A NOTE ON SPACE CURVE GIVEN BY ITS INTRINSIC EQUATION

OLEG ZUBELEVICH

DEPT. OF THEORETICAL MECHANICS,
MECHANICS AND MATHEMATICS FACULTY,
M. V. LOMONOSOV MOSCOW STATE UNIVERSITY
RUSSIA, 119899, MOSCOW, VOROB'EVY GORY, MGU
E-MAIL: OZUBEL@YANDEX.RU

ABSTRACT. We consider a curve in the three dimensional Euclidean space and provide sufficient conditions on the curvature and the torsion for the curve to be unbounded.

1. Introduction

In this short note we concern a smooth curve γ in the standard three dimensional Euclidean space E. Let this curve be defined (up to translations and rotations of E) by its curvature $\kappa(s)$ and its torsion $\tau(s)$, the argument s is the arc-length parameter. The pair $(\kappa(s), \tau(s))$ is called the intrinsic equation of the curve.

In the sequel we assume that $k, \tau \in C[0, +\infty)$.

To obtain the radius-vector of the curve γ one must solve the system of Frenet-Serret equations:

$$\begin{aligned} \mathbf{v}'(s) &= \kappa(s)\mathbf{n}(s), \\ \mathbf{n}'(s) &= -\kappa(s)\mathbf{v}(s) + \tau(s)\mathbf{b}(s), \\ \mathbf{b}'(s) &= -\tau(s)\mathbf{n}(s). \end{aligned} \tag{1.1}$$

The vectors $\mathbf{v}(s), \mathbf{n}(s), \mathbf{b}(s)$ stand for the Frenet-Serret frame at the point with parameter s. Then the radius-vector of the curve is computed as follows $\mathbf{r}(s) = \int_0^s \mathbf{v}(\xi)d\xi + \mathbf{r}(0)$.

So we obtain very natural and pretty problem: having the curvature $\kappa(s)$ and the torsion $\tau(s)$ to restore the properties of the curve γ .

For example, which conditions should be imposed on the functions κ, τ so that the curve γ is closed or helix? There may by another question: on which conditions does the curve lie on a sphere? Such a type questions have been discussed in [3], [2], [4].

2000 *Mathematics Subject Classification.* 53A04.
Key words and phrases. curves, intrinsic equation.
Partially supported by grants RFBR 08-01-00681, Science Sch.-8784.2010.1.

The curve γ is a planar curve if and only if $\tau(s)=0$. In this situation system (1.1) is integrated explicitly. This case is not very interesting.

In the general case, (1.1) is a linear system of ninth order with matrix depending on s. To describe the properties of γ one must study this system.

In this note we formulate and prove some sufficient conditions for unboundedness of the curve γ.

2. Main Theorem

We shall say that γ is unbounded if $\sup_{s\ge 0}|\mathbf{r}(s)|=\infty$.

Theorem 1. *Suppose there exists a function $\lambda(s)$ such that functions*

$$k(s)=\lambda(s)\kappa(s),\quad t(s)=\lambda(s)\tau(s)$$

are monotone, but not necessarily strictly monotone.

Introduce a function $T(s)=\int_0^s t(\xi)d\xi$.

Suppose also that the following equalities hold

$$\lim_{s\to\infty}T(s)=\infty,\quad \lim_{s\to\infty}\frac{k(s)}{T(s)}=\lim_{s\to\infty}\frac{t(s)}{T(s)}=0. \tag{2.1}$$

Then the curve γ is unbounded.

Putting in this Theorem $\lambda=1/\tau$, we deduce the following corollary.

Corollary 1. *Suppose that a function $\kappa(s)/\tau(s)$ is monotone and*

$$\lim_{s\to\infty}\frac{\kappa(s)}{s\cdot\tau(s)}=0.$$

Then the curve γ is unbounded.

For example, a curve with $\kappa(s)=s$ and $\tau(s)=\sqrt{s}$ is unbounded.

Consider a system which consists of (1.1) and equation $\mathbf{r}'(s)=\mathbf{v}(s)$. From the stability theory viewpoint Theorem 1 states that under certain conditions this system is unstable.

Since $|\mathbf{r}(s)|=O(s)$ as $s\to\infty$, this instability is too mild to study it by standard methods such as the Lyapunov exponents method.

3. Proof of Theorem 1

Let us write the formula

$$\mathbf{r}(s)=r_1(s)\mathbf{v}(s)+r_2(s)\mathbf{n}(s)+r_3(s)\mathbf{b}(s).$$

Differentiating this formula we obtain

$$\mathbf{v}(s)=r_1'(s)\mathbf{v}(s)+r_2'(s)\mathbf{n}(s)+r_3'(s)\mathbf{b}(s)$$
$$+r_1(s)\mathbf{v}'(s)+r_2(s)\mathbf{n}'(s)+r_3(s)\mathbf{b}'(s).$$

From this formula and by virtue of system (1.1) it follows that

$$r'(s)=\begin{pmatrix}0&\kappa(s)&0\\-\kappa(s)&0&\tau(s)\\0&-\tau(s)&0\end{pmatrix}r(s)+\begin{pmatrix}1\\0\\0\end{pmatrix},\qquad r=\begin{pmatrix}r_1\\r_2\\r_3\end{pmatrix}. \tag{3.1}$$

About system (3.1) the author has known from Professor Ya. V. Tatarinov.

Let us multiply both sides of system (3.1) on a vector-row $\lambda(s)(\tau(s), 0, \kappa(s))$ from the left :

$$t(s)r_1'(s) + k(s)r_3'(s) = t(s).$$

Then we integrate this equality:

$$\int_0^s t(a)r_1'(a)da + \int_0^s k(a)r_3'(a)da = T(s). \tag{3.2}$$

From the Second Mean Value Theorem [1], we know that there is a parameter $\xi \in [0, s]$ such that

$$\begin{aligned}\int_0^s t(a)r_1'(a)da &= t(0)\int_0^\xi r_1'(a)da + t(s)\int_\xi^s r_1'(a)da \\ &= t(0)\big(r_1(\xi) - r_1(0)\big) + t(s)\big(r_1(s) - r_1(\xi)\big)\end{aligned}$$

By the same argument for some $\eta \in [0, s]$ we have

$$\int_0^s k(a)r_3'(a)da = k(0)\big(r_3(\eta) - r_3(0)\big) + k(s)\big(r_3(s) - r_3(\eta)\big).$$

Thus formula (3.2) takes the form

$$\begin{aligned}t(0)\big(r_1(\xi) - r_1(0)\big) &+ t(s)\big(r_1(s) - r_1(\xi)\big) \\ &+ k(0)\big(r_3(\eta) - r_3(0)\big) + k(s)\big(r_3(s) - r_3(\eta)\big) = T(s).\end{aligned} \tag{3.3}$$

Since the Frenet-Serret frame is orthonormal we have

$$|\mathbf{r}(s)|^2 = r_1^2(s) + r_2^2(s) + r_3^2(s).$$

Suppose that the curve γ is bounded: $\sup_{s\geq 0} |\mathbf{r}(s)| < \infty$. Then due to conditions (2.1) the left side of formula (3.3) is $o(T(s))$ as $s \to \infty$. This is the contradiction.

The Theorem is proved.

References

[1] R. Courant, Differential and Integral Calculus. vol. 1, John Wiley and Sons, 1988.

[2] W. Frenchel, On the differential geometry of closed space curves, Bull. Amer. Math. Soc., 57(1951), 44-54.

[3] P.W. Gifford, Some refinements in theory of specialized space curves, Amer. Math. Monthly, 60(1953), 384-393.

[4] Yung-Chow Wong and Hon-Fei Lai, A Critical Examination of the Theory of Curves in Three Dimensional Differential Geometry. Tohoku Math. Journ. Vol. 19, No. 1, 1967.

E-mail address: `ozubel@yandex.ru`