Eigenvalues of Laplacian with constant magnetic field on noncompact hyperbolic surfaces with finite area

March 14, 2010

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Abstract

We consider a magnetic Laplacian $-\Delta_A = (id + A)^\star(id + A)$ on a noncompact hyperbolic surface $\mathbf{M}$ with finite area. A is a real one-form and the magnetic field dA is constant in each cusp. When the harmonic component of A satifies some quantified condition, the spectrum of $-\Delta_A$ is discrete. In this case we prove that the counting function of the eigenvalues of $-\Delta_A$ satisfies the classical Weyl formula, even when $dA = 0$. [1]

1 Introduction

We consider a smooth, connected, complete and oriented Riemannian surface $(\mathbf{M}, g)$ and a smooth, real one-form A on $\mathbf{M}$. We define the magnetic

[1]*Keywords* : spectral asymptotics, magnetic field, Aharanov-Bohm, hyperbolic surface.

Laplacian, the Bochner Laplacian

$$\begin{gathered} -\Delta_A \;=\; (i\; d + A)^\star (i\; d + A)\;, \\ (\;(i\; d + A)u = i\; du + uA\;,\; \forall\; u \;\in\; C_0^\infty(\mathbf{M};\mathbb{C})\;)\;. \end{gathered} \tag{1.1}$$

The magnetic field is the exact two-form $\rho_B \;=\; dA$.

If dm is the Riemannian measure on $\mathbf{M}$, then

$$\rho_B \;=\; \widetilde{\mathbf{b}}\, dm\;, \quad \text{with} \quad \widetilde{\mathbf{b}} \;\in\; C^\infty(\mathbf{M};\mathbb{R})\;. \tag{1.2}$$

The magnetic intensity is $\mathbf{b} \;=\; |\widetilde{\mathbf{b}}|$.

It is well known, (see [Shu]), that $-\Delta_A$ has a unique self-adjoint extension on $L^2(\mathbf{M})$, containing in its domain $C_0^\infty(\mathbf{M};\mathbb{C})$, the space of smooth and compactly supported functions. The spectrum of $-\Delta_A$ is gauge invariant : for any $f \in C^1(\mathbf{M};\mathbb{R})$, $-\Delta_A$ and $-\Delta_{A+df}$ have the same spectrum.

We are interested in constant magnetic fields on $\mathbf{M}$ in the case when $(\mathbf{M},\; g)$ is a non-compact geometrically finite hyperbolic surface of finite area; (see [Per] or [Bor] for the definition and the related references). More precisely

$$\mathbf{M} \;=\; \bigcup_{j=0}^{J} M_j \tag{1.3}$$

where the M_j are open sets of $\mathbf{M}$, such that the closure of M_0 is compact, and $(J \geq 1)$ the other M_j are cuspidal ends of $\mathbf{M}$.

This means that, for any $j,\; 1 \;\leq\; j \;\leq\; J$, there exist strictly positive constants a_j and L_j such that M_j is isometric to $\mathbb{S}\times]a_j^2, +\infty[$, equipped with the metric

$$ds_j^2 \;=\; y^{-2}(\; L_j^2\; d\theta^2 \;+\; dy^2\;)\;; \tag{1.4}$$

$(\mathbb{S} \;=\; \mathbb{S}^1$ is the unit circle and $M_j \cap M_k \;=\; \emptyset$ if $j \neq k$) .
Let us choose some $z_0 \;\in\; M_0$ and let us define

$$d\;:\;\mathbf{M} \;\rightarrow\; \mathbb{R}_+\;; \quad d(z) \;=\; d_g(z, z_0)\;; \tag{1.5}$$

$d_g(\;.\;,\;.\;)$ denotes the distance with respect to the metric g.

It is not possible to have a constant magnetic field on $\mathbf{M}$, but for any $b \;\in\; \mathbb{R}^J$, there exists a one-form A , such that the corresponding magnetic field dA satisfies

$$dA \;=\; \widetilde{\mathbf{b}}(z) dm \quad \text{with} \quad \widetilde{\mathbf{b}}(z) \;=\; b_j\; \forall\; z \;\in\; M_j\;. \tag{1.6}$$

The following statement on the essential spectrum is proven in [Mo-Tr1] :

Theorem 1.1 *Assume (1.3) and (1.6). Then for any* j , $1 \leq j \leq J$ *and for any* $z \in M_j$ *there exists a unique closed curve through* z , $\mathcal{C}_{j,z}$ *in* $(M_j,\ g)$, *not contractible and with zero* $g-$*curvature. The following limit exists and is finite:*

$$[A]_{M_j} \ = \lim_{d(z)\to+\infty} \int_{\mathcal{C}_{j,z}} A\ . \tag{1.7}$$

If $J^A \ = \ \{j \in \mathbb{N}\ ,\ 1 \leq j \leq J\ s.t.\ [A]_{M_j} \in 2\pi\mathbb{Z}\ \} \ \neq \ \emptyset$, *then*

$$\mathrm{sp}_{ess}(-\Delta_A) \ = \ [\frac{1}{4} + \min_{j\in J^A} b_j^2\ ,\ +\infty[\ . \tag{1.8}$$

If $J^A \ = \ \emptyset$, *then* $\mathrm{sp}_{ess}(-\Delta_A) \ = \ \emptyset$:
$-\Delta_A$ *has purely discrete spectrum, (its resolvent is compact).*

When the magnetic Laplacian $-\Delta_A$ has purely discrete spectrum, it is called a magnetic bottle, (see [Col]).

If $A = df + A^H + A^\delta$ is the Hodge decomposition of A with A^H harmonic, ($dA^H = 0$ and $d^\star A^H = 0$) , then $\forall\ j$, $[A]_{M_j} = [A^H]_{M_j}$, so the discreteness of the spectrum of $-\Delta_A$ depends only on the harmonic component of A . So one can see the case $J^A = \emptyset$ as an Aharonov-Bohm phenomenon [Ah-Bo], a situation where the magnetic field dA is not sufficient to describe $-\Delta_A$ and the use of the magnetic potential A is essential : we can have magnetic bottle with null intensity.

2 The Weyl formula in the case of finite area with a non-integer class one-form

Here we are interested in the pure point part of the spectrum. We assume that $J^A \ = \ \emptyset$, then the spectrum of $-\Delta_A$ is discrete. In this case, we denote by $(\lambda_j)_j$ the increasing sequence of eigenvalues of $-\Delta_A$, (each eigenvalue is repeated according to its multiplicity). Let

$$N(\lambda, -\Delta_A) \ = \ \sum_{\lambda_j<\lambda} 1\ . \tag{2.1}$$

We will show that the asymptotic behavior of $N(\lambda)$ is given by the Weyl formula :

Theorem 2.1 *Consider a geometrically finite hyperbolic surface* $(\mathbf{M}, g)$ *of finite area, and assume (1.6) with* $J^A = \emptyset$, *(see (1.7 for the definition).*

Then

$$N(\lambda, -\Delta_A) \;=\; \lambda\frac{|\mathbf{M}|}{4\pi} \;+\; \mathbf{O}(\frac{\lambda}{\ln\lambda}) \; . \tag{2.2}$$

Remark 2.2 *As* J^A *depends only on the harmonic component of* A , J^A *is not empty when* $\mathbf{M}$ *is simply connected. In [Go-Mo] there are some results close to Theorem 2.1, but for simply connected manifolds.*

The cases where the magnetic field prevails were studied in [Mo-Tr1] and in [Mo-Tr2].

Proof of Theorem 2.1. Any constant depending only on the b_j and on $\min_{1\leq j\leq J}\inf_{k\in\mathbb{Z}} |[A]_{M_j} - 2k\pi|$ will be denoted invariably C .

Consider a cusp $M = M_j = \mathbb{S}\times]\alpha^2, +\infty[$ equipped with the metric $ds^2 \;=\; L^2e^{-2t}d\theta^2 \;+\; dt^2$ for some $\alpha > 0$ and $L > 0$.

Let us denote by $-\Delta_A^M$ the Dirichlet operator on M , associated to $-\Delta_A$.

The first step will be to prove that

$$N(\lambda, -\Delta_A^M) \;=\; \lambda\frac{|M|}{4\pi} \;+\; \mathbf{O}(\frac{\lambda}{\ln\lambda}) \; . \tag{2.3}$$

Since $-\Delta_A^M$ and $-\Delta_{A+d\varphi+kd\theta}^M$ are gauge equivalent for any $\varphi \in C^\infty(\overline{M};\mathbb{R})$ and any $k \in \mathbb{Z}$, we can assume that

$$-\Delta_A^M = L^{-2}e^{2t}(D_\theta - A_1)^2 + D_t^2 + \frac{1}{4} \; , \quad \text{with} \quad A_1 = -\xi \pm bLe^{-t} \; , \; \xi \in]0,1[\; ,$$

$(b = b_j \; , \; 2\pi\xi - [A]_M \;\in\; 2\pi\mathbb{Z})$. Then we get that

$$\mathrm{sp}(-\Delta_A^M) = \bigcup_{\ell\in\mathbb{Z}} \mathrm{sp}(P_\ell) \; ; \; P_\ell = D_t^2 + \frac{1}{4} + \left(e^t\frac{(\ell+\xi)}{L} \pm b\right)^2 \quad ,$$

for the Dirichlet condition on $L^2(I; dt)$; $I =]\alpha^2, +\infty[$. This implies that

$$N(\lambda, -\Delta_A^M) \;=\; \sum_{\ell\in\mathbb{Z}} N(\lambda, P_\ell) = \sum_{\ell\in X_\lambda} N(\lambda, P_\ell) \tag{2.4}$$

with $X_\lambda = \{\ell \; / \; e^{\alpha^2}\frac{|\ell+\xi|}{L} < \sqrt{\lambda - 1/4} - b \; \}$.

Denoting by Q_ℓ the Dirichlet operator on I associated to

$$Q_\ell = D_t^2 + \frac{1}{4} + \frac{(\ell+\xi)^2}{L^2}e^{2t} \; ,$$

we easily get that

$$Q_\ell - C\sqrt{Q_\ell} \ \leq\ P_\ell \ \leq\ Q_\ell + C\sqrt{Q_\ell}\ . \tag{2.5}$$

Therefore one can find a constant $C(b)$, depending only on b , such that, for any $\lambda >> 1 + C(b)$,

$$N(\lambda - \sqrt{\lambda}C(b), Q_\ell) \ \leq\ N(\lambda, P_\ell) \ \leq\ N(\lambda + \sqrt{\lambda}C(b), Q_\ell)\ ; \tag{2.6}$$

Applying the Weyl formula we thus get the following

Lemma 2.3 *There exists* $C_0 > 1$ *such that, for any* $\lambda >> 1$ *and any* $\ell \in X_\lambda$,

$$w_\ell(\lambda - C_0\sqrt{\lambda}) \ \leq\ \pi N(\lambda, P_\ell) \ \leq\ w_\ell(\lambda + C_0\sqrt{\lambda})\ ,$$

with

$$\begin{aligned} w_\ell(\mu) \ &=\ \int_{\alpha^2}^{+\infty} \left[\mu - \frac{(\ell+\xi)^2}{L^2}e^{2t}\right]_+^{1/2} dt \qquad (2.7) \\ &=\ \int_{\alpha^2}^{T(\mu,L)} \left[\mu - \frac{(\ell+\xi)^2}{L^2}e^{2t}\right]_+^{1/2} dt\ ; \end{aligned}$$

$(e^{T(\mu,L)} = L\sqrt{\mu}/(\inf_{k\in\mathbb{Z}}|\xi - k|)\ )$.

In view of (2.4) we now compute $\sum_{\ell\in\mathbb{Z}} w_\ell(\mu)$. We first get the following

Lemma 2.4 *There exists* $C_0 > 1$ *such that, for any* $\mu >> 1$ *and any* $t \in [\alpha^2, T(\mu, L)]$,

$$\left| \int_{\mathbb{R}} \left[\mu - \frac{(x+\xi)^2}{L^2}e^{2t}\right]_+^{1/2} dx \ - \sum_{\ell\in\mathbb{Z}} \left[\mu - \frac{(\ell+\xi)^2}{L^2}e^{2t}\right]_+^{1/2} \right| \ \leq\ C_0(\sqrt{\mu} + \frac{e^t}{L})\ .$$

This leads to

Lemma 2.5 *There exists* $C_0 > 1$ *such that, for any* $\mu >> 1$,

$$\left| \int_{\alpha^2}^{T(\mu,L)} \int_{\mathbb{R}} \left[\mu - \frac{(x+\xi)^2}{L^2}e^{2t}\right]_+^{1/2} dxdt \ - \sum_{\ell\in\mathbb{Z}} w_\ell(\mu) \right| \ \leq\ C_0\sqrt{\mu}\ln\mu\ .$$

Changing variables in the integral in the right-hand side we get

Lemma 2.6 *There exists $C_0 > 1$ such that, for any $\mu >> 1$,*

$$\left| \int_{\alpha^2}^{T(\mu,L)} \int_{\mathbb{R}} \left[\mu - \frac{(x+\xi)^2}{L^2} e^{2t} \right]_+^{1/2} dxdt \;-\; \mu L e^{-\alpha^2} \int_{\mathbb{R}} \left[1 - x^2\right]_+^{1/2} dx \right| \;\leq\; C_0 \frac{\mu}{\ln \mu} \;.$$

Noticing that $|M| = 2\pi L e^{-\alpha^2}$ we deduce from Lemmas 2.5 and 2.6 that

Lemma 2.7

$$\frac{1}{\pi} \sum_{\ell \in \mathbb{Z}} w_\ell(\mu) \;=\; \frac{|M|}{4\pi}\mu \;+\; \mathbf{O}(\frac{\mu}{\ln \mu}) \;, \quad \text{as} \quad \mu \to +\infty \;.$$

In view of (2.4) this ends the proof of (2.3).

Now it remains to consider the whole surface **M**.

We have : $\mathbf{M} \;=\; \left(\bigcup_{j=0}^{J} M_j \right)$

where the M_j are open sets of **M**, such that the closure of M_0 is compact, and the other M_j are cuspidal ends of **M** and

$M_j \cap M_k = \emptyset$, if $j \neq k$. We denote $M_0^0 = \mathbf{M} \setminus (\bigcup_{j=1}^{J} \overline{M_j})$, then

$$\mathbf{M} \;=\; \overline{M_0^0} \bigcup \left(\bigcup_{j=1}^{J} \overline{M_j} \right) \;. \tag{2.8}$$

Let us denote by $-\Delta_A^\Omega$ the Dirichlet operator on an open set Ω of **M** associated to $-\Delta_A$.

The minimax principle and (2.8) imply that

$$N(\lambda, -\Delta_A^{M_0^0}) + \sum_{1 \leq j \leq J} N(\lambda, -\Delta_A^{M_j}) \leq N(\lambda, -\Delta_A) \tag{2.9}$$

To get an upper bound we use a partition of unity. Let us consider, for any $1 \leq j \leq J$ the diffeorphism

$$\Phi_j \;:\; \overline{M_j} \;\to\; \mathbb{S} \times [\alpha_j^2, +\infty[$$

and define the sets

$$O_j^\lambda \;=\; \Phi_j^{-1}(\mathbb{S} \times [\alpha_j^2, \alpha_j^2 + \frac{1}{\ln \lambda}[) \;.$$

Then we have the following covering of $\mathbf{M}$ with open sets

$$\mathbf{M} \;=\; M_0^\lambda \bigcup \left(\bigcup_{j=0}^{J} M_j \right) \;, \quad \text{with} \quad \mathrm{M}_0^\lambda \;=\; \mathrm{M}_0^0 \bigcup \left(\bigcup_{\mathrm{j}=0}^{\mathrm{J}} \mathrm{O}_{\mathrm{j}}^\lambda \right) \;.$$

We associate to this covering, a smooth partition of unity $(\psi_{j,\lambda})_{j=0,1,\dots,J}$, $\sum_{0 \le j \le J} \psi_{j,\lambda}^2(x) = 1, \; \forall x \in \mathbf{M}$, such that

$$\begin{cases} \psi_{j,\lambda} = 1 \text{ on } M_j \setminus O_j^\lambda \;, \; (1 \le k \le J) \\ \psi_{j,\lambda} = 0 \text{ on } M_k \text{ if } k \ne j, \\ \psi_{0,\lambda} = 1 \text{ on } M_0^0 \\ |\nabla \psi_{j,\lambda}(x)| \le C \ln \lambda \end{cases}$$

Using again minimax principle as in [Mo-Tr1], we get the following upper bound

$$N(\lambda, -\Delta_A) \le N(\lambda + C \ln^2 \lambda, -\Delta_A^{M_0^\lambda}) + \sum_{1 \le j \le J} N(\lambda + C \ln^2 \lambda, -\Delta_A^{M_j}) \quad (2.10)$$

The Weyl formula with remainder, (see [Hor] for smooth boundary and [Ivr] for boundary with cone-like singularities), gives that

$$\left\{ \begin{array}{l} N(\lambda, -\Delta_A^{M_0^0}) = (4\pi)^{-1} |M_0^0| \lambda + \mathbf{O}(\sqrt{\lambda}) \\ N(\lambda + C \ln^2 \lambda, -\Delta_A^{M_0^\lambda}) = (4\pi)^{-1} |M_0^\lambda| (\lambda + C \ln^2 \lambda) + \mathbf{O}(\sqrt{\lambda}) \end{array} \right\} \quad (2.11)$$

Noticing that $|M_0^\lambda|(\lambda + C \ln^2 \lambda) = |M_0^0| \lambda + \mathbf{O}(\lambda / \ln \lambda \,)$,
we get (2.2) from (2.3) (with $M = M_j \;,\; j = 1, \dots, J$), (2.9), (2.10) and (2.11) □

Remark 2.8 *Theorem 2.1 still holds if the metric of* $\mathbf{M}$ *is modified in a compact set.*

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