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Lack of Gromov-Hyperbolicity in Colored Random Networks

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Abstract

The geometry of complex networks has a close relationship with their structure and function. In this paper, we introduce an inhomogeneous random network $G(n, \{c_i\}, \{p_i\})$, called the colored random network, and investigate its Gromov-hyperbolicity. We show that the colored random networks are non-hyperbolic in the regime $\sum_{i=1}^{m} c_i^2 p_i = c/n$ for $c > 1$, by approximation to binomial random graphs. Numerical simulations are provided to illustrate our results.

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1 Introduction

Generally speaking, trees are graphs with some very basic interconnection structures which stand for low dimensionality, low complexity, efficient information decomposition and often turns out to be the borderline between tractable and intractable cases [1]. A natural generalization of the concept of tree is that of Gromov hyperbolic graph [6]. Gromov hyperbolicity is strikingly important not only in coarse geometry or large-scale geometry [3, 7] but also in many applied fields such as communication networks [10, 12], cyber security [8], phylogenetics [4, 5] and statistical physics [11].

Hyperbolicity is observed in many real world scale-free networks like Internet [8, 9] and data networks at the IP layer [12]. For the classical $G(n, p)$ model of random graphs [2], it is well-known that $G(n, p)$ are tree-like when $p = o(1/n)$

for large enough n, which roughly implies that they are hyperbolic within this regime. However, the recent work [13] shows that, with positive probability, they are not δ-hyperbolic (in the sense of Gromov) for any positive δ when $p = \Theta(1/n)$. The definition of *δ-hyperbolic* will be developed in detail in Section 2. This phenomenon is further confirmed by extensive simulations and is interpreted as "lack of hyperbolicity" in random graphs by the authors of [13].

In this paper, we introduce a kind of inhomogeneous random graph model, $G(n, \{c_i\}, \{p_i\})$, which we referred to as *colored random graph*, and study its Gromov-hyperbolicity. Each vertex in this random network is assigned with a color i with probability c_i, and possible links are established among vertices with same colors (see Section 2 for details). Therefore, the edges in $G(n, \{c_i\}, \{p_i\})$ are interdependent and the network topology is separated clique-like, which resembles the moving neighborhood network models [14, 15] arisen in social network modeling. Based on an approximation to Erdős-Rényi random graph, we prove that the colored random network is not δ-hyperbolic for any positive δ. Numerical results reveal a surprising degree of closeness between these two models.

The rest of the paper is organized as follows. In Section 2, we define our colored random network and present the tree-likeness parameter, Gromov-hyperbolicity. Section 3 contains the analytical result of non-hyperbolicity for classical random graphs and numerical simulations in Section 4 illustrate and validate our approximation. We conclude the paper in Section 5.

2 Model and preliminaries

In this section, we present some necessary preliminaries leading to the notion of hyperbolicity and colored random networks.

Any simple, but not necessarily finite, graph $G = (V, E)$ is defined by its vertex set V and edge set E. A graph G together with the usual shortest-path metric on it, $d : V \times V \to \{0, 1, 2, \cdots\}$, gives rise to a metric space, or metric graph (G, d). Thus, for a pair of vertices $x, y \in V$, $d(x, y)$ is the distance between them. Note that x and y form an edge if and only if $d(x, y) = 1$. For $S, T \subseteq V$, we write $d(S, T)$ for $\min_{x\in S, y\in T} d(x, y)$. We often omit the brackets and adopt the convention that x stands for the singleton set $\{x\}$ when no confusion can be caused. A path of length k in G is a sequence of distinct vertices $x_0, x_1, x_2, \cdots, x_k$ such that $d(x_{i-1}, x_i) = 1$ for $i = 1, \cdots, k$. A cycle of length k, or k-cycle, is a cyclic sequence of k distinct vertices $x_1, x_2, \cdots, x_k \in V$ such that $d(x_i, x_j) = 1$ whenever $j = i+1 \pmod k$; we will reserve the notation $(x_1 x_2 \cdots x_k)$ for this cycle.

Suppose the metric graph (G, d) has bounded local geometry, i.e., the degree of its vertices is uniformly bounded. Hence, all vertex pairs in G have at least one shortest paths, or geodesic, between them. We denote a geodesic between a pair of vertices x and y by $[xy]$, which may be regarded as just one of possibly many shortest paths between them. In such a situation, (G, d) is said to be a geodesic metric space [6]. A geodesic triangle Δxyz on vertices x, y, z is defined as $[xy] \cup [yz] \cup [zx]$, and it is called δ-thin if each of the geodesics $[xy]$, $[yz]$ and $[zx]$ is contained within the δ-neighborhoods of the other two geodesics for the metric d. More specifically, $[xy] \subseteq N_\delta([yz]) \cup N_\delta([zx])$, and similarly for $[yz]$ and $[zx]$. A geodesic triangle Δxyz is δ^*-fat if δ^* is the smallest δ for which Δxyz is δ-thin. The notion of Gromov-hyperbolicity is then defined as follows.

Definition 1. *A geodesic metric graph is δ-hyperbolic if all geodesic triangles are δ-thin, for some fixed $\delta \geq 0$.*

It is straightforward to check that all tree graphs are δ-hyperbolic with $\delta = 0$. A hyperbolic graph is also said to have negative or hyperbolic curvature. This concept of hyperbolicity comes from the work of Gromov in geometric group theory encapsulating many of the global features of the geometry of complete, simply connected manifolds of negative curvature [3]. Other definitions of curvature, such as Gauss-Bonnet curvature, count the triangles or polygons that meet at each vertex defining a local curvature, not incorporating the global performance of networks. In Section 4, an equivalent characterization of Gromov-hyperbolicity will be provided.

Next, we introduce our random network model. Let m be a natural number and $c_i, p_i \in [0, 1]$ for $i = 1, \cdots, m$. Suppose that $\sum_{i=1}^m c_i = 1$, and that V is a set of n vertices. The colored random graph $G(n, \{c_i\}, \{p_i\})$ on V is defined as follows.

Let $\{1, 2, \cdots, m\}$ be m types of colors, and we consider a random coloring of the vertices in V by

$$f : V \to \{1, 2, \cdots, m\}. \tag{1}$$

For each vertex $v \in V$, we define $P(f(v) = i) = c_i$ and the coloring of a vertex is independent with that of other vertices. In other words, n vertices are assigned colors independently and identically distributed according to the probability distribution $\{c_1, \cdots, c_m\}$. For each pair of different vertices (v_i, v_j), an edge occurs with probability p_k if and only if $f(v_i) = f(v_j) = k$. There is no edges between v_i and v_j if $f(v_i) \neq f(v_j)$.

Clearly, the binomial random graph model $G(n, p)$ can be viewed as the special case of $m = 1$ (and $c_1 = 1$), i.e., $G(n, 1, p)$. It is straightforward to show that, for a pair of different vertices (v_i, v_j), the edge is present with probability $\sum_{k=1}^m c_k^2 p_k$. In the sequel, our analysis will based on the binomial random graph model $G(n, \sum_{k=1}^m c_k^2 p_k)$, and our simulation results in Section 4 implies that it is a sharp approximation for large n as far as the hyperbolicity is concerned.

3 Positive measure of δ-fat triangles

From Definition 1, if a graph G is δ-hyperbolic, then for any geodesic triangle $\Delta \subset G$, there is some $\delta^* \leq \delta < \infty$ such that Δ is δ^*-fat. We will show that with positive probability the random network contains δ-fat triangles for arbitrary large δ as $n \to \infty$, which implies non-hyperbolicity. The following result is more or less implicit in [13], and we include the complete proofs here not only for the convenience of the reader but also to make the argument self-contained.

Theorem 2. *Suppose that $\delta \geq 0$ and $\sum_{k=1}^{m} c_k^2 p_k = c/n$ for some $c > 1$. We have*

$$\liminf_{n\to\infty} P\Big(G\Big(n, \sum_{k=1}^{m} c_k^2 p_k\Big) \textit{ contains a } \delta\textit{-fat triangle}\Big) \geq \frac{1}{4}e^{-6c\delta}c^{6\delta+1}(2 - e^{-6c\delta}c^{6\delta+1}) > 0. \quad (2)$$

Proof. The random graph $G(n, c/n)$ for $c > 1$ almost surely has a unique giant component of size γn, where γ is the unique solution of $1 - \gamma = e^{-c\gamma}$ in the interval $(0, 1)$, c.f. [2]. Without loss of generality, we assume the giant component has size n and consider the hyperbolicity/non-hyperbolicity in the giant component.

Let $a = 6\delta$. For any a vertices $x_1, x_2 \cdots, x_a$, it is clear that there exists a δ-fat triangle on the cycle $(x_1 x_2 \cdots x_a)$. The probability q that $G(n, c/n)$ contains an induced a-cycle on vertices $x_1, x_2 \cdots, x_a$ with a single connection to the giant component is shown to be given by

$$\begin{aligned} q &= \frac{(a-1)!}{2} a(n-a)\Big(\frac{c}{n}\Big)^{a+1}\Big(1 - \frac{c}{n}\Big)^{\binom{n}{2} - \binom{n-a}{2} - a - 1} \\ &= \frac{a!}{2}(n-a)\Big(\frac{c}{n}\Big)^{a+1}\Big(1 - \frac{c}{n}\Big)^{\frac{a(2n-3)-a^2-2}{2}}. \end{aligned} \quad (3)$$

Let X be a random variable that counts the number of induced a-cycles on vertices $y_1, y_2 \cdots, y_a$ in $G\backslash\{x_1, x_2 \cdots, x_a\}$ with a single connection to the giant component. We denote by P_1 the probability that $G(n, c/n)$ contains an induced cycle $(x_1 x_2 \cdots x_a)$ and another distinct induced cycle $(y_1 y_2 \cdots y_a)$, both of which have a single connection to the giant component. Therefore, by using the Markov inequality and (3), we have

$$P_1 = P(X \geq 1|A) \cdot P(A) \leq EX \cdot q = \binom{n-a}{a} q^2, \quad (4)$$

where A represents the event that graph $G(n, c/n)$ contains an induced cycle $(x_1 x_2 \cdots x_a)$ having a single connection to the giant component.

Next, we want to compute the probability P_2 that $G(n, c/n)$ contains only one induced cycle $(x_1 x_2 \cdots x_a)$ with a single connection to the giant component. It follows from (3) and (4) that

$$P_2 = q - P_1 \geq q - \binom{n-a}{a} q^2. \quad (5)$$

Hence, utilizing (5), the probability that $G(n, c/n)$ contains at least one induced cycle $(x_1 x_2 \cdots x_a)$ which has a single connection to the giant component is bounded below by

$$\binom{n}{a} P_2 \geq \binom{n}{a}\Big(q - \binom{n-a}{a} q^2\Big) > \rho(1-\rho), \quad (6)$$

where $\rho := \binom{n}{a}q$. From the comment just before Eq. (3), the probability on the left-hand side of (2) is also lower bounded by the above probability. Consequently,

$$P\Big(G\Big(n,\frac{c}{n}\Big) \text{ contains a } \delta\text{-fat triangle}\Big) > \rho(1-\rho). \tag{7}$$

By (3) and the inequality $n^a/a! > \binom{n}{a} > (n-a)^a/a!$, we get

$$\begin{aligned}\Big(\frac{n-a}{2n}\Big)\Big(1-\frac{c}{n}\Big)^{an}\Big(1-\frac{c}{n}\Big)^{-\frac{(a+1)(a+2)}{2}}c^{a+1} &> \rho \\ &> \frac{(n-a)^{a+1}}{2n^{a+1}}\Big(1-\frac{c}{n}\Big)^{an}\Big(1-\frac{c}{n}\Big)^{-\frac{(a+1)(a+2)}{2}}c^{a+1}.\end{aligned} \tag{8}$$

Since $\lim_{n\to\infty}\rho = e^{-ca}c^{a+1}/2 \in (0,1)$ for all $a, c > 1$, we have

$$\begin{aligned}\liminf_{n\to\infty} P\Big(G\Big(n,\frac{c}{n}\Big) \text{ contains a } \delta\text{-fat triangle}\Big) &\geq \liminf_{n\to\infty}\rho(1-\rho) \\ &= \frac{1}{2}e^{-ca}c^{a+1}\Big(1-\frac{1}{2}e^{-ca}c^{a+1}\Big) \\ &> 0,\end{aligned} \tag{9}$$

which readily concludes the proof. ■

4 Numerical simulations

We have shown that the random graph $G(n, \sum_{k=1}^m c_k^2 p_k)$ are not hyperbolic for $\sum_{k=1}^m c_k^2 p_k = c/n$ with $c > 1$ due to existence of a non-zero probability of arbitrary fat triangles. In this section, we will illustrate that $G(n, \sum_{k=1}^m c_k^2 p_k)$ is a nice approximation to the colored random network $G(n, \{c_i\}, \{p_i\})$, which has similar non-hyperbolic properties.

We first provide an equivalent characterization of Gromov-hyperbolicity and introduce the curvature plot of a network, which has been successfully applied to some physical networks [12]. For any geodesic triangle Δxyz in a graph (G, d), we define

$$\delta_{\Delta xyz} = \min_{w\in G}\max\{d(w,[xy]), d(w,[yz]), d(w,[zx])\}, \tag{10}$$

where, as defined in Section 2, $d(w,[xy])$ is the shortest distance between vertex w and all the other vertices on geodesic $[xy]$. Let

$$\delta = \max_{\Delta xyz\subseteq G}\delta_{\Delta xyz}. \tag{11}$$

Then δ is finite if and only if the graph G is δ-hyperbolic [6]. Instead of the maximum taken in (11), the curvature plot Fig. 1 shows the average value δ_a of $\delta_{\Delta xyz}$ for all triangles whose shortest side is L, as a function of L. That is, $\delta_a = \delta_a(L)$, where $L = \min\{d(x,y), d(y,z), d(z,x)\}$. Results for random graphs

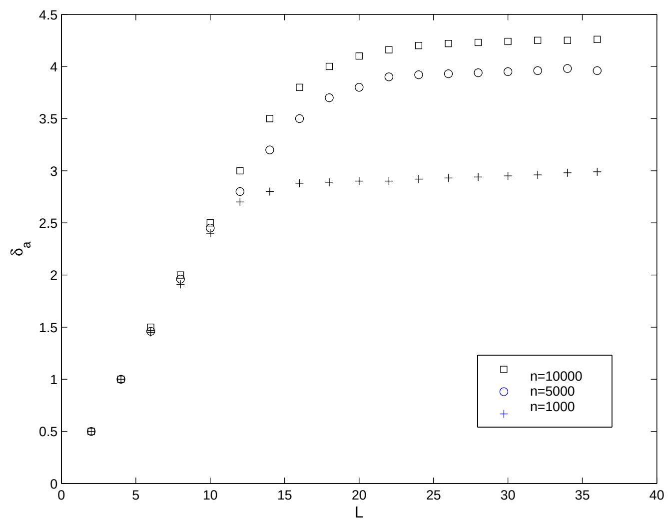

Figure 1: Curvature plots for random graphs $G(n, c/n)$ with $c = 1.5$ and different sizes n. Each quantity is an average over 50 realizations.

$G(n, c/n)$ with $c = 1.5$ of different sizes are shown. We can see from Fig. 1 that a linear increase in $\delta_a(L)$ saturating at a plateau whose height increases as the size of the graph is increased. In addition, all the three curves coincide before their plateaus. Therefore, we are in a position to infer that $\delta_a(L)$ is unbounded with the increase of L for $n \to \infty$, which obviously is a stronger demonstration of non-hyperbolicity than the probabilistic lower bound obtained in Theorem 2.

Now we want to compare the binomial random graph $G(n, \sum_{k=1}^{m} c_k^2 p_k)$ with our colored random network $G(n, \{c_i\}, \{p_i\})$. Take $m = 3$ colors, $c_1 = 0.6$, $c_2 = 0.3$ and $c_3 = 0.1$. Choose $p_1 = 2c/n$, $p_2 = c/2n$ and $p_3 = 135c/2n$ so that $\sum_{k=1}^{m} c_k^2 p_k = c/n$ holds. For $c = 1.5$, results for different size n are shown in Fig. 2, Fig. 3 and Fig.4. We observe that the approximation is sharper for larger n, which indicates that the colored random networks are also non-hyperbolic in the regime of $\sum_{k=1}^{m} c_k^2 p_k = c/n$ with $c > 1$.

Let $m = 3$, $c_1 = 0.5$, $c_2 = 0.3$, $c_3 = 0.2$, $p_1 = 3c/n$, $p_2 = c/n$ and $p_3 = 21c/n$. Hence, we have $\sum_{k=1}^{m} c_k^2 p_k = c/n$. Fig. 5 shows the similar result for $c = 2$. The approximation is nice indeed.

5 Conclusion

In this paper we propose a heterogeneous random graph model, the colored random network, and study its Gromov-hyperbolicity. We demonstrate the lack of hyperbolicity in colored random networks by approximations to classical random graphs. Since our approximation is based on numerical simulations, the

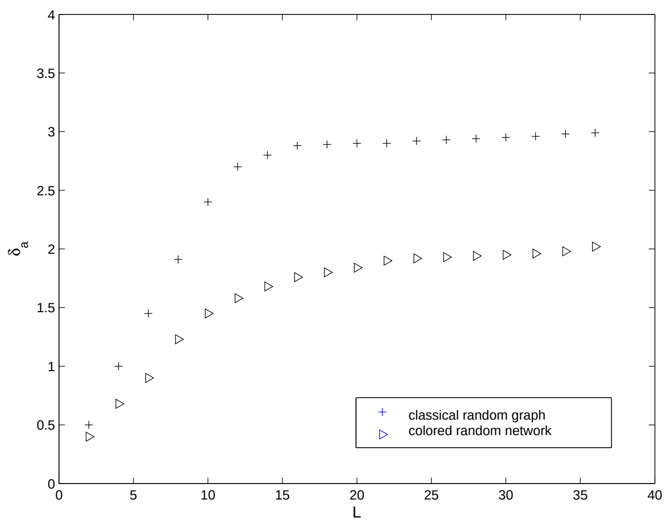

Figure 2: Curvature plots for random graphs $G(n, \sum_{k=1}^{m} c_k^2 p_k)$ and $G(n, \{c_i\}, \{p_i\})$ with $c = 1.5$ and $n = 1000$. Each quantity is an average over 50 realizations.

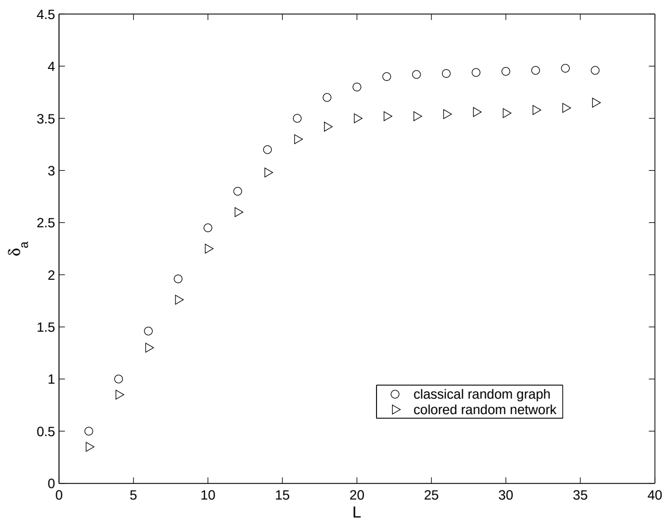

Figure 3: Curvature plots for random graphs $G(n, \sum_{k=1}^{m} c_k^2 p_k)$ and $G(n, \{c_i\}, \{p_i\})$ with $c = 1.5$ and $n = 5000$. Each quantity is an average over 50 realizations.

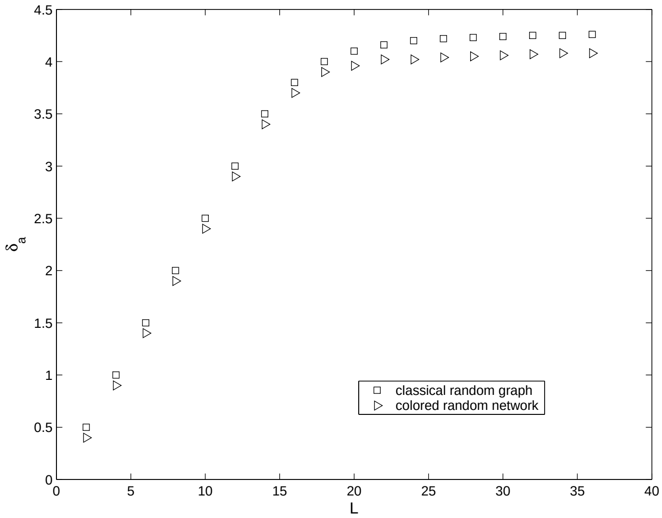

Figure 4: Curvature plots for random graphs $G(n, \sum_{k=1}^{m} c_k^2 p_k)$ and $G(n, \{c_i\}, \{p_i\})$ with $c = 1.5$ and $n = 10000$. Each quantity is an average over 50 realizations.

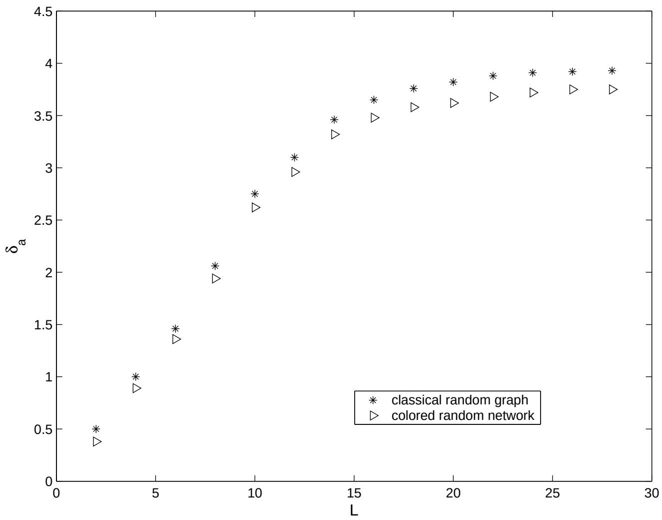

Figure 5: Curvature plots for random graphs $G(n, \sum_{k=1}^{m} c_k^2 p_k)$ and $G(n, \{c_i\}, \{p_i\})$ with $c = 2$ and $n = 10000$. Each quantity is an average over 50 realizations.

corresponding analytical results would be interesting. This issue remains open for future research.

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