ON THE GROBMAN-HARTMAN THEOREM IN α-HÖLDER CLASS FOR BANACH SPACES

GENRICH BELITSKII AND VICTORIA RAYSKIN

Abstract. We consider a hyperbolic diffeomorphism in a Banach space with a hyperbolic fixed point 0 and a linear part Λ. We define $\sigma(\Lambda) \in (0, 1]$, and prove that for any $\alpha < \sigma(\Lambda)$ the diffeomorphism admits local α-Hölder linearization.

1. Introduction

In this paper, we extend the well-known Grobman-Hartman Theorem (see [10] and [9]) for topological linearization of local dynamical systems to Banach spaces. The extension to Banach spaces is valuable for applications, in particular because of its relevance to partial differential equations. The technique of C^∞ linearization in a neighborhood of hyperbolic fixed point in Banach spaces was developed in [2]. It served as a basis for the proof of existence of smooth conjugation for non-resonant diffeomorphisms in Banach spaces. (See [11]). As it was noted in [2] and in [11], in the case of smooth conjugation, the Banach space itself has to satisfy certain "smoothness" condition. However, for α-Hölder linearization we do not need this condition. Nonetheless, even for linearization of lower order of smoothness there is a qualitative difference between linearizations in Banach spaces and in $\mathbb{R}^n$. It is known that if the spectrum of linear part Λ is not separated by a unit sphere, the diffeomorphism is C^1-linearizable in $\mathbb{R}^n$. (See [7] or [3]). However, it was shown in the work of Rodrigues and Solà-Morales [12] that certain non-resonance assumption in Banach space is essential even for contractions. They construct an example of a contraction diffeomorphism in infinite dimensions that is not C^1 linearizable. In our paper, we define a constant $\alpha < 1$ and prove the existence of α-Hölder linearization for hyperbolic diffeomorphisms in Banach spaces (without non-resonance assumption).

Also, recently an interest to α-Holder transformations arose. The result, which this paper extends, (i.e., α-Holder transformation in $\mathbb{R}^n$) was available in pre-print format ([4]). It found numerous applications and was frequently sited. (See [1], [6] and [8].) Also, we do not make any additional assumptions on smoothness of the given diffeomorphism. This distinguishes our work from other proofs of low order of smoothness linearizations. (See, for example, [6] and Theorem 11.9 in [5].)

Therefore, we study a possibility of local α-Holder linearization of diffeomorphisms in Banach spaces.

We assume that the linear part Λ of the diffeomorphism is hyperbolic, but we do not make a non-resonance assumption. Our proof mimics the proof of existence of a topological linearization by Z. Nitecki ([9]) up to our Lemma 2, which guaranties α-Holder solutions of corresponding equations. In Lemma 2, we write Banach space as a direct sum of subspaces, on which certain functional equations have solutions.

These solutions are defined as fixed points of corresponding operators, which map some closed subsets of the subspaces into themselves. Since these closed subsets belong to the space of α-Holder maps, the fixed points are α-Holder as well. All these fixed points together form linearizations to the diffeomorphism and its inverse (See Section 4). They belong to the class of α-Hölder maps if the spectrum of Λ satisfies conditions of Lemma 2, and Lipschitz constants of non-linear terms of the diffeomorphism and its inverse are small. The key estimate that guaranties α-Hölder smoothness and defines the value of $\sigma(\Lambda)$ can be found in Lemma 2.

Lemma 2 is proven in a greater generality, than our proof requires. It applies to all diffeomorphisms in a Banach space E, for which the splitting $E = E_1 \oplus \ldots \oplus E_n$ produces properties (2)-(3) of equations (1).

2. Statement of the result

Let E be a Banach space. A local homeomorphism $\Phi : (E, 0) \to (E, 0)$ will be called *α-Holder* if

$$\|\Phi(x') - \Phi(x'')\| \leq C\|x' - x''\|^{\alpha}$$

and

$$\|\Phi^{-1}(x') - \Phi^{-1}(x'')\| \leq C\|x' - x''\|^{\alpha}$$

in a neighborhood of the origin.

Let $\Lambda : E \to E$ be a hyperbolic linear operator. It divides E into the direct sum $E = E_- \oplus E_+$ of Λ-invariant sub-spaces such that the restrictions $\Lambda_{\mp} = \Lambda|E_{\mp}$ is a contraction and an expansion, respectively. We denote by $r(\Lambda)$ the spectral radius of an operator Λ. Further we set

$$\sigma(\Lambda) = \min\left(-\frac{\ln r(\Lambda_-)}{\ln r(\Lambda^{-1})}, -\frac{\ln r(\Lambda_+^{-1})}{\ln r(\Lambda)}\right).$$

We assume

$$\sigma(\Lambda) = -\frac{\ln r(\Lambda)}{\ln r(\Lambda^{-1})}$$

if Λ is a contraction, i.e. $E_- = E$, and

$$\sigma(\Lambda) = -\frac{\ln r(\Lambda^{-1})}{\ln r(\Lambda)}$$

if Λ is an expansion, i.e. $E_+ = E$. In any case $\sigma(\Lambda) \subset (0, 1]$.

Theorem 1. *Let*

$$F(x) = \Lambda x + f(x), \quad f(0) = 0, \; f'(0) = 0$$

be a local hyperbolic diffeomorphism in E. Then for every $\alpha < \sigma(\Lambda)$ there is a local α-Holder homeomorphism conjugating F with its linear part Λ.

The proof repeats arguments proving an existence of a topological linearization (see [9], Theorem 2, p. 80) up to our Lemma 2, which guaranties α-Holder solutions of corresponding equations. Also, our Lemma 2 provides conditions for α-Holder solvability of a wide class of equations. This Lemma can be applied to all functional equations of the form (1), satisfying conditions (2)-(3).

3. The main lemma

Let

$$E = E_1 \oplus \ldots \oplus E_n$$

be a direct decomposition. Consider the system of equations

$$\varphi_i(x) = \Lambda_i \varphi_i(G_i x) + h_i(x, \varphi(H_i x)), \quad i \in \overline{1,n}. \tag{1}$$

Here $\Lambda_i : E_i \to E_i$ are linear maps, while $G_i, H_i : E \to E$ are maps satisfying the Lipschitz conditions

$$\|G_i(x') - G_i(x'')\| \le L_i \|x' - x''\|$$

and

$$\|H_i(x') - H_i(x'')\| \le L \|x' - x''\|.$$

Also, maps h_i, are "small" in the sense

$$\sup \|h_i(u)\| \le \delta, \quad \|h_i(u') - h_i(u'')\| \le \delta \|u' - u''\|$$

for all $u, u', u'' \in E \times E$.

We consider (1) as a system with respect to unknown maps $\varphi = (\varphi_1, \ldots, \varphi_n)$ with $\varphi_i : E \to E_i$, $i \in \overline{1,n}$.

Lemma 2. *Assume that*

$$\max_i \|\Lambda_i\|_i + \delta < 1 \tag{2}$$

and

$$\max_i \|\Lambda_i\|_i L_i^\alpha + \delta\, L^\alpha < 1. \tag{3}$$

Then the equation (1) *has a unique bounded solution* $\varphi : E \to E$. *This solution is* α*-Holder.*

Proof. Denote by $C_b^0(E)$ the Banach space of bounded continuous maps $\varphi : E \to E$ endowed with the norm

$$\|\varphi\| = \sup_x \|\varphi(x)\|.$$

Denote by T the map from the right of (1). It acts in the space $C_b^0(E)$. Moreover,

$$\|T\varphi - T\psi\| \le (\max_i \|\Lambda_i\|_i + \delta)\|\varphi - \psi\|.$$

Because of (2), the map T is a contraction in $C_b^0(E)$. Therefore, (1) has a unique solution $\varphi \in C_b^0(E)$.

In order to show that φ is α-Holder, denote by $K(M)$ the closed subset of maps $\varphi \in C_b^0(E)$, satisfying

$$\|\varphi(x') - \varphi(x'')\| \le M \|x' - x''\|^\alpha.$$

Show that $K(M)$ is T-invariant under an appropriate choice of M. Indeed, let $\varphi \in K(M)$. Then

$$\|T\varphi(x') - T\varphi(x'')\| \le (\max_i \|\Lambda_i\| \cdot L_i^\alpha) M \|x' - x''\|^\alpha +$$

$$+ \max_i \|h_i(x', \varphi(H_i x')) - h_i(x'', \varphi(H_i x''))\|.$$

It follows that if $\|x' - x''\| \ge 1$, then

$$\|T\varphi(x') - T\varphi(x'')\| \le (\max_i \|\Lambda_i\| \cdot L_i^\alpha M + 2\delta)\|x' - x''\|^\alpha.$$

If $\|x' - x''\| \le 1$, then

$\|T\varphi(x') - T\varphi(x'')\| \le \max_i(\|\Lambda_i\| L_i^\alpha) M \|x' - x''\|^\alpha + \delta \|x' - x''\|^\alpha + \delta L^\alpha M \|x' - x''\|^\alpha.$

Choose

$$M > \frac{2\delta}{1 - \delta L^\alpha - \max_i \|\Lambda_i\| L_i^\alpha}.$$

Then

$$\varphi \in K(M) \Rightarrow T\varphi \in K(M).$$

Being closed, the set $K(M)$ contains the solution φ. □

4. Proof of Theorem 1

Proof. By Proposition (p. 69) and Lemma (p. 79) from [9], given $\delta > 0$, we can assume that $\tilde{F}$ is a homeomorphism defined on E, which coincides with F in a neighborhood of the origin and such that the residues

$$f(x) = \tilde{F}(x) - \Lambda x, \quad f_1(x) = \tilde{F}^{-1}(x) - \Lambda^{-1} x$$

are $\mathcal{L}ip$ 1-maps with a constant $\le \delta$. (To simplify notations, we will drop tilde sign.)

Let us look for a linearization transformation in the form $\Phi(x) = x + \varphi(x)$. Then we have the equation

$$\varphi(Fx) = \Lambda\varphi(x) - f(x).$$

Taking into account the decomposition $E = E_- \oplus E_+$, we write the equation in the form (1):

$$\varphi_-(x) = \Lambda_- \varphi_-(F^{-1}x) - f_-(F^{-1}x)$$

(4)

$$\varphi_+(x) = \Lambda_+^{-1} \varphi_+(Fx) + \Lambda_+^{-1} f_+(x).$$

Here $f_\mp : E \to E_\mp$, $f = (f_-, f_+)$ and $\varphi_\mp : E \to E_\mp$. Accordingly, we look for a solution $\varphi = (\varphi_-, \varphi_+)$.

We can choose a norm in E such that

$$\max(\|\Lambda_-\| \cdot \|\Lambda^{-1}\|^\alpha, \quad \|\Lambda_+^{-1}\| \cdot \|\Lambda\|^\alpha) < 1.$$

Moreover, we can choose $\delta > 0$ to be small enough so that Lemma 2 can be applied to the system (4). By the lemma, the system has a unique bounded continuous solution φ. The solution belongs to Lip α. The map $\Phi(x) = x + \varphi(x)$ satisfies

$$\Phi(Fx) = \Lambda\Phi(x).$$

In order to show that Φ is a α-Holder homeomorphism, consider the equations

$$\psi_-(x) = \Lambda_- \psi_-(\Lambda^{-1}x) + f_-(\Lambda^{-1}x + \psi(\Lambda^{-1}x))$$

(5)

$$\psi_+(x) = \Lambda_+^{-1} \psi_+(\Lambda x) - \Lambda_+^{-1} f_+(x + \psi(x)).$$

Under an appropriate choice of $\delta > 0$ the lemma is applicable to the system (5). As a consequence it has a α-Holder solution $\psi = (\psi_-, \psi_+)$, which is unique in $\mathbb{C}_b^0(E)$. The map $\Psi(x) = x + \psi(x)$ satisfies

$$F(\Psi(x)) = \Psi(\Lambda x).$$

Furthermore, the map $H(x) = \Phi(\Psi(x)) = x + h(x)$ is α-Holder; and the bounded continuous map h satisfies the equation $h(\Lambda x) = \Lambda h(x)$. By uniqueness, $h = 0$. Similarly, the map $\tilde{H}(x) = \Psi(\Phi(x)) = x + \tilde{h}(x)$ satisfies

$$\tilde{h}_-(x) = -\Lambda_- \tilde{h}_-(F^{-1}x) + f_-(F^{-1}x + \tilde{h}(F^{-1}x)) - f_-(F^{-1}x),$$

$$\tilde{h}_+(x) = \Lambda_+^{-1}\tilde{h}_+(Fx) - \Lambda_+^{-1}f_+(x + \tilde{h}(x)) + \Lambda_+^{-1}f_+(x).$$

Therefore, $\tilde{H}(x)$ is the identity map by the same arguments. Hence, Φ is a α-Holder homeomorphism, and Ψ is its inverse. □

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Department of Mathematics, Ben-Gurion University of the Negev, P.O.B. 653, Beer-Sheva 84105, Israel

E-mail address: genrich@math.bgu.ac.il