pendulum has a homoclinic orbit $(p^0(\sigma), q^0(\sigma))$ to $(0,0)$, with $\sigma \in \mathbb{R}$. We also assume that h_0 satisfies a uniform twist condition $\partial^2 h_0/\partial I^2 > \theta$, for some $\theta > 0$, and for all I in some interval (I^-, I^+), with $I^- < I^+$ independent of ε.

The Melnikov potential for the homoclinic orbit $(p^0(\sigma), q^0(\sigma))$ is defined by

$$\begin{aligned}\mathcal{L}(I,\phi,t) = \quad -\int_{-\infty}^{\infty} \quad &\big[h(p^0(\sigma), q^0(\sigma), I, \phi+\omega(I)\sigma, t+\sigma; 0) \\ &-h(0,0,I,\phi+\omega(I)\sigma, t+\sigma;0)\big]\, d\sigma,\end{aligned}$$

where $\omega(I) = (\partial h_0/\partial I)(I)$.

We assume the following non-degeneracy conditions on the Melnikov potential:

(i) For each $I \in (I^-, I^+)$, and each (ϕ,t) in some open set in $\mathbb{T}^1\times\mathbb{T}^1$, the map

$$\tau \in \mathbb{R} \to \mathcal{L}(I, \phi-\omega(I)\tau, t-\tau) \in \mathbb{R}$$

has a non-degenerate critical point τ^*, which can be parameterized as

$$\tau^* = \tau^*(I,\phi,t).$$

(ii) For each (I,ϕ,t) as above, the function

$$(I,\phi,t) \to \frac{\partial \mathcal{L}}{\partial \phi}(I, \phi-\omega(I)\tau^*, t-\tau^*)$$

is non-constant, negative in the case of P_-, and positive in the case of P_+.

This example and the above conditions are considered in [16]. There are some additional non-degeneracy conditions on h and $\partial h/\partial\varepsilon$ that are required in [16]; we do not need to assume those conditions here.

Now we verify the conditions (A1)-(A6) from Section 2 for this model. We will rely heavily on the estimates from [16].

Condition (A1). The time-dependent Hamiltonian in (3.1) is transformed into an autonomous Hamiltonian by introducing a new variable A, symplectically conjugate with t obtaining

$$\tilde{H}_\varepsilon(p,q,I,\phi,A,t) = h_0(I) \pm (\frac{1}{2}p^2 + V(q)) + A + \varepsilon h(p,q,I,\phi,t;\varepsilon), \tag{3.2}$$

where $(p,q,I,\phi,A,t) \in (\mathbb{R}\times\mathbb{T}^1)^3$. We fix an energy manifold $\{\tilde{H}_\varepsilon = \tilde{h}\}$ for some $\tilde{h}$, and restrict to the Poincaré section $\{t=1\}$ for the Hamiltonian flow. The resulting manifold is a 4-dimensional manifold M_ϵ parametrized by some coordinates $(p_\varepsilon, q_\varepsilon, I_\varepsilon, \phi_\varepsilon)$. The first return map to M_ε of the Hamiltonian flow is a smooth map f_ϵ.

Condition (A2). In the unperturbed case $\varepsilon = 0$, the manifold

$$\Lambda_0 := \{(p, q, I, \phi) \,|\, p = q = 0\}$$

is a normally hyperbolic invariant manifold for f_0. The dynamics on Λ_0 is given by an integrable twist map, and Λ_0 is foliated by invariant 1-dimensional tori. In the perturbed system, the standard theory of normal hyperbolicity (see [29]), implies that the manifold Λ_0 can be continued to a normally hyperbolic invariant manifold Λ_ε diffeomorphic to Λ_0, for all ε sufficiently small. Each point in Λ_ε has 1-dimensional stable and unstable manifolds. The regularity of h_0 and the uniform twist condition allows one to apply the KAM theorem and conclude the existence of a KAM family of primary invariant tori in Λ_ε that survives the perturbation, for all ε sufficiently small.

Condition (A3). The map f_ε is symplectic. This plus the twist condition on h_0 implies that f_ε restricted to Λ_ε is an area preserving, monotone twist map.

Condition (A4). The non-degeneracy conditions on the Melnikov function imply that $W^u(\Lambda_\varepsilon)$ and $W^s(\Lambda_\varepsilon)$ have a transverse intersection along a homoclinic manifold Γ_ε, provided ε is sufficiently small. Moreover, it is shown in [16, 17] that the transversality condition (A3)-(i) is satisfied, and that the manifold Γ_ε can be restricted further so that the maps $\Omega^\pm_\varepsilon : \Gamma_\varepsilon \to \Lambda_\varepsilon$ are diffeomorphisms onto their images; thus the scattering map $S_\varepsilon : U^-_\varepsilon \to U^+_\varepsilon$ is a diffeomorphism between some two open sets $U^-_\varepsilon, U^+_\varepsilon \subseteq \Lambda_\varepsilon$, of size $O(1)$. For ε fixed to some sufficiently small value, we let $\Gamma := \Gamma_\varepsilon$ and $S := S_\varepsilon$.

Conditions (A5),(A6) and (A6′) The paper [16] applies an averaging procedure to reduce the dynamics on Λ_ε to a normal form up to $O(\varepsilon^2)$ away from resonances. The averaging procedure fails within the resonant regions, corresponding to the values $I_\varepsilon(k, l)$ of the action variable where $k\omega(I) + l = 0$. A resonance is said to be of order j if the j-th order averaging cannot be applied about the corresponding action level set.

Since h is a trigonometric polynomial, one has to deal with only finitely many resonant regions. Outside the resonant regions one applies the KAM theorem and obtain KAM tori that are at a distance of order $O(\varepsilon^{3/2})$ from one another. The resonant regions yield gaps between KAM tori of size $O(\varepsilon^{j/2})$, where j is the order of the resonance. Only the resonances of order 1 and 2 are of interest, as they produce gaps of size $O(\varepsilon)$ and $O(\varepsilon^{1/2})$ respectively. Inside each resonant region, the system can be approximated by a system similar to a pendulum. In such a region, under appropriate non-degeneracy conditions, it is shown that there exist primary KAM tori close to the separatrices of the pendulum, secondary KAM tori (homotopically trivial), and stable and unstable manifolds of periodic orbits that pass close to the separatrices of the pendulum. Moreover, these objects can be chosen to be $O(\varepsilon^{3/2})$ from one another. In the generic case when the stable and unstable manifolds of such a periodic orbit intersect transversally, the resonant region determines a BZI. In the non-generic case when the stable and unstable manifolds of periodic orbits coincide, the resonant region is as described in (A6′-i-b). The estimates from [16] imply that there exist primary KAM tori that are within $O(\varepsilon^{3/2})$ from the boundaries of the gap, or to the stable and unstable manifolds of the hyperbolic periodic orbits inside the resonant regions. These estimates do not allow one to precisely locate the boundaries of the BZI's or to say anything about their dynamics.

The Melnikov conditions imply that the scattering map S_ε associated to this homoclinic channel Γ_ε can be computed in terms of the Melnikov potential $\mathcal{L}$. If $S_\varepsilon(x^-) = x^+$, then the change in the I_ε-coordinate under S_ε is given by

$$I_\varepsilon(x^+) - I_\varepsilon(x^-) = -\varepsilon \frac{\partial \mathcal{L}}{\partial \phi}(I_\varepsilon, \phi_\varepsilon - \omega(I_\varepsilon)\tau^*, t - \tau^*) + O_{C^1}(\varepsilon^{1+\varrho}), \tag{3.3}$$

for some $\varrho > 0$. Condition (ii) implies that there are points in the domain of the scattering map S whose I_ε-coordinate is increased by $O(\varepsilon)$ under S.

We can use these estimates to construct transition chains of invariant primary tori alternating with gaps, as in (A5) and (A6), or in (A5) and (A6$'$). For ε sufficiently small and fixed, we let $M := M_\varepsilon$, $f := f_\varepsilon$, and Λ be the annulus in Λ_ε bounded by a pair of tori T_{I_a}, T_{I_b} with $I^- < I_a < I_b < I^+$.

First, we choose a sequence of resonant regions and non-resonant regions that intersect the domain U^- and the range U^+ of the scattering map. Since the KAM primary tori are within $O(\varepsilon^{3/2})$ from one another, and the scattering map makes jumps of order $O(\varepsilon)$ in the increasing direction of I_ε, then we can find smooth KAM primary tori $\{T_{i_k+2}, T_{i_k+2}, \ldots, T_{i_{k+1}-1}\}$ such that $W^u(T_i)$ has a transverse intersection with $W^s(T_{i+1})$ for all $i \in \{i_k+2, i_k+3, \ldots, i_{k+1}-1\}$, and that T_{i_k+2} and $T_{i_{k+1}-1}$ are within $O(\varepsilon^{3/2})$ from the separatrices of the penduli corresponding to two consecutive resonant gaps of orderer 1 or 2. The dynamics on each such a torus is quasi-periodic, so is topologically transitive. This ensures condition (A5)-(iii). Moreover, we can choose these KAM tori so that they are 'interior' to the Cantor family of tori, i.e. they can be approximated from both sides by other KAM primary tori. This ensures condition (A5)-(iv). To the transition chain $\{T_{i_k+2}, T_{i_k+3}, \ldots, T_{i_{k+1}-1}\}$ we add, at each end, a torus T_{i_k+1} and a torus $T_{i_{k+1}}$. These end tori bound resonant gaps that are either BZI's or consist of periodic orbits together with their invariant manifolds. Since $T_{i_k+1}, T_{i_{k+1}}$ are within $O(\varepsilon^{3/2})$ from $T_{i_k+2}, T_{i_{k+1}-1}$, respectively, and the scattering map S_ε makes jumps by order $O(\varepsilon)$, it follows that $S(T_{i_k+1})$ topologically crosses T_{i_k+2}, and $S(T_{i_{k+1}-1})$ topologically crosses $T_{i_{k+1}}$. This ensures condition (A5)-(ii). Condition (A1)-(i) is ensured automatically by our initial choice of the resonant regions and the non-resonant regions so that they intersect the domain U^- and the range U^+ of S_ε. The end tori T_{i_k+1} and $T_{i_{k+1}}$ are at the boundaries of two consecutive resonant gaps. This construction is continued for all resonant and non-resonant regions. Thus, for ε fixed and sufficiently small, we obtain sequences of tori $\{T_{i_k+1}, T_{i_k+2}, \ldots, T_{i_{k+1}}\}$ as in (A5), interspersed with gaps between T_{i_k} and T_{i_k+1}, and also between $T_{i_{k+1}}$ and $T_{i_{k+1}+1}$, as in (A6).

We are under the assumption of Theorem 2.1. Then there exists a diffusing orbit that shadows the transition chains of invariant primary tori and crosses the prescribed gaps. In particular, if we choose an initial and a final torus that are $O(1)$ apart, we obtain diffusing orbits whose action variable changes by $O(1)$. We note our theorem applies even for the choice of the pendulum P_-, when the unperturbed Hamiltonian does not have positive-definite normal torsion. The assumption of positive definiteness seems to be very important for variational methods.

We emphasize that, although we are using many of the estimates from [16], we obtain a different mechanism of diffusion. Our mechanism still involves transition chains of invariant primary tori, but uses connecting orbits within the normally hyperbolic invariant manifold to cross over the large gaps. It does not use secondary tori, therefore we do not need to assume the additional non-degeneracy conditions corresponding to these objects as in [16]. Moreover, one can combine the topological mechanism in this paper with the one in [22] and obtain diffusing orbits that visit any given collection of primary tori, secondary tori, invariant manifolds of lower dimensional tori, and Aubry-Mather sets, in any prescribed order.

4. Background on twist maps and Aubry-Mather sets

Let $\tilde{A} = \mathbb{T}^1 \times [0,1] = \{(x,y) \in \mathbb{T}^1 \times [0,1]\}$ be an annulus, and let $A = \mathbb{R} \times [0,1]$ be its universal cover with the natural projection $\pi : A \to \tilde{A}$ given by $\pi(x,y) = (\tilde{x}, \tilde{y})$, where $\tilde{x} = x(\text{mod } 1)$ and $\tilde{y} = y$. Let π_x be the projection onto the first component, and π_y be the projection onto the second component. Let $\tilde{f} : \tilde{A} \to \tilde{A}$ be a continuous mapping on $\tilde{A}$, and let $f : A \to A$ be the unique lift of $\tilde{f}$ to A satisfying $\pi_x(f(0,0)) \in [0,1)$ and $\pi \circ f = \tilde{f} \circ \pi$. Let $\tilde{f}$ be an orientation preserving and boundary preserving mapping of $\tilde{A}$. The map $\tilde{f}$ is called an area preserving, monotone twist map if it satisfies the following properties:

(i) $\tilde{f}$ preserves the area induced by $dx \wedge dy$ on $\tilde{A}$;
(ii) $|\partial(\pi_x \circ \tilde{f})/\partial \tilde{y}| > 0$.

We note that the above properties imply that $\tilde{f}$ is exact symplectic, i.e. $\tilde{f}$ has zero flux, meaning that for any rotational curve γ the area of the regions above γ and below $f(\gamma)$ equals the area below γ and above $f(\gamma)$.

In the sequel we will assume that $\tilde{f}$ is an area preserving, monotone twist map of the annulus. We will also assume that is a positive twist, meaning that $\partial(\pi_x \circ \tilde{f})/\partial \tilde{y} > 0$ for all $(\tilde{x}, \tilde{y})$. In order to simplify the notation, we will not make distinction between $\tilde{A}$ and A, and between $\tilde{f}$ and f.

By a invariant primary torus (essential invariant circle) we mean a 1-dimensional torus $T \subseteq A$ invariant under f in A that cannot be homotopically deformed into a point inside A. Since f is a monotone twist map, each invariant primary torus T is the graph of some Lipschitz function (see [4, 5]).

A region in A between two invariant primary tori T_1 and T_2 is called a Birkhoff Zone of Instability (BZI) provided that there is no invariant primary torus in the interior of the region.

It is known that, for an area preserving monotone twist map f of A, given a BZI, there exist Birkhoff connecting orbits that go from any neighborhood of one boundary torus to any neighborhood of the other boundary torus (see [4, 5, 36]). We have the following results:

Theorem 4.1 (Birkhoff Connecting Theorem)**.** *Suppose that T_1 and T_2 bound a BZI. For every pair of neighborhoods U of T_1 and V of T_2 there exist a point $z \in U$ and an integer $N > 0$ such that $f^N(z) \in V$.*

Corollary 4.2. *Suppose that T_1 and T_2 bound a BZI, and that the restrictions of f to T_1 and T_2 are topologically transitive. For every $\zeta_1 \in T_1, \zeta_2 \in T_2$ and every pair of neighborhoods U of ζ_1 and V of ζ_2, there exist a point $z \in U$ and an integer $N > 0$ such that $f^N(z) \in V$.*

A subset $M \subseteq A$ is said to be monotone (cyclically ordered) if $\pi_x(z_1) < \pi_x(z_2)$ implies $\pi_x(f(z_1)) < \pi_x(f(z_2))$ for all $z_1, z_2 \in M$. For $z \in A$ the extended orbit of z is the set $EO(z) = \{f^n(z) + (j,0) : n,j \in \mathbb{Z}\}$. The orbit of z is said to be monotone (cyclically ordered) if the set $EO(z)$ is monotone. If the orbit of $z \in A$ is monotone, then the rotation number $\rho(z) = \lim_{n\to\infty}(\pi_x(f^n(z))/n)$ exists. We denote $\mathrm{Rot}(\omega) = \{z \in A : \rho(z) = \omega\}$. All points in the same monotone set have the same rotation number.

Definition 4.3. An Aubry-Mather set for $\omega \in \mathbb{T}^1$ is a minimal, monotone, f-invariant subset of $\mathrm{Rot}(\omega)$.

Here by a minimal set we mean a closed invariant set that does not contain any proper closed invariant subsets. (Equivalently, the orbit of every point in the set is dense in the set.) This should not be confused with action-minimizing or h-minimal sets, where h is a generating function for f.

Since the restrictions of f to the boundary components of the annulus A are orientation preserving homeomorphisms, the rotation numbers ω^- and ω^+ of these restrictions are well defined. We denote ω^- the lowest and ω^+ the highest of the two rotation numbers.

Theorem 4.4 (Aubry-Mather Theorem)**.** *For every $\omega \in [\omega^-, \omega^+]$, there exists a non-empty Aubry-Mather set Σ_ω in Rot(ω).*

Aubry-Mather sets defined as above can be obtained as limits of monotone Birkhoff periodic orbits [33]. There may be many Aubry-Mather sets with the same rotation number [41]. On the other hand, if one requires Aubry-Mather sets to be action minizing, there exists a unique recurrent Aubry-Mather set for any given irrational rotation number.

In the sequel we will use the following result on the vertical ordering of Aubry-Mather sets from [26].

Theorem 4.5. *There exists a family of essential circles C_ω in A for $\omega \in [\omega^-, \omega^+]$ such that:*

(i) *Each C_ω is a graph over $y = 0$;*
(ii) *The circles C_ω are mutually disjoint, and if $\omega' > \omega$ then $C_{\omega'}$ is above C_ω;*
(iii) *Each C_ω contains an Aubry-Mather set Σ_ω.*

The above circles have Lipschitz regularity, and are projections of so called 'ghost circles' that are objects in $\mathbb{R}^{\mathbb{Z}}$. See [26] for details. A similar result to Theorem 4.5 appears in [34] who find Aubry-Mather sets lying on pseudo-graphs that are (not strictly) vertically ordered.

There are some analogues of the Birkhoff Connecting Theorem for Aubry-Mather sets. The following lemma is used in [32] to provide a topological proof for Mather Connecting Theorem stated below.

Lemma 4.6. *Suppose that T_1 and T_2 bound a BZI. Let Σ_ω be an Aubry-Mather set of rotation number ω inside the BZI. Let p be a recurrent point in Σ_ω and $V_\epsilon(p)$ be an ε-neighborhood of p, for some $\varepsilon > 0$. The following hold true:*

(i) *For some positive number $n^+ = n^+(p, \varepsilon)$ (resp. $n^- = n^-(p, \varepsilon)$) the set $\bigcup_{j=0}^{n^+} f^j(V_\varepsilon(p))$ (resp. $\bigcup_{j=0}^{n^-} f^{-j}(V_\varepsilon(p))$) separates the cylinder.*
(ii) *The set $V_\varepsilon^+ := \bigcup_{j=0}^{\infty} f^j(V_\varepsilon(p))$ (resp. the set $V_\varepsilon^- := \bigcup_{j=0}^{\infty} f^{-j}(V_\varepsilon(p))$), is connected and open.*
(iii) *The closure of V_ε^+ (resp. V_ε^-) contains both boundary tori T_1 and T_2.*
(iv) *The set $V_\varepsilon^\infty := \bigcup_{j=-\infty}^{\infty} f^j(V_\varepsilon(p))$ is invariant, and both V_ε^+ and V_ε^- are open and dense in V_ε^∞.*

The following result says that there exist orbits that visit any prescribed bi-infinite sequence of Aubry-Mather sets inside a BZI (see [42, 47, 28, 34, 32]).

Theorem 4.7 (Mather Connecting Theorem)**.** *Suppose that T_1 and T_2 bound a BZI, and $\{\Sigma_{\omega_i}\}_{i\in\mathbb{Z}}$ is a bi-infinite sequence of Aubry-Mather sets inside the BZI. Let $\varepsilon_i > 0$ for $i \in \mathbb{Z}$. Then there exist a point z inside the BZI and an increasing bi-infinite sequence of integers $\{j_i\}_{i\in\mathbb{Z}}$ such that $f^{j_i}(z)$ is within ε_i from Σ_{ω_i} for all $i \in \mathbb{Z}$.*

The Aubry-Mather sets in Theorem 4.7 are action minimizing. The following topological version of Mather Connecting Theorem, due to Hall [28], provides shadowing orbits of Aubry-Mather sets that are not necessarily action minimizing. This approach can be implemented in rigorous computer experiments [31].

Theorem 4.8. *Suppose that T_1 and T_2 bound a BZI, and $\{z_s\}_{s\in\mathbb{Z}}$ is a bi-infinite sequence of monotone (p_s/q_s)-periodic points, with the rotation numbers p_s/q_s mutually distinct, inside the BZI. Given a bi-infinite sequence $\{n_s\}_{s\in\mathbb{Z}}$ of positive integers, then there exist a point z and a bi-infinite sequence $\{m_s\}_{s\in\mathbb{Z}}$ of positive integers such that, for each $s \geq 0$,*

$$\begin{gathered}\pi_x(f^j(w_s)) < \pi_x(f^j(z)) < \pi_x(f^j(\bar{w}_s)) \text{ for} \\ \sum_{t=0}^{s-1} n_t + \sum_{t=0}^{s-1} m_t \leq j \leq \sum_{t=0}^{s} n_t + \sum_{t=0}^{s-1} m_t,\end{gathered} \tag{4.1}$$

where $w_s, \bar{w}_s$ are some points in the extended orbit of z_s.

A similar statement holds for each $s < 0$.

In the above, n_s represents the number of iterates for which the orbit of z shadows – in the sense of the cyclical ordering – the extended orbit of z_s, and m_s represents the number of iterates it takes the orbit of z to pass from the extended orbit of z_s to the extended orbit of z_{s+1}.

They main tool used in Hall's arguments is that of a positive (negative) diagonal. Assume that the tori T_1 and T_2 at the boundary of the BZI are given by $y = 0$ and $y = 1$, respectively. Let

$$I_z = \{w \,|\, \pi_x(w) = \pi_x(z)\}, \tag{4.2}$$

$$I_z^+ = \{w \in I_z \,|\, \pi_y(w) \geq \pi_y(z)\}, \tag{4.3}$$

$$I_z^- = \{w \in I_z \,|\, \pi_y(w) \leq \pi_y(z)\}, \tag{4.4}$$

$$B_{z_1,z_2} = \{w \,|\, \pi_x(z_1) < \pi_x(w) < \pi_x(z_2)\}, \tag{4.5}$$

where z, z_1, z_2 are points in the annulus.

A positive diagonal D in B_{z_1,z_2} is a set $D \subseteq \mathrm{cl}(B_{z_1,z_2})$ such that

(i) D is simply connected and the closure of its interior;
(ii) $\mathrm{bd}(D) \cap \mathrm{cl}(B_{z_1,z_2}) \subseteq I_{z_1}^- \cup I_{z_2}^+ \cup \{y = 0\} \cup \{y = 1\}$;
(iii) $\mathrm{bd}(D) \cap I_{z_1}^- \neq \emptyset$ and $\mathrm{bd}(D) \cap I_{z_2}^+ \neq \emptyset$.

The set $\partial D \cap B_{z_1,z_2}$ has exactly two components connecting $I_{z_1}^- \cup \{y = 0\}$ to $I_{z_2}^+ \cup \{y = 1\}$, which are called the upper and lower edges of D, respectively. We say that these components 'stretch across' B_{z_1,z_2}.

A negative diagonal and its upper and lower edges are defined similarly.

An important feature of positive diagonals is the following hereditary property. Given z_1, z_2 such that $\pi_x(z_1) < \pi_x(z_2)$ and $\pi_x(f(z_1)) < \pi_x(f(z_2))$, if D is a positive diagonal in B_{z_1,z_2}, then $f(D) \cap B_{f(z_1),f(z_2)}$ has a component D' that is a positive diagonal in $B_{f(z_1),f(z_2)}$. Moreover, if D has the upper edge contained in $f^k(I_{w_0}^+)$

and the lower edge contained in $f^k(I^-_{w_1})$, and $\partial D\cap B_{z_1,z_2}\subseteq f^k(I^+_{w_0}\cup I^-_{w_1})$, for some w_0, w_1 with $\pi_x(w_0)<\pi_x(w_1)$ and some $k>0$, then D' can be chosen so that its upper edge is contained in $f^{k+1}(I^+_{w_0})$ and its lower edge is contained in $f^{k+1}(I^-_{w_1})$. A similar property holds for negative diagonals.

The proof in [28] is an inductive argument which, for a given pair of adjacent points $w_0,\bar w_0$ in the extended orbit of z_0, and for each $\sigma\geq 0$, produces a nested sequence $D_0\supseteq D_1\supseteq\ldots\supseteq D_\sigma$ of negative diagonals of $B_{w_0,\bar w_0}$ such that, for each $s\in\{0,\ldots,\sigma\}$, the following hold: (a) the orbit of each point $z\in D_s$ satisfies the ordering relation (4.1), and (b) there is a sufficiently large $j_s>0$ such that $f^{j_s+j}(D_s)$ contains a component that is a positive diagonal in $B_{f^j(w_s),f^j(\bar w_s)}$, for some adjacent points $w_s,\bar w_s\in EO(z_s)$, and for all $j=1,\ldots,n_s$. In the above, $j_s=\sum_{t=0}^{s-1}n_t+\sum_{t=0}^{s-1}m_t$. Moreover, in this inductive argument one can choose the diagonal sets D_s so that $f^{j_s+j}(D_s)$ has the upper edge contained in $f^{j_s+j}(I^+_{w_0})$, lower edge contained in $f^{j_s+j}(I^-_{\bar w_0})$, and $\partial f^{j_s+j}(D_s)\cap B_{f^{j_s+j}(w_0),f^{j_s+j}(\bar w_0)}\subseteq f^{j_s+j}(I^+_{w_0}\cup I^-_{\bar w_0})$.

For the basis step, starting with $w_0,\bar w_0$ and applying the hereditary property from above n_0 times, one obtains a negative diagonal set D_0 of $B_{w_0,\bar w_0}$ with the properties that each point $z\in D_0$ satisfies the ordering relation (4.1) for $s=0$, and $f^{n_0}(D_0)$ has a component that is a positive diagonal of $B_{f^{n_0}(w_0),f^{n_0}(\bar w_0)}$.

For the inductive step, one assumes a negative diagonal D_σ of $B_{w_0,\bar w_0}$ as above, and wants to produce a negative diagonal $D_{\sigma+1}\subseteq D_\sigma$ of $B_{w_0,\bar w_0}$ which fulfils the corresponding properties. The key idea is to use the existence of points near $y=0$ that get near $y=1$, and of points near $y=1$ that get near $y=0$, as provided by Theorem 4.1, in order to show that for some j_σ sufficiently large $f^{j_\sigma}(D_\sigma)$ contains a component that stretches all the way across a fundamental interval of the annulus. Hence $f^{j_\sigma}(D_\sigma)$ contains a subset that is a positive diagonal of $B_{w_{\sigma+1},\bar w_{\sigma+1}}$ for two adjacent points $w_{\sigma+1},\bar w_{\sigma+1}\in EO(z_{\sigma+1})$. From this it follows that $f^{j_\sigma+j}(D_\sigma)$ contains a component that is a positive diagonal in $B_{f^j(w_{\sigma+1}),f^j(\bar w_{\sigma+1})}$ for all $j=1,\ldots,n_{\sigma+1}$. This completes the inductive step.

Applying a similar argument for the negative iterates of f produces a nested sequence of positive diagonals of $B_{w_0,\bar w_0}$. A positive diagonal of $B_{w_0,\bar w_0}$ always has a non-empty intersection with a negative diagonal of $B_{w_0,\bar w_0}$. This implies the existence of points z whose forward orbits satisfy the ordering conditions in (4.1) and whose backwards orbits satisfy similar ordering conditions.

Using limit arguments as in [33], one can obtain shadowing of Aubry-Mather sets of irrational rotation numbers as well. These topological ideas will be used in the proof of Theorem 5.2 below.

Remark 4.9. The results in this section hold if we replace conditions (i) and (ii) from the definition of an area preserving, monotone twist map with the following weaker conditions:

(i') f satisfies the following condition B: for every $\varepsilon>0$ there exist $z_1,z_2\in A$ and $n_1,n_2>0$ such that $\pi_y(z_1)<\varepsilon$ and $\pi_y(f^{n_1}(z_1))>1-\varepsilon$, and $\pi_y(z_2)>1-\varepsilon$ and $\pi_y(f^{n_2}(z_2))<\varepsilon$.

(ii') f satisfies the following positive tilt condition: if we denote by θ_z the angle deviation from the vertical, measured from $(0,1)$ to $Df_z(0,1)$, with the clockwise direction taken as the positive direction, and defined in such a way that $\theta_{(x,0)}\in[-\pi/2,\pi/2]$ and θ is continuous, then $\theta_z>0$ at all points.

Compositions of positive twist maps, are for example, positive tilt maps. We shall note that the Aubry-Mather theory applies for positive tilt maps as well (see [30]).

Remark 4.10. Aubry-Mather theory and the above shadowing result also hold for generalized twist maps of the higher dimensional annulus $S^1 \times \mathbb{R}^n$. See [1].

Remark 4.11. The topological approach in this section does not yield trajectories that get close to each set in the prescribed collection of Aubry-Mather sets, as in Theorem 4.7. It seems possible, however, that these topological methods can be combined with the variational methods in [42] to obtain trajectories that go very close to each Aubry-Mather set in the given collection.

5. Existence of Birkhoff connecting orbits

In this section we state and prove an extension of the Corollary 4.2 of the Birkhoff connecting orbit theorem, and an extension of Mather's theorem on shadowing of Aubry-Mather sets. The methodology is based on the topological approach of Hall and on the Jordan curve theorem. The statements below will be used in the proof of the main theorem.

In Corollary 4.2, it was assumed that the restrictions of the map to the boundary tori of the BZI are topologically transitive, and it was inferred the existence of connecting orbits from an arbitrarily small neighborhood of some prescribed point on one boundary torus to an arbitrarily small neighborhood of some prescribed point on the other boundary torus. In the statements below we prove the same result without the topological transitivity assumption.

Theorem 5.1. *Suppose that T_1 and T_2 bound a BZI $\mathcal{Z}$. Assume that $\zeta_1 \in T_1$ and $\zeta_2 \in T_2$. For every pair of neighborhoods U of ζ_1 and V of ζ_2, there exists a point $z \in U$ and an integer $N > 0$ such that $f^N(z) \in V$ for some $N > 0$ that can be chosen arbitrarily large. Moreover, there exists a point $z' \in bd(U)$ such that $f^N(z') \in bd(V)$.*

Proof. We consider a one-sided compact neighborhood $U_0 \subseteq U$ of ζ_1 that is homeomorphic to a closed disk, i.e., $U_0 = \mathrm{cl}(B_{\varepsilon_1}(\zeta_1) \cap \mathcal{Z})$ where $\mathrm{cl}(B_{\varepsilon_1}(\zeta_1)) \subseteq U$. The boundary of U_0 consists of a curve segment in T_1 and a simple curve γ_0 contained in $\mathcal{Z}$. Similarly, we consider a one-sided compact neighborhood V_0 of ζ_2 that is homeomorphic to a closed disk, whose boundary consists of a curve segment in T_1 and a simple curve η_0 contained in $\mathcal{Z}$.

Assume first that the interior of U_0 meets some Aubry-Mather set $\Sigma_{\rho_1} \subseteq \mathcal{Z}$, and that the interior of V_0 meets some Aubry-Mather set $\Sigma_{\rho_2} \subseteq \mathcal{Z}$. Since U_0 and V_0 are neighborhoods of points in the Aubry-Mather sets Σ_{ρ_1} and Σ_{ρ_2} respectively, Theorem 4.7 yields the existence of a forward orbit that goes from U_0 to V_0, as well as one from U to V. By the Jordan curve theorem, there also exists a forward orbit that goes from ∂U to ∂V.

Assume now that the interiors of U_0, V_0 do not meet any Aubry-Mather set.

We choose three Aubry-Mather sets Σ_{ρ_1}, $\Sigma_{\rho_1'}$, $\Sigma_{\rho_1''}$ in $\mathcal{Z}$ near T_1, lying on three essential circles $C_{\rho_1}, C_{\rho_1'}, C_{\rho_1''}$, respectively, with $\rho_1 < \rho_1' < \rho_1''$ irrational rotation numbers, and $C_{\rho_1} \prec C_{\rho_1'} \prec C_{\rho_1''}$. The existence of such vertically ordered Aubry-Mather sets follows from Theorem 4.5.

Let p_1 be a point in $\Sigma_{\rho_1''}$. Let $W(p_1)$ be a small neighborhood of p_1 inside the BZI, which does not intersect Σ_{ρ_1} and $\Sigma_{\rho_1'}$. By assumption, U_0 does not meet any of the sets Σ_{ρ_1}, $\Sigma_{\rho'_1}$, $\Sigma_{\rho''_1}$. By Lemma 4.6 (iii) the closure of $\bigcup_{j=0}^{\infty} f^{-j}(W(p_1))$ contains T_1, and in particular ζ_1. Since U_0 is a one-sided neighborhood of ζ_1, there exists $j_1 > 0$ such that $f^{j_1}(U_0) \cap W(p_1) \neq \emptyset$. In the covering space of the annulus, $f^{j_1}(U_0)$ intersects some copy $W^{h_1} := W(p_1) + (h_1, 0)$ of $W(p_1)$, where h_1 is some positive integer. Since U_0 does not intersect Σ_{ρ_1} and $\Sigma_{\rho'_1}$, it follows that $f^{j_1}(U_0)$ does not intersect Σ_{ρ_1} and $\Sigma_{\rho'_1}$. On the other hand, $f^{j_1}(U_0)$ intersects the essential circles $C_{\rho_1}, C_{\rho'_1}$, containing the Aubry-Mather seta $\Sigma_{\rho_1}, \Sigma_{\rho'_1}$, respectively.

Let $\gamma_1 : [0, 1] \to U_0$ be a vertical curve, i.e., $\pi_x(\gamma_1(t)) = x_1$ for some $x_1 \in T_1$ and all t, such that $f^{j_1}(\gamma_1(1))$ is an intersection point of $f^{j_1}(U_0)$ with W^{h_1}. The curve $f^{j_1}(\gamma_1(t))$ crosses both essential circles C_{ρ_1} and $C_{\rho'_1}$. Since $f^{j_1}(U_0)$ is disjoint from Σ_{ρ_1}, the intersections between $f^{j_1}(\gamma_1(t))$ and C_{ρ_1} occur within the 'gaps' of Σ_{ρ_1}, i.e. within the interval components of $C_{\rho_1} \setminus \Sigma_{\rho_1}$. We can assign and oriented intersection number for each intersection point between $f^{j_1}(\gamma_1(t))$ and C_{ρ_1}. We set the intersection number to be $+1$ at a point where the curve moves from below C_{ρ_1} to above C_{ρ_1} as t increases, and to be -1 at a point where the curve moves from above C_{ρ_1} to below C_{ρ_1} as t increases. We can also assign an oriented intersection number between $f^{j_1}(\gamma_1(t))$ and each gap of Σ_{ρ_1}, by adding the oriented intersection numbers for all of the intersection points within that gap. Let $a^1_{\rho_1}, b^1_{\rho_1}$ be the endpoints of the leftmost gap of Σ_{ρ_1} that is crossed by $f^{j_1}(\gamma_1(t))$ with oriented intersection number equal to $+1$. Similarly, let $a^1_{\rho'_1}, b^1_{\rho'_1}$ be the endpoints of the leftmost gap of $\Sigma_{\rho'_1}$ that is crossed by $f^{j_1}(\gamma_1(t))$ with oriented intersection number equal to $+1$. Note that the curve $f^{j_1}(\gamma_1(t))$ is a positively tilted curve, i.e., the angle deviation from the vertical $\theta(t)$ along the curve is always positive (see Remark 4.9). This implies that $\pi_x(a^1_{\rho_1}) < \pi_x(b^1_{\rho'_1})$.

The conclusion of this step is that the curve $f^{j_1}(\gamma_1(t))$ passes through the gap between $a^1_{\rho_1}$ and $b^1_{\rho_1}$ of Σ_{ρ_1}, from below C_{ρ_1} to above C_{ρ_1}, and then passes through the gap between $a^1_{\rho'_1}$ and $b^1_{\rho'_1}$ of $\Sigma_{\rho'_1}$, from below $C_{\rho'_1}$ to above $C_{\rho'_1}$.

Now we consider a one-sided rectangular neighborhood $U_1 \subseteq U_0$ of some point in T_1, bounded below by T_1, to the left by γ_1, and to the right by some other vertical curve segment γ'_1. If γ'_1 is sufficiently close to γ_1, then, by continuity, the image of each vertical curve in U_1 under f^{j_1} crosses the gap between $a^1_{\rho_1}$ and $b^1_{\rho_1}$) of Σ_{ρ_1} with oriented intersection number $+1$, and crosses the gap between $a^1_{\rho'_1}$ and $b^1_{\rho'_1}$ of $\Sigma_{\rho'_1}$ with oriented intersection number $+1$. We choose and fix a set $U_1 \subseteq U_0$ with these properties. By Lemma 4.6 (iii) the closure of $\bigcup_{j=0}^{\infty} f^{-j}(W(p_1))$ contains T_1, so there exists $j_2 > j_1$ such that $f^{j_2}(U_1) \cap W(p_1) \neq \emptyset$. In the covering space of the annulus, $f^{j_2}(U_1)$ intersects some copy $W^{h_2} := W(p_1) + (h_2, 0)$ of $W(p_1)$ for some positive integer $h_2 > h_1$.

Then there exists a vertical curve $\gamma_2 : [0, 1] \to U_1$, such that $f^{j_2}(\gamma_2(1))$ is an intersection point of $f^{j_2}(U_1)$ with W^{h_2}. The curve $f^{j_2}(\gamma_2(t))$ crosses C_{ρ_1} and $C_{\rho'_1}$. Let $a^2_{\rho_1}, b^2_{\rho_1}$ be the endpoints of the leftmost gap of Σ_{ρ_1} that is crossed by $f^{j_2}(\gamma_2(t))$ with oriented intersection number equal to $+1$, and let $a^2_{\rho'_1}, b^2_{\rho'_1}$ be the endpoints of the leftmost gap of $C_{\rho'_1}$ that is crossed by $f^{j_2}(\gamma_2(t))$ with oriented intersection number equal to $+1$. The image curve $f^{j_2}(\gamma_2)$ is a positively tilted curve located on the 'right side' of the positively tilted curve $f^{j_2}(\gamma_1)$, in the sense that any graph over x that intersects both $f^{j_2}(\gamma_1)$ and $f^{j_2}(\gamma_2)$ has the left-most intersection point with the $f^{j_1}(\gamma_1)$. Therefore, the gap endpoints $a^2_{\rho_1}, b^2_{\rho_1}$ are either the image under

$f^{j_2-j_1}$ of the gap endpoints $a^1_{\rho_1}, b^1_{\rho_1}$ found at the previous step, or are the image under $f^{j_2-j_1}$ of some other gap endpoints of Σ_{ρ_1} located to the right of the gap between $a^1_{\rho_1}$ and $b^1_{\rho_1}$). Similarly, the gap endpoints $a^2_{\rho'_1}, b^2_{\rho'_1}$ are either the image under $f^{j_2-j_1}$ of the gap endpoints $a^1_{\rho'_1}, b^1_{\rho'_1}$ from the previous step, or are the image under $f^{j_2-j_1}$ of some other gap of $\Sigma_{\rho'_1}$ located to the right of the gap between $a^1_{\rho'_1}$ and $b^1_{\rho'_1}$.

Then, there exists a one-sided rectangular neighborhood $U_2 \subseteq U_1$ of some point in T_1, bounded below by T_1, to the left by γ_2, and to the right by some other vertical curve segment γ'_2, such that the image of each vertical curve in U_2 under f^{j_2} crosses the gap between $a^2_{\rho_1}$ and $b^2_{\rho_1}$ of Σ_{ρ_2} with oriented intersection number $+1$, and crosses the gap between $a^2_{\rho'_1}$ and $b^2_{\rho'_1}$ of $\Sigma_{\rho'_1}$ with oriented intersection number $+1$.

Inductively, we obtain a nested sequence of one-sided neighborhoods of points in T_1, denoted $U_1 \supseteq U_2 \supseteq \ldots U_m \supseteq \ldots$, all contained in U_0, and two sequences of positive integers $j_1 < j_2 < \ldots < j_m < \ldots$ and $h_1 < h_2 < \ldots < h_m < \ldots$ with the following properties:

(i) each set U_m is a topological rectangle consisting of vertical curves starting from T_1, bounded on the left-side by a vertical curve γ_m and on the right by a vertical curve γ'_m;
(ii) $f^{j_m}(U_m) \cap W^{h_m} \neq \emptyset$, where $W^{h_m} := W(p_1) + (h_m, 0)$;
(iii) the image of each vertical curve in U_m under f^{j_m} crosses C_{ρ_1} with oriented intersection number $+1$ through a gap between $a^m_{\rho_1}$ and $b^m_{\rho_1}$ of Σ_{ρ_1} , and it crosses $C_{\rho'_1}$ with oriented intersection number $+1$ through a gap between $a^m_{\rho'_1}$ and $b^m_{\rho'_1}$) of $\Sigma_{\rho'_1}$;
(iv) $\pi_x(a^m_{\rho_1}) < \pi_x(b^m_{\rho'_1})$;
(v) the gap endpoints of $a^m_{\rho_1}, b^m_{\rho_1}$ are the images under $f^{j_m-j_{m-1}}$ of the gap endpoints of $a^{m-1}_{\rho_1}, b^{m-1}_{\rho_1}$, or of the endpoints of some other gap in Σ_{ρ_1} located to the right side of this gap; the gap endpoints of $a^m_{\rho'_1}, b^m_{\rho'_1}$ are the images under $f^{j_m-j_{m-1}}$ of the endpoints of the gap between $a^{m-1}_{\rho_1}$ and $b^{m-1}_{\rho'_1}$, or of the endpoints of some other gap in $\Sigma_{\rho'_1}$ located to the right side of this gap.

The endpoints of a gap of Σ_{ρ_1} or $\Sigma_{\rho'_1}$ are mapped by f into the endpoints of some other gap of Σ_{ρ_1} or $\Sigma_{\rho'_1}$, respectively. Also, the order of the gaps is preserved under iteration. The endpoints of the gap in Σ_{ρ_1} are iterated with rotation number ρ_1, and the endpoints of the gap in $\Sigma_{\rho'_1}$ are iterated with rotation number $\rho'_1 > \rho_1$. Then, for some sufficiently large iterate j_m the order of the gaps gets reversed in the annulus. That is, we have the following ordering in terms of the angle coordinate in the annulus:

(i) $a^m_{\rho_1} < b^m_{\rho_1} < a^1_{\rho_1} < b^1_{\rho_1}$,
(ii) $a^1_{\rho'_1} < b^1_{\rho'_1} < a^m_{\rho'_1} < b^m_{\rho'_1}$.

This implies that $f^{j_1}(U_1) \cup f^{j_m}(U_m)$ forms an 'arch' over some part of Σ_{ρ_1}. That is, $f^{j_1}(U_1) \cup f^{j_m}(U_m)$ determines a closed neighborhood $\mathcal{U}$ of some point $\xi_1 \in \Sigma_{\rho_1}$, homoeomorphic to a closed disk, whose boundary consists of a curve segment of T_1 and a finite union of sub-arcs of the boundaries of $f^{j_1}(U_1)$ and $f^{j_m}(U_m)$. See Figure 1.

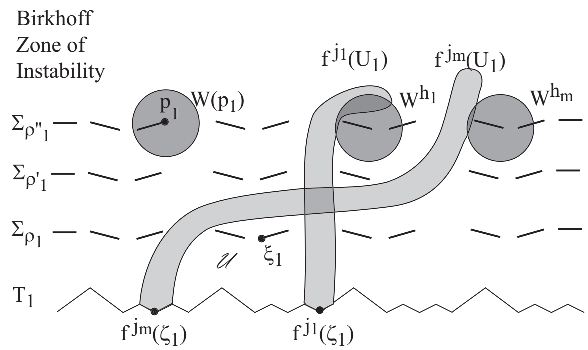

FIGURE 1. An arch over some part of an Aubry-Mather set

We carry on an analogous argument about T_2 starting with the one-sided neighborhood V_0 of $\zeta_2 \in T_2$ and iterating it backwards in time. We obtain an 'arch' over some part of an Aubry-Mather set Σ_{ρ_2} near T_2. The arch is a closed neighborhood $\mathcal{V}$ of some point $\xi_2 \in \Sigma_{\rho_2}$, homoeomorphic to a closed disk, whose boundary consists of a curve segment of T_2 and a finite union of sub-arcs in the boundaries of $f^{-l_1}(V_1)$ and $f^{-l_m}(V_p)$, for some $l_1 < l_2 < \ldots < l_p$ and $V_1 \supseteq V_2 \supseteq \ldots \supseteq V_p \supseteq \ldots$, all contained in V_0.

By Theorem 4.7 there is an orbit that goes from the interior of $\mathcal{U}$ to the interior of $\mathcal{V}$. By the Jordan curve theorem there is another orbit from the boundary of $\mathcal{U}$ to the boundary of $\mathcal{V}$. Since the boundary of $\mathcal{U}$ is made of pieces of the boundaries of $f^{j_1}(U_1)$ and $f^{j_m}(U_m)$, and the boundary of $\mathcal{V}$ is made of pieces of the boundaries of $f^{-l_1}(V_1)$ and $f^{-l_p}(V_p)$, it means that there is a forward orbit from the boundary of U_1 or of U_m to the boundary of V_1 or of V_p. Therefore there is a forward orbit from the neighborhood U of $\zeta_1 \in T_1$ to the neighborhood V of $\zeta_2 \in T_2$. Applying the Jordan curve theorem again yields an orbit from $\mathrm{bd}(U)$ to $\mathrm{bd}(V)$.

The remaining case of the proof, when the interior of U_0 does intersects some Aubry-Mather set and the interior of V_0 does not, or when the interior of U_0 does intersect some Aubry-Mather set and the interior of V_0 does not, follows easily from the above arguments. □

The next statement says that given two points on the boundary tori of a BZI, and a finite sequence of Aubry-Mather sets inside the zone, there exists an orbit that starts in a prescribed neighborhood of the point on the lower boundary torus, then moves on and shadows, in the sense of the ordering of the orbit, each Aubry-Mather set in the sequence, and ends in a prescribed neighborhood of the point on the upper boundary torus. This result extends Theorem 4.8, and relies on the topological argument of Hall. As in the previous theorem, we do not need any extra conditions on the dynamics on the boundary tori. The resulting shadowing orbits are not necessarily minimal.

Theorem 5.2. *Suppose that T_1 and T_2 bound a BZI $\mathcal{Z}$. Let $\zeta_1 \in T_1, \zeta_2 \in T_2$, U be a neighborhood of ζ_1, and V a neighborhood of ζ_2. Let $\{\Sigma_{\omega_s}\}_{s \in \{1,\ldots,\sigma\}}$ be a finite sequence of Aubry-Mather sets inside $\mathcal{Z}$ such that each Σ_{ω_s} lies on some essential circle C_{ω_s} that is a graph over the x-coordinate, with $C_{\omega_s} \prec C_{\omega'_s}$ provided $\omega_s < \omega'_s$.*

Let $\{n_s\}_{s=1,\dots,\sigma}$ be sequence of positive integers. Then there exist a point $z \in U$, and a sequence of positive integers $\{m_s\}_{s=0,\dots,\sigma}$, such that, for each $s \in \{1,\dots,\sigma\}$,

$$\pi_x(f^j(w_s)) < \pi_x(f^j(z)) < \pi_x(f^j(\bar{w}_s)) \text{ for}$$
$$\sum_{t=1}^{s-1} n_t + \sum_{t=0}^{s-1} m_t \leq j \leq \sum_{t=1}^{s} n_t + \sum_{t=0}^{s-1} m_t, \tag{5.1}$$

where w_s and $\bar{w}_s$ are some points in the Aubry-Mather set Σ_{ω_s}, and $f^N(z) \in V$ for $N = \sum_{t=1}^{\sigma} n_t + \sum_{t=0}^{\sigma} m_t$. The number N can be chosen arbitrarily large.

Moreover, there exists a point $z' \in bd(U)$ satisfying the ordering condition (5.1) such that $f^N(z') \in bd(V)$.

Proof. We use the construction of diagonals described in the sketch of the proof of Theorem 4.8; for details see [28].

Part 1. We first prove the existence of a point $z \in U$ satisfying (5.1) and such that $f^N(z) \in V$. We choose a one-sided neighborhood U_0 of $\zeta_1 \in T_1$ that is homeomorphic to a closed disk and whose closure is contained in U. Let C_{ω_1} be the essential circle containing Σ_{ω_1}. We choose an Aubry-Mather set Σ_{ρ_1} lying on some essential circle C_{ρ_1}, such that $C_{\omega_1} \prec C_{\rho_1}$. We choose a point $p_1 \in \Sigma_{\rho_1}$ and a small neighborhood $W(p_1)$ of p_1 which does not intersect Σ_{ω_1}. We assume that the interior of U_0 does not meet Σ_{ω_1} and Σ_{ρ_1}, otherwise the proof follows as in [28]. By Lemma 4.6 (iii) the closure of $\bigcup_{j=0}^{\infty} f^{-j}(W(p_1))$ contains T_1, and in particular ζ_1. Proceeding as in the proof of Theorem 5.1, we obtain a nested sequence $U_1 \supseteq U_2 \supseteq \ldots \supseteq U_i$ of one-sided neighborhoods of points in T_1, all contained in U_0, and two sequences of positive integers $j_1 < j_2 < \ldots < j_i < \ldots$ and $h_1 < h_2 < \ldots < h_i < \ldots$ with the following properties:

- (i) each set U_i is a topological rectangle consisting of vertical curves starting from T_1, bounded on the left-side by a vertical curve γ_i and on the right by a vertical curve γ_i';
- (ii) $f^{j_i}(U_i) \cap W^{h_i} \neq \emptyset$, where $W^{h_i} := W(p_1) + (h_i, 0)$;
- (iii) the image of each vertical curve in U_i under f^{j_i} crosses C_{ω_1} with oriented intersection number $+1$ through a gap in Σ_{ω_1} of endpoints $a^i_{\omega_1}$ and $b^i_{\omega_1}$; the gap is chosen as the leftmost gap in Σ_{ω_1} that is crossed over with intersection number $+1$;
- (iv) the endpoints of the gap between $a^i_{\omega_1}$ and $b^i_{\omega_1}$ are the images under $f^{j_i - j_{i-1}}$ of either the endpoints of the gap between $a^{i-1}_{\omega_1}$ and $b^{i-1}_{\omega_1}$, or of a gap in Σ_{ω_1} located to the right side of that gap.

Since the rotation number of Σ_{ω_1} is smaller than the rotation number of Σ_{ρ_1}, any pair of points chosen on these two sets shift apart from one another under positive iterations. Therefore there exists some i large enough so that the gap of endpoints $a^i_{\omega_1}$ and $b^i_{\omega_1}$ is on the left side of the copy of W^{h_i}, in the sense that $\pi_x(b^i_{\omega_1}) < \pi_x(z)$ for all $z \in W^{h_i}$.

We claim that, by choosing i large enough and γ_i' sufficiently close to γ_i, we can ensure that the set $f^{j_i}(U_i)$ has a part which is a positive diagonal set in $B_{a^i_{\omega_1}, b^i_{\omega_1}}$. Now we justify the claim. The image of the left-side γ_i of U_i is mapped by f^{j_i} onto a positively tilted curve that crosses the gap between $a^i_{\omega_1}$ and $b^i_{\omega_1}$ with intersection number $+1$. Thus, the intersection between $f^{j_i}(\gamma_i)$ and this gap should consist of an odd number of intersection points, with the first intersection point q_1 and

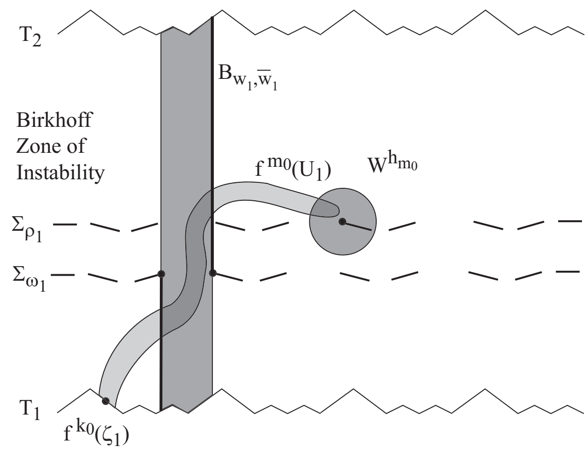

FIGURE 2. A positive diagonal set

the last intersection point q_1' having both intersection numbers $+1$. Indeed, the first intersection point of $f^{j_i}(\gamma_i)$ with the gap cannot have intersection number -1 since this would imply that there exists another gap, to the left of the gap between $a^i_{\omega_1}$ and $b^i_{\omega_1}$ that is crossed over by $f^{j_i}(\gamma_i)$ with intersection number $+1$, and this would contradict condition (iii) from above. Following the curve $f^{j_i}(\gamma_i)$ backwards starting from q_1, $f^{j_i}(\gamma_i)$ needs to intersect the left side $I_{a^i_{\omega_1}}$ of the strip $B_{a^i_{\omega_1},b^i_{\omega_1}}$ at a point r_1 below $a^i_{\omega_1}$, i.e. $r_1 \in I^-_{a^i_{\omega_1}}$; otherwise q_1 does not have intersection number $+1$ with the gap between $a^i_{\omega_1}$ and $b^i_{\omega_1}$. Following the curve $f^{j_i}(\gamma_i)$ forward starting from q_1', $f^{j_i}(\gamma_i)$ needs to intersect the right side $I_{b^i_{\omega_1}}$ of the strip $B_{a^i_{\omega_1},b^i_{\omega_1}}$ at a point r_1' above $b^i_{\omega_1}$, i.e. $r_1' \in I^+_{b^i_{\omega_1}}$; otherwise q_1' does not have intersection number $+1$ with the gap between $a^i_{\omega_1}$ and $b^i_{\omega_1}$. The curve segment of $f^{j_i}(\gamma_i)$ between r_1 and r_1' cannot intersects $I_{a^i_{\omega_1}}$ above r_1, and cannot intersect $I_{b^i_{\omega_1}}$ below r_1'; otherwise it would violate the positive tilt condition on $f^{j_i}(\gamma_i)$. Therefore, the curve segment $f^{j_i}(\gamma_i)$ between r_1 and r_1' has a component in $B_{a^i_{\omega_1},b^i_{\omega_1}}$ that goes from $I^-_{a^i_{\omega_1}}$ to $I^+_{b^i_{\omega_1}}$ without intersecting again $I^+_{a^i_{\omega_1}}$ or $I^-_{b^i_{\omega_1}}$. Now taking a curve γ_i' sufficiently close to γ_i results in a set U_i with the property that $f^{j_i}(U_i)$ has a part which is a positive diagonal set in $B_{a^i_{\omega_1},b^i_{\omega_1}}$.

We change notation at this point: we denote $m_0 := j_i$, $w_1 := a^i_{\omega_1}$, and $\bar{w}_1 := b^i_{\omega_1}$, $U' = U_i$ for i fixed as above. Thus, the points w_1 and $\bar{w}_1$ are the endpoints of a gap in Σ_{ω_1}, and $f^{m_0}(U')$ has a part which is a positive diagonal in $B_{w_1,\bar{w}_1}$. The positive integer m_0 is the first term of the sequence $\{m_s\}_{s=0,\dots,\sigma}$ in the statement of the theorem. Note that U' consists of a union of vertical segments emerging from T_1. See Figure 2.

Using the construction described in the sketch of the proof of Theorem 4.8, we obtain a nested sequence $D_0 \supseteq D_1 \supseteq \dots \supseteq D_\sigma$ of negative diagonals of $B_{w_1,\bar{w}_1}$ and a sequence of positive integers $\{m_s\}_{s=0,\dots,\sigma-1}$ such that for each $s \in \{1,\dots,\sigma\}$ and each $z \in D_s$ we have

$$\pi_x(f^j(w_s)) < \pi_x(f^j(z)) < \pi_x(f^j(\bar{w}_s)) \text{ for } j_s \le j \le j_s + n_s, \tag{5.2}$$

where $j_s := \sum_{t=1}^{s} n_t + \sum_{t=0}^{s-1} m_t$, w_s and $\bar{w}_s$ are the endpoints of some gap in the Aubry-Mather set Σ_{ω_s}, and $f^{(j_s+n_s)}(D_s)$ is a positive diagonal in $B_{f^{n_s}(w_s),f^{n_s}(\bar{w}_s)}$. In particular, $f^{(j_\sigma+n_\sigma)}(D_\sigma)$ is a positive diagonal in $B_{f^{n_\sigma}(w_\sigma),f^{n_\sigma}(\bar{w}_\sigma)}$, where w_σ and $\bar{w}_\sigma$ are the endpoints of some gap in the Aubry-Mather set Σ_{ω_σ}.

Since $f^{m_0}(U')$ has a part which is a positive diagonal in $B_{w_1,\bar{w}_1}$ and D_σ is a negative diagonal of $B_{w_1,\bar{w}_1}$, then $f^{m_0}(U')$ and D_σ have a non-empty intersection. Also, $f^{(j_\sigma+n_\sigma)}(U')$ has a part which is a positive diagonal in $B_{f^{n_\sigma}(w_\sigma),f^{n_\sigma}(\bar{w}_\sigma)}$ that stretches across $f^{(j_\sigma+n_\sigma)}(D_\sigma)$.

Now we start with the one-sided ε-neighborhood V_0 of $\zeta_2 \in T_2$. Let C_{ω_σ} be an essential circle containing Σ_{ω_σ}. We choose an Aubry-Mather set Σ_{ρ_2} lying on an essential circle C_{ρ_2}, such that C_{ρ_2} is below C_{ω_σ}. We assume that the interior of V_0 does not meet Σ_{ω_σ} and Σ_{ρ_2}, otherwise the proof follows as in [28]. Using Lemma 4.6 (iii) and following the procedure described above for negative iterations, we produce a one-sided neighborhood V' of a point in T_2, with $V' \subseteq V$, and a positive integer m'_σ such that $f^{-m'_\sigma}(V')$ contains a part which is a negative diagonal in $B_{w'_\sigma,\bar{w}'_\sigma}$, where w'_σ and $\bar{w}'_\sigma$ are the endpoints of a gap in Σ_{ω_σ}.

By using the existence of orbits passing from near T_2 to near T_1, and of orbits passing from near T_1 to near T_2, as in the sketch of the proof of Theorem 4.8, we can further iterate the positive diagonal $f^{(j_\sigma+n_\sigma)}(U')$ from above, so that we obtain an iterate $f^{(j_\sigma+n_\sigma+m''_\sigma)}(U')$ which contains a component that stretches all the way across a fundamental interval of the annulus. In particular, $f^{(j_\sigma+n_\sigma+m''_\sigma)}(U')$ contains a component that is a positive diagonal of $B_{w'_\sigma,\bar{w}'_\sigma}$. Since $f^{-m'_\sigma}(V')$ is a negative diagonal in $B_{w'_\sigma,\bar{w}'_\sigma}$, then $f^{(j_\sigma+n_\sigma+m''_\sigma)}(U')$ has a nonempty intersection with $f^{-m'_\sigma}(V)$. Equivalently, $f^{(j_\sigma+n_\sigma+m_\sigma)}(U')$ has a non-empty intersection with V', where $m_\sigma := m'_\sigma + m''_\sigma$.

Thus, each point $z \in U' \cap f^{-(j_\sigma+n_\sigma+m_\sigma)}(V')$ goes from the neighborhood U of ζ_1 to the neighborhood V of ζ_2 and it shadows, in the sense of the ordering, each of the Aubry-Mather set Σ_{ω_s}, $s = 1, \ldots, \sigma$, along the way.

Part 2. Now we explain how to modify the above proof to show that there exists a point $z' \in \mathrm{bd}(U)$ that satisfies (5.1) and $f^N(z') \in \mathrm{bd}(V)$. This does not follow immediately from the above argument since the image of $\mathrm{bd}(U)$ under iteration may fail being a positively tilted curve; hence we cannot infer that $f^{m_0}(\mathrm{bd}(U))$ intersects the negative diagonal set D_σ from above.

By Theorem 5.1, there exist $l > 0$ and point $q \in U$ depending on l such that $f^l(q)$ is in some prescribed neighborhood of a point $r \in T_2$, where l can be chosen arbitrarily large. Since the points of T_1 and T_2 have different rotation numbers hence move apart under iteration, there exists l_0 sufficiently large such that $f^{l_0}(T_1 \cap U)$ and $f^{l_0}(q)$ are separated by a fundamental interval of the annulus, i.e., $\pi_x(f^{l_0}(r)) - \pi_x(f^{l_0}(q)) > 1$ for all $r \in T_1 \cap U$. Let $x_0, \bar{x}_0$ be such that $\pi_x(f^{l_0}(r)) < x_0 < \bar{x}_0 < \pi_x(f^{l_0}(q))$ and $1 < \bar{x}_0 - x_0$. Let $w_0 \in I_{x_0}$ be a point on the vertical line $x = x_0$ whose y-coordinate is larger than that of any point in $f^{l_0}(U) \cap I_{x_0}$. Similarly, let $\bar{w}_0 \in I_{\bar{x}_0}$ be a point on the vertical line $x = \bar{x}_0$ whose y-coordinate is smaller than that of any point in $f^{l_0}(U) \cap I_{\bar{x}_0}$. Then $f^{l_0}(U) \cap \mathrm{cl}(B_{w_0,\bar{w}_0})$ is a positive diagonal in $B_{w_0,\bar{w}_0}$.

Now we want to show that a certain iterate of this diagonal set has a component that is a positive diagonal in $B_{w_1,\bar{w}_1}$, for some points $w_1, \bar{w}_1 \in \Sigma_1$, the first Aubry-Mather set in the prescribed sequence. There exists x'_0 sufficiently close to x_0 such that for all x between x_0 and x_0', the point $(x, \pi_y(w_0))$ has the y-coordinate larger than that of any point in $f^{l_0}(\mathrm{cl}(U)) \cap I_x$. Then the set $W_0 = \{(x, y) \,|\, x_0 < x < x_0', \pi_y(w_0) < y\}$ is a neighborhood of an arc in T_2, with the property that each point $(x, y) \in W_0$ has the y-coordinate larger than that of any point in $f^{l_0}(\mathrm{cl}(U)) \cap I_x$. Similarly, there is $\bar{x}_0'$ sufficiently close to $\bar{x}_0$ such that for all x between $\bar{x}_0$ and $\bar{x}_0'$, the point $(x, \pi_y(\bar{w}_0))$ has the y-coordinate smaller than that of any point in $f^{l_0}(\mathrm{cl}(U)) \cap I_x$. Then the set $\bar{W}_0 = \{(x, y) \,|\, \bar{x}_0 < x < \bar{x}_0', \pi_y(\bar{w}_0) > y\}$ is a neighborhood of an arc in T_1, with the property that each point $(x, y) \in \bar{W}_0$ has the y-coordinate smaller than that of any point in $f^{l_0}(\mathrm{cl}(U)) \cap I_x$.

Let Σ_{ρ_1} be an Aubry-Mather set lying on an essential circle C_{ρ_1} that is below the essential circle C_{ω_1} containing Σ_{ω_1}, let $p_1 \in C_{\omega_1}$, and let $W(p_1)$ be a small neighborhood of p_1 that does not intersect Σ_{ω_1}. Then there exists $(x_0'', y_0'') \in W_0$ and j sufficiently large such that $f^j(x_0'', y_0'') \in W(p_1)$. Since the curve $f^j(I^+_{(x_0'', y_0'')})$ is a positively tilted curve emerging from T_2, the argument based on Lemma 4.6 that was used in Part 1 shows that there is a gap of the Aubry-Mather set Σ_{ω_1}, between a pair of points $w_1, w_1' \in \Sigma_{\omega_1}$, such that $f^j(I^+_{(x_0'', y_0'')})$ crosses this gap with intersection number -1 and has its first intersection with I_{w_1} below the point w_1, provided j is chosen large enough. This implies that the image of the set $f^{l_0}(\mathrm{cl}(U)) \cap B_{(x_0'', y_0''), (\bar{x}_0'', \bar{y}_0'')}$ under f^j has a component that satisfies the positive diagonal set conditions relative to its left side. Moreover, for all $j' > 0$, the image of the set $f^{l_0}(\mathrm{cl}(U)) \cap B_{(x_0'', y_0''), (\bar{x}_0'', \bar{y}_0'')}$ under $f^{j+j'}$ also has a component that satisfies the positive diagonal set conditions relative to the left side of $B_{f^{j'}(w_1), f^{j'}(w_1')}$. In a similar fashion, there exists $j_0 > j$ sufficiently large such that the curve $f^{j_0}(I^-_{(\bar{x}_0'', \bar{y}_0'')})$ is a positively tilted curve emerging from T_1 and crosses a gap of the Aubry-Mather set Σ_{ω_1}, between a pair of points $\bar{w}_1, \bar{w}_1' \in \Sigma_{\omega_1}$, such that $f^{j_0}(I^+_{(x_0'', y_0'')})$ crosses this gap with intersection number $+1$ and has its first intersection with $I_{\bar{w}_1'}$ above the point $\bar{w}_1'$. Thus, the image $f^{l_0}(\mathrm{cl}(U)) \cap B_{(x_0'', y_0''), (\bar{x}_0'', \bar{y}_0'')}$ under f^{j_0} satisfies the positive diagonal set conditions relative to its right side. The conclusion is that the image $f^{l_0}(\mathrm{cl}(U)) \cap B_{(x_0'', y_0''), (\bar{x}_0'', \bar{y}_0'')}$ under f^{j_0}, i.e. the set $f^K(\mathrm{cl}(U)) \cap B_{(x_0'', y_0''), (\bar{x}_0'', \bar{y}_0'')}$ with $K = l_0 + j_0$, contains a component that is a positive diagonal in $B_{w_1, \bar{w}_1'}$, where w_1, w_1' are two points in the Aubry-Mather set Σ_{ω_1}. Moreover, the upper edge and the lower edge of the positive diagonal component of $f^K(\mathrm{cl}(U)) \cap B_{w_1, \bar{w}_1'}$ are contained in $f^K(\mathrm{bd}(U))$.

We apply an analogous argument at the other boundary torus T_2. Given V a neighborhood of a point $\zeta_2 \in T_2$, there exist $L > 0$ and a pair of points $w_\sigma, \bar{w}_\sigma \in \Sigma_{\omega_\sigma}$ such that $f^{-L}(\mathrm{cl}(V)) \cap B_{w_\sigma, \bar{w}_\sigma}$ has a component D_σ'' that is a negative diagonal in $B_{w_\sigma, \bar{w}_\sigma}$. The upper edge and the lower edge of the this positive diagonal set are contained in $f^{-L}(\mathrm{bd}(V))$.

Now we apply the argument from Part 1. There exists D_σ a negative diagonal set in $B_{w_1, \bar{w}_1}$ such that all points $z \in D_\sigma$ satisfy (5.1). The negative diagonal set D_σ intersects the positive diagonal component of $f^K(\mathrm{cl}(U)) \cap B_{w_1, \bar{w}_1'}$, and in particular intersects its upper and lower edges that are contained in $f^K(\mathrm{bd}(U))$. Iterating for $j_\sigma = \sum_{s=0}^{\sigma} n_s + \sum_{s=0}^{\sigma-1} m_s$ times, the positive diagonal component of $f^K(\mathrm{cl}(U)) \cap B_{w_1, \bar{w}_1'}$ yields a positive diagonal set D_σ' in $B_{w_\sigma, \bar{w}_\sigma}$. The upper and lower edge of D_σ' are contained in $f^{K+j_\sigma}(\mathrm{bd}(U))$. The positive diagonal set D_σ' intersects the negative diagonal component D_σ'' of $f^{-L}(\mathrm{cl}(V)) \cap B_{w_\sigma, \bar{w}_\sigma}$. In particular the upper and lower edges of D_σ', that are contained in $f^{j_\sigma+K}(\mathrm{bd}(U))$, intersect the upper and lower edges of D''_σ that are contained in $f^{-L}(\mathrm{bd}(V))$. Thus, there exists a point $z' \in \mathrm{bd}(U)$ that is taken by f^{K+j_σ} to $\mathrm{bd}(V)$ and satisfies the ordering relations (5.1). □

6. Scattering map

The scattering map acts on the normally hyperbolic invariant manifold Λ and relates the past asymptotic trajectory of each orbit in the homoclinic manifold to its future asymptotic behavior. We review its properties following [17].

As before, we consider a normally hyperbolic invariant manifold Λ for a map $F : M \to M$. Since the stable and unstable manifolds of Λ are foliated by stable and unstable manifolds of points, respectively, we have that for each $x \in W^s(\Lambda)$ there exists a unique $x^+ \in \Lambda$ such that $x \in W^s(x^+)$, and for each $x \in W^u(\Lambda)$ there exists a unique $x^- \in \Lambda$ such that $x \in W^u(x^-)$.

We define the wave maps $\Omega^+ : W^s(\Lambda) \to \Lambda$ by $\Omega^+(x) = x^+$, and $\Omega^- : W^u(\Lambda) \to \Lambda$ by $\Omega^-(x) = x^-$. The maps Ω^+ and Ω^- are C^ℓ-smooth.

We now describe the scattering map. Assume that $W^u(\Lambda)$ and $W^s(\Lambda)$ have a differentiably transverse intersection along a homoclinic l-dimensional $C^{\ell-1}$-smooth manifold Γ. This means that $\Gamma \subseteq W^u(\Lambda) \cap W^s(\Lambda)$ and, for each $x \in \Gamma$, we have

$$\begin{aligned} T_xM &= T_xW^u(\Lambda) + T_xW^s(\Lambda), \\ T_x\Gamma &= T_xW^u(\Lambda) \cap T_xW^s(\Lambda). \end{aligned} \tag{6.1}$$

We assume the additional condition that for each $x \in \Gamma$ we have

$$\begin{aligned} T_xW^s(\Lambda) &= T_xW^s(x^+) \oplus T_x(\Gamma), \\ T_xW^u(\Lambda) &= T_xW^u(x^-) \oplus T_x(\Gamma), \end{aligned} \tag{6.2}$$

where x^-, x^+ are the uniquely defined points in Λ corresponding to x.

The restrictions $\Omega^+_\Gamma, \Omega^-_\Gamma$ of Ω^+, Ω^- to Γ are local $C^{\ell-1}$-diffeomorphisms. By replacing Γ to a submanifold of it we can ensure that $\Omega^+_\Gamma, \Omega^-_\Gamma$ are $C^{\ell-1}$-diffeomorphisms.

Definition 6.1. A homoclinic manifold Γ satisfying (6.1) and (6.2), and for which the corresponding restrictions of the wave maps are $C^{\ell-1}$-diffeomorphisms, is referred as a homoclinic channel.

Definition 6.2. Given a homoclinic channel Γ, the scattering map associated to Γ is the $C^{\ell-1}$-diffeomorphism $S_\Gamma = \Omega^+_\Gamma \circ (\Omega^-_\Gamma)^{-1}$ from the open subset $U^- := \Omega^-_\Gamma(\Gamma)$ in Λ to the open subset $U^+ := \Omega^+_\Gamma(\Gamma)$ in Λ.

In the sequel we will regard S_Γ as a partially defined map, so the image of a set A by S_Γ means the set $S_\Gamma(A \cap U^-)$.

We list below some remarkable properties of the scattering map. The first property states that, in a Hamiltonian context, the scattering map is symplectic. The second property says that the scattering map takes invariant submanifolds of the center transversality across some other invariant submanifolds of the center, under some appropriate non-degeneracy condition.

Proposition 6.3. *Assume that* $\dim M = 2n + l$ *is even (i.e.,* l *is even) and* M *is endowed with a symplectic (respectively exact symplectic) form* ω *and that* $\omega_{|\Lambda}$ *is also symplectic.*

(i) *If* F *is symplectic (respectively exact symplectic), then the scattering map* S_Γ *is symplectic (respectively exact symplectic).*

(ii) *If T_1 and T_2 are two invariant submanifolds of complementary dimension in Λ, and $W^u(T_1)$ has a transverse intersection with $W^s(T_2)$ inside Γ, then $S_\Gamma(T_1)$ has a transverse intersection with T_2 in Λ.*

6.1. **Topological method of correctly aligned windows.** We describe briefly the topological method of correctly aligned windows. We follow [49]. See also [23, 22, 39].

Definition 6.4. An (m_1, m_2)-window in an m-dimensional manifold M, where $m_1 + m_2 = m$, is a compact subset W of M together with a parametrization given by a C^0-parametrization χ from some open neighborhood of $[0,1]^{m_1} \times [0,1]^{m_2}$ in $\mathbb{R}^{m_1} \times \mathbb{R}^{m_2}$ to an open subset of M, with $W = \chi([0,1]^{m_1} \times [0,1]^{m_2})$, and with a choice of an 'exit set'

$$W^{\text{exit}} = \chi\left(\partial[0,1]^{m_1} \times [0,1]^{m_2}\right)$$

and of an 'entry set'

$$W^{\text{entry}} = \chi\left([0,1]^{m_1} \times \partial[0,1]^{m_2}\right).$$

We adopt the following notation: $W_\chi = (\chi)^{-1}(W)$, $(W^{\text{exit}})_\chi = (\chi)^{-1}(W^{\text{exit}})$, $(W^{\text{entry}})_\chi = (\chi)^{-1}(W^{\text{entry}})$. Also denote $\pi_{m_1} : \mathbb{R}^{m_1} \times \mathbb{R}^{m_2} \to \mathbb{R}^{m_1}$ the orthogonal projection onto the first component, and $\pi_{m_2} : \mathbb{R}^{m_1} \times \mathbb{R}^{m_2} \to \mathbb{R}^{m_2}$ the orthogonal projection onto the second component. (Note that $W_\chi = [0,1]^{m_1} \times [0,1]^{m_2}$, $(W^{\text{exit}})_\chi = \partial[0,1]^{m_1} \times [0,1]^{m_2}$, and $(W^{\text{entry}})_\chi = [0,1]^{m_1} \times \partial[0,1]^{m_2}$.) When the local parametrization χ is evident from context, we suppress the subscript χ from the notation.

Definition 6.5. Let W_1 and W_2 be (m_1, m_2)-windows, and let χ_1 and χ_2 be the corresponding local parametrizations. Let f be a continuous map on M with $f(\text{im}(\chi_1)) \subseteq \text{im}(\chi_2)$. We say that W_1 is correctly aligned with W_2 under f if the following conditions are satisfied:

(i) $W_1^{\text{exit}} \cap (W_2) = \emptyset$ and $W_1 \cap W_2^{\text{entry}} = \emptyset$;

(ii) There exists $y_0 \in [0,1]^{m_2}$ such that the curve $\hat{f}(x, y_0)$ for $x \in [0,1]^{m_1}$, where $\hat{f} := \chi_2^{-1} \circ f \circ \chi_1$, has the following properties:

$$\begin{aligned} \hat{f}_{y_0}\left([0,1]^{m_1}\right) &\subseteq \mathbb{R}^{m_1} \times (0,1)^{m_2}, \\ \hat{f}_{y_0}\left(\partial[0,1]^{m_1}\right) &\subseteq \left(\mathbb{R}^{m_1} \setminus [0,1]^{m_1}\right) \times (0,1)^{m_2} = \emptyset, \\ \deg(\pi_{m_1} \circ \hat{f}_{y_0}, 0) &= w \neq 0. \end{aligned}$$

We call the integer $w \neq 0$ in the above definition the degree of the alignment.

The following result is a topological version of the Shadowing Lemma.

Theorem 6.6. *Let W_i be a collection of (m_1, m_2)-windows in M, where $i \in \mathbb{Z}$ or $i \in \{0, \dots, d-1\}$, with $d > 0$ (in the latter case, for convenience, we let $W_i = W_{(i \bmod d)}$ for all $i \in \mathbb{Z}$). Let f_i be a collection of continuous maps on M. If W_i is correctly aligned with W_{i+1}, for all i, then there exists a point $p \in W_0$ such that*

$$(f_i \circ \dots \circ f_0)(p) \in W_{i+1},$$

Moreover, if $W_{i+k} = W_i$ for some $k > 0$ and all i, then the point p can be chosen periodic in the sense

$$(f_{k-1} \circ \dots \circ f_0)(p) = p.$$

The correct alignment of windows is robust, in the sense that if two windows are correctly aligned under a map, then they remain correctly aligned under a sufficiently small perturbation of the map. Robustness makes the method of correctly aligned windows appropriate for rigorous numerical experiment.

Also, the correct alignment satisfies a natural product property. Given two windows and a map, if each window can be written as a product of window components, and if the components of the first window are correctly aligned with the corresponding components of the second window under the appropriate components of the map, then the first window is correctly aligned with the second window under the given map. For example, if we consider a pair of windows in a neighborhood of a normally hyperbolic invariant manifold, if the center components of the windows are correctly aligned and the hyperbolic components of the windows are also correctly aligned, then the windows are correctly aligned. Although the product property is quite intuitive, its rigorous statement is rather technical, so we will omit it here. The details can be found in [22].

7. A shadowing lemma in normally hyperbolic invariant manifolds

In this section we present a shadowing lemma-type of result saying that, given a sequence of windows in a normally hyperbolic invariant manifold, consisting of pairs of windows correctly aligned under the scattering map, alternating with pairs of windows correctly aligned under some iterate of the inner map, then there exists a true orbit in the full space dynamics that follows these windows. This reduces the construction of windows in the full dimensional phase space to construction of windows, of lower dimension, in the normally hyperbolic invariant manifold.

Lemma 7.1. *Let $\{R_i, R'_i\}_{i\in\mathbb{Z}}$ be a bi-infinite sequence of l-dimensional windows contained in Λ. Assume that the following properties hold for all $i \in \mathbb{Z}$:*

(i) *$R_i \subseteq U^-$ and $R'_i \subseteq U^+$.*
(ii) *R_i is correctly aligned with R'_{i+1} under the scattering map S.*
(iii) *for each $L > 0$ there exists $L' > L$ such that the window R'_{i+1} is correctly aligned with the window R_{i+1} under the iterate $f^{L'}_{|\Lambda}$ of the restriction $f_{|\Lambda}$ of f to Λ.*

Given any bi-infinite sequence of positive real numbers $\{\varepsilon_i\}_{i\in\mathbb{Z}}$. Then there exist an orbit $(f^n(z))_{n\in\mathbb{Z}}$ of some point $z \in M$, an increasing sequence of integers $(n_i)_{i\in\mathbb{Z}}$, and some sequences of positive integers $\{N_i\}_{i\in\mathbb{Z}}, \{K_i\}_{i\in\mathbb{Z}}, \{M_i\}_{i\in\mathbb{Z}}$, such that, for all $i \in \mathbb{Z}$:

$$
\begin{aligned}
d(f^{n_i}(z), \Gamma) &< \varepsilon_i, \\
d(f^{n_i+N_{i+1}}(z), f^{N_{i+1}}_{|\Lambda}(R'_{i+1})) &< \varepsilon_{i+1}, \\
d(f^{n_i-M_i}(z), f^{-M_i}_{|\Lambda}(R_i)) &< \varepsilon_i, \\
n_{i+1} = n_i + N_{i+1} &+ K_{i+1} + M_{i+1}.
\end{aligned}
$$

Proof. The idea of this proof is to 'thicken' some appropriate iterates of the windows R_i, R'_i in Λ to full dimensional windows W_i, W'_i in M, so that $\{W_i, W'_i\}_{i\in\mathbb{Z}}$ form a sequence of windows that are correctly aligned under some appropriate maps.

Step 1. Let $\bar{R}_i = (\Omega^-_\Gamma)^{-1}(R_i)$ and $\bar{R}'_{i+1} = (\Omega^+_\Gamma)^{-1}(R'_{i+1})$ be the copies of R_i and R'_{i+1}, respectively, in the homoclinic channel Γ. By making some arbitrarily small changes in the sizes of their exit and entry directions, we can alter the windows R_i

and R'_{i+1} such that R_i is correctly aligned with $\bar{R}_i$ under $(\Omega^-_\Gamma)^{-1}$, $\bar{R}_i$ is correctly aligned with $\bar{R}'_{i+1}$ under the identity mapping, and $\bar{R}'_{i+1}$ is correctly aligned with R_{i+1} under Ω^+_Γ.

We 'thicken' the l-dimensional windows $\bar{R}_i$ and $\bar{R}'_{i+1}$ in Γ, which are correctly aligned under the identity mapping, to $(l+2n)$-dimensional windows $\bar{W}_i$ and $\bar{W}'_{i+1}$ that are correctly aligned under the identity map as well. We now explain the 'thickening' procedure.

First, we describe how to thicken $\bar{R}_i$ to a full dimensional window $\bar{W}_i$. We choose some $0 < \bar{\delta}_i < \varepsilon_i$ and $0 < \bar{\eta}_i < \varepsilon_i$. At each point $x \in \bar{R}_i$ we choose an n-dimensional closed ball $\bar{B}_{\bar{\delta}_i}(x)$ of radius $\bar{\delta}_i$ centered at x and contained in $W^u(x^-)$, where $x^- = \Omega^-_\Gamma(x)$. We take the union $\bar{\Delta}_i := \bigcup_{x\in\bar{R}_i} \bar{B}^u_{\bar{\delta}_i}(x)$. Note that $\bar{\Delta}_i$ is contained in $W^u(\Lambda)$ and is homeomorphic to an $(l+n)$-dimensional rectangle. We define the exit set and the entry set of this rectangle as follows:

$$(\bar{\Delta}_i)^{\text{exit}} := \bigcup_{x\in(\bar{R}_i)^{\text{exit}}} \bar{B}^u_{\bar{\delta}_i}(x) \cup \bigcup_{x\in\bar{R}_i} \partial\bar{B}^u_{\bar{\delta}_i}(x),$$
$$(\bar{\Delta}_i)^{\text{entry}} := \bigcup_{x\in(\bar{R}_i)^{\text{entry}}} \bar{B}^u_{\bar{\delta}_i}(x).$$

We consider the normal bundle N to $W^u(\Lambda)$. At each point $y \in \bar{\Delta}_i$, we choose an n-dimensional closed ball $\bar{B}_{\bar{\eta}_i}(y)$ centered at y and contained in the image of N_y under the exponential map $\exp_y : T_yM \supseteq N_y \to M$. We let $\bar{W}_i := \bigcup_{y\in\bar{\Delta}_i} \bar{B}^s_{\bar{\eta}_i}(y)$. By the Tubular Neighborhood Theorem (see, for example [6]), we have that for $\bar{\eta}_i$ sufficiently small, the set $\bar{W}_i$ is a homeomorphic copy of an $(l+2n)$-dimensional rectangle. We now define the exit set and the entry set of $\bar{R}_i$ as follows:

$$(\bar{W}_i)^{\text{exit}} := \bigcup_{y\in(\bar{\Delta}_i)^{\text{exit}}} \bar{B}^s_{\bar{\eta}_i}(y),$$
$$(\bar{W}_i)^{\text{entry}} := \bigcup_{y\in(\bar{\Delta}_i)^{\text{entry}}} \bar{B}^s_{\bar{\eta}_i}(y) \cup \bigcup_{y\in(\bar{\Delta}_i)} \partial\bar{B}^s_{\bar{\eta}_i}(y).$$

Second, we describe in a similar fashion how to thicken $\bar{R}'_{i+1}$ to a full dimensional window $\bar{W}'_{i+1}$. We choose $0 < \bar{\delta}'_{i+1} < \varepsilon_i$ and $0 < \bar{\eta}'_{i+1} < \varepsilon_i$. We consider the $(l+n)$-dimensional rectangle $\bar{\Delta}'_{i+1} := \bigcup_{x\in\bar{R}'_{i+1}} \bar{B}^s_{\bar{\eta}'_{i+1}}(x) \subseteq W^s(\Lambda)$, where $\bar{B}^s_{\bar{\eta}'_{i+1}}(x)$ is the n-dimensional closed ball of radius $\bar{\eta}'_{i+1}$ centered at x and contained in $W^s(x^+)$, with $x^+ = \Omega^+_\Gamma(x)$. The exit set and entry set of this window are defined as follows:

$$(\bar{\Delta}'_{i+1})^{\text{exit}} := \bigcup_{x\in(\bar{R}'_{i+1})^{\text{exit}}} \bar{B}^s_{\bar{\eta}'_{i+1}}(x),$$
$$(\bar{\Delta}'_{i+1})^{\text{entry}} := \bigcup_{x\in(\bar{R}'_{i+1})^{\text{entry}}} \bar{B}^s_{\bar{\eta}'_{i+1}}(x) \cup \bigcup_{x\in(\bar{R}'_{i+1})} \partial\bar{B}^s_{\bar{\eta}'_{i+1}}(x).$$

We define $\bar{W}'_{i+1} := \bigcup_{y\in\bar{\Delta}'_{i+1}} \bar{B}^u_{\bar{\delta}'_{i+1}}(y)$, where $\bar{B}^u_{\bar{\delta}'_{i+1}}(y)$ is the n-dimensional closed ball centered at y and contained in the image of N'_y under the exponential map $\exp_y : T_yM \supseteq N'_y \to M$, with N' being the normal bundle to $W^s(\Lambda)$. The Tubular Neighborhood Theorem implies that for $\bar{\delta}'_{i+1} > 0$ sufficiently small the set $\bar{W}'_{i+1}$ is a homeomorphic copy of a $(l+2n)$-dimensional rectangle. The exit set and the entry set of $\bar{R}'_{i+1}$ are defined by:

$$(\bar{W}'_{i+1})^{\rm exit} := \bigcup_{y\in(\bar{\Delta}'_{i+1})^{\rm exit}} \bar{B}^u_{\bar{\delta}'_{i+1}}(y) \cup \bigcup_{y\in(\bar{\Delta}'_{i+1})} \partial\bar{B}^u_{\bar{\delta}'_{i+1}}(y),$$
$$(W'_{i+1})^{\rm entry} := \bigcup_{y\in(\bar{\Delta}'_{i+1})^{\rm entry}} \bar{B}^u_{\bar{\delta}'_{i+1}}(y).$$

This completes the description of the thickening of the l-dimensional window $\bar{R}_i$ into an $(l+2n)$-dimensional window $\bar{W}_i$, and of the thickening of the l-dimensional window $\bar{R}'_{i+1}$ into an $(l+2n)$-dimensional window $\bar{W}'_{i+1}$. Note that by construction $\bar{W}_i$ and $\bar{W}'_{i+1}$ are both contained in an ε_i-neighborhood of Γ.

Now we want to make $\bar{W}_i$ correctly aligned with $\bar{W}'_{i+1}$ under the identity map. This is achieved by choosing $\bar{\delta}'_{i+1}$ sufficiently small relative to $\bar{\delta}_i$, and by choosing $\bar{\eta}_i$ sufficiently small relative to $\bar{\eta}'_{i+1}$.

Step 2. We take a negative iterate $f^{-M}(\bar{R}_i)$ of $\bar{R}_i$, where $M > 0$. We have that $f^{-M}(\Gamma)$ is ε_i-close to Λ on a neighborhood in the C^1-topology, for all M sufficiently large. The vectors tangent to the fibers $W^u(x^-)$ in $\bar{R}_i$ are contracted, and the vectors transverse to $W^u(\Lambda)$ along $\bar{R}_i$ are expanded by the derivative of f^{-M}. We choose and fix $M_i = M$ sufficiently large; thus $f^{-M_i}(\bar{R}_i)$ is ε_i-close to $f^{-M_i}(R_i)$.

We now construct a window W_i about $f^{-M_i}(R_i)$ that is correctly aligned with $f^{-M_i}(\bar{W}_i)$ under the identity. Note that each closed ball $\bar{B}^u_{\bar{\delta}_i}(x)$, which is a part of $\bar{\Delta}_i$, gets exponentially contracted as it is mapped into $W^u(f^{-M_i}(x^-))$ by f^{-M_i}. By the Lambda Lemma (see the version in [40]), each closed ball $\bar{B}^s_{\bar{\eta}_i}(y)$ with $y \in \bar{\Delta}_i$, which is a part of $\bar{R}_i$, C^1-approaches a subset of $W^s(f^{-M_i}(y^-))$ under f^{-M_i}, as $M_i \to \infty$. For M_i sufficiently large, we can assume that $f^{-M_i}(\bar{B}^s_{\bar{\eta}_i^-}(y))$ is ε_i-close to some ball in $W^s(f^{-M_i}(y^-))$ in the C^1-topology, for all $y \in \bar{\Delta}_i$. As R_i is correctly aligned with $\bar{R}_i$ under $(\Omega^-_\Gamma)^{-1}$, we have that $f^{-M_i}(R_i)$ is correctly aligned with $f^{-M_i}(\bar{R}_i)$ under $(\Omega^-{}_{f^{-M_i}(\Gamma)})^{-1}$. In other words, $f^{-M_i}(R_i)$ is correctly aligned under the identity mapping with the projection of $f^{-M_i}(\bar{R}_i)$ onto Λ along the unstable fibres. Let us consider $0 < \delta_i < \varepsilon_i$ and $0 < \eta_i < \varepsilon_i$.

To define the window W_i we use a local linearization of the normally hyperbolic invariant manifold. By Theorem 1 in [44], there exists a homeomorphisms h from an open neighborhood of $T\Lambda \times \{0\} \times \{0\}$ in $T\Lambda \oplus E^s \oplus E^u$ to a neighborhood of Λ in M such that $h \circ Df = f \circ h$. At each point $x \in f^{-M_i}(\bar{R}_i)$ we consider a rectangle $H_i(x)$ of the type $h(\{\tilde{x}\} \times \bar{B}^u_{\delta_i}(0) \times \bar{B}^s_{\eta_i}(0))$, where $\tilde{x} \in T\Lambda$ is such that $h(\{\tilde{x}\} \times \{0\} \times \{0\}) = x$, $\bar{B}^u_{\delta_i}(0)$ is the closed ball centered at 0 of radius δ_i in the unstable bundle E^u, and $\bar{B}^s_{\eta_i}$ is the closed ball centered at 0 of radius η_i in the stable bundle E^s. We define the exit and entry sets of $H_i(x)$ as $(H_i(x))^{\rm exit} = h(\{\tilde{x}\} \times \partial\bar{B}^u_{\delta_i}(0) \times \bar{B}^s_{\eta_i}(0))$ and $(H_i(x))^{\rm entry} = h(\{\tilde{x}\} \times \bar{B}^u_{\delta_i}(0) \times \partial\bar{B}^s_{\eta_i}(0))$.

Then we define the window W_i as follows:

$$W_i = \bigcup_{x\in f^{-M_i}(\bar{R}_i)} H_i(x),$$
$$(W_i)^{\rm exit} = \bigcup_{x\in (f^{-M_i}(\bar{R}_i))^{\rm exit}} H_i(x) \cup \bigcup_{x\in f^{-M_i}(\bar{R}_i)} (H_i(x))^{\rm exit},$$
$$(W_i)^{\rm entry} = \bigcup_{x\in (f^{-M_i}(\bar{R}_i))^{\rm entry}} H_i(x) \cup \bigcup_{x\in f^{-M_i}(\bar{R}_i)} (H_i(x))^{\rm entry}.$$

The exit and entry sets of $f^{-M_i}(\bar{R}_i)$ are those corresponding to the exit and entry sets of $\bar{R}_i$, through the map f^{-M_i}.

In order to ensure the correct alignment of W_i with $f^{-M_i}(\bar{W}_i)$ under the identity map, it is sufficient to choose δ_i, η_i such that $\bigcup_{x\in f^{-M_i}(\bar{R}_i)} h(\{\tilde{x}\}\times \bar{B}^u_{\delta_i}(0)\times\{0\})$ is correctly aligned with $f^{-M_i}(\bar{\Delta}_i)$ under the identity map (the exit sets of both windows being in the unstable directions), and that each closed ball $f^{-M_i}(\bar{B}^s_{\eta_i})$ intersects W_i in a closed ball that is contained in the interior of $f^{-M_i}(\bar{B}^s_{\eta_i})$. The existence of suitable δ_i, η_i follows from the exponential contraction of $\bar{\Delta}_i$ under negative iteration, and from the Lambda Lemma applied to $\bar{B}^s_{\eta_i}(y)$ under negative iteration.

In a similar fashion, we construct a window W'_{i+1} contained in an ε_{i+1}-neighborhood of Λ such that $\bar{W}'_{i+1}$ is correctly aligned with W'_{i+1} under $f^{N_{i+1}}$. The window W'_{i+1}, and its entry and exit sets, are defined by:

$$W'_{i+1} = \bigcup_{x\in f^{N_{i+1}}(R'_{i+1})} H'_{i+1}(x),$$
$$(W'_{i+1})^{\rm exit} = \bigcup_{x\in (f^{N_{i+1}}(R'_{i+1}))^{\rm exit}} H'_{i+1}(x) \cup \bigcup_{x\in f^{N_{i+1}}(R'_{i+1})} (H'_{i+1}(x))^{\rm exit},$$
$$(W'_{i+1})^{\rm entry} = \bigcup_{x\in (f^{N_{i+1}}(R'_{i+1}))^{\rm entry}} H'_{i+1}(x) \cup \bigcup_{x\in f^{N_{i+1}}(R'_{i+1})} (H'_{i+1}(x))^{\rm entry},$$

where $H'_{i+1}(x) = h(\{\tilde{x}\}\times \bar{B}^u_{\delta_{i+1}}(0)\times \bar{B}^s_{\eta_{i+1}}(0))$, $(H'_{i+1}(x))^{\rm exit}$, and $(H'_{i+1}(x))^{\rm entry}$ are defined as before for some appropriate choices of radii $\delta_{i+1}, \eta_{i+1} > 0$.

Step 3. Suppose that we have constructed the window W'_{i+1} about the l-dimensional rectangle $f^{N_{i+1}}(R'_{i+1}) \subseteq \Lambda$, and the window W_{i+1} about the l-dimensional rectangle $f^{-M_{i+1}}(R_{i+1}) \subseteq \Lambda$. Under positive iterations, the rectangle $\bar{B}^u_{\delta'_{i+1}}(0)\times \bar{B}^s_{\eta'_{i+1}}(0) \subseteq E^u \oplus E^s$ gets exponentially expanded in the unstable direction and exponentially contracted in the stable direction by Df. Thus $\bar{B}^u_{\delta'_{i+1}}(0)\times \bar{B}^s_{\eta'_{i+1}}(0)$ is correctly aligned with $\bar{B}^u_{\delta_{i+1}}(0)\times \bar{B}^s_{\eta'_{i+1}}(0)$ under the power $Df^{L_{i+1}}$ of Df, provided L_{i+1} is sufficiently large. This implies that $f^{L_{i+1}}(h(\{\tilde{x}\}\times \bar{B}^u_{\delta'_{i+1}}(0)\times \bar{B}^s_{\eta'_{i+1}}(0)))$ is correctly aligned with $h(f^{L_{i+1}}(\tilde{x})\times \bar{B}^u_{\delta_{i+1}}(0)\times \bar{B}^s_{\eta_{i+1}}(0))$ under the identity map (both rectangles are contained in $h(f^{L_{i+1}}(\tilde{x})\times E^u\times E^s)$).

By assumption (iii), there exists $L'_{i+1} > \max\{L_{i+1}, N_{i+1}+M_{i+1}\}$ such that R'_{i+1} is correctly aligned with R_{i+1} under $f^{L'_{i+1}}$. This means that $f^{N_{i+1}}(R'_{i+1})$ is correctly aligned with $f^{-M_{i+1}}(R_{i+1})$ under $f^{K_{i+1}}$ with $K_{i+1} := L'_{i+1} - N_{i+1} - M_{i+1} > 0$.

By the product property of correctly aligned windows implies that W'_{i+1} is correctly aligned with W_{i+1} under $f^{K_{i+1}}$, provided that K_{i+1} is chosen as above.

Step 4. We will now describe how to construct, based on the constructions in Step 1, 2, and 3, two bi-infinite sequences of windows and $\{W_i, W'_i\}_{i\in\mathbb{Z}}$ and $\{\bar{W}_i, \bar{W}'_i\}_{i\in\mathbb{Z}}$ such that, for all $i \in \mathbb{Z}$, W_i is correctly aligned with $\bar{W}_i$ under f^{N_i}, $\bar{W}_i$ is correctly aligned with $\bar{W}'_{i+1}$ under the identity mapping, $\bar{W}'_{i+1}$ is correctly aligned with W_{i+1} under f^{M_i}, and W_{i+1} is correctly aligned with W_{i+1} under $f^{K_{i+1}}$.

We start with the l-dimensional windows R_0 and R'_1. These windows have corresponding copies $\bar{R}_0$ and $\bar{R}'_1$ in Γ, and $\bar{R}_0$ is correctly aligned with $\bar{R}'_1$ under the identity. As in Step 1, we construct a pair of windows $\bar{W}_0$ about $\bar{R}_0$, and $\bar{W}_1$ about $\bar{R}_1$. The size of $\bar{W}_0$ in the hyperbolic directions is given by some disks radii $\bar{\delta}_0, \bar{\eta}_0 < \varepsilon_0$, and that of $\bar{W}'_1$ by some disk radii $\bar{\delta}'_1, \bar{\eta}'_1 < \varepsilon_0$. We choose the quantities $\bar{\delta}_0, \bar{\eta}_0, \bar{\delta}'_1, \bar{\eta}'_1$ such that $\bar{W}_0$ is correctly aligned with $\bar{W}'_1$ under the identity.

Then, we consider the pair of l-dimensional windows R_1 and R'_2 in Λ, and their corresponding copies $\bar{R}_1$ and $\bar{R}'_2$ in Γ. As in Step 1, we construct a pair of windows $\bar{W}_1$ about $\bar{R}_1$, and $\bar{W}'_2$ about $\bar{R}'_2$. By choosing the quantities $\bar{\delta}_1, \bar{\eta}_1, \bar{\delta}'_2, \bar{\eta}'_2$ as in Step 1 we can ensure that $\bar{W}_1$ is correctly aligned with $\bar{W}'_2$.

We choose N_1, M_1 large enough so that $f^{N_1}(\bar{W}'_1)$ is contained in an ε_1-neighborhood of $f^{N_1}(R'_1)$, and $f^{-M_1}(\bar{W}_1)$ is contained in an ε_1-neighborhood of $f^{N_1}(R_1)$. As in Step 2, we construct the window W'_1 about $f^{N_1}(R'_1)$ such that $\bar{W}'_1$ correctly aligned with W'_1 under f^{N_1}, and the window W_1 about $f^{-M_1}(R_1)$ such that W_1 correctly aligned with $\bar{W}_1$ under f^{M_1}. This amounts to choosing the quantities $\delta'_1, \eta'_1, \delta_1, \eta_1$ as in Step 2 in order to ensure the correct alignment of the windows.

Then, we choose K_1 sufficiently large, and at least as large as $N_1 + M_1$, such that W'_1 is correctly aligned with W_1 under f^{K_1}. At this point, we have that $\bar{W}'_1$ is correctly aligned with W'_1 under f^{N_1}, W'_1 is correctly aligned with W_1 under f^{K_1}, and W_1 is correctly aligned with $\bar{W}_1$ under f^{M_1}.

This construction can be continued forward, by induction for all $i \geq 0$. Suppose that at the i-th step of the construction we have obtained the windows $\bar{W}_i$ and $\bar{W}'_i$ that are correctly aligned under the identity, and the window W'_i about $f^{N_i}(R'_i)$ such that $\bar{W}'_i$ is correctly aligned with W'_i under f^{N_i}. We consider the l-dimensional windows R_i and R'_{i+1} in Λ that are correctly aligned under S, and their corresponding copies $\bar{R}_i$ and $\bar{R}'_{i+1}$ in Γ that are correctly aligned under the identity map. As in Step 1, we 'thicken' the l-dimensional windows $\bar{R}_i$ and $\bar{R}'_{i+1}$ to $(l+2n)$-dimensional windows $\bar{W}_i$ and $\bar{W}'_{i+1}$ that are correctly aligned under the identity mapping. Then, as in Step 2, we construct the window W'_{i+1} about $f^{N_{i+1}}(R'_{i+1})$ such that $\bar{W}'_{i+1}$ correctly aligned with W'_{i+1} under $f^{N_{i+1}}$, and the window W_{i+1} about $f^{-M_{i+1}}(R_{i+1}i)$ such that W_{i+1} correctly aligned with $\bar{W}_{i+1}$ under $f^{M_{i+1}}$. Since for each $L > 0$ there exists $L' > L$ such that R'_{i+1} is correctly aligned with R_{i+1} under $f^{L'}$, it follows as in Step 3 that there exists K_{i+1} sufficiently large, and at least as large as $N_{i+1} + M_{i+1}$, such that W'_{i+1} is correctly aligned with W_{i+1} under $f^{K_{i+1}}$. Thus, $\bar{W}'_{i+1}$ is correctly aligned with W'_{i+1} under $f^{N_{i+1}}$, W'_{i+1} is correctly aligned with W_{i+1} under $f^{K_{i+1}}$, and W_{i+1} is correctly aligned with $\bar{W}_{i+1}$ under $f^{M_{i+1}}$. This completes the induction step.

We obtain two sequences of windows $\{W_i, W'_i\}_{i\geq 0}$ and $\{\bar{W}_i, \bar{W}'_i\}_{i\geq 0}$ that satisfy the desired correct alignment conditions for all $i \geq 0$. A similar inductive construction of windows can be done backwards starting with W_0, W'_1.

In the end, we obtain in two sequence of windows $\{W_i, W'_i\}_{i\in\mathbb{Z}}$ and $\{\bar{W}_i, \bar{W}'_i\}_{i\geq 0}$ that satisfy the desired correct alignment conditions stated for all $i \in \mathbb{Z}$. By Theorem 6.6, there exists an orbit $\{f^n(z)\}_{n\in\mathbb{Z}}$ with $f^{n_i}(z) \in \bar{W}_i \cap \bar{W}'_{i+1}$, $f^{n_i+N_{i+1}}(z) \in W'_{i+1}$, $f^{n_i+N_{i+1}+K_{i+1}}(z) \in W_{i+1}$, $f^{n_i+N_{i+1}+K_{i+1}+M_{i+1}}(z) \in \bar{W}_{i+1} \cap \bar{W}'_{i+2}$, for all $i \in \mathbb{Z}$. Thus $n_{i+1} = n_i + N_{i+1} + K_{i+1} + M_{i+1}$. Since the windows are all ε_i-small in the hyperbolic directions, the orbit $\{f^n(z)\}_{n\in\mathbb{Z}}$ satisfies the properties required by Lemma 7.1. □

8. Construction of correctly aligned windows

In this section we prove Theorem 2.1. The methodology consists of constructing 2-dimensional windows in Λ about the prescribed invariant primary tori, BZI's, and Aubry-Mather sets inside the BZI's. The successive pairs of windows are correctly aligned under the scattering map alternatively with powers of the inner map. Lemma 7.1 implies that there exist trajectories that follow these windows.

8.1. **Construction of correctly aligned windows across a BZI.** In this section we will construct correctly aligned windows across a BZI between two successive transition chains of tori. On each side of the BZI we will choose a one-sided neighborhood of a point on the boundary torus, and we will use Theorem 5.1 or Theorem 5.2 to cross over the BZI. These one-sided neighborhoods are of a special type: their boundaries are images of some transition tori under the inner or outer dynamics. Then we will construct some windows about the boundaries of these one-sided neighborhoods, and some other windows about the corresponding transition tori; the sides of these latter windows lie on nearby tori. We will use this feature later to connect sequence of windows across the BZI's with sequences of windows along the transition chains.

Consider an annular region Λ_k in Λ that is a BZI, and is between two transition chains of invariant tori, as in (A5). To simplify notation, we denote the tori at the boundary of Λ_k by T_a and T_b. We choose a pair of transition tori T_i, T_j in Λ as in (A5), ordered as follows: $T_j \prec T_i \prec T_a$. These tori are outside of the BZI Λ_k and on the same side of it as T_a. By (A5-iv) there exist $T_{i'} \prec T_i$ and $T_{j'} \prec T_j$ such that $T_{i'}$ is ε_i-close to T_i and $T_{j'}$ is ε_j-close to T_j, in the C^0 topology. By (A5-ii) $S(T_j)$ intersects T_i in a topologically transverse manner, so both $S(T_j)$ and $S(T_{j'})$ intersect both T_i and $T_{i'}$ in a topologically transverse manner, provided $T_{i'}, T_{j'}$ are sufficiently close to T_i, T_j, respectively.

Since S is a diffeomorphism, $T_i, T_{i'}$ and the images of $T_j, T_{j'}$ under S form a topological rectangle $D_{iji'j'}$ in Λ. This rectangle may not be contained in the domain U^- of the scattering map S. Provided that we choose the tori $T_i, T_{i'}$ sufficiently C^0-close to one another, and also $T_j, T_{j'}$ sufficiently C^0-close to one another, the rectangle $D_{iji'j'}$ will be sufficiently small so that some iterate $f_{|\Lambda}^{K_a}$ of $f_{|\Lambda}$ takes the rectangle $D_{iji'j'}$ into a rectangle $f_{|\Lambda}^{K_a}(D_{iji'j'})$ inside U^-. This is possible since each torus intersects U^- by (A5-i), and the motion on the tori is topologically transitive by (A5-iii). The rectangle $f_{|\Lambda}^{K_a}(D_{iji'j'})$ has a pair of sides lying on the tori $T_i, T_{i'}$, and the other pair of sides on the images $(f_{|\Lambda}^{K_a} \circ S)(T_j), (f_{|\Lambda}^{K_a} \circ S)(T_{j'})$ under $f_{|\Lambda}^{K_a} \circ S$ of $T_j, T_{j'}$ respectively. The curves $(f_{|\Lambda}^{K_a} \circ S)(T_j), (f_{|\Lambda}^{K_a} \circ S)(T_{j'})$ are topologically transverse to both $T_i, T_{i'}$. By assumption (A5-ii), $S(T_j)$ topologically crosses T_a, so $S^{-1}(T_a)$ topologically crosses T_j. We can ensure that the interior of

$f^{K_a}_{|\Lambda}(D_{iji'j'})$ intersects $S^{-1}(T_a)$ by choosing K_a sufficiently large, and the tori in each pair $T_i, T_{i'}$ and $T_j, T_{j'}$ sufficiently C^0-close to one another. This implies that the image of $f^{K_a}_{|\Lambda}(D_{iji'j'})$ under S is a topological rectangle in Λ which intersects T_a, and the intersection of $S(f^{K_a}_{|\Lambda}(D_{iji'j'}))$ with Λ_k forms a one-sided neighborhood in Λ_k of some part of T_a. The boundary of $S(f^{K_a}_{|\Lambda}(D_{iji'j'}))\cap\Lambda_k$ consists of arcs of the curves $T_a, S(T_i), S(T_{i'}), S\circ f^{K_a}_{|\Lambda}\circ S(T_j), S\circ f^{K_a}_{|\Lambda}\circ S(T_{j'})$.

We make a similar construction on the other side of the BZI Λ_k. We choose a pair of transition tori T_k, T_l in Λ as in (A5), ordered as follows: $T_b \prec T_k \prec T_l$, outside of the BZI Λ_k and on the same side as T_b. There exist $T_k \prec T_{k'}$ and $T_l \prec T_{l'}$ such that $T_{k'}$ is ε_k-close to T_k and $T_{l'}$ is ε_l-close to T_l; moreover, we require that $S^{-1}(T_l), S^{-1}(T_{l'})$ are topologically transverse to both $T_k, T_{k'}$. Provided that the tori in each pair $T_k, T_{k'}$ and $T_l, T_{l'}$ are chosen sufficiently C^0-close to one another, $T_k, T_{k'}, S^{-1}(T_l), S^{-1}(T_{l'})$ form a topological rectangle $D_{klk'l'}$, and there exists some iterate $f^{-K_b}_{|\Lambda}$ of $f_{|\Lambda}$ such that $f^{-K_b}_{|\Lambda}(D_{klk'l'})$ is contained in the domain U^+ of S^{-1}. The rectangle $f^{-K_b}_{|\Lambda}(D_{klk'l'})$ has a pair of sides lying on the tori $T_k, T_{k'}$, and the other pair of sides on the images $(f^{-K_b}_{|\Lambda}\circ S^{-1})(T_l), (f^{-K_b}_{|\Lambda}\circ S^{-1})(T_{l'})$ under $f^{-K_b}_{|\Lambda}\circ S^{-1}$ of $T_l, T_{l'}$ respectively, which are transverse to both $T_k, T_{k'}$. Additionally, we can ensure that the interior of $f^{-K_b}_{|\Lambda}(D_{klk'l'})$ intersects $S(T_b)$, by choosing K_b sufficiently large, and the tori in each pair $T_k, T_{k'}$ and $T_l, T_{l'}$ sufficiently C^0-close to one another. Thus the image of $f^{-K_b}_{|\Lambda}(D_{klk'l'})$ under S^{-1} is a topological rectangle in Λ which intersects T_b, and the intersection of $S^{-1}(f^{-K_b}_{|\Lambda}(D_{klk'l'}))$ with Λ_k forms a one-sided neighborhood in Λ_k of some part of T_b. The boundary of $S^{-1}(f^{K_b}_{|\Lambda}(D_{klk'l'}))\cap\Lambda_k$ consists of arcs of the curves $T_b, S(T_k), S(T_{k'}), S^{-1}\circ f^{K_b}_{|\Lambda}\circ S^{-1}(T_l), S^{-1}\circ f^{K_b}_{|\Lambda}\circ S^{-1}(T_{l'})$.

At this stage we have obtained a one-sided neighborhood $(S\circ f^{K_a}_{|\Lambda})(D_{iji'j'})$ in Λ_k of an arc in T_a, and a one-sided neighborhood $(S^{-1}\circ f^{-K_b}_{|\Lambda})(D_{klk'l'})$ of an arc in T_b.

If we are under the assumptions (A1)-(A6), Theorem 5.1 yields a point $x_a\in \operatorname{bd}(S\circ f^{K_a}_{|\Lambda})(D_{iji'j'})$ whose image $x_b = f^{K_{ab}}_{|\Lambda}(x_a)$ under $f^{K_{ab}}_{|\Lambda}$ lies on $\operatorname{bd}(S^{-1}\circ f^{-K_b}_{|\Lambda})(D_{klk'l'})$.

If we also assume (A7), applying Theorem 5.2 yields a point $x_a \in \operatorname{bd}(S\circ f^{K_a}_{|\Lambda})(D_{iji'j'})$ with $x_b = f^{K_{ab}}_{|\Lambda}(x_a)\in \operatorname{bd}(S^{-1}\circ f^{-K_b}_{|\Lambda})(D_{klk'l'})$ as above, satisfying the additional conditions

$$\pi_\phi(f^j(w^k_s)) < \pi_\phi(f^j(x_a)) < \pi_\phi(f^j(\bar{w}^k_s)),$$

for each $s\in\{1,\ldots,s_k\}$, where $w^k_s, \bar{w}^k_s\in\Sigma_{\omega^k_s}$, and for all j within a certain interval of integers. The trajectories of all points sufficiently close to x_a will satisfy these conditions.

In either case, there exist an arc $\bar{e}'_a\subseteq \operatorname{bd}(S\circ f^{K_a}_{|\Lambda})(D_{iji'j'})$ containing x_a, and another arc $\bar{e}_b\subseteq \operatorname{bd}(S^{-1}\circ f^{-K_b}_{|\Lambda})(D_{klk'l'})$ containing x_b, such that $f^{K_{ab}}_{|\Lambda}(\bar{e}'_a)$ is topologically transverse to $\bar{e}_b$ at x_b. (Indeed, if all intersections between the images under $f^{K_{ab}}_{|\Lambda}$ of the edges of $(S\circ f^{K_a}_{|\Lambda})(D_{iji'j'})$ and $(S^{-1}\circ f^{-K_b}_{|\Lambda})(D_{klk'l'})$ would be

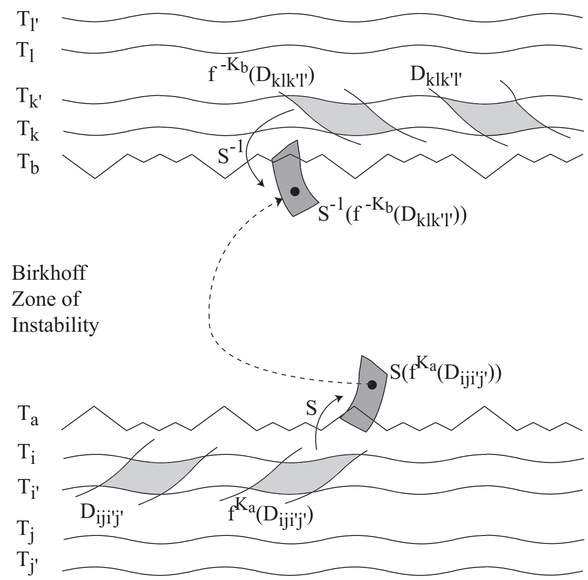

FIGURE 3. Orbits across a BZI

tangential, then no interior point of $(S\circ f^{K_a}_{|\Lambda})(D_{iji'j'})$ will be mapped by $f^{K_{ab}}_{|\Lambda}$ onto an interior point of $(S^{-1}\circ f^{-K_b}_{|\Lambda})(D_{klk'l'})$.)

The arc $\bar{e}'_a$ lies on one of the sets $S(T_i), S(T_{i'}), (S\circ f^{K_a}_{|\Lambda}\circ S)(T_j), (S\circ f^{K_a}_{|\Lambda}\circ S)(T_{j'})$. Similarly, the arc $\bar{e}_b$ lies on one of the sets $S^{-1}(T_k), S^{-1}(T_{k'}), (S^{-1}\circ f^{-K_b}_{|\Lambda}\circ S^{-1})(T_l), (S^{-1}\circ f^{-K_b}_{|\Lambda}\circ S^{-1})(T_{l'})$.

We define a 2-dimensional window R'_a about $\bar{e}'_a$, and a 2-dimensional window R_b about $\bar{e}_b$ such that R'_a is correctly aligned with R_b under $f^{K_{ab}}_{|\Lambda}$. Informally, the exit direction of R'_a is along $\bar{e}'_a$, and the exit direction of R_b is across $\bar{e}_b$. The formal construction now follows. Since the arc $\bar{e}'_a$ is an embedded 1-dimensional C^0-submanifold of Λ, there exists a C^0-local parametrization $\chi'_a:\mathbb{R}^2\to\Lambda$ such that $\chi'_a([0,1]\times\{0\})=\bar{e}'_a$, provided $\bar{e}'_a$ is sufficiently small. Then

$$R'_a=\chi_a([0,1]\times[-\eta'_a,\eta'_a]),$$

is a topological rectangle. We define the exit set of R_a as

$$R'^{\rm exit}_a=\chi_a(\partial[0,1]\times[-\eta'_a,\eta'_a]).$$

Similarly, there exists a C^0-local parametrization $\chi_b:\mathbb{R}^2\to\Lambda$ such that $\chi_b(\{0\}\times[0,1])=\bar{e}_b$, and

$$R_b=\chi_b([-\delta_b,\delta'_b]\times[0,1]),$$

is a topological rectangle. We define the exit set of R_b as

$$R^{\rm exit}_b=\chi_b(\partial[-\delta_b,\delta_b]\times[0,1]).$$

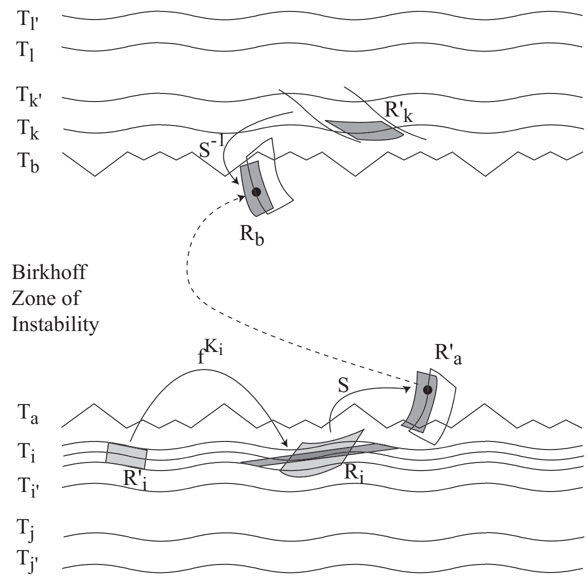

FIGURE 4. Construction of windows near the boundaries of a BZI – case 1

By choosing η'_a, δ_b sufficiently small, we ensure that R'_a is correctly aligned with R_b under $f^{K_{ab}}_{|\Lambda}$ (see Definition 6.5). See Figure 4.

We construct other windows outside the BZI Λ_k. We consider two cases: first case, when the arc $\bar{e}'_a$ is a part of $S(T_i)$ or $S(T_{i'})$; second case, when $\bar{e}'_a$ is a part of $(S \circ f^{K_a}_{|\Lambda} \circ S)(T_j)$ or $(S \circ f^{K_a}_{|\Lambda} \circ S)(T_{j'})$.

Case 1. In the first case, when $\bar{e}'_a$ is a part of $S(T_i)$ or $S(T_{i'})$ we proceed with the construction as follows. For simplicity, we assume that $\bar{e}'_a$ is a part of $S(T_i)$. We take the inverse image of R'_a by S. This is a topological rectangle $S^{-1}(R'_a)$ about the torus T_i. We construct a window R_i about an arc $\bar{e}_i$ in T_i, such that R_i is correctly aligned with $S^{-1}(R'_a)$ under the identity map, and with the exit direction of R_i in the direction of T_i. Let $\bar{e}_i$ be an arc in T_i, and $\chi_i : \mathbb{R}^2 \to \Lambda$ be a local parametrization with $\chi_i([0,1] \times \{0\}) = \bar{e}_i$. We define

$$R_i = \chi_i([0,1] \times [-\eta_i, \eta_i])$$
$$R_i^{\text{exit}} = \chi_i(\partial[0,1] \times [-\eta_i, \eta_i]),$$

where $\eta_i > 0$ is sufficiently small. By choosing the arc $\bar{e}_i$ sufficiently large so that $\bar{e}_i \supseteq S^{-1}(\bar{e}'_a)$, and by choosing $\eta_i > 0$ sufficiently small, we can ensure that R_i is correctly aligned with $S^{-1}(R'_a)$ under the identity map, or, equivalently, R_i is correctly aligned with R'_a under S. See Figure 4.

We will now construct another window R'_i about T_i such that R'_i is correctly aligned with R_i under some power $f^{K_i}_{|\Lambda}$ of $f_{|\Lambda}$, with $K_i > 0$. The window R'_i will have its exit direction is across the torus T_i. We choose a pair of invariant primary tori $T_i^{\text{lower}} \prec T_i \prec T_i^{\text{upper}}$, with $T_i^{\text{lower}}, T_i^{\text{upper}}$ located ε_i-close to T_i in the C^0-topology. Such tori exist due to the assumption that T_i is not an end torus in the transition chain, as in (A5-iv). Moreover, we choose the tori $T_i^{\text{lower}}, T_i^{\text{upper}}$ sufficiently close to T_i so that the components of the entry set of R_i are outside the annulus between $T_i^{\rm lower}$ and $T_i^{\rm upper}$, with one component on one side and the other component on the other side of this annulus. We choose an arc $\bar{e}'_i \subseteq T_i$ that lies on an edge of the topological rectangle $D_{iji'j'}$. Let $\chi'_i : \mathbb{R}^2 \to \Lambda$ be a C^0-local parametrization with $\chi'_i(\{0\} \times [0,1]) = \bar{e}'_i$, such that for some δ'_i sufficiently small, $\chi'_i(\{-\delta'_i\} \times [0,1]) \subseteq T_i^{\rm lower}$ and $\chi'_i(\{\delta'_i\} \times [0,1]) \subseteq T_i^{\rm upper}$. We define R'_i by:

$$R'_i = \chi'_i([-\delta'_i, \delta'_i] \times [0,1]),$$
$$R'^{\rm exit}_i = \chi'_i(\partial[-\delta'_i, \delta'_i] \times [0,1]).$$

The exit set components of R'_i lie on the tori $T_i^{\rm upper}, T_i^{\rm lower}$ neighboring T_i.

Since the motion on the tori is topologically transitive, by (A5-iii), there exists $K_i > 0$ such that R'_i is correctly aligned with R_i under $f_{|\Lambda}^{K_i}$. Indeed, to achieve correct alignment of these windows in the covering space of the annulus we only have to choose K_i sufficiently large so that the two components of $R'^{\rm exit}_i$, which are two arcs in $T_i^{\rm lower}$ and $T_i^{\rm upper}$, are mapped by $f_{|\Lambda}^{K_i}$ on the opposite sides of the part of R_i between $T_i^{\rm lower}$ and $T_i^{\rm upper}$. We note that the number of iterates K_i of $f_{|\Lambda}$ needed to make R'_i correctly aligned with R_i may be different than the number of iterates K_a which takes the topological rectangle $D_{iji'j'}$ onto $f_{|\Lambda}^{K_a}(D_{ihi'j'})$. See Figure 4. The conclusion of this step is that we obtain the window R'_i around T_i, with its exit direction across T_i, such that R'_i is correctly aligned with R_i under $f_{|\Lambda}^{K_i}$. Both windows R'_i, R_i are contained in an ε_i-neighborhood of T_i.

In the case when the edge $\bar{e}'_a$ of $D_{iji'j'}$ is a part of $S(T_{i'})$ instead of $S(T_i)$, we construct in a similar fashion a pair of windows $R'_{i'}$ and $R_{i'}$ about $T_{i'}$, such that $R'_{i'}$ is correctly aligned with $R_{i'}$ under some iterate $f_{|\Lambda}^{K_{i'}}$ of $f_{|\Lambda}$, and $R_{i'}$ is correctly aligned with R'_a under S. The exit direction of $R'_{i'}$ is across the torus $T_{i'}$, and the exit direction of $R_{i'}$ is in the direction of the torus $T_{i'}$. Since $T_{i'}$ is ε_i-close to T_i, we can choose the windows $R'_i, R_{i'}$ so that they are both contained in an ε_i-neighborhood of T_i.

Case 2. We now consider the second case, when the arc $\bar{e}'_a$ of $\mathrm{bd}(S \circ f_{|\Lambda}^{K_a})(D_{iji'j'})$ is a part of $(S \circ f_{|\Lambda}^{K_a} \circ S)(T_j)$ or $(S \circ f_{|\Lambda}^{K_a} \circ S)(T_{j'})$. For simplicity, we assume that $\bar{e}'_a$ is a part of $(S \circ f_{|\Lambda}^{K_a} \circ S)(T_j)$. We construct a window R'_a in Λ about $\bar{e}'_a$ as before; the exit set of R'_a is in a direction along $\bar{e}'_a$, and the size of R'_a in the direction across R_a is given by some parameter η'_a. See Figure 5. We consider the inverse image $S^{-1}(R'_a)$ of R'_a by S, which is a topological rectangle about $(f_{|\Lambda}^{K_a} \circ S)(T_j)$. Let $\bar{e}''_j$ be an arc in $(f_{|\Lambda}^{K_a} \circ S)(T_j)$ such that $S(\bar{e}''_j) \supset \bar{e}'_a$, and let $\chi''_j : \mathbb{R}^2 \to \Lambda$ be a local parametrization with $\chi''_j([0,1] \times \{0\}) = \bar{e}''_j$. We define R''_j by:

$$R''_j = \chi''_j([0,1] \times [-\eta''_j, \eta''_j]),$$
$$R''^{\rm exit}_j = \chi''_j(\partial[0,1] \times [-\eta''_j, \eta''_j]).$$

By choosing $\eta''_j > 0$ sufficiently small, we can ensure that R''_j is correctly aligned with aligned with R'_a under S. The window R''_j is in a neighborhood of an edge of the topological rectangle $f_{|\Lambda}^{K_a}(D_{iji'j'})$. The exit set of R''_j is in the direction of $(f_{|\Lambda}^{K_a} \circ S)(T_j)$. See Figure 5.

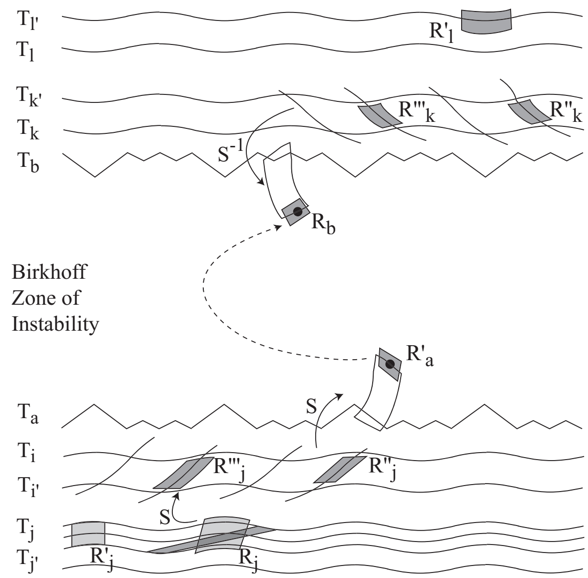

FIGURE 5. Construction of windows near the boundaries of a BZI – case 2

We construct a window R_j''' about $S(T_j)$ such that R_j''' is correctly aligned with R_j'' under $f_{|\Lambda}^{K_a}$. We define

$$R_j''' = \chi_j'''([0,1]\times[-\eta_j''',\eta_j''']),$$
$$R'''^{\text{exit}}_j = \chi_j'''(\partial[0,1]\times[-\eta_j''',\eta_j''']),$$

where $\bar{e}_j'''$ is an arc of $S(T_j)$ with $f_{|\Lambda}^{K_a}(\bar{e}_j''')\supseteq\bar{e}_j''$, $\chi_j''':\mathbb{R}^2\to\Lambda$ is a local parametrization with $\chi_j'''([0,1]\times\{0\})=\bar{e}_j'''$, and $\eta_j'''>0$ is sufficiently small. The arc $\bar{e}_j'''$ is contained in one of the edges of the topological rectangle $D_{iji'j'}$. For suitable $\eta_j'''>0$, we can ensure that R_j''' is correctly aligned with aligned with R_j'' under $f_{|\Lambda}^{K_a}$. Moreover, we choose R_j''', R_j'' so that these windows are both contained in an ε_j neighborhood of T_j. The exit set of R_j''' is in a direction along $S(T_j)$.

We take the inverse image $S^{-1}(R_j''')$ of R_j''' under S, which is a topological rectangle about an arc in T_j. In a fashion similar to Case 1, we construct two windows R_j', R_j about T_j such that R_j' is correctly aligned with R_j under some power $f_{|\Lambda}^{K_j}$ of $f_{|\Lambda}$, and R_j is correctly aligned with R_j''' under S. The exit of R_j' is chosen in a direction across T_j, and the exit set components R'^{exit}_j of R_j' lie on two invariant primary tori T_j^{upper}, T_j^{lower} neighboring T_j, that are ε_j-close to T_j and contain T_j between them. The exit of R_j is in a direction along T_j. The windows R_j', R_j are chosen to lie in an ε_j neighborhood of T_j relative to the C^0-topology.

The case when the arc $\bar{e}_a'$ is a part of $(S\circ f_{|\Lambda}^{K_a}\circ S)(T_{j'})$ is treated similarly.

This concludes the construction of correctly aligned windows in Λ, starting with the window R_a' about T_a, and moving backwards along transition tori that are on the same side as T_a of the BZI Λ_k.

This construction yields a sequence of windows of the type

$$R'_i, R_i, R'_a, \quad (\text{ or } R'_{i'}, R_{i'}, R'_a) \tag{8.1}$$

in the first case, or

$$R'_j, R_j, R'''_j, R''_j, R'_a, (\text{ or } R'_{j'}, R_{j'}, R'''_{j'}, R''_{j'}, R'_a), \tag{8.2}$$

in the second case. In the first case, R'_i is correctly aligned with R_i under some power $f_{|\Lambda}^{K_i}$ of the inner map, and R_i is correctly aligned with R'_a under the outer map S. The exit direction of R'_i is across the torus T_i, and its exit set components lie on some invariant primary tori that are ε_i-close to T_i. The windows R'_i, R_i are contained in an ε_i neighborhood of T_i. Similar relations hold in the case of $R'_{i'}, R_{i'}$. In the second case, R'_j is correctly aligned with R_j under some power $f_{|\Lambda}^{K_j}$ of the inner map, R_j is correctly aligned with R'''_j under the outer map S, R'''_j is correctly aligned with R''_j under some power $f_{|\Lambda}^{K_a}$ of the inner map, and R''_j is correctly aligned with R'_a under the outer map S. The exit direction of R_j is across the torus T_j, and its exit set components lie on some invariant primary tori that are ε_j-close to T_j. The windows R'_j, R_j are contained in an ε_j neighborhood of T_j, and the windows R'''_j, R''_j are contained in an ε_i neighborhood of T_i. Similar relations hold in the case of $R'_{j'}, R_{j'}, R'''_{j'}, R''_{j'}$.

We proceed with a similar construction on the other side of the BZI between Λ_k, that is, on the same side of the BZI as T_b. We have already defined the window R_b about T_b that R'_a is correctly aligned with R_b under $f_{|\Lambda}^{K_{ab}}$. Starting with the window R_b and moving forward along the transition chain T_b, T_k, T_l, we construct a sequence of windows of the type

$$R_b, R'_k, \quad (\text{ or } R_b, R'_{k'}), \tag{8.3}$$

or of the type

$$R_b, R'''_k, R''_k, R'_l, (\text{ or } R_b, R'''_{k'}, R''_{k'}, R'_{l'}), \tag{8.4}$$

satisfying the correct alignment conditions below. In the first case, R_b is correctly aligned with R'_k under the outer map S. The exit direction of R'_k is across T_k, and the exit set components lie on two invariant primary tori ε_k-close to T_k. Moreover, R'_k is contained in an ε_k-neighborhood of T_k. In the second case, R'_b is correctly aligned with R'''_k under the outer map S, R'''_k is correctly aligned with R''_k under some power $f_{|\Lambda}^{K_b}$ of the inner map, and R''_k is correctly aligned with R'_l under the outer map S. The exit direction of R'_l is across T_l, and its exit set components lie on some invariant primary tori that are ε_l-close to T_l. The windows R'''_k, R''_k are contained in an ε_k-neighborhood of T_k, and R'_l is contained in an ε_l-neighborhood of T_l. Similar statements apply when instead of windows around T_k or T_l we construct windows about $T_{k'}$ or $T_{l'}$, respectively.

The conclusion of this section is that, by combining a sequence of correctly aligned window of the type (8.1) or (8.2) with a sequence of correctly aligned window of the type (8.3) or (8.4) we obtain a finite sequence of correctly aligned windows that crosses the BZI Λ_k. The shadowing lemma-type of result Lemma 7.1 yields an orbit that visits some prescribed neighborhood in the phase space of each window in Λ. In particular, the shadowing orbit has points that go close to the transition tori.

8.2. **Construction of correctly aligned windows across annular regions separated by invariant tori.** We consider an annular region in Λ between two transition chains of invariant tori. Inside this annular region, we assume the existence of a finite collection of invariant tori that separate the region, as in (A6′-i). Thus, the annular region between the transition chains is not a BZI. We also assume that the scattering map satisfies the stronger non-degeneracy condition (A6′-ii) on these invariant tori. The situation presented in this section is non-generic.

Assume that the annular region in Λ is bounded by a pair of invariant Lipschitz tori T_a and T_b. Let T_a be the end torus of the transition chain of one side, and T_b be the end torus of the transition chain on the other side. We assume that inside the region in the annulus bounded by T_a and T_b there exist a finite collection of invariant tori $\{\Upsilon_h\}_{h=1,\dots,k-1}$. Each Υ_h is either an isolated invariant primary torus, or a hyperbolic periodic orbit together with its stable and unstable manifolds that are assumed to coincide. In either case, the regions between T_a and Υ_1, Υ_h and Υ_{h+1} for $h=1,\dots,k$, and between Υ_k and T_b are all BZI's.

The constructions in Subsection 8.1 provide a one-sided neighborhood of the type $(S\circ f_{|\Lambda}^{K_a})(D_{iji'j'})$ of some point in T_a, and a one-sided neighborhood $(S^{-1}\circ f_{|\Lambda}^{-K_b})(D_{klk'l'})$ of some point in T_b. Let us denote $D_0=(S\circ f_{|\Lambda}^{K_a})(D_{iji'j'})$ and $D_{k+1}=(S^{-1}\circ f_{|\Lambda}^{-K_b})(D_{klk'l'})$.

Assume that Υ_1 is an isolated invariant primary torus. We have that $S^{-1}(\Upsilon_1)$ forms with Υ_1 a topological disk D_1 between T_a and Υ_1, which is mapped by S onto a topological disk $S(D_1)$ between Υ_1 and Υ_2. Theorem 5.1 provides us a trajectory that starts from $\mathrm{bd}(D_0)$ and ends at $\mathrm{bd}(D_1)$. In particular, there exist $K_1>0$ and a component of $f^{K_1}(D_0)\cap D_1$ that is a topological disk D_1' whose boundary contains an arc of $S^{-1}(T_1)$. The image of D_1' under S is a one-sided neighborhood $D_1''\subseteq S(D_1)$ of some point in Υ_1 that is contained in the region between Υ_1 and Υ_2. The boundary of D_1'' consists of an arc in Υ_1 and an arc in $(S\circ f^{K_1}(\mathrm{bd}(D_0)))$.

Now assume that Υ_1 is a hyperbolic periodic orbit, together with its stable and unstable manifolds that are assumed to coincide. Then Υ_1 is the union of two curves $\Upsilon_1^{\mathrm{lower}}$, $\Upsilon_1^{\mathrm{upper}}$ that have in common only the points of the periodic orbit. Excepting the common points, $\Upsilon_1^{\mathrm{lower}}$ is below $\Upsilon_1^{\mathrm{upper}}$. We have that $S^{-1}(\Upsilon_1^{\mathrm{lower}})$ forms with $\Upsilon_1^{\mathrm{lower}}$ a topological disk D_1^{lower} between T_a and $\Upsilon_1^{\mathrm{lower}}$, which is mapped by S onto a topological disk $S(D_1^{\mathrm{lower}})$ between $\Upsilon_1^{\mathrm{lower}}$ and $\Upsilon_1^{\mathrm{upper}}$. We also have that $S^{-1}(\Upsilon_1^{\mathrm{upper}})$ forms with $\Upsilon_1^{\mathrm{upper}}$ a topological disk D_1^{upper} between T_a and $\Upsilon_1^{\mathrm{upper}}$, which is mapped by S onto a topological disk $S(D_1^{\mathrm{upper}})$ between $\Upsilon_1^{\mathrm{upper}}$ and Υ_2. By Theorem 5.1 there exist $K_1>0$ and a component of $f^{K_1}(D_0)\cap D_1^{\mathrm{lower}}$ that is a topological disk ${D'}_1^{\mathrm{lower}}$ whose boundary contains an arc of $S^{-1}(\Upsilon_1)$. The image of ${D'}_1^{\mathrm{lower}}$ under S is a one-sided neighborhood ${D''}_1^{\mathrm{lower}}\subseteq S(D_1)$ of some point in $\Upsilon_1^{\mathrm{lower}}$, contained in the region between $\Upsilon_1^{\mathrm{lower}}$ and $\Upsilon_2^{\mathrm{upper}}$. The boundary of ${D''}_1^{\mathrm{lower}}$ consists of an arc in $\Upsilon_1^{\mathrm{lower}}$ and an arc in $S(f^{K_1}(\mathrm{bd}(D_0)))$. By the argument in the Homoclinic Orbit Theorem (see e.g. [7]), there exists N_1 such that $f^{N_1}({D''}_1^{\mathrm{lower}})$ intersects D_1^{upper}. Let ${D'}^{\mathrm{upper}}$ be a component of this intersection that is a topological disk, and whose boundary contains an arc of $S^{-1}(\Upsilon_1^{\mathrm{upper}})$. Then the image ${D'}^{\mathrm{upper}}$ under S is a one sided neighborhood D_1'' of some point in $\Upsilon_1^{\mathrm{upper}}$ that is contained in the region between $\Upsilon_1^{\mathrm{upper}}$ and Υ_2. The boundary of D_1'' consists of an arc in $\Upsilon_1^{\mathrm{upper}}$ and an arc in $(S\circ f^{N_1}\circ S\circ f^{K_1})(\mathrm{bd}(D_0))$. See Figure 6.

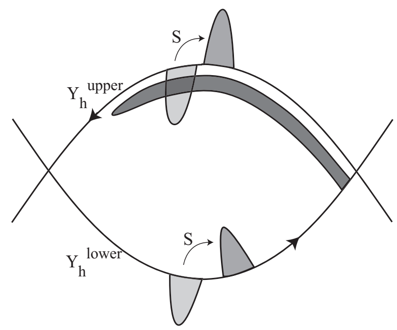

FIGURE 6. Construction of windows across a resonance

The main point of this construction is that using the inner and outer dynamics, one can cross the BZI between T_a and Υ_1, and obtain a one-sided neighborhood of some point in Υ_1, contained in the BZI between Υ_1 and Υ_2, such that boundary of that neighborhood contains the image of $\mathrm{bd}(D_0)$ under a suitable composition of S and powers of f.

This argument can be repeated for each invariant torus Υ_h with $2 \leq h \leq k$, yielding a point $x_a \in \mathrm{bd}(D_0)$ that is mapped by some appropriate composition of S and powers of f onto a point $x_b \in \mathrm{bd}(D_{k+1})$. Then the constructions from Subsection 8.1 can be applied to obtain a window R'_a about $\mathrm{bd}(D_0)$, and a window R'_b about $\mathrm{bd}(D_{k+1})$, such that R'_a is correctly aligned with R_b under the appropriate composition of S and powers of f. Then there exists a shadowing orbit that goes from a neighborhood of R_a in M to a neighborhood of R'_b in M.

If the region between T_a and T_b contains some prescribed collection of Aubry-Mather sets, then Theorem 5.2 yields a shadowing orbit that crosses the region between T_a and T_b and follows the prescribed Aubry-Mather sets as in (2.2).

8.3. **Construction of correctly aligned windows along transition chains of tori.** We consider a finite transition chain of invariant primary tori $\{T_1, T_2, \ldots, T_n\}$ in the annulus Λ satisfying (A5). All tori T_k, $k = 1, \ldots, n$, intersect the subset U^-_+ of the domain of the scattering map S where S moves points upwards in the annulus Λ. The motion on each torus T_k is topologically transitive. Each torus T_k with $2 \leq k \leq n-1$ is an 'interior' torus, i.e. it can be C^0-approximated from both sides in Λ by other invariant primary tori. For each $k \in \{1, \ldots, n-1\}$, the image $S(T_k)$ of the torus T_k under the scattering map S has a topologically transverse intersection with the torus T_{k+1}. We would like to show that there exists an orbit that visits some ε_k-neighborhood of each torus T_k, $k = 1, \ldots, n$, in the prescribed order. To this end we will construct a sequence of 2-dimensional windows in Λ that are correctly aligned one with another under the scattering map alternatively with some iterates of the inner map, as in Lemma 7.1. Then the lemma will imply the existence of a shadowing orbit to the transition chain.

For each $k \in \{1, \ldots, n-1\}$, we choose and fix a pair of points $x^-_{k,k+1} \in T_k$ and $x^+_{k,k+1} \in T_{k+1}$ such that $S(x^-_{k,k+1}) = x^+_{k,k+1}$ and $S(T_k)$ intersects T_{k+1} transversally at $x^+_{k,k+1}$.

We construct inductively a sequence of correctly aligned windows in Λ along the tori $\{T_1, T_2, \ldots, T_n\}$, such that each window is correctly aligned with the next window in the sequence either by the outer map or by some sufficiently large power of the inner map. Moreover, each window will be contained in some ε-neighborhood of some transition torus. We start the inductive construction at T_1. Consider the point $x^-_{1,2} \in T_1$ with $S(x^-_{1,2}) = x^+_{1,2} \in T_2$ as above.

We construct a window R'_1 about T_1 as follows. Let $\bar{e}'_1$ be an arc contained in T_1, and $\chi'_1 : \mathbb{R}^2 \to \Lambda$ a C^0-local parametrization such that $\chi'_1([0,1] \times \{0\}) = \bar{e}'_1$. Then we define

$$R'_1 = \chi'_1([0,1] \times [-\delta'_1, \delta'_1]),$$
$$R'^{\text{exit}}_1 = \chi'_1(\partial[0,1] \times [-\delta'_1, \delta'_1]),$$

where $0 < \delta'_1 < \varepsilon_1$. We choose $\bar{e}'_1$ and δ'_1 sufficiently small so that and $S(\bar{e}'_1)$ intersects T_2 only at $x^+_{1,2}$, and also so that R'_1 defined as above is contained in U^-_+. The exit direction of R'_1 is in the direction of the torus T_1.

The image $S(R'_1)$ of R'_1 under the scattering map is a topological rectangle. Since $S(\bar{e}'_1)$ is transverse to T_2 at $x^+_{1,2}$, then by choosing δ'_1 sufficiently small we ensure that the components of $S(R'^{\text{exit}}_1)$ lie on opposite sides of T_2. Thus, the exit direction of $S(R'_1)$ is across the torus T_2.

Next we consider the point $x^-_{2,3} \in T_2$ with $S(x^-_{2,3}) = x^+_{2,3} \in T_3$ as above. We construct a window R'_2 about T_2 in a manner similar to R'_1:

$$R'_2 = \chi'_2([0,1] \times [-\delta'_2, \delta'_2]),$$
$$R'^{\text{exit}}_2 = \chi'_2(\partial[0,1] \times [-\delta'_2, \delta'_2]),$$

where $\bar{e}'_2$ is an arc contained in T_2, $\chi'_2 : \mathbb{R}^2 \to \Lambda$ is a C^0-local parametrization such that $\chi'_2([0,1] \times \{0\}) = \bar{e}'_2$, and $0 < \delta'_2 < \varepsilon_2$ is chosen sufficiently small.

Now we construct a window R_2 about the point $x^+_{1,2}$ such that R'_1 is correctly aligned with R_2 under S and R_2 is correctly aligned with R'_2 under some power of $f_{|\Lambda}$. We choose an arc $\bar{e}_2$ in T_2 containing $x^+_{1,2}$ such that $\bar{e}_2 \supseteq S(R'_1) \cap T_2$. We choose a pair of invariant tori T^{lower}_2 and T^{upper}_2 such that $T^{\text{lower}}_2 \prec T_2 \prec T^{\text{upper}}_2$ and the exit set components of $S(R'_1)$, as well as the entry set components of R'_2, are outside of the annulus bounded by T^{lower}_2 and T^{upper}_2, on the both sides of the annulus. Furthermore, we require that T^{lower}_2 and T^{upper}_2 should be ε_2-close to T_2 in the C^0-topology. The existence of such neighboring tori T^{lower}_2 and T^{upper}_2 to T_2 is guaranteed by (A5-iv). Let $\chi_2 : \mathbb{R}^2 \to \Lambda$ be a C^0-local parametrization such that $\chi_2(\{0\} \times [0,1]) = \bar{e}_2$, $\chi_2(\{-\delta_2\} \times [0,1]) \subseteq T^{\text{lower}}_2$ and $\chi_2(\{\delta_2\} \times [0,1]) \subseteq T^{\text{upper}}_2$, for some $0 < \delta_2 < \varepsilon_2$ sufficiently small. Define

$$R_2 = \chi_2([-\delta_2, \delta_2] \times [0,1]),$$
$$R^{\text{exit}}_2 = \chi_2(\partial[-\delta_2, \delta_2] \times [0,1]).$$

The exit set components of R_2 lie on the invariant tori $T^{\text{lower}}_2, T^{\text{upper}}_2$ neighboring T_2.

Since the motion on the torus T_2 is topological transitive, there exists a power $f^{K_2}_{|\Lambda}$ of $f_{|\Lambda}$ such that R_2 is correctly aligned with R'_2 under $f^{K_2}_{|\Lambda}$. See Figure 7. We have obtained the windows R'_1 about T_1 and R'_2, R_2 about T_2 such that R'_1 is correctly aligned with R_2 under S and R_2 is correctly aligned with R'_2 under some power of $f_{|\Lambda}$. This ends the initial step of the inductive construction.

The inductive step goes on similarly. Assume that we have arrived at a torus T_k with a window R'_k, about the point $x^-_{k,k+1}$, with the exit direction along the torus T_k.

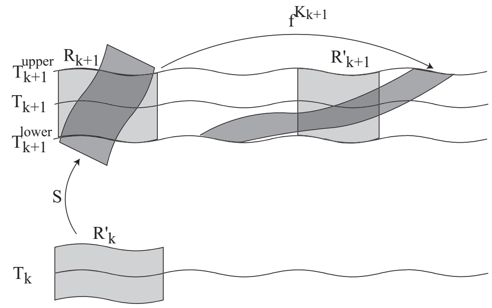

FIGURE 7. Construction of windows along a transition chain

Let $S(x^-_{k,k+1}) = x^+_{k,k+1} \in T_{k+1}$. The image $S(R'_k)$ of R'_k under the scattering map is a topological rectangle about $x^+_{k,k+1}$, with its exit direction across the torus T_{k+1}. Consider a point $x^-_{k+1,k+2} \in T_{k+1}$ such that $S(x^-_{k+1,k+2}) = x^+_{k+1,k+2} \in T_{k+2}$. We construct a window R'_{k+1} about T_{k+1} with it exit direction along the torus T_{k+1}. We choose a pair of invariant tori T^{lower}_{k+1} and T^{upper}_{k+1} such that $T^{\text{lower}}_{k+1} \prec T_{k+1} \prec T^{\text{upper}}_{k+1}$ and the exit set components of $S(R'_k)$, as well as the entry set components of R'_{k+1}, are outside of the annulus bounded by T^{lower}_{k+1} and T^{upper}_{k+1}, on the opposite sides of the annulus. Furthermore, we require that T^{lower}_{k+1} and T^{upper}_{k+1} should be ε_{k+1}-close to T_{k+1} in the C^0-topology. We define a window R_{k+1} about the point $x^+_{k,k+1}$such that its exit set components lie on T^{lower}_{k+1} and T^{upper}_{k+1}. If the size R_{k+1} is chosen small enough in its entry direction, then we can ensure that R'_k is correctly aligned with R_{k+1} under S. There exists a power $f^{K_{k+1}}_{|\Lambda}$ of $f_{|\Lambda}$ such that R_{k+1} is correctly aligned with R'_{k+1} under $f^{K_{k+1}}_{|\Lambda}$. We have obtained the windows R'_k about T_k and R'_{k+1}, R_{k+1} about T_{k+1} such that R'_k is correctly aligned with R_{k+1} under S and R_{k+1} is correctly aligned with R'_{k+1} under some power of $f_{|\Lambda}$. See Figure 7. This ends the inductive proof of the construction.

The construction proceeds inductively until a sequence of windows in Λ of the following type is obtained

$$R'_1, R_2, R'_2, \ldots, R_k, R'_k, R_{k+1}, R'_{k+1}, \ldots, R_n,$$

where for each $k \in \{1, \ldots, n-1\}$ we have that R'_k is correctly aligned with R_{k+1} under S, and R_{k+1} is correctly aligned with R'_{k+1} under $f^{K_{k+1}}_{|\Lambda}$ for some $K_{k+1} > 0$. Each window R_k, R'_k is contained in an ε_k-neighborhood of the torus T_k. Applying the shadowing lemma-type of result Lemma 7.1 provides an orbit that visits an ε_k-neighborhood in the phase space of each window in the sequence, and in particular of each torus in the transition chain.

8.4. **Gluing correctly aligned windows across BZI's with correctly aligned windows along transition chains of tori.** We consider three transition chains of tori $\{T_i\}_{i=i_{k-1}+1,\ldots,i_k}$, $\{T_i\}_{i=i_k+1,\ldots,i_{k+1}}$, $\{T_i\}_{i=i_{k+1}+1,\ldots,i_{k+2}}$, with the property that each one of the regions between T_{i_k} and T_{i_k+1}, and between $T_{i_{k+1}}$ and $T_{i_{k+1}+1}$, is either a BZI as in (A6), or it contains a finite number of invariant tori that separate the region as in (A6′). Inside each region there is a prescribed collection of Aubry-Mather sets as in (A7)

The constructions in Subsection 8.1 and in Subsection 8.2 yield correctly aligned windows in Λ that cross the region between T_{i_k} and T_{i_k+1}, and correctly aligned windows in Λ that cross the region between $T_{i_{k+1}}$ and $T_{i_{k+1}+1}$. The construction in Subsection 8.3 yield sequences of correctly aligned windows along the adjacent transition chains. The choices of the windows constructed along the transition chains depend on the choices of the windows that cross the region between T_{i_k} and T_{i_k+1}, and of the windows that cross the region between $T_{i_{k+1}}$ and $T_{i_{k+1}+1}$. Propagating the construction of windows starting from T_{i_k+1} and moving forward along the transition chain $\{T_i\}_{i=i_k+1,\dots,i_{k+1}}$, and the construction of windows starting from $T_{i_{k+1}}$ and moving backward along the same transition chain $\{T_i\}_{i=i_k+1,\dots,i_{k+1}}$, may result in a pair of windows about some intermediate torus that are not correctly aligned. We would like to glue this sequences of windows in a manner that is correctly aligned, without having to revise the windows constructed to that point.

Assume that T_j is one of the tori $\{T_i\}_{i=i_k+1,\dots,i_{k+1}}$, with $j\in\{i_k+2,\dots,i_{k+1}-1\}$. Assume that one has already constructed a window R_j about T_j by propagating the construction from T_{i_k+1} and moving forward along the transition chain, and another window R'_j about T_j by propagating the construction from $T_{i_{k+1}}$ and moving backwards along the transition chain. The window R_j is of the form

$$\begin{aligned} R_j &= \chi_j([-\delta_j,\delta_j]\times[0,1]),\\ R_j^{\text{exit}} &= \chi_j(\partial[-\delta_j,\delta_j]\times[0,1]), \end{aligned}$$

where $\chi_j:\mathbb{R}^2\to\Lambda$ is a C^0-local parametrization with $\chi_j(\{0\}\times[0,1])\subseteq T_j$, and $\chi_j(\{-\delta_j\}\times[0,1])\subseteq T_j^{\text{lower}}$ and $\chi_j(\{\delta_j\}\times[0,1])\subseteq T_j^{\text{upper}}$, where T_j^{lower} and T_j^{upper} are two primary invariant tori on the opposite sides of T_j. The window R'_j is of the form

$$\begin{aligned} R'_j &= \chi'_j([0,1]\times[-\delta'_j,\delta'_j]),\\ {R'}_j^{\text{exit}} &= \chi'_j(\partial[0,1]\times[-\delta'_j,\delta'_j]), \end{aligned}$$

where $\chi'_j:\mathbb{R}^2\to\Lambda$ is a C^0-local parametrization with $\chi'_j([0,1]\times\{0\})\subseteq T_j$, and $\chi'_j([0,1]\times\{-\delta'_j\})\subseteq T_j^{\text{lower}}$ and $\chi'_j([0,1]\times\{\delta'_j\})\subseteq {T'}_j^{\text{upper}}$, where ${T'}_j^{\text{lower}}$ and ${T'}_j^{\text{upper}}$ are two primary invariant tori on the opposite sides of T_j.

Let us assume that the annular region between ${T'}_j^{\text{lower}}$ and ${T'}_j^{\text{upper}}$ is inside the region between T_j^{lower} and T_j^{upper}. We construct a new window R''_j about T_j, such that R_j is correctly aligned with R''_j under the identity map, and R''_j is correctly aligned with R'_j under some power of f. We let R''_j is of the form

$$\begin{aligned} R''_j &= \chi''_j([-\delta''_j,\delta''_j]\times[0,1]),\\ {R''}_j^{\text{exit}} &= \chi''_j(\partial[-\delta''_j,\delta''_j]\times[0,1]), \end{aligned}$$

where $\chi''_j:\mathbb{R}^2\to\Lambda$ is a C^0-local parametrization with $\chi''_j(\{0\}\times[0,1])\supseteq\chi'_j(\{0\}\times[0,1])$, and $\chi''_j(\{-\delta''_j\}\times[0,1])\subseteq{T'}_j^{\text{lower}}$ and $\chi''_j(\{\delta''_j\}\times[0,1])\subseteq{T'}_j^{\text{upper}}$, for some $\delta''_j>0$. Since the motion on the torus T_j is topological transitive, there exists a power $f_{|\Lambda}^{K_j}$ of $f_{|\Lambda}$ such that R''_j is correctly aligned with R'_j under $f_{|\Lambda}^{K_j}$. See Figure 8. We have obtained that R_j is correctly aligned with R''_j under the identity map, and R''_j is correctly aligned with R'_j under some power $f_{|\Lambda}^{K_j}$ of $f_{|\Lambda}$.

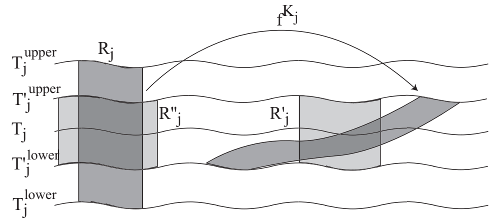

FIGURE 8. Gluing sequences of correctly aligned windows along a transition chain

The case when the annular region between T_j^{lower} and T_j^{upper} is inside the region between T'^{lower}_j and T'^{upper}_j results in the window R_j correctly aligned with R'_j under some power of f, without the need of creating an intermediate window R''_j. The case when neither annular region is contained in the other annular region can be dealt similarly by constructing an intermediate window R''_j.

Through this process, a sequence of correctly windows constructed forward along a transition chain can be glued, in a correctly aligned manner, with a sequence of correctly windows constructed backwards along the same transition chain.

8.5. **Proof of Theorem 2.1.** To summarize, in Subsection 8.1 and in Subsection 8.2, we described the construction of correctly aligned windows that cross an annular regions between two consecutive transition chains of invariant tori. These annular regions are BZI's or else they are separated by some finite collection of invariant tori. If each annular region has some prescribed collection od Aubry-Mather sets, the construction yields windows that follow these Aubry-Mather sets. In Subsection 8.3 yield sequences of correctly aligned windows each transition chain. In Subsection 8.4 the construction of a sequence of correctly windows constructed forward along a transition chain can be glued, in a correctly aligned manner, with a the construction of a sequence of correctly windows constructed backwards along the same transition chain. Thus, starting from some initial annular region, one can construct sequences of correctly aligned windows, forward and backwards, along infinitely many topological transition chains interspersed with annular regions. The Shadowing Lemma-type of result Theorem 7.1 implies the existence of an orbit that gets ε_i-close to each transition torus T_i, and also follows each Aubry-Mather set Σ_i for a prescribed time n_i.

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