NATURAL BOUNDARIES AND SPECTRAL THEORY

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ABSTRACT. We present and exploit an analogy between lack of absolutely continuous spectrum for Schrödinger operators and natural boundaries for power series. Among our new results are generalizations of Hecke's example and natural boundary examples for random power series where independence is not assumed.

1. INTRODUCTION

In this paper, we'll present and exploit a powerful analogy between spectral theory and the question of when a power series $f(z) = \sum_{n=0}^{\infty} a_n z^n$ defining an analytic function on $\mathbb{D} = \{z \mid |z| < 1\}$ has a natural boundary on $\partial\mathbb{D}$, in that for no $z_0 = e^{i\theta}$ does f have an analytic continuation to $\{z \mid |z - z_0| < \delta\}$ for some $\delta > 0$. In particular, we shall import two notions ("reflectionless" and "right limit") from the spectral theory of Jacobi matrices and obtain a general theorem regarding the consequence of the possibility to analytically continue f across an arc. While spectral theory ideas motivated our approach to natural boundaries, what we develop doesn't require any spectral theory. The reader not knowledgeable in spectral theory and not interested in the background should skip to the paragraph containing (1.4).

To set the stage, consider a Jacobi matrix,

$$J = \begin{pmatrix} b_1 & a_1 & 0 & \cdots \\ a_1 & b_2 & a_2 & \cdots \\ 0 & a_2 & b_3 & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix} \tag{1.1}$$

where $\{a_n, b_n\}_{n=1}^{\infty}$ are bounded. By a right limit of J, we mean a two-sided Jacobi matrix, $J^{(r)}$, with parameters $\{a_n^{(r)}, b_n^{(r)}\}_{n=-\infty}^{\infty}$ given by

$$a_n^{(r)} = \lim_{k \to \infty} a_{n+n_k} \qquad b_n^{(r)} = \lim_{k \to \infty} b_{n+n_k} \qquad (1.2)$$

where $n_k \to \infty$ is a subsequence. By compactness, right limits exist. This definition is from Last–Simon [22], who used it in

Theorem 1.1 ([22]). *If* $\Sigma_{\mathrm{ac}}(A)$ *is the essential support of the a.c. spectrum of an operator, A, then*

$$\Sigma_{\mathrm{ac}}(J) \subset \Sigma_{\mathrm{ac}}(J^{(r)}) \qquad (1.3)$$

for every right limit $J^{(r)}$ of J.

Right limits are also relevant to essential spectrum where they have been exploited by several authors (e.g., [9, 10, 17, 22, 23, 25, 35]), but their relevance for a.c. spectrum goes back to Last–Simon [22]. Recently, Remling [36] found a much stronger property of right limits when there is a.c. spectrum. It depends on the notion of reflectionless two-sided Jacobi matrix. The precise definition is irrelevant for our discussion here (see, e.g., [4, 36, 43, 44]), but we note that if $\tilde{J}$ is a two-sided Jacobi matrix, reflectionless on some $\mathfrak{e} \subset \mathbb{R}$ with the Lebesgue measure, $|\mathfrak{e}|$, of $\mathfrak{e}$ nonzero, then $\mathfrak{e} \subset \Sigma_{\mathrm{ac}}(\tilde{J})$ and $\{\tilde{a}_n, \tilde{b}_n\}_{n=-\infty}^{-1}$ determine $\{\tilde{a}_n, \tilde{b}_n\}_{n=0}^{\infty}$. Remling [36] proved

Theorem 1.2 ([36]). *If $\mathfrak{e}$ is $\Sigma_{\mathrm{ac}}(J)$ for a Jacobi matrix, then any right limit, $J^{(r)}$, is reflectionless on $\mathfrak{e}$.*

With this result, Remling was able to recover and extend virtually every result on the absence of a.c. spectrum for situations where $\{a_n, b_n\}_{n=1}^{\infty}$ are bounded, and the lack of a.c. spectrum is due to a part of the a's and b's that is dominant at infinity (rather than a perturbation that goes to zero at infinity).

Our work here began by our noticing that the major classes of Jacobi matrices with no a.c. spectrum have analogs in the major classes of results on the occurrence of natural boundaries (see Remmert [37, Ch. 11] for a summary of classical results on natural boundaries), as seen in

- Gap theorems [18, 16, 15, 37] $\sim$ sparse potentials [32, 36]
- Finite-valued power series [47, 37] $\sim$ finite-valued Jacobi matrices [21, 36]
- Random power series [46, 30, 20] $\sim$ Anderson localization [7, 31]

Let us describe our major abstract results on natural boundaries motivated by Last–Simon [22] and Remling [36]. Given a power series

$f(z) = \sum_{n=0}^{\infty} a_n z^n$ with

$$\sup_n |a_n| < \infty \tag{1.4}$$

we define a right limit of $\{a_n\}_{n=0}^{\infty}$ to be a two-sided sequence $\{b_n\}_{n=-\infty}^{\infty}$ with

$$b_n = \lim_{j \to \infty} a_{n+n_j} \tag{1.5}$$

for some $n_j \to \infty$. By compactness and (1.4), right limits exist.

Given a two-sided bounded sequence, $\{b_n\}_{n=-\infty}^{\infty}$, we consider two functions, $f_+(z)$ on $\mathbb{D}$ and $f_-(z)$ on $\mathbb{C} \cup \{\infty\} \setminus \overline{\mathbb{D}}$, defined by

$$f_+(z) = \sum_{n=0}^{\infty} b_n z^n \qquad f_-(z) = \sum_{n=-\infty}^{-1} b_n z^n \tag{1.6}$$

where the series are guaranteed to converge on the indicated sets. Let I be an open interval in $\partial \mathbb{D}$. We say $\{b_n\}_{n=-\infty}^{\infty}$ is *reflectionless* on I if and only if f_+ has an analytic continuation from $\mathbb{D}$ to $\mathbb{C} \cup \{\infty\} \setminus (\partial \mathbb{D} \setminus I)$, so that on $\mathbb{C} \cup \{\infty\} \setminus \overline{\mathbb{D}}$, we have that

$$f_+(z) + f_-(z) = 0 \tag{1.7}$$

Obviously, it suffices that f_+ have a continuation to a neighborhood of I so that (1.7) holds in the intersection of that neighborhood and $\mathbb{C} \setminus \overline{\mathbb{D}}$.

Example. Let $b_n \equiv 1$. Then $f_+(z) = (1-z)^{-1}$ and $f_-(z) = -(1-z)^{-1}$. This series is reflectionless on $I = \{e^{i\theta} \mid 0 < \theta < 2\pi\}$. Similarly, it is easy to see that a periodic b_n of period p is reflectionless on any I in $\partial \mathbb{D}$ with all the p-th roots of unity removed.

Our first main theorem is:

Theorem 1.3. *Let $f(z) = \sum_{n=0}^{\infty} a_n z^n$ be a power series with (1.4). Suppose $I \subset \partial \mathbb{D}$ is an open interval so that $f(z)$ has an analytic continuation to a neighborhood of I. Then every right limit of $\{a_n\}_{n=0}^{\infty}$ is reflectionless on I.*

We will actually prove a stronger result (we state Theorem 1.3 both for conceptual reasons and because we need it in the proof of the stronger result):

Theorem 1.4. *Let $f(z) = \sum_{n=0}^{\infty} a_n z^n$ be a power series with (1.4). Suppose $I \subset \partial \mathbb{D}$ is an open interval so that*

$$\sup_{0 < r < 1} \int_{e^{i\theta} \in I} |f(re^{i\theta})| \frac{d\theta}{2\pi} < \infty \tag{1.8}$$

Then every right limit of $\{a_n\}$ is reflectionless on I.

These theorems imply natural boundaries. We say f has a *strong natural boundary* on $\partial \mathbb{D}$ if (1.8) fails for every $I \subset \partial \mathbb{D}$. In particular, in that case, f is unbounded in every sector $\{re^{i\theta} \mid 0 < r < 1,\ e^{i\theta} \in I\}$.

Corollary 1.5. *Let $f(z) = \sum_{n=0}^{\infty} a_n z^n$ be such that for any open interval $I \subset \partial \mathbb{D}$, there is a right limit of $\{a_n\}_{n=0}^{\infty}$ which is not reflectionless on I. Then f has a strong natural boundary on $\partial \mathbb{D}$.*

In some ways, our results are stronger than most classical results but, in some ways, also weaker. They are stronger in that we obtain strong natural boundaries (we note that Boas [3] long ago had results on what we call strong natural boundaries, and earlier, Duffin–Schaeffer [12] and Agmon [1] had results on $f(z)$ being unbounded on sectors). They are weaker in that they have nothing to say, for example, if $a_n \to 0$. (In Section 9, we'll see how to modify the methods to accommodate some cases where $a_n \to 0$.) Of course, if $a_n \to 0$, it can happen that there are natural boundaries which are not strong natural boundaries. For example, by the Hadamard gap theorem [37],

$$f(z) = \sum_{n=1}^{\infty} \frac{z^{n!}}{(n!)^n} \tag{1.9}$$

has a natural boundary on $\partial \mathbb{D}$ but, for all k, $f^{(k)}(z)$ is bounded on $\mathbb{D}$. As with Remling's theory, our results are optimal in the class where (1.4) holds and where the natural boundary comes from leading behavior.

For us, the point of these theorems is conceptual: they provide a unified framework that makes many theorems transparent. That said, the following results seem to be new:

Theorem 1.6. *Suppose $\{a_n\}_{n=0}^{\infty}$ obeys (1.4) and there exists $n_j \to \infty$ so that for all $k < 0$,*

$$\lim_{j \to \infty} a_{n_j + k} = 0 \tag{1.10}$$

$$\liminf |a_{n_j}| > 0 \tag{1.11}$$

Then $f(z) = \sum_{n=0}^{\infty} a_n z^n$ has a strong natural boundary on $\partial \mathbb{D}$.

Remarks. 1. This result, while not stated explicitly in the literature, follows from results of Agmon [1].

2. The result is true if (1.10) is replaced by the assumption for all $k > 0$ or for all $k < K$, some $K < 0$ or for all $k > K$, some $K > 0$. The proofs are essentially identical. In addition, (1.10) can be replaced by an exponential decay condition on the limit; see Theorem 4.3.

3. This includes the famous examples $\sum_{n=1}^{\infty} z^{n!}$ of Weierstrass and $\sum_{n=0}^{\infty} z^{n^2}$ of Kronecker.

4. This allows gaps where the set of zeros has zero density. At first sight, this seems a violation of the result of Pólya [33] and Erdös [14] that the Fabry gap theorem is optimal. We'll explain this apparent discrepancy in Section 4.

Theorem 1.7. *Let $\{a_n(\omega)\}_{\omega \in \Omega}$ be a translation invariant, ergodic, stochastic process, which is nondeterministic, so that*

$$\sup_{n,\omega} |a_n(\omega)| < \infty \tag{1.12}$$

Then for a.e. ω, $\sum_{n=0}^{\infty} a_n(\omega) z^n$ has a strong natural boundary.

Remarks. 1. So far as we know, all previous results on random power series rely on independence and only obtain natural boundaries, not strong natural boundaries.

2. Recall that a stochastic process $\{a_n(\omega)\}$ is deterministic if $a_0(\omega)$ is a measurable function of $\{a_n(\omega)\}_{n=-\infty}^{-1}$ for a.e. ω, and nondeterministic if it is not deterministic.

3. By taking an average of a deterministic and a nondeterministic process, it is easy to see that ergodicity is essential for this theorem to be true.

Theorem 1.8. *Let $f \colon \partial \mathbb{D} \to \mathbb{C}$ be a bounded and piecewise continuous function with only a finite number of discontinuities, at one of which the one-sided limits exist and are unequal. Then for any irrational number q and every $\theta \in \mathbb{R}$, we have that*

$$\sum_{n=0}^{\infty} f\big(e^{2\pi i(qn+\theta)}\big) z^n \tag{1.13}$$

has a strong natural boundary.

Remark. This includes Hecke's famous example [19],

$$f(z) = \sum_{n=0}^{\infty} \{nq\} z^n \tag{1.14}$$

where $\{x\}$ is the fractional part of $x \in \mathbb{R}$.

The following is elementary and does not require our full machinery. It was suggested to us by the Wonderland theorem of Simon [45] and seems to be new.

Theorem 1.9. *Let $\Omega \subset \mathbb{C}$ be a compact set with more than one point. Let Ω^∞ be a countable product of copies of Ω in the weak topology. Then $\{\{a_n\}\} \in \Omega^\infty \mid \sum_{n=0}^{\infty} a_n z^n$ has a natural boundary on $\partial \mathbb{D}\}$ is a dense G_δ in Ω^∞.*

In Section 2, we prove Theorem 1.3. The key is a lemma of M. Riesz whose proof we include because we need a modified version of it in Section 9. In Section 3, we prove Theorem 1.4 using the theory of H^p spaces on a sector (Duren [13]). In Section 4, we discuss gap theorems, including Theorem 1.6. In Section 5, we discuss Szegő's theorem on finite-valued power series using Theorem 1.4. In Section 6, we discuss random power series, including Theorem 1.7. In Section 7, we prove Theorem 1.8, following a spectral theory analysis of Damanik–Killip [11]. In Section 8, we prove Theorem 1.9. In Section 9, we discuss extensions, including some cases where a_n has leading behavior of $O(n^\alpha)$ for some $\alpha \in \mathbb{R}$.

We believe our work here opens up numerous new directions in the study of power series and of spectral theory. In particular, there is a dynamical view of reflectionless in spectral theory (see [4]) and there is the distinction in spectral theory between pure point and singular continuous spectra. What are the analogs for power series?

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2. Classical Natural Boundaries

In this section, we prove Theorem 1.3. The key will be a lemma of M. Riesz [38] used by many other authors in the study of natural boundaries. The use of right limits and reflectionless power series can be viewed as a tool for squeezing maximum benefit from Riesz's lemma.

Given a power series $f(z) = \sum_{n=0}^\infty a_n z^n$, where the a_n obey (1.4), we define

$$f_+^{(N)}(z) = \sum_{n=0}^\infty a_{n+N} z^n \qquad f_-^{(N)}(z) = \sum_{n=-N}^{-1} a_{n+N} z^n \qquad (2.1)$$

so for $z \in \mathbb{D} \setminus \{0\}$,

$$f_+^{(N)}(z) + f_-^{(N)}(z) = z^{-N} f(z) \qquad (2.2)$$

Clearly, f_- is defined and analytic on $\mathbb{C} \setminus \{0\}$, f_+ is defined initially on $\mathbb{D}$, but by (2.2), has analytic continuation to any region that f does.

Theorem 2.1 (M. Riesz's Lemma). *Suppose $\{a_n\}_{n=0}^\infty$ obeys (1.4) and that f has an analytic continuation to a neighborhood of $\mathbb{D} \cup S$ where*

$$S = \{re^{i\theta} \mid 0 < r \le R,\ \alpha \le \theta \le \beta\} \qquad (2.3)$$

for some $R > 1$ and $\alpha < \beta$. Then

$$\sup_{\substack{z \in S \\ N=0,1,2,\dots}} |f_+^{(N)}(z)| < \infty \qquad (2.4)$$

Remark. Riesz's lemma is usually in terms of $z^{-N}(f(z) - \sum_{n=0}^{N-1} a_n z^n)$, but this is exactly $f_+^{(N)}(z)$.

Proof. While the result is classical and appears in many places (e.g., [37]), we sketch the proof since we'll need a modified version of it in Section 9.

By comparing with a geometric series, for $z \in \mathbb{D}$ and all N,

$$|f_+^{(N)}(z)| \leq (1 - |z|)^{-1} \sup_n |a_n| \tag{2.5}$$

and similarly, for $z \in \mathbb{C} \setminus \overline{\mathbb{D}}$,

$$|f_-^{(N)}(z)| \leq (1 - |z|^{-1})^{-1} \sup_n |a_n| \tag{2.6}$$

Let $\tilde{S}$ have the form of S with α, β replaced by $\tilde{\alpha}, \tilde{\beta}$ and $\tilde{\alpha} < \alpha < \beta < \tilde{\beta}$ so that $\tilde{S}$ lies in the neighborhood of analyticity of f. Let $z_1 = e^{i\tilde{\alpha}}$, $z_2 = e^{i\tilde{\beta}}$, and define

$$g_\pm^{(N)}(z) = (z - z_1)(z - z_2) f_\pm^{(N)}(z) \tag{2.7}$$

Clearly, (2.4) is implied by

$$\sup_{\substack{z \in \tilde{S} \\ N=0,1,2,\dots}} |g_+^{(N)}(z)| < \infty \tag{2.8}$$

By the maximum principle, we need only check this on $\partial \tilde{S} \setminus \{z_1, z_2\}$.

Because of (2.5) and the zeros of $g_+^{(N)}$, we have

$$\sup_{\substack{z \in \partial \tilde{S} \cap \mathbb{D} \\ N=0,1,2,\dots}} |g_+^{(N)}(z)| < \infty \tag{2.9}$$

Similarly, by (2.6),

$$\sup_{\substack{z \in \partial \tilde{S} \cap \mathbb{C} \setminus \overline{\mathbb{D}} \\ N=0,1,2,\dots}} |g_-^{(N)}(z)| < \infty \tag{2.10}$$

Since $|z^{-N}| < 1$ on $\mathbb{C} \setminus \overline{\mathbb{D}}$, (2.2) and (2.10) imply

$$\sup_{\substack{z \in \partial \tilde{S} \cap \mathbb{C} \setminus \overline{\mathbb{D}} \\ N=0,1,2,\dots}} |g_+^{(N)}(z)| < \infty \tag{2.11}$$

proving (2.8). $\qquad \square$

Proof of Theorem 1.3. Suppose $a_{N_j+n} \to b_n$ for $n \in \mathbb{Z}$ and let $f_\pm$ be the functions in (1.6). Then, by estimating Taylor series, we have

$$f_+^{(N_j)}(z) \to f_+(z) \qquad f_-^{(N_j)}(z) \to f_-(z) \tag{2.12}$$

uniformly on compact subsets of $\mathbb{D}$ and $\mathbb{C} \cup \{\infty\} \setminus \overline{\mathbb{D}}$, respectively.

If f has an analytic continuation to a neighborhood of $I = \{e^{i\theta} \mid \alpha_0 < \theta < \beta_0\}$, we can apply Riesz's lemma for any S of the form (2.3) with $\alpha_0 < \alpha < \beta < \beta_0$ and some suitable $R > 1$ (depending on α, β). Thus, by the Vitali convergence theorem, $f_+^{(N_j)}$ converges uniformly on S, so $f_+(z)$ has an analytic continuation to S. Moreover, by the analytic continuation of (2.2) to S and $z^{-N_j} \to 0$ on $\mathbb{C} \setminus \mathbb{D}$, we see on $S \setminus \overline{\mathbb{D}}$,

$$f_+(z) + f_-(z) = 0 \tag{2.13}$$

Thus, $\{b_n\}_{n=-\infty}^{\infty}$ is reflectionless across I. $\square$

The proof shows that if $\Omega_I = \mathbb{C} \cup \{\infty\} \setminus (\partial\mathbb{D} \setminus I)$ and if $\mathcal{R}$ is the family of all right limits and $f_b(z)$ the function on Ω_I equal to $\sum_{n=0}^{\infty} b_n z^n$ on $\mathbb{D}$, then

Theorem 2.2. *Fix I an interval in $\partial\mathbb{D}$ and $A \in (0, \infty)$. Let $\mathcal{R}$ be the set of two-sided sequences reflectionless across I with*

$$\sup_{-\infty < n < \infty} |b_n| \le A \tag{2.14}$$

Then, one has that, for any compact $K \subset \Omega_I$,

$$\sup_{b \in \mathcal{R}} \sup_{z \in K} |f_b(z)| < \infty \tag{2.15}$$

Moreover, $\mathcal{R}$ is compact in the topology of uniform convergence on compacts of Ω_I and on

$$\{\{b_n\}_{n=-\infty}^{-1} \mid \{b_n\}_{n=-\infty}^{\infty} \in \mathcal{R}\} \tag{2.16}$$

$b_0, b_1, \dots$ are continuous functions of $\{b_n\}_{n=-\infty}^{-1}$. In fact, for any $k < \ell$, there is a homeomorphism of $\{b_n\}_{n=-\infty}^{k}$ to $\{b_n\}_{n=\ell}^{\infty}$ by associating the two ends of a two-sided sequence.

Proof. If $K \subset \mathbb{D}$ or $K \subset \mathbb{C} \setminus \overline{\mathbb{D}}$, one can use (2.5) or (2.6) for f_b and $\sup_n |b_n|$. For K straddling I, use the argument in the proof of Riesz's lemma and the fact that f_b is $\sum_{n=0}^{\infty} b_n z^n$ on $\mathbb{D}$ and $-\sum_{n=-\infty}^{-1} b_n z^n$ on $\mathbb{C} \setminus \overline{\mathbb{D}}$. This proves (2.15).

Compactness then follows from Montel's theorem and the fact that, by these bounds, the set of reflectionless functions is closed. The continuity is immediate if one notes that $\{b_n\}_{n=-\infty}^{-1}$ determines f near ∞ and f near 0 determines an $\{b_n\}_{n=0}^{\infty}$. The "in fact" statement uses that if $\{b_n\}_{n=0}^{\infty}$ is reflectionless, so are its translates. $\square$

Since Theorem 1.3 says that if some right limit is not reflectionless, then f has a natural boundary, and since reflectionless $\{b_n\}_{n=-\infty}^{\infty}$ have one half determining the other half, we have the following, which implies almost all the natural boundary results of this paper:

Theorem 2.3. *Let $\{a_n\}_{n=0}^{\infty}$ be a bounded sequence with two right limits, $\{b_n\}_{n=-\infty}^{\infty}$ and $\{c_n\}_{n=-\infty}^{\infty}$, that obey*

$$b_0 \neq c_0 \tag{2.17}$$

and either for some $k > 0$ and all $j \geq k$ or for some $k < 0$ and all $j \leq k$,

$$b_j = c_j \tag{2.18}$$

Then $\sum_{n=0}^{\infty} a_n z^n$ has a natural boundary.

Remarks. 1.   For example, if $k > 0$, we look at the right limits $\{b_{n-k}\}_{n=-\infty}^{\infty}$ and $\{c_{n-k}\}_{n=-\infty}^{\infty}$ which have the same f_+'s but unequal f_-'s.

2.   In the next section, we extend this to conclude strong natural boundaries.

3. STRONG NATURAL BOUNDARIES

In this section, we prove Theorem 1.4. We suppose that f obeys (1.8) where $I = (\alpha, \beta)$. Since (1.8) implies $\int_0^1 (\int_\alpha^\beta |f(re^{i\theta})| \frac{d\theta}{2\pi}) \, dr < \infty$, for a.e. θ_0 in (α, β), we have $\int_0^1 |f(re^{i\theta})| \, dr < \infty$. So, for a sequence $\varepsilon_n \downarrow 0$, we have $f \in E(S_n)$, where $S_n = \{re^{i\theta} \mid 0 < r < 1, \, \alpha + \varepsilon_n < \theta < \beta - \varepsilon_n\}$ and $E(S_n)$ is the space introduced in [13, Sect. 10.1]. It follows that:
 (a) $\lim_{r\uparrow 1} f(re^{i\theta}) = f(e^{i\theta})$ exists for a.e. $\theta \in (\alpha, \beta)$ ([13, Thm. 10.3]).
 (b) For all ε, $\int_{\alpha+\varepsilon}^{\beta-\varepsilon} |f(e^{i\theta})| \frac{d\theta}{2\pi} < \infty$ ([13, Thm. 10.3]).
 (c) For every $\varepsilon > 0$,

$$\lim_{r\uparrow 1} \int_{\alpha+\varepsilon}^{\beta-\varepsilon} |f(re^{i\theta}) - f(e^{i\theta})| \frac{d\theta}{2\pi} = 0 \tag{3.1}$$

 (d) If we define

$$F(z) = \int_{\alpha+\varepsilon}^{\beta-\varepsilon} f(e^{i\theta})(e^{i\theta} - z)^{-1} \frac{d\theta}{2\pi i} \tag{3.2}$$

then F is analytic in $\mathbb{C} \setminus \{e^{i\theta} \mid \alpha + \varepsilon < \theta < \beta - \varepsilon\}$ and

$$\lim_{r\uparrow 1} F(re^{i\theta}) - \lim_{r\downarrow 1} F(re^{i\theta}) = f(e^{i\theta}) \tag{3.3}$$

(follows from [13, Thm. 10.4]).
 We also need the following Painlevé-type theorem, which follows easily from Morera's theorem:
 (e) If f_+ is analytic in $\mathbb{D}$, f_- in $\mathbb{C} \setminus \overline{\mathbb{D}}$, and

$$\sup_{0<r<1} \int_\alpha^\beta |f_+(re^{i\theta})| \frac{d\theta}{2\pi} + \sup_{1<r<2} \int_\alpha^\beta |f_-(re^{i\theta})| \frac{d\theta}{2\pi} < \infty \tag{3.4}$$

and if for a.e. $\theta \in (\alpha, \beta)$,

$$f_+(e^{i\theta}) = f_-(e^{i\theta}) \qquad (3.5)$$

then there is G analytic in $\mathbb{C} \setminus [\partial\mathbb{D} \setminus \{e^{i\theta} \mid \alpha < \theta < \beta\}]$ so that $G = f_+$ on $\partial\mathbb{D}$ and f_- on $\mathbb{C} \setminus \overline{\mathbb{D}}$.

Proof of Theorem 1.4. Let F be given by (3.2) and define

$$b_n = \int_{\alpha+\varepsilon}^{\beta-\varepsilon} e^{-in\theta} f(e^{i\theta}) \frac{d\theta}{2\pi} \qquad (3.6)$$

Then, by expanding $(e^{i\theta} - z)^{-1}$ in suitable geometric series, the Taylor expansion of F near zero is $\sum_{n=0}^{\infty} b_n z^n$ and near ∞ is $\sum_{n=-\infty}^{-1} b_n z^n$. Moreover, by the Riemann–Lebesgue lemma and (b) above,

$$\lim_{|n|\to\infty} b_n = 0 \qquad (3.7)$$

Let $c_n = a_n - b_n$ and

$$f_+(z) = \sum_{n=0}^{\infty} c_n z^n \qquad (3.8)$$

Then, by (a), for a.e. $\theta \in (\alpha + \varepsilon, \beta - \varepsilon)$,

$$\lim_{r\uparrow 1} f_+(re^{i\theta}) = f(e^{i\theta}) - \lim_{r\uparrow 1} F(re^{i\theta}) \qquad (3.9)$$

and if

$$f_-(z) = - \sum_{n=-\infty}^{-1} b_n z^n \qquad (3.10)$$

then, for a.e. $\theta \in (\alpha + \varepsilon, \beta - \varepsilon)$,

$$\lim_{r\downarrow 1} f_-(z) = - \lim_{r\downarrow 1} F(re^{i\theta}) \qquad (3.11)$$

It follows that (3.5) holds, so f_+ has a classical analytic continuation across $(\alpha + \varepsilon, \beta - \varepsilon)$. By Theorem 1.3, every right limit of c_n is reflectionless on $(\alpha + \varepsilon, \beta - \varepsilon)$.

But, by (3.7), the right limits of c_n and a_n are the same! Thus, each right limit of a_n is reflectionless on each $(\alpha + \varepsilon, \beta - \varepsilon)$, and so on (α, β). $\qquad \square$

Theorem 3.1. *If the hypotheses of Theorem 2.3 hold, then $\sum_{n=0}^{\infty} a_n z^n$ has a strong natural boundary on $\partial\mathbb{D}$.*

Example 3.2. The Rudin–Shapiro [39, 42] sequence is defined by defining polynomials P_n and Q_n recursively by $P_0(z) = Q_0(z) = 1$, $P_{n+1}(z) = P_n(z) + z^{2^n} Q_n(z)$, $Q_{n+1}(z) = P_n(z) - z^{2^n} Q_n(z)$. As power series with bounded coefficients, $\lim P_n(z)$ exists and defines a series

$f(z) = \sum_{n=0}^{\infty} a_n z^n$, where each a_n is $= 1$ or -1. By using Szegő's theorem, Brillhart [6] proved this function had a natural boundary. Here is an elementary direct proof. P_{n+1} is $P_{n-1}Q_{n-1}P_{n-1}(-Q_{n-1})$. Taking right limits at the end of the P_{n-1} in $P_{n-1}Q_{n-1}$ and in $P_{n-1}(-Q_{n-1})$ yields right limits which agree at negative index but have opposite signs at positive index. □

4. Gap Theorems

In this section, we'll prove Theorem 1.6 and resolve the apparent contradiction to Pólya [33]–Erdös [14]. The following proof shows the power of reflectionless theorems.

Proof of Theorem 1.6. By compactness, we can find a subsubsequence, n_{j_ℓ}, so $\lim_{\ell \to \infty} a_{n_{j_\ell}+k} = b_k$ exists for all k and $b_0 \neq 0$, $b_k = 0$ for $k < 0$. For such a right limit, $f_-(z) = 0$, but $f_+(0) = b_0 \neq 0$. Thus, f_- cannot be an analytic continuation of f_+ through any I, and this $\{b_n\}_{n=-\infty}^{\infty}$ is not reflectionless on any I. Theorem 1.4 completes the proof. □

It is simplest to resolve the apparent contradiction with [33, 14] in the context of an example.

Example 4.1. Let $U = \cup_{j=2}^{\infty}\{n \mid j! \leq n \leq j!+j\}$. We want to consider bounded power series with $a_n = 0$ if $n \in U$. If $a_{j!+j+1} = 1$ for $j \geq 2$ and a_n is arbitrary but bounded for $n \notin U \cup \{j!+j+1\}_{j=2}^{\infty}$, then $f(z)$ has a strong natural boundary on $\partial \mathbb{D}$. The zero values may be only on U, which is a set with zero density.

On the other hand, [33, 14] say that since U does not have density 1, there must be $\{a_n\}_{n=1}^{\infty}$ with $a_n = 0$ for $n \in U$ so that $\partial \mathbb{D}$ is not a natural boundary.

The resolution is that our natural boundary examples have hard edges, that is, a_n jumps at the edges of U, while the examples of [14] have soft edges. $\mathbb{Z}_+ \setminus U$ has longer and longer nonzero intervals and Erdös' examples ramp up slowly and down slowly to be 1 in the center of these intervals. It is easy to see that these examples have right limits which are constant, and so reflectionless!

These examples of Erdös are reminiscent of the sparse potentials of Molchanov [27] and Remling [36], where approximate solitons are placed in between long gaps. In fact, given the chronology, we should say the examples of [27, 36] are reminiscent of Erdös [14]! □

As noted by Agmon [1], it isn't important that $a_{N_j+n} \to 0$ as $j \to 0$ from gaps, only that its $\limsup$ decays exponentially fast. Using right limits, this is easy to see since

Theorem 4.2. *Let $\{b_n\}_{n=-\infty}^{\infty}$ be a two-sided series which is reflectionless on some interval, $I \subset \partial\mathbb{D}$. Suppose that for some $C, D > 0$, we have that*

$$|b_n| \le Ce^{-Dn} \qquad for\ n > 0 \tag{4.1}$$

Then $b_n \equiv 0$.

Proof. By (4.1), $f_+(z)$ has an analytic continuation to the circle $\{z \mid |z| < e^D\}$. Since $f_+ = -f_-$ in a neighborhood of I, we conclude that f_+ defines an entire function. Since $|f_-(z)| \to 0$ as $|z| \to \infty$, $f_+ \equiv 0$, so $f_- \equiv 0$ also, and then $b_n \equiv 0$. $\square$

This immediately implies the following extension of Theorem 1.6:

Theorem 4.3. *Suppose $\{a_n\}_{n=0}^{\infty}$ obeys (1.4) and there exists $n_j \to \infty$ so that for some $C, D > 0$ and for all $k < 0$,*

$$\limsup_{j\to\infty} |a_{n_j+k}| \le Ce^{-D|k|} \tag{4.2}$$

$$\liminf |a_{n_j}| > 0 \tag{4.3}$$

Then $f(z) = \sum_{n=0}^{\infty} a_n z^n$ has a strong natural boundary on $\partial\mathbb{D}$.

Proof. By compactness, there exists a right limit, so (4.1) holds for $n < 0$ and $b_0 \ne 0$. If $\{b_n\}_{n=-\infty}^{\infty}$ is reflectionless on I, then $\{b_{-n}\}_{n=-\infty}^{\infty}$ is reflectionless on $\bar{I} = \{z \mid \bar{z} \in I\}$, so Theorem 4.2 implies $b_n \equiv 0$ if this right limit is reflectionless. But $b_0 \ne 0$. $\square$

5. Szegő's Theorem

In this section, we'll prove

Theorem 5.1. *Let $\sum_{n=0}^{\infty} a_n z^n$ where the values of $\{a_n\}$ lie in a finite set, F. Then either $f(z) = \sum_{n=0}^{\infty} a_n z^n$ has a strong natural boundary or a_n is eventually periodic, in which case f is a rational function with poles at roots of unity.*

Remark. This result for ordinary natural boundaries is due to Szegő [47]. That f is unbounded on any sector is due to Duffin–Schaeffer [12], and that there is a strong natural boundary is a result of Boas [3]. This is an analogy of spectral theory results of Kotani [21] and Remling [36]. One could use an argument of Kotani and our reflectionless machinery to prove Theorem 5.2, but instead we'll borrow part of Boas' argument and note one could use that to find an alternate proof of the spectral theory results.

Proof. Let V be the finite set of possible values of a_n. Suppose $\{a_n\}$ is not eventually periodic. Fix $p = 1, 2, \ldots$ and consider p blocks, $\{a_j\}_{j=\ell p+1}^{(\ell+1)p}$ for $\ell = 0, 1, 2, \ldots$. Since there are only $(\#V)^p$ possible p blocks, some value must recur, that is, there exist $Q_p > P_p$ so that

$$a_{Q_p+j} = a_{P_p+j} \qquad j = 1, \ldots, p \tag{5.1}$$

If (5.1) holds for all $j \geq 1$, then for $k = P_p + j \geq P_p + 1$, we have

$$a_{(Q_p-P_p)+k} = a_k \qquad k = 1, \ldots$$

that is, a is eventually periodic. Since we are assuming the contrary, there is $L_p \geq p + 1$, so $a_{Q_p+L_p} \neq a_{P_p+L_p}$.

   Let $N_p = P_p + L_p \to \infty$ as $p \to \infty$ since $L_p \geq p$ and $M_p = Q_p + L_p > N_p + L$. Then

$$a_{N_p} + j = a_{M_p} + j \qquad j = -p, \ldots, -1 \tag{5.2}$$

$$a_{N_p} \neq a_{M_p} \tag{5.3}$$

By compactness, we get right limits, b, c, obeying $(2.16)/(2.17)$, so by Theorem 3.1, $\sum_{n=0}^{\infty} a_n z^n$ has a strong natural boundary. $\square$

   We note the following extension of Szegő's theorem due to Bieberbach [2] (who only proved classical natural boundary):

Theorem 5.2. *Let $\{a_n\}_{n=0}^{\infty}$ be a bounded sequence with finitely many limit points. Then either $\sum_{n=0}^{\infty} a_n z^n$ has a strong natural boundary or there is a periodic sequence, c_n, with*

$$|a_n - c_n| \to 0 \qquad as\ n \to \infty \tag{5.4}$$

Proof. Let V be the finite set of limit points. Let $\gamma = \frac{1}{2}\min_{x,y \in V,\, x \neq y}|x - y|$. Eventually, for all n, there is $c_n \in V$ with $|a_n - c_n| \leq \gamma$. It follows that (5.4) holds. If c_n is eventually periodic, it can be modified for small n to be periodic and (5.4) still holds.

   If c_n is not eventually periodic, by the above, it has a right limit which is not reflectionless. But a_n and c_n have the same right limits. $\square$

6. RANDOM POWER SERIES

   In this section, we'll prove Theorem 1.7 as well as

Theorem 6.1. *Let $\{a_n(\omega)\}_{n=0}^{\infty}$ be a sequence of independent random variables so that*
   (i) $\sup_{n,\omega}|a_n(\omega)| = K < \infty$
   (ii) *For some sequence $n_j \to \infty$,*

$$\limsup_{j \to \infty} \left[\mathbb{E}(|a_{n_j}(\omega)|^2) - |\mathbb{E}(a_{n_j}(\omega))|^2\right] > 0 \tag{6.1}$$

   Then for a.e. ω, $\sum_{n=0}^{\infty} a_n(\omega)z^n$ has a strong natural boundary.

Remark. $\mathbb{E}$ is expectation. We'll use $\mathbb{P}$ for probability and $\mathbb{V}$ for variation, so (6.1) is $\limsup_{j\to\infty} \mathbb{V}(a_{n_j}(\omega)) > 0$.

Lemma 6.2. (i) *For any K and m, there exists $K_m > 0$ so that for any random variable f with $\|f\|_\infty \leq K$, there exists $z \in \mathbb{C}$ so that*

$$\mathbb{P}\left(|f(\omega) - z| \leq \frac{1}{m}\right) \geq K_m \tag{6.2}$$

(ii) *For any K, m, and $\sigma > 0$, there exists $\widetilde{K}_m > 0$ so that for any random variable f with $\|f\|_\infty \leq K$ and $\mathbb{V}(f) \geq \sigma$, there exists $z, w \in \mathbb{C}$ with*

$$|z - w| \geq \left(\frac{\sigma}{2}\right)^{1/2} \tag{6.3}$$

so that

$$\mathbb{P}\left(|f(\omega) - z| \leq \frac{1}{m}\right) \geq \widetilde{K}_m \qquad \mathbb{P}\left(|f(\omega) - w| \leq \frac{1}{m}\right) \geq \widetilde{K}_m \tag{6.4}$$

Proof. (i) Cover $\{z \mid |z| \leq K\}$ by disks of radius $\frac{1}{m}$. By compactness (or simple geometry), one can arrange for a finite number N_m. One of these disks must have probability at least $K_m = (N_m)^{-1}$. (6.2) holds for the center of that disk.

(ii) For any $c > 0$ and bounded random variable g,

$$\mathbb{E}(|g|^2) \leq c^2 \mathbb{P}(|g| \leq c) + \|g\|_\infty^2 \mathbb{P}(|g| \geq c) \tag{6.5}$$

Thus, for any α with $|\alpha| \leq \|f\|_\infty$,

$$\mathrm{Var}(f) \leq c^2 + (2\|f\|_\infty)^2 \mathbb{P}(|f - \alpha| > c) \tag{6.6}$$

Picking $c \leq (\frac{\sigma}{2})^{1/2}$ with $\sigma = \mathrm{Var}(f)$, we see that

$$\mathbb{P}(|f - \alpha| > c) \geq \frac{1}{8\|f\|_\infty^2} \mathrm{Var}(f) \tag{6.7}$$

Use (i) to find z so $\mathbb{P}(|f(\omega) - z| \leq \frac{1}{m}) \geq K_m$ where, if necessary, m is increased so $\frac{1}{m} \leq (\frac{\sigma}{2})^{1/2}$. By repeating (i) using $\mathbb{P}(|f - z| \geq (\frac{\sigma}{2})^{1/2}) \geq \frac{1}{8\|f\|_\infty}\sigma$, we get a w outside the disk $\{\zeta \mid |\zeta - z| \leq (\frac{\sigma}{2})^{1/2}\}$ so that $\mathbb{P}(|f(\omega) - w| \leq \frac{1}{m}) \geq \widetilde{K}_m$ for some $\widetilde{K}_m \leq K_m$. $\square$

Proof of Theorem 6.1. Pick $\sigma > 0$ and $n_j \to \infty$ so that $\mathbb{V}(a_{n_j}(\omega)) \geq \sigma$ for all j and so that $n_{j+1} \geq n_j$ and $n_{j+1} - n_j \to \infty$. For each $n \neq n_j$, for all j, use (i) of Lemma 6.2 to pick $z_n^{(m)}$ so that

$$\mathbb{P}\left(|a_n(\omega) - z_n^{(m)}| \leq \frac{1}{m}\right) \geq K_m \tag{6.8}$$

By compactness and using the diagonalization trick, one can pass to a subsequence of the n_j, which we'll still denote by n_j, so for all m and all $k \neq 0$, $z_{n_j+k}^{(m)} \to z_k$ as $j \to \infty$.

By using (ii) of Lemma 6.2, find $w_j^{(m)}, \zeta_j^{(m)}$ so that

$$
\begin{aligned}
\mathbb{P}\left(|a_{n_j}(\omega) - w_j^{(m)}| \leq \frac{1}{m}\right) \geq \widetilde{K}_m \\
\mathbb{P}\left(|a_{n_j}(\omega) - \zeta_j^{(m)}| \leq \frac{1}{m}\right) \geq \widetilde{K}_m
\end{aligned}
\tag{6.9}
$$

and $|w_j^{(m)} - \zeta_j^{(m)}| \geq (\frac{\sigma}{2})^{1/2}$. Again, by passing to a subsequence, we may assume that $w_j^{(m)} \to w$, $\zeta_j^{(m)} \to \zeta$ where, of course, $|\zeta - w| \geq (\frac{\sigma}{2})^{1/2}$.

By the Borel–Cantelli lemma, for each m and each Q,

$$
|a_{n_j}(\omega) - w_j^{(m)}| \leq \frac{1}{m} \qquad |a_{n_j+k}(\omega) - z_{n_j+k}^{(m)}| \leq \frac{1}{m} \tag{6.10}
$$

for all k with $0 < |k| \leq Q$, occurs infinitely often for a.e. ω, and the same for $\zeta_j^{(m)}$. Thus, for a.e. ω, the right limits include b, c with

$$
b_k = c_k = z_k \text{ for } k \neq 0 \qquad b_0 = w \qquad c_0 = \zeta \neq w \tag{6.11}
$$

It follows from Theorem 3.1 that $a_n(\omega)$ has a strong natural boundary for all ω. $\qquad\square$

Remark. The use of the Borel–Cantelli lemma to get nonreflectionless limits (in the context of spectral theory of CMV matrices) is taken from work of Breuer–Ryckman–Zinchenko [5].

For our proof of Theorem 1.7, we need two lemmas whose proof we defer to the end of the section.

Lemma 6.3. *Let $\alpha < \beta$ in $(-\pi, 2\pi)$ with $|\beta - \alpha| < 2\pi$. Then $\{\{a_n\}_{n=0}^\infty \mid \sup_n|a_n| \leq A$ for all $\varepsilon > 0$; $\sup_{0<r<1}(\int_{\alpha+\varepsilon}^{\beta-\varepsilon}|\sum_{n=0}^\infty a_n r^n e^{in\theta}|\frac{d\theta}{2\pi}) < \infty\}$ is a measurable set (in the product topology) invariant under $a_n \to a_{n+1}$.*

Lemma 6.4. *Let μ be a probability measure on $\{\{a_n\}_{n=-\infty}^\infty \mid \sup_n|a_n| \leq A\}$ that defines an ergodic invariant stochastic process. Let $\mathcal{R}_\mu$ be the set of two-sided series $\{b_n\}_{n=-\infty}^\infty$ so that $\{b_n\}_{n=-\infty}^\infty$ is a right limit of $\{a_n\}_{n=0}^\infty$ with positive probability. Then $\mathcal{R}_\mu$ is the support of the measure μ.*

Proof of Theorem 1.7. By Lemma 6.3, Theorem 1.4, and ergodicity (letting α, β run through rational multiples of 2π), either $\sum_{n=0}^\infty a_n z^n$ has a strong natural boundary with probability 1 or there is an interval, I, in $\partial\mathbb{D}$ so that with probability 1, all right limits of $\sum_{n=0}^\infty a_n z^n$

are reflectionless across I. In that case, by Lemma 6.4, all $\{b_n\}_{n=-\infty}^\infty$ in the support of μ are reflectionless across I. By Theorem 2.2, on $\text{supp}(\mu)$, b_0 is a continuous, and so measurable, function of $\{b_n\}_{n=-\infty}^{-1}$. Thus, the process is deterministic (in the strong sense of there being a continuous, rather than merely a measurable function). $\square$

Proof of Lemma 6.3. By Theorem 10.1 of Duren [13], the space E (sector) can be defined by a countable family of approximating curves. Thus, letting $z_1^{(n)} = \exp(i(\alpha + \frac{1}{n}))$, $z_2^{(n)} = \exp(i(\beta - \frac{1}{n}))$, the sup condition can be replaced by

$$\sup_{m,n} \left(\int_{\alpha + \frac{1}{n}}^{\beta - \frac{1}{n}} \left| (z - z_1^{(n)})(z - z_2^{(n)}) \sum_{n=0}^\infty a_n z^n \right| \Big|_{z = (1 - \frac{1}{m})e^{i\theta}} \frac{d\theta}{2\pi} \right) < \infty$$

$$(6.12)$$

Since it is described by a countable sup of uniformly convergent sums, this set is clearly measurable.

If $f(z) = \sum_{n=0}^\infty a_n z^n$ and $\tilde{f}(z) = \sum_{n=0}^\infty a_{n+1} z^n$, then

$$\tilde{f}(z) = z^{-1}(f(z) - a_0 z) \qquad (6.13)$$

and finiteness of the sup in (6.12) implies finiteness of the sup for $\tilde{f}$ replacing f. This proves the claimed invariance. $\square$

Proof of Lemma 6.4. If b_n is in $\text{supp}(d\mu)$, then for any ε, N, $\mu(\{a_n \mid |a_{N+n} - b_n| < \varepsilon, \ n = 0, \pm 1, \ldots, \pm N\}) > 0$. By the ergodic theorem for a.e. ω,

$$\lim_{M \to \infty} \frac{1}{M} \#\{j < M \mid |a_{n+j+N} - b_j| < \varepsilon, \ n = 0, \pm 1, \ldots, \pm N\} > 0$$

$$(6.14)$$

so for a.e. ω, there exists $N_\ell \to \infty$ with $|a_{n+N_\ell} - b_n| < \varepsilon$ for all n with $|n| < N$. By a diagonalization trick, b_n is a right limit for a.e. ω.

If b_n is not in the support of μ, pick N and ε so that $\mu(\{a_n \mid |a_{N+n} - b_n| \leq \varepsilon, \ |n| \leq N\}) = 0$. By translation invariance for all k,

$$\mu(\{a_n \mid |a_{N+k+n} - b_n| \leq \varepsilon, \ n \leq N\}) = 0 \qquad (6.15)$$

which implies b_n is not a right limit with probability 1. $\square$

7. HECKE'S EXAMPLE

Motivated by a spectral theory result of Damanik–Killip [11], we prove the following that includes Theorem 1.8 and so, Hecke's example.

Theorem 7.1. *Let $T: \partial\mathbb{D} \to \partial\mathbb{D}$ be a homeomorphism so that for any $e^{i\theta} \in \partial\mathbb{D}$, $\{T^k(e^{i\theta}) \mid k = 0, 1, \ldots\}$ is dense in $\partial\mathbb{D}$. Let $f: \partial\mathbb{D} \to \mathbb{C}$ be a*

bounded and piecewise continuous function with a finite number of discontinuities so that at, at least, one discontinuity, the right and left limits exist and are not equal. Then for any $e^{i\theta} \in \partial\mathbb{D}$, $\sum_{n=0}^{\infty} f(T^n(e^{i\theta}))z^n$ has a strong natural boundary on $\partial\mathbb{D}$.

Remark. For other papers on other extensions of Hecke's example, see [40, 29, 28, 41, 26, 34, 8].

Proof. By rotating, we suppose $e^{i\theta} = 1$ is a point of a discontinuity with $\lim_{\theta\downarrow 0} f(e^{i\theta}) = r \neq s = \lim_{\theta\uparrow 0} f(e^{i\theta})$. By the density of any orbit for any $e^{i\theta}$, we can find $n_j \to \infty$, so $T^{n_j}(e^{i\theta}) \to 1$ with $T^{n_j}(e^{i\theta}) = e^{i\psi_j}$ with $-\pi < \psi_j < 0$ and $m_j \to \infty$, so $T^{m_j}(e^{i\theta}) \to 1$ and $T^{m_j}(e^{i\theta}) = e^{i\eta_j}$ with $\pi > \eta_j > 0$ (for find $n_{j+1} > n_j$ so $|T^{n_{j+1}}(e^{i\theta}) - e^{-i/(j+1)}| \leq (j+1)^{-2}$).

Thus, $a_{n_j} \to s$ and $a_{m_j} \to r$. On the other hand, look at the orbit $\{T^\ell(1) \mid \ell = 1, 2, \dots\}$. Since this orbit is dense, the $T^\ell(1)$ must be distinct, so for some L and all $\ell > L$, $T^\ell(1)$ must be a point of continuity. Thus, for $\ell > L$, $a_{n_j+\ell} - a_{m_j+\ell} \to 0$. The hypotheses of Theorem 2.3 hold for the right limits defined by a_{n_j+n} and a_{m_j+n}. So, by Theorem 3.1, we have a strong natural boundary. □

8. Baire Genericity

Proof of Theorem 1.9. Let $\{a_n\}_{n=0}^\infty \in \Omega^\infty$. Pick two distinct points, $z_0, z_1 \in \Omega$. For any ℓ, define

$$a_n^{(\ell)} = \begin{cases} a_n & n \leq \ell \\ z_0 & n = \ell + k! \text{ for } k = 1, 2, \dots \\ z_1 & n = \ell + m, \ m \geq 1, \ m \neq k! \text{ for any } k \end{cases}$$

Then, by the gap theorem, $\{a_n^{(\ell)}\}$ has a natural boundary on $\partial\mathbb{D}$, indeed, by Weierstrass' original direct arguments. But $\lim_{\ell\to\infty} a_n^{(\ell)} = a_n$ for each fixed n, so the set in the theorem is dense in the weak topology.

For every rational multiple α, β of 2π with $\alpha < \beta$, and every $K = 1, 2, \dots$, and any $n = 1, \dots$, let $F_{\alpha,\beta,K,n} = \{\{a_n\}_{n=1}^\infty \in \Omega^\infty \mid f(z) = \sum_{n=0}^\infty a_n z^n$ has an analytic continuation to $\{z \mid |z| < 1 + n^{-1}, \alpha < \arg(z) < \beta\}$ with $|f(z)| \leq K$ there$\}$. It is easy to see that

(i) Each $F_{\alpha,\beta,K,n}$ is closed in the topology of pointwise convergence of the a_n's (by the Vitali theorem).

(ii) $\cup F_{\alpha,\beta,K,n}$ is the set of power series for which $\partial\mathbb{D}$ is not a natural boundary.

Thus, the complement of the set in the theorem is an F_δ, so the set is a G_δ. □

9. Extensions

Thus far, our results were limited to bounded a_n with $\liminf |a_n| \neq 0$. Here we want to show, still within the class of power series whose disk of convergence is $\mathbb{D}$, that one can handle some cases where one or the other of these conditions fails.

Theorem 9.1. *Let $\{a_n\}_{n=0}^\infty$ be Taylor coefficients of a series so that $f(z) = \sum_{n=0}^\infty a_n z^n$ has an analytic continuation through some interval I in $\partial \mathbb{D}$. Suppose that for some $\alpha \in \mathbb{R}$, we have*

(i)
$$|a_n| \leq C(n+1)^\alpha \qquad \text{for all } n \tag{9.1}$$

(ii) *For some two-sided sequence $\{c_n\}_{n=-\infty}^\infty$, we have for some $n_j \to \infty$,*
$$c_m = \lim_{j \to \infty} \frac{a_{n_j + m}}{n_j^\alpha} \tag{9.2}$$

exists (and is finite). Then $\{c_m\}_{m=-\infty}^\infty$ is reflectionless on I.

Remark. We note in (9.2) that the limit is the same if n_j^α is replaced by $(n_j + m)^\alpha$.

As usual, this implies

Corollary 9.2. *Let $\{a_n\}_{n=0}^\infty$ obey (9.1) for some $\alpha \in \mathbb{R}$ and let $\{c_m\}_{n=-\infty}^\infty$ obey (9.2). If $\{c_m\}_{m=-\infty}^\infty$ is not reflectionless on any interval in $\partial \mathbb{D}$, then $\sum_{n=0}^\infty a_n z^n$ has a natural boundary on $\partial \mathbb{D}$.*

Remark. If $\alpha < 0$, it can happen that f does not have a strong natural boundary (e.g., if a_n is summable, which always happens if $\alpha < -1$, but can happen if a_n is sparse and $-1 \leq \alpha < 0$, then f has a continuation to $\overline{\mathbb{D}}$), but using our methods, it is easy to prove a strong natural boundary for $\alpha > 0$, and that if $\alpha \geq -k$, then $f^{(k)}$ has a strong natural boundary.

As a warmup, we note that Theorem 9.1 is especially simple if α is an integer. If $\alpha = -k$, with $k > 0$ and integral, then $f^{(k)}(z) = \sum_{n=0}^\infty a_{n+k}(n+k)\dots(n+1)z^n$ has c_m as a right limit, and if f has an analytic continuation past I, so does $f^{(k)}$. Thus, Theorem 1.3, applied to $f^{(k)}$, implies this case of Theorem 9.1.

Similarly, if $\mathcal{I}(f)(z)$ is defined by
$$\mathcal{I}(f)(z) = \int_0^1 f(tz)\, dt$$
then
$$\mathcal{I}(f)(z) = \sum_{n=0}^\infty (n+1)^{-1} a_n z^n$$

So a similar argument applied to $\mathcal{I}^k(f)$ shows that if $\alpha = k$ with $k > 0$ and integral, then c_m is a right limit of $\mathcal{I}^k(f)$ and Theorem 1.3 for $\mathcal{I}^k(f)$ implies this special case of Theorem 9.1.

We turn now to the general case of Theorem 9.1. Fix $\alpha \in \mathbb{R}$. For $N = 1, 2, \ldots$, define

$$f_+^{(N,\alpha)} = N^{-\alpha} \sum_{n=0}^{\infty} a_{n+N} z^n \qquad f_-^{(N,\alpha)} = N^{-\alpha} \sum_{n=-N}^{-1} a_{n+N} z^n$$

Notice that for $z \in \mathbb{C} \setminus \{0\}$, we have

$$f_+^{(N,\alpha)}(z) + f_-^{(N,\alpha)}(z) = N^{-\alpha} z^{-N} f(z) \tag{9.3}$$

Lemma 9.3. *Let $\alpha > 0$. Suppose $\{a_n\}_{n=}^{\infty}$ obeys (9.1) and f has an analytic continuation to a neighborhood of the set $\mathbb{D} \cup S$ with S given by (2.3). Then (2.4) holds with $f_\pm^{(N)}$ replaced by $f_\pm^{(N,\alpha)}$.*

Proof. Since $\alpha > 0$, $(N + n)^\alpha \leq [2\max(N,n)]^\alpha \leq 2^\alpha(N^\alpha + n^\alpha)$. By (9.1), for $n \geq 0$,

$$|N^{-\alpha} a_{n+N}| \leq CN^{-\alpha}(2^\alpha(N^\alpha + n^\alpha))$$
$$\leq 2^\alpha C(1 + n^\alpha) \tag{9.4}$$

For $-N \leq n < 0$, since $\alpha > 0$,

$$|N^{-\alpha} a_{N+n}| \leq CN^{-\alpha}(N + n)^\alpha$$
$$\leq C \tag{9.5}$$

Next, note that if $0 < \alpha \leq k$, an integer, and $0 < r < 1$,

$$\sum_{n=1}^{\infty} n^\alpha r^n \leq \sum_{n=1}^{\infty} n^k r^n = \left(r \frac{d}{dr}\right)^k (1-r)^{-1}$$
$$\leq C_k (1-r)^{-1-k} \tag{9.6}$$

for some constant C_k.

By (9.4) and (9.6), for a constant, D_α, depending on α and C, for all N and $|z| < 1$,

$$|f_+^{(N,\alpha)}(z)| \leq D_k (1 - |z|)^{-1-k} \tag{9.7}$$

By (9.5) and the argument that led to (2.6),

$$|f_-^{(N,\alpha)}(z)| \leq C(1 - |z|^{-1})^{-1} \tag{9.8}$$

Now just follow the argument leading to Theorem 2.1, but with $g_\pm^{(N)}$ replaced by

$$g_\pm^{(N,\alpha)}(z) = (z - z_1)^{k+1}(z - z_2)^{k+1} f_\pm^{(N,\alpha)}(z) \tag{9.9}$$

$\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

Proof of Theorem 9.1. Suppose first that $\alpha > 0$. By (9.2), for $z \in \mathbb{D}$, uniformly on compacts, $f_+^{(N,\alpha)}(z) \to \sum_{m=0}^{\infty} c_m z^m$. Similarly, for $z \in \mathbb{C} \setminus \overline{\mathbb{D}}$, $f_-^{(N,\alpha)}(z) \to \sum_{m=-\infty}^{-1} c_m z^m$. Now just follow the argument used in Section 2 to prove Theorem 1.3.

If $\alpha < 0$, pick $k \in \{1, 2, \dots\}$ so $\alpha > -k$. Then

$$f^{(k)}(z) = \sum_{n=0}^{\infty} a_{n+k}(n+k)\dots(n+1)z^n$$

If $d_n = a_{n+k}(n+k)\dots(n+1)$, then d_n obeys the hypotheses of Theorem 9.1 with $\tilde{\alpha} = k + \alpha > 0$ and c_m is the same. The result for $\alpha > 0$ implies the one for $\alpha < 0$. $\qquad\square$

Agmon [1], using in part ideas of Levinson [24], allows more growth than polynomials. It is possible that such extensions can be obtained using our methods by relying on functions with infinite-order zeros at z_1 and z_2 but which are still bounded in the sector S.

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