Collision of invariant bundles of quasi-periodic attractors in the dissipative standard map

Renato Calleja
Department of Mathematics and Statistics
McGill University
805 Sherbrooke St West
Montreal Qc, H3A 2K6 (Canada)
(calleja@math.mcgill.ca)

Jordi-Lluís Figueras
Department of Mathematics
Uppsala University
Box 480, 75106 Uppsala (Sweden)
(figueras@math.uu.se)

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Abstract

We perform a numerical study of the breakdown of hyperbolicity of quasi-periodic attractors in the dissipative standard map. In this study, we compute the quasi-periodic attractors together with their stable and tangent bundles. We observe that the loss of normal hyperbolicity comes from the collision of the stable and tangent bundles of the quasi-periodic attractor. We provide numerical evidence that, close to the breakdown, the angle between the invariant bundles has a linear behavior with respect to the perturbing parameter. This linear behavior agrees with the universal asymptotics of the general framework of breakdown of hyperbolic quasi-periodic tori in skew product systems.

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1 Introduction

The existence of quasi-periodic orbits for the dissipative standard map can be established by several results in the literature (see for example [BHS96, CCdlL11a]). Whenever the existence proofs are constructive they give rise to very efficient numerical algorithms that allow us to continue the quasi-periodic orbits and to explore their breakdown of analyticity [CC10]. It is known that in the dissipative setting quasi-periodic attractors lie in normally attracting manifolds [CCdlL11b]. The goal of this paper is to study numerically the breakdown of hyperbolicity and the scaling relations of hyperbolicity that are present close to the breakdown.

The proof of existence of quasi-periodic attractors in [CCdlL11a] is based on an approximate reducibility method that allows to write a KAM theorem in an *a-posteriori* format. The *a-posteriori* format of the KAM theorem means that if, for a fixed Diophantine number ω, there is parametrization (K, μ) of the quasi-periodic orbit that satisfies very approximately an invariance equation and certain smoothness and non-degeneracy conditions, then there exists a nearby true solution corresponding to a quasi-periodic attractor. Moreover, the parameterization of the invariant torus is analytic whenever the map is analytic. The analyticity of the invariant torus also holds when the approximate solution (K, μ) is not analytic but only belongs to a Sobolev space H^s with a Sobolev exponent s large enough.

The proof of this *a-posteriori* theorem is constructive and allows to produce very effective numerical algorithms to continue the parameterization of the quasi-periodic orbits in the perturvative parameter ε of the dissipative standard map. The technique at hand also justifies a numerically implementable criterion for the breakdown of analyticity of the quasi-periodic orbit as the Sobolev norms of the parameterization K blow up to infinity as the continuing parameter ε approaches the value where the quasi-periodic solution breaks down [CCdlL11a, CdlL10]. The breakdown of quasi-periodic solutions via this mechanism has been explored numerically by one of the authors of the present work and A. Celletti in [CC10].

In [CCdlL11a], the authors note that the approximate reducibility method in the dissipative setting also allows to find changes of variables that make the linear cocycle around the quasi-periodic orbit reducible, see [Fig11]. The geometric meaning of this reducibility is that we can construct a linear coordinate system around the torus in which there is a contracting space in the perpendicular direction. Since the whole construction is done explicitly the procedure can be implemented producing algorithms that also allow us to compute the tangent and normal bundles to the parameterization (K, μ).

On the other hand, the construction of the parameterization and parameter (K, μ), produces a smooth invariant torus. Since the change of variables that makes the linear cocycle around the torus is reducible, we obtain that the tangent and normal bundles are as smooth as the invariant torus. Moreover, Fenichel theory [Fen74, Fen77] states that, for more general maps, the smoothness of the invariant bundles depends on the ratio of the Lyapunov exponents. However, for the case of the dissipative standard map, the Lyapunov multipliers associated with the invariant bundles remain constant. Therefore, the breakdown of hyperbolicity can only occur when the invariant directions of the transversal and tangential dynamics come close to each other. We observe that the collision of these bundles is nonsmooth, i.e., the minimum distance between them approaches to zero while the maximum distance stays away from zero.

In the present work we use the reducibility construction presented in Section 2.2, to compute numerically the tangent and normal bundles to a quasi-periodic orbit of the dissipative standard map. This technique allows us to continue the invariant torus very close to the breakdown. In this way, we are able to approximate the smoothness of the torus and of the map conjugating the dynamics on the torus to a rigid rotation. We also compute the angle between the two bundles and observe numerically that as the perturbing parameter ε approaches the breakdown of analyticity, the angle between the bundles goes linearly to zero. We also observe that the minimum distance of the invariant bundles near the breakdown is reached always at the same point of the invariant torus.

The linear behavior of the minimum distance was first observed in the pioneering works [HdlL06a, HdlL07] in the context of quasi-periodic skew products. In [Fig11, FH11] the authors study the relation between these asymptotics and the possible breakdown scenarios in the context of quasi-periodic skew products. The fact that the minimum is reached always at the same point agrees with the theoretical results of [BS08], where the authors explore this type of collision between invariant bundles in the context of Schrödinger operators in the perturbative regime.

2 Setup of the problem

We consider the dissipative standard map defined on the cylinder $\mathcal{M} = \mathbb{T} \times \mathbb{R}$ given by the equations $f_{\mu,\varepsilon}(x_n, y_n) = (x_{n+1}, y_{n+1})$ and

$$\begin{aligned} y_{n+1} &= \lambda y_n + \mu + \varepsilon\, V'(x_n) \\ x_{n+1} &= x_n + y_{n+1}\ , \end{aligned} \tag{2.1}$$

where $0 < \lambda < 1$ is fixed, and μ, ε are parameters.

As it is noted in [CCdlL11a], the map (2.1) is a conformally symplectic system. Let $\Omega = dy \wedge dx$ be a symplectic form on the cylinder, then the map $f_{\mu,\varepsilon}$ satisfies that

$$f_{\mu,\varepsilon}^{*}\Omega = \lambda\Omega. \tag{2.2}$$

For certain values of the parameter μ it is known that conformally symplectic maps of the form (2.2) have solutions that are quasi-periodic with a Diophantine rotation number ω (see for instance [BHS96, CL09]).

2.1 Quasi-periodic orbits

In this paper, we will approximate quasi-periodic orbits by computing their parameterization K defined as folllows.

Definition 2.1 *For a Diophantine number ω satisfying*

$$|\omega q - p| \ \geq \ \nu |q|^{-\tau} \ , \qquad p \in \mathbb{Z} \ , \quad q \in \mathbb{Z}\backslash\{0\} \ , \tag{2.3}$$

the parameterization pair (K,μ) will be a function $K : \mathbb{T} \to \mathbb{T} \times \mathbb{R}$ and $\mu \in \mathbb{R}$ solving the invariance equation

$$f_{\mu,\varepsilon} \circ K(\theta) = K(\theta + \omega). \tag{2.4}$$

In [CCdlL11a], the authors prove an *a-posteriori* theorem by means of a Newton iteration in the spirit of Nash-Noser theory [Zeh75, Zeh76, CdlL10]. The Newton iteration starts form an approximate solution (K,μ) satisfying some non-degeneracy conditions and satisfying the invariance equation (2.4) very approximately. Namely, the function

$$e(\theta) = f_{\mu,\varepsilon} \circ K(\theta) - K(\theta + \omega) \tag{2.5}$$

is small. Starting from the approximate solution (K,μ), we implement a Newton step $K + \Delta$ and $\mu + \sigma$ satisfying the Newton step equation

$$Df_{\mu}(K(\theta))\Delta(\theta) - \Delta(\theta + \omega) + \sigma \cdot \partial_{\mu} f_{\mu}(\theta) = -e(\theta). \tag{2.6}$$

If we are able to solve for Δ and σ from equation (2.6) then the new norm of the error,

$$\tilde{e}(\theta) = f_{\mu+\sigma,\varepsilon} \circ (K(\theta) + \Delta(\theta)) - (K(\theta + \omega) + \Delta(\theta + \omega)),$$

of the new solution will be comparable to the square of the norm of the original error, i.e.,

$$\|\tilde{e}\| \approx \|e\|^2. \tag{2.7}$$

The problem that one runs into is that solving for Δ and σ from (2.6), involves solving difference equations with non-constant coeficients. As it is standard in Nash-Moser theory, we will not try to solve equation (2.6) directly, but we will introduce approximate reduction of $Df_\mu(K(\theta))$ to an upper triangular matrix (see equation (2.11)) that will allow us to solve for Δ and σ from the Newton step equation (2.6) approximately while keeping the quadratic character of the iteration (2.7).

Remark 2.2 *Let us identify some functions related to the invariant torus. We summarize the standard notation (see for instance [OP08]) for completeness. We will consider the function $R : \mathbb{T} \to \mathbb{R}$ such that the invariant circle $\mathcal{K} = K(\mathbb{T})$ is the graph of R, i.e.,*

$$\mathcal{K} = \{(\theta, y) \in \mathbb{T} \times \mathbb{R} : y = R(\theta)\}.$$

We will also consider the advance map $g : \mathbb{T} \to \mathbb{T}$ *defined by*

$$f_{\mu,\varepsilon}(\theta, R(\theta)) = (g(\theta), R \circ g(\theta)).$$

Then, the function K defined by the invariance equation (2.4) *is the* hull map *parametrizing the invariant torus. In particular if we consider the two components of $K : \mathbb{T} \to \mathbb{T} \times \mathbb{R}$,*

$$K(\theta) = (K_1(\theta), K_2(\theta)),$$

then K_1 conjugates the advance map g to a rigid rotation by ω

$$g \circ K_1(\theta) = K_1(\theta + \omega).$$

Later in the paper, when we explore the regularity of K_1, it will be more convenient to consider the periodic function

$$u(\theta) = K_1(\theta) - \theta. \tag{2.8}$$

For convenience we will denote this function u as the conjugacy*. We also remark that both functions K_1 and u are a byproduct of the computation of the parameterization K.*

2.2 Reducibility of the cocycle

The conformally symplectic character of (2.1) then provides the existence of a reduction of the dynamics around (K, μ) solving the invariance equation

$$f_{\mu,\varepsilon} \circ K(\theta) = K(\theta + \omega). \tag{2.9}$$

That is, we have that the matrix

$$M(\theta) = [DK(\theta)|J^{-1}DK(\theta)N(\theta)] \tag{2.10}$$

where

$$J = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \qquad \text{and} \qquad N(\theta) = (DK(\theta)^T DK(\theta))^{-1},$$

reduces the derivative of (2.1) around K to an upper triangular matrix

$$Df_{\mu,\varepsilon}(K(\theta))M(\theta) = M(\theta+\omega)\begin{pmatrix} 1 & S(\theta) \\ 0 & \lambda \end{pmatrix}. \tag{2.11}$$

Furthermore, if (K, μ) only satisfies the invariance equation approximately in the sense of (2.5), then the reduction (2.11) is valid approximately, i.e.

$$Df_{\mu,\varepsilon}(K(\theta))M(\theta) = M(\theta+\omega)\begin{pmatrix} 1 & S(\theta) \\ 0 & \lambda \end{pmatrix} + O(e). \tag{2.12}$$

Remark 2.3 *See [CCdlL11a] for details about this reduction and [dlLGJV05] for a similar reduction in the symplectic setting.*

Equation (2.12) allows us to introduce a quasi-Newton method to find the Newton step (Δ, σ) by means of solving for $W(\theta) = M(\theta)\Delta(\theta)$ and σ from

$$\begin{pmatrix} 1 & S(\theta) \\ 0 & \lambda \end{pmatrix} W(\theta) - W(\theta+\omega) = -M^{-1}(\theta+\omega)[E(\theta) + \partial_\mu f_\mu(\theta)\sigma]. \tag{2.13}$$

Using this new form of the Newton step equation allows us to obtain estimates for the existence theorem and to implement an efficient method for the computation and continuation of quasi-periodic attractors for the dissipative standard map.

Remark 2.4 *To solve for $W : \mathbb{T} \to \mathbb{T} \times \mathbb{R}$ form (2.13) we separate the function in two components $W = (W_1, W_2)$. If we denote by $\eta(\theta, \sigma) = -M^{-1}(\theta+\omega)[E(\theta) + \partial_\mu f_\mu(\theta)\sigma]$ and consider the components of $\eta = (\eta_1, \eta_2)$, then we obtain the equivalent set of equations for the components $W_{1,2}$ and $\eta_{1,2}$,*

$$\begin{aligned} W_1(\theta) - W_1(\theta+\omega) &= \eta_1(\theta,\sigma) - S(\theta)W_2(\theta), \\ \lambda W_2(\theta) - W_2(\theta+\omega) &= \eta_2(\theta,\sigma). \end{aligned} \tag{2.14}$$

The difference equations for $W_{1,2}$ in (2.14) are usually called cohomology equations in the literature. The second equation has a solution W_2 for any ω and $|\lambda| \neq 1$. On the other hand, the first eqution involves small divisors (see [dlL01]) and has a solution W_1 when ω satisfies (2.3) and

$$\int_{\mathbb{T}} \eta_1(\theta,\sigma) - S(\theta)W_2(\theta)d\theta = 0.$$

To compute the solution W numerically it is useful to notice that the the equations for $W_{1,2}$ in (2.14) are diagonal in Fourier space. Let us write the functions in terms of Fourier coefficients, i.e., $W_{1,2}(\theta) = \sum_{k\in\mathbb{Z}}(W_{1,2})_k e^{2\pi ik\theta}$, $\eta_{1,2}(\theta,\sigma) = \sum_{k\in\mathbb{Z}}(\eta_{1,2}(\sigma))_k\, e^{2\pi ik\theta}$, and $S(\theta) = \sum_{k\in\mathbb{Z}} S_k e^{2\pi ik\theta}$. In Fourier space, the equation for W_2 in (2.14) can be solved for any σ,

$$(W_2)_k = \frac{(\eta_2(\sigma))_k}{\lambda - e^{2\pi ik\omega}}.$$

Note that the solution W_2 depends on σ, i.e., $W_2 = W_2(\sigma)$. Then, the solution $W_2(\sigma)$ can be used to solve for W_1,

$$(W_1)_k = \frac{(\eta_1(\sigma))_k - (S \cdot W_2(\sigma))_k}{1 - e^{2\pi ik\omega}},$$

by choosing a σ^* so that $(\eta_1(\sigma^*))_0 - (S \cdot W_2(\sigma^*))_0 = 0$ and obtain W_1.

Therefore, if we use the FFT to solve for the Newton step $(M \cdot W, \sigma *)$, then the methods are $O(N \log N)$ operations and $O(N)$ storage, where N is the number of Fourier modes used to discretize the functions.

To compute the invariant bundles of the torus we notice that a further change of variables can be performed. Given an invariant torus (K, μ), its stable and tangent bundles, $E^s(\theta)$ and $E^c(\theta)$, satisfy the reducibility equation

$$Df_{\mu,\varepsilon}(K(\theta))P(\theta) = P(\theta + \omega) \begin{pmatrix} 1 & 0 \\ 0 & \lambda \end{pmatrix}, \tag{2.15}$$

where

$$P(\theta) = [E^c(\theta)|E^s(\theta)]. \tag{2.16}$$

Since the matrix-valued map $Df_{\mu,\varepsilon}(K(\theta))$ can be reduced, via the linear change of variables $M(\theta)$ in (2.10), to the upper diagonal matrix-valued map

$$\hat{S}(\theta) = \begin{pmatrix} 1 & S(\theta) \\ 0 & \lambda \end{pmatrix},$$

it is enough to reduce $\hat{S}(\theta)$ to the diagonal matrix. We reduce $\hat{S}(\theta)$ by finding a matrix $\hat{P}(\theta)$ such that

$$\begin{pmatrix} 1 & S(\theta) \\ 0 & \lambda \end{pmatrix} \hat{P}(\theta) = \hat{P}(\theta + \omega) \begin{pmatrix} 1 & 0 \\ 0 & \lambda \end{pmatrix}. \tag{2.17}$$

Remark 2.5 *We are able to find such a $\hat{P}(\theta)$ since the system is dissipative. In this case the dissipative character is given by $0 < \lambda < 1$. In the case of expanding maps, i.e. $\lambda > 1$, such a reduction is also possible, but the reduction breaks down in the conservative case ($\lambda = 1$).*

In order to solve for $\hat{P}(\theta)$ we will separate the matrix in two components

$$\hat{P}(\theta) = [V_1(\theta)|V_2(\theta)]. \tag{2.18}$$

Solving the equation (2.17) for $V_1(\theta)$, and $V_2(\theta)$ we obtain the following

$$V_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad \text{and} \quad V_2 = \begin{pmatrix} v_2(\theta) \\ 1 \end{pmatrix},$$

where the function $v_2(\theta)$ will have to satisfy the cohomology equation

$$v_2(\theta) - \lambda v_2(\theta + \omega) = S(\theta). \tag{2.19}$$

Remark 2.6 *Equation* (2.19) *has a solution for every* ω *and* $|\lambda| \neq 1$*. In terms of the Fourier coefficients, if* $v_2(\theta) = \sum_{k\in\mathbb{Z}} v_{2,k} e^{2\pi i k\theta}$ *and* $S(\theta) = \sum_{k\in\mathbb{Z}} S_k e^{2\pi i k\theta}$*, then*

$$v_{2,k} = \frac{S_k}{1 - \lambda e^{2\pi i k\omega}}.$$

If $\lambda < 1$*, the function* $v_2(\theta)$ *can also be computed by evaluating the following sum*

$$v_2(\theta) = \sum_{j=0}^{\infty} \lambda^j S(\theta + j\omega). \tag{2.20}$$

¿From Lusin's theorem and Poincaré recurrence theorem it follows that if $S(\theta)$ *is measurable, then the sum* (2.20) *always converges to a measurable function* $v_2(\theta)$*. Furthermore, since* (2.19) *does not involve solving a small divisors problem* ($|\lambda| \neq 1$)*, then the solution* $v_2(\theta)$ *is as smooth as the function* $S(\theta)$*.*

Finally, we recover the matrix-valued map $P(\theta)$ as $M(\theta) \cdot \hat{P}(\theta)$ and, following [Fig11], we can compute the minimum angle between the stable and tangent bundle around $K(\theta)$ using the parameterization of the stable and tangent bundles given by the columns of $P(\theta)$.

3 Breakdown of hyperbolicity

The fact that the matrix $P(\theta)$ in (2.15) exists implies that we can find a frame where the dynamics are given by a diagonal matrix with 1 and λ as its Lyapunov multipliers. In fact, for every $k > 0$

$$Df^k_{\mu,\varepsilon} \circ K(\theta) = P(\theta + k\omega) \begin{pmatrix} 1 & 0 \\ 0 & \lambda^k \end{pmatrix} P^{-1}(\theta). \tag{3.21}$$

where $P(\theta)$ is the same matrix in (2.15). In particular, this tells us that for every $\theta \in \mathbb{T}$, there is a continuous decomposition of the tangent space of $\mathcal{M}$ at $K(\theta)$

$$T_{K(\theta)}\mathcal{M} = E^c_{K(\theta)} \oplus E^s_{K(\theta)} \tag{3.22}$$

with $E^c_{K(\theta)} = \text{Range}(DK(\theta))$ and $E^s_{K(\theta)}$ the corresponding eigenspace to the eigenvalues λ. Therefore, the splitting (3.22) is continuous and invariant under $Df_{\mu,\varepsilon}$ and there is a constant C such that

$$C^{-1}\lambda^k|v| \leq |Df^k_{\mu,\varepsilon} \circ K(\theta)v| \leq C\lambda^k|v| \quad \text{if and only if} \quad v \in E^s_{K(\theta)} \tag{3.23}$$

and

$$C^{-1}|v| \leq |Df^k_{\mu,\varepsilon} \circ K(\theta)v| \leq C|v| \quad \text{if and only if} \quad v \in E^c_{K(\theta)}. \tag{3.24}$$

Therefore $\mathcal{K} = K(\mathbb{T})$ is a Normally Hyperbolic Invariant Manifold. Moreover, we can verify that $\mathcal{K}$ is a C^r (one dimensional) manifold for any $r \in \mathbb{N}$. Indeed, this is due to the fact that the tangent space (3.22) has two different growth rates $\rho_c = 1$ and $\rho_s = \lambda$ (using the terminology in [Fen74, Fen77]). Thus, the regularity theory in [Fen77] implies that $\mathcal{K}$ should be as regular as the map $f_{\mu,\varepsilon}$ which is C^r, $\forall r \in \mathbb{N}$.

Now since our curve $\mathcal{K}$ is C^r, one dimensional and since ω satisfies the Diophantine condition (2.3), we can use the powerful results of [Her79, SK89, KO89] to verify that the map, say u, conjugating the dynamics in K to a rigid rotation, is in $C^{r-\tau-\delta}$ for a small $\delta > 0$. Moreover, the dissipative standard map $f_{\mu,\varepsilon}$ is not only C^r but also analytic. Therefore, by the bootstrap of analyticity result (Theorem 44 in [CCdlL11a]), we conclude that the conjugacy u (see equation (2.8)) is an analytic function and remains analytic up to the breakdown. Since the computation of the matrix-valued map P is done by analytic changes of variables, the tangent and stable bundles are also anaylitic.

The Lyapunov multipliers (1 and λ) are constant along the family of invariant tori. Hence, the only mechanism in which the hyperbolicity can break down is if the angle between the bundles goes to zero and the bundles collide. Therefore, we explore the breakdown of hyperbolicity by monitoring the minimum angle in θ between the stable and tangent bundles and also the value of θ, $\theta_{\min}$, and the value $x_{\min} = K_1(\theta_{\min})$, where that minimum angle is reached. We remark that measuring this angle numerically is not difficult once we have computed parameterized representations of the bundles. The algorithm to compute such parameterizations is discussed in the following section.

4 An algorithm to compute (K, μ) with stable and tangent bundles

An algorithm to find an approximation to (K, μ) together with its stable and tangent bundles comes from solving the simplified Newton step equation (2.13) and then using the computation of the function $S(\theta)$ to find a matrix $P(\theta)$ that satisfies the reduction (2.17). As we mentioned above the solution $P(\theta)$ is constructed by first solving the equation (2.19) and then performing the product $M(\theta) \cdot \hat{P}(\theta)$. Equation (2.19) is solved numerically by means of an FFT.

An algorithm to compute (K, μ), with a desired accuracy *tol*, together with its stable and tangent bundles is described below. We start from an approximation to K and μ stored on a set of $N+1$ points and compute Δ and σ from the approximate reduction obtained from the N points representing K.

Algorithm 4.1 *1) Compute the error* $E = f_{\mu,\varepsilon} \circ K - K \circ T_\omega$, *where* $K \circ T_\omega$ *is the translation by* ω *of* K.

If $\|E\| < tol$, *stop the algorithm and the pair* (K, μ) *is the approximation of the invariant torus with the desired accuracy.*

2) Compute the derivative $\alpha = DK$.

3) From α *construct the function* $N = [\alpha^T \alpha]^{-1}$.

4) Construct the matrix $M = [\alpha, J^{-1} \circ K\alpha N]$.

5) From M *construct its translated inverse. Call it* β, *then multiply it times the error* E. *I.e.,* $\beta = M^{-1} \circ T_\omega$ *and* $\tilde{E} = \beta E$.

7) From α, N *and* K, *construct the functions* B, S, *and* A. *Namely,* $B = \alpha N$ $S = (B \circ T_\omega)^T Df_{\mu,\varepsilon} \circ K J^{-1} B$, *and* $A = \beta D_\mu \circ K$.

8) From S, *construct the function* v_1 *solving from* $v_1 - \lambda v_1 \circ T_\omega = S$.

9) Construct $\hat{W}_2^0$ *and* $\hat{W}_2$ *by solving* $\lambda W_2^0 - W_2^0 \circ T_\omega = -\widetilde{E}_2$ *and* $\lambda \widehat{W}_2 - \widehat{W}_2 \circ T_\omega = -\tilde{A}_2$.

10) Find σ *so that* $\langle \widetilde{E}_1 + \gamma^0 \rangle = \langle \hat{\gamma} + \tilde{A}_1 \rangle \sigma$.

11) Compute W_1 *by solving the small divisors problem*
$W_1 - W_1 \circ T_\omega = -\widetilde{E}_1 - \gamma^0 + [S\widehat{W}_2 + \tilde{A}_1]\sigma$.

12) $K \leftarrow K + MW$

$\mu \leftarrow \mu + \sigma$.

13) If $\|MW\| < tol$, *stop the algorithm. The pair* (K, μ) *is the approximation of the invariant torus with the desired accuracy. If* $\|MW\| \geq tol$, *repeat the process starting from step 1).*

Once (K, μ) are computed via the previous iterations, the parameterization of the invariant bundles is computed following the next steps:

14) v_2 *solves* $v_2 - \lambda v_2 \circ T_\omega = S$.

15) With v_2 *construct* $\hat{P}$ *as in* (2.18).

16) Compute the parameterization $P(\theta)$ *of the stable and tangent bundles as* $M(\theta) \cdot \hat{P}(\theta)$.

We notice that all the steps in the algorithm are multiplications, computation of derivatives, and solving difference equations with constant coefficients. All of these operations can be performed by means of FFT. In such a case, the total cost of computing a new approximate solution $(K + \Delta, \mu + \sigma)$ together with its stable and tangent bundle is $O(N \log N)$ operations. In the following section we show some implementations of Algorithm 4.1 that we performed to approximate and continue the invariant circle $K(\mathbb{T})$, the parameter μ and the tangent and stable bundle for the dissipative standard map (2.1) close to the breakdown.

4.1 Numerical results on the breakdown of hyperbolicity

We implemented Algorithm 4.1 in the case of the dissipative standard map (2.1) with $V(x) = -\frac{1}{(2\pi)^2} \cos(2\pi x)$ and for several values of the dissipative constant λ. With the help of the previous algorithm, we studied the breakdown of analycity of the invariant tori and also the behavior of their invariant stable and tangent bundles near the breakdown. The continuation is done as follows: First, we fix the value of the parameter λ with modulus less than 1 and the Diophantine rotation number ω (the golden mean $\frac{\sqrt{5}-1}{2}$). For $\varepsilon = 0$ we have that the invariant torus of (2.1) is $K(\theta) = (\theta, \omega)$ and $\mu = (1 - \lambda)\omega$. Then, we perform a continuation, with respect to the parameter ε, to study the breakdown of the invariant torus. At every step of the continuation, we compute (K, μ) using the Algorithm 4.1 together with its invariant stable and tangent bundles. At each step we also compute the minimum distance between these invariant bundles and the value of the angle θ and the (x, y) coordinates of the torus where the minimum distance is reached.

4.1.1 Collision between invariant bundles close to the breakdown

Figure 1 shows the evolution of the invariant bundles for $\lambda = 0.4$ with respect diferent values of ε. Note that in this case the breakdown value of ε is near 0.9808192038, where we can observe that the invariant bundles collide. Figure 1 also shows a very remarkable fact, the collision between this invariant bundles is non-uniform. As the parameter ε approaches the breakdown, the minimum distance between the bundles goes to zero, while the maximum distance stays away from zero. Another important observation in this

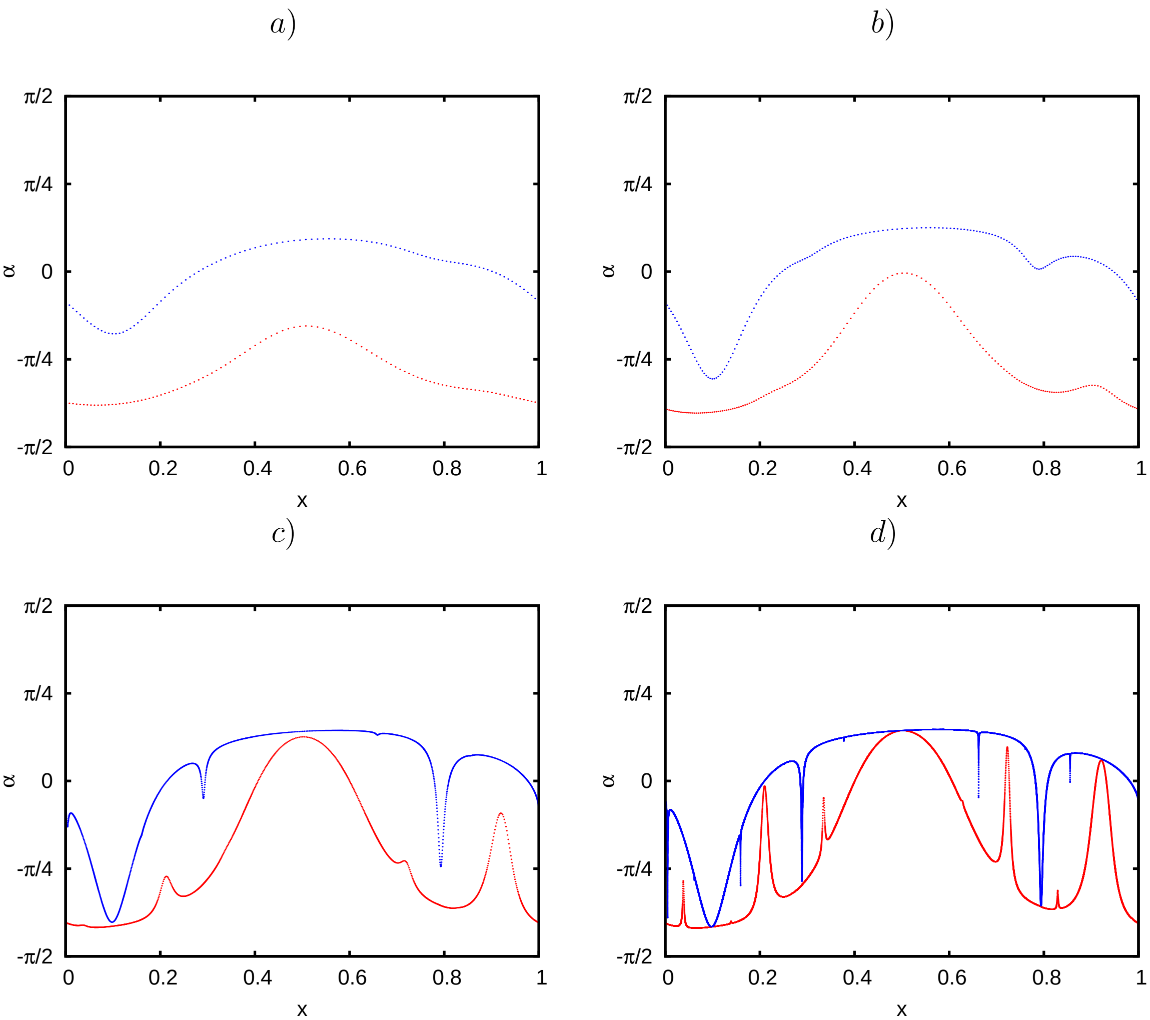

Figure 1: Stable (dashed line) and tangent (solid line) bundle of the invariant tori for $\lambda = 0.4$ and for different values of ε. In the horizontal axis we plot the x coordinate of the invariant torus and in the vertical axis we plot the angle α between the bundle and the $\{x > 0\}$ semiaxis. a) $\varepsilon = 0.5$, b) $\varepsilon = 0.75$, c) $\varepsilon = 0.9502$, d) $\varepsilon = 0.98081920384$.

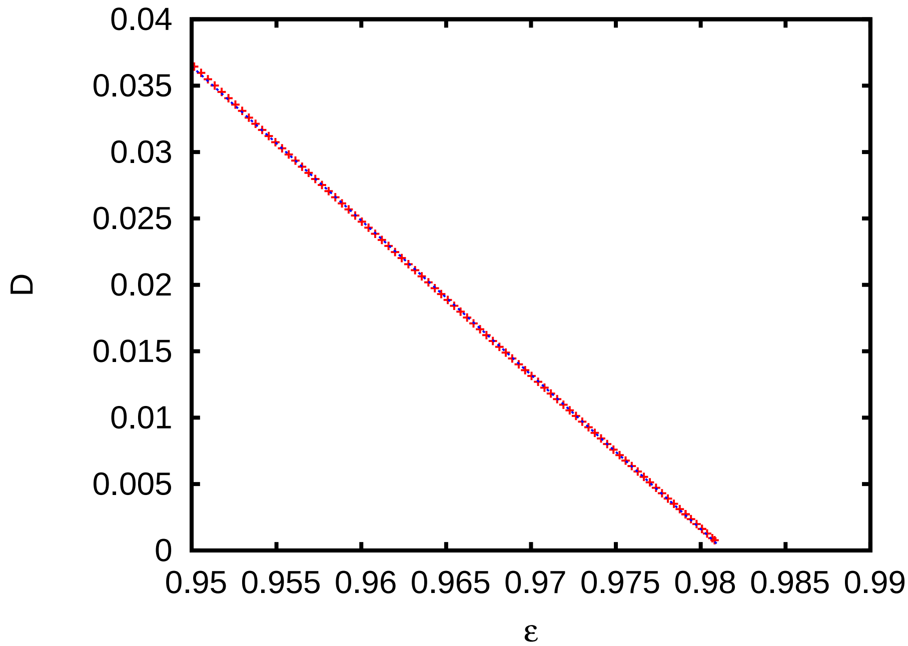

Figure 2: Minimum distance between the bundles near the breakdown for $\lambda = 0.4$. In the horizontal axis we plot the parameter ε and in the vertical axis the minimum distance between the invariant bundles.

figure is that, for values of ε close to the breakdown, the x coordinate of the invariant torus where the minimum distance is reached is continuous with respect ε and has a limit when the breakdown occurs.

Remark 4.2 *We note that at the breakdown value, whenever there is a point $x(\theta)$ where the invariant bundles collide, then the bundles will collide on every point of the orbit $x(\theta + k\omega)$ for $k \in \mathbb{Z}$. A similar phenomenon was observed in the context of the creation of Strange Nonchaotic Attractors in [BS08, Bje09, Jäg09].*

4.1.2 Asymptotic behavior close to the breakdown

We also explored the asymptotic behavior of the minimum distance of the invariant bundles close to the breakdown. As we observe in figure 2, the minimum distance behaves asymptotically like a straight line.

Remark 4.3 *In the context of skew products, linear asymptotics of the minimum distance between the angles close to the breakdown of invariant bundles have been observed [HdlL06b, HdlL06a, HdlL07]. In [FH10], one of the authors of the present work and A.*

Haro observe the similar linear asymptotics and prove the existence of the fiberwise hyperbolic torus using computer assisted methods. A rigorous proof for the linear asymptotics appears in [BS08] for a particular case of skew products.

We verified numerically that

i) The asymptotics of the minimum distance between the invariant bundles is linear.

ii) The minimum distance is reached at a single point ($x_{\min}$) for several values of λ.

In order to verify *i*) and *ii*), we have considered a grid of 1170 equidistant points with λ in the interval $[0.05, 0.95]$. For each of the values in the grid, we have approximated both the minimum distance and the corresponding angle.

Let D be the minimum distance between the tangent and the stable bundle. For every λ, we have computed qualitative measures of the linear behavior D as follows. If for small D and ε inside an interval, we find that the function $D(\varepsilon)$ is very close to linear, then we approximate the linear model

$$D(\varepsilon) = a(\varepsilon - b) \tag{4.25}$$

and compute the coefficients a and b by linear fitting. We note that the fitted value of b is a prediction of the value of the parameter ε where the breakdown occurs.

In figure 3, we show the values of b and a as functions of λ with λ inside the inverval $[0.05, 0.95]$. Figure 4, shows the value of the angle $x_{\min}(\lambda)$ as which the minimum distance is reached from every λ in our grid. We remark that $x_{\min}(\lambda)$ appears to be a smooth function of λ for almost every λ, but has a corner close to $\lambda = 0.4958$. In figure 3 we notice that close to the point of discotnuity of $x_{\min}(\lambda)$ there is a change in the behavior of the coefficient $a(\lambda)$.

4.1.3 Regularity of the torus at the breakdown

We have also explored the regularity of the parameterization of the invariant torus at the breakdown. We remark that our methods allow us to approximate the parameterization $K(\theta)$ and the conjugacy $u(\theta)$ with $\approx 10^6$ Fourier modes (see figure 5). Using the CLP method [dlLP08, dlLP02, AdlLP05, OP08], we approximated the regularity of the functions.

Remark 4.4 *The CLP method is based on the Littlewood-Paley theorem. A periodic function f on the torus is C^r, $r \geq 0$, with its rth derivative α-Hölder continuous if and only if for every $\eta \geq 0$ there exists a constant C such that for any $t > 0$*

$$\left\| \left(\frac{\partial}{\partial t}\right)^{\eta} e^{-t\sqrt{-\Delta}} f(\theta) \right\|_{L^\infty(\mathbb{T})} \leq C t^{r+\alpha-\eta}.$$

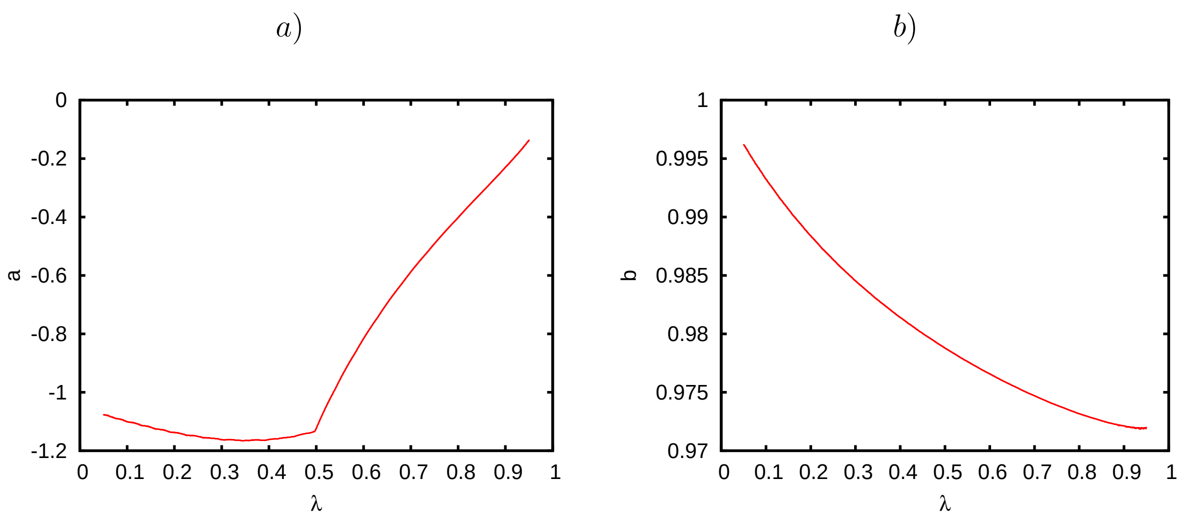

Figure 3: In a) we show the coefficient a obtained from the linear fitting of (4.25) for different values of λ. In b) we show the coefficient b obtained form the linear fitting of (4.25) for different values of λ.

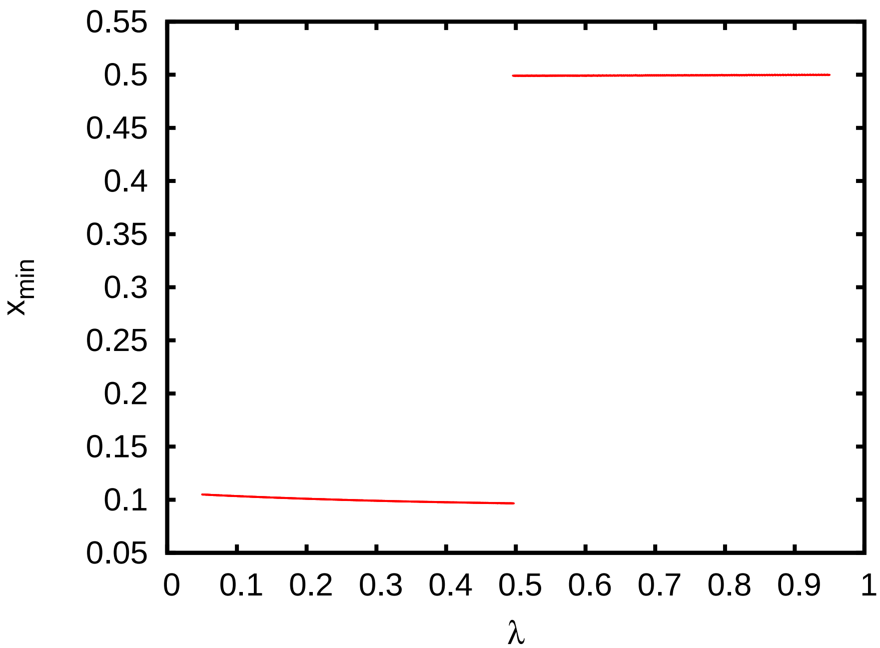

Figure 4: For several values of λ, the x coordinate of the invariant torus where the minimun distance between the invariant bundles is achieved.

Note that the linear operator $\left(\frac{\partial}{\partial t}\right)^{\eta} e^{-t\sqrt{-\Delta}}$ *is diagonal in the Fourier base. Then, the CLP method is based on the estimation of the quantity* $r+\alpha$ *by fitting. See the previous cited works for more details on the technical implementation of this method.*

We have obtained that, for every value of λ in $(0,1)$, the parameterization $u(\theta)$ is $\simeq 0.6$ Hölder continuous, and the graph of the invariant curve $(x, R(x))$ is differentiable with derivative $\simeq 0.1$ Hölder continuous. Figure 6 shows the conjugacy u in (2.8) of the invariant torus with respect to θ and the invariant torus in the phase space near the breakdown when $\lambda = 0.4$. Note that the conjugacy u is less regular that the graph R of the invariant curve.

Remark 4.5 *The regularity of the conjugacy and parameterization is given up to one digit of precision. This is due to the fact that our method is still not close enough to the breakdown to predict the regularity up to higher accuracy.*

Remark 4.6 *In figure 5 a) we can obvserve that it appears some periodicity on the peaks of the Fourier coefficients. See also figure 7. This is predicted by the renormalization theory of dissipative maps and their relation with the limit case* $\lambda = 0$*, the circle maps. For more details see [ROSS82, ORSS83] and references therein.*

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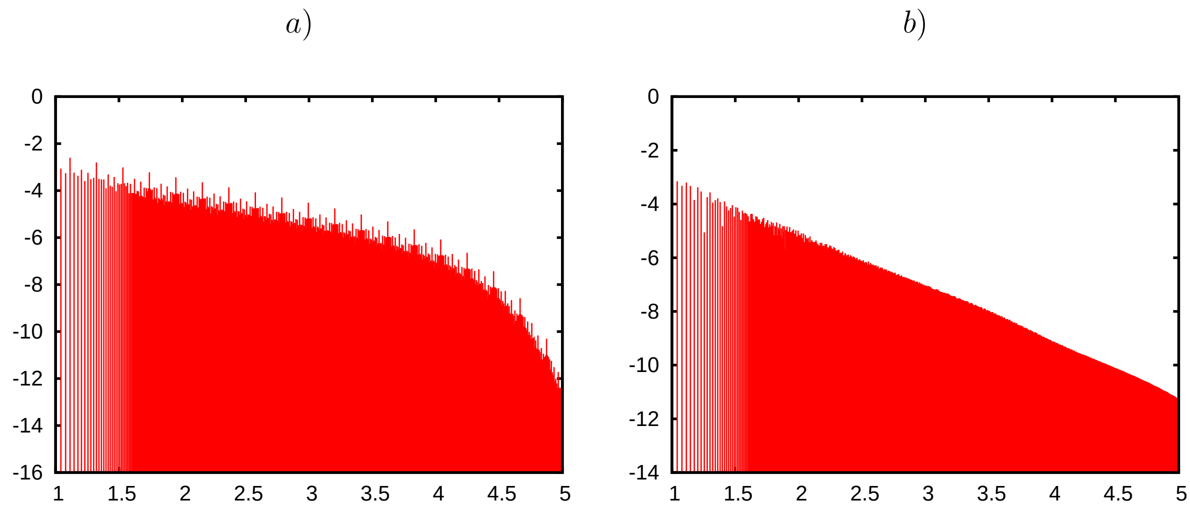

Figure 5: Modulus of the Fourier coefficients of the invariant torus for the parameter $\lambda = 0.4$ near the breakdown. The figures are in $\log_{10} - \log_{10}$ scale. Here $\varepsilon = 0.981421$ and $\mu = 0.36661891732$. a) Fourier coefficients u_k of the parameterization $u(\theta)$. Note that the decay is linear for Fourier coefficients less than $k \approx 10^4$ and that is nonlinear for bigger k. This reflects that the invariant curve is not at the breakdown, but close to it (at a distance $\approx 10^{-4}$). b) Fourier coefficients of the graph of the torus.

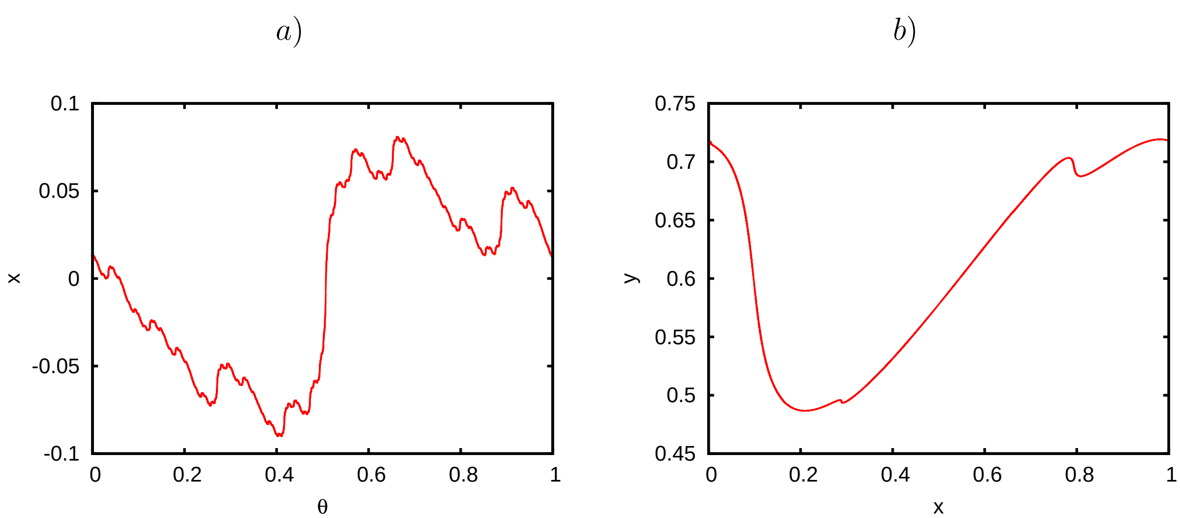

Figure 6: Invariant torus for the parameter $\lambda = 0.4$ near the breakdown. Here $\varepsilon = 0.98081920384$ and $\mu = 0.36661891732$. a) Conjugacy $u(\theta)$ of the invariant torus with respect θ (see equation (2.8)). b) Invariant torus in the phase space.

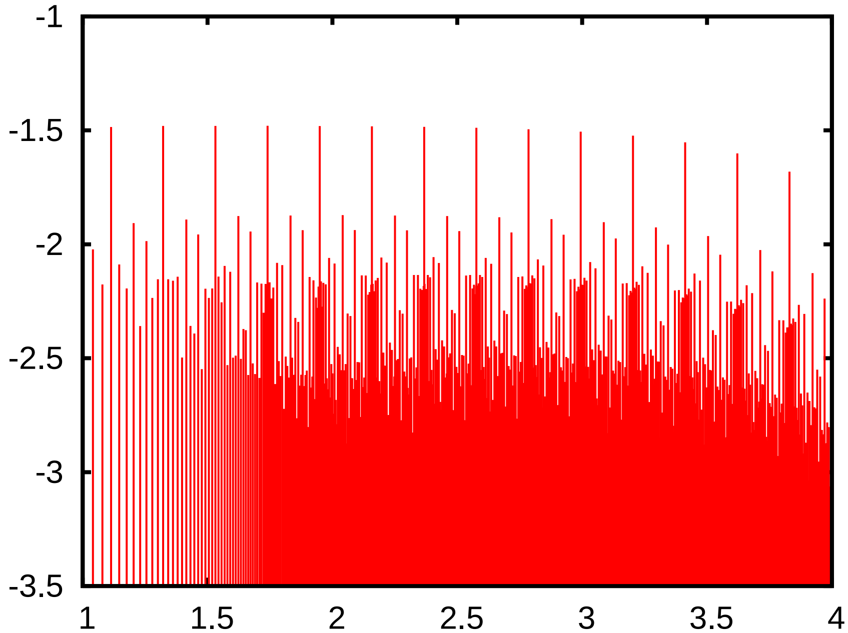

Figure 7: Plot of $|k \cdot u_k|$ versus k. The figure is in $\log_{10} - \log_{10}$ scale. See figure 5 for the parameters of the computation.

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